

Gauss map of rank zero

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$X \subset \mathbb{P}^N$: a proj variety of dim n over $k = \bar{k}$ with $p = \text{char}(k) > 0$.

Def (Gauss map). For an **embedding** $\iota : X \hookrightarrow \mathbb{P}^M$,

$$\gamma = \gamma_\iota : X \dashrightarrow \mathbb{G}(n, \mathbb{P}^M) \quad (\text{Gauss map of } \iota)$$

is given by sending $x \in X$ to the embedded tangent $T_x X$ in $\mathbb{P}^M$.

(We treat γ only for the case when $\iota(X)$ is non-linear in $\mathbb{P}^M$.)

Define **rk** γ (rank of the Gauss map) to be $\text{rk}(d_x \gamma : t_x X \rightarrow t_{\gamma(x)} \mathbb{G}(n, \mathbb{P}^M))$ with gen $x \in X$.

• $p = 0 \Rightarrow \gamma$ is birational (thus, $\text{rk } \gamma = n$) for any embedding ι if X is smooth.

In $p > 0$, birationality of γ is not guaranteed, and **rk** $\gamma < n$ is possible.

We study the extreme case:

Def (GMRZ). We say that a proj variety X satisfies **(GMRZ)** if

$$\exists \iota : X \hookrightarrow \mathbb{P}^M \text{ s.t. } \mathbf{rk} \gamma_\iota = 0 \text{ (Gauss map of } \iota \text{ is of rank zero).}$$

Example (Fermat hypersurf). Let $X = (x_0^{p+1} + \cdots + x_N^{p+1} = 0) \subset \mathbb{P}^N$ ($p > 0$).

Then $\gamma : \mathbb{P}^N \rightarrow \widetilde{\mathbb{P}}^N : (x_0, \dots, x_N) \mapsto (x_0^p, \dots, x_N^p)$ (inseparable & **rk** $\gamma = 0$).

Approach. Suppose that X has a **rational curve** param by $f : \mathbb{P}^1 \rightarrow X$.

If f is unramified, then the normal bdl N_f is given by $(\ker(f^* \Omega_X^1 \rightarrow \Omega_{\mathbb{P}^1}^1))^\vee$.

If X satisfies (GMRZ), then N_f has the following significant property:

Theorem 1 (Splitting type of the normal bundle).

X : proj variety, $f : \mathbb{P}^1 \rightarrow X$: unramified. Assume that X is smooth along $f(\mathbb{P}^1)$, and $N_f^\vee \simeq \bigoplus_{i \geq -1} \mathcal{O}_{\mathbb{P}^1}(i)^{r_i}$ for some $r_i \in \mathbb{Z}_{\geq 0}$ ($i \geq -1$). Then,

X satisfies (GMRZ) $\Rightarrow r_{i-1} r_i = 0$ for any $i \geq 0$.

Moreover, $r_i > 0$ for some $i \geq 0 \Rightarrow p = 2$ or $p | i + 1$.

Theorem 2. (1) $\prod_{1 \leq i \leq r} \mathbb{P}^{n_i}$ satisfies (GMRZ) $\Leftrightarrow p = 2$ & $n_i = 1$ ($\forall i$).

(2) $\mathbb{G}(l, \mathbb{P}^m)$ satisfies (GMRZ) $\Leftrightarrow l = 0$ or $l = m - 1$.

(3) A smooth **quad** hypersurf in $\mathbb{P}^N$ ($N \geq 3$) satisfies (GMRZ)

$$\Leftrightarrow p = 2 \text{ \& } N = 3.$$

(4) A smooth **cubic** hypersurf in $\mathbb{P}^N$ ($N \geq 3$) satisfies (GMRZ) $\Rightarrow p = 2$.

Theorem 3 (Cubic hypersurf). A smooth **cubic** hypersurf $X \subset \mathbb{P}^N$ with $N \geq 4$ satisfies **(GMRZ)**
 $\Leftrightarrow p = 2$ and X is proj equivalent to a **Fermat** hypersurf.

Theorem 4. $X \subset \mathbb{P}^N$: a smooth **cubic** hypersurf with $N \geq 3$ in $p = 2$. Denote by $\gamma_0 : X \rightarrow \widetilde{\mathbb{P}}^N$ the Gauss map of the **original** embedding $X \subset \mathbb{P}^N$. Then, **rk** $\gamma_0 = 0 \Leftrightarrow X$ is proj equivalent to a **Fermat** hypersurf.

Rem. In order to prove Theorem 3, it is sufficient to show that “ $\exists \iota : X \hookrightarrow \mathbb{P}^M$ s.t. **rk** $\gamma_\iota = 0 \Rightarrow \mathbf{rk} \gamma_0 = 0$ ” due to Theorem 4. The case $N \geq 5$ was shown in [1], and the case $N = 4$ was recently shown in [2].

Theorem 5 (General hypersurf).

A **general** hypersurf in $\mathbb{P}^N$ of deg d with $3 \leq d \leq 2N - 3$ satisfies **(GMRZ)** $\Rightarrow p = 2$ and $d = 2N - 3$.

Rem. We have $N_{L/X}^\vee \simeq \mathcal{O}^{2N-3-d} \oplus \mathcal{O}(1)^{d-N+1}$ or $\mathcal{O}(-1)^{N-1-d} \oplus \mathcal{O}^{d-1}$ for a general **line** $L \subset X$ ([3, V, (4.4.2)]). Thus Thm 1 implies $d = 2N - 3$ or $d = N - 1$. By considering **conics** on X , we can show that $d = 2N - 3$.

Application (Absence of minimal free rational curves). $f : \mathbb{P}^1 \rightarrow X$ is said to be **free** if $f^* T_X$ is generated by its global sections, and a free f **minimal** if $f^* T_X \simeq \mathcal{O}(2) \oplus \mathcal{O}(1)^{d-2} \oplus \mathcal{O}^{n-d+1}$ with $d = \deg(-f^* K_X)$.

Theorem A [3, IV.2.10]. X : a smooth proj variety in $p = 0$. Then,
 $\exists$ a **free** rational curve on $X \Rightarrow \exists$ a **minimal free** rational curve on X .

On the other hand, by using Theorem 1, we have:

Theorem 6. Assume $p > 0$, and let $X \subset \mathbb{P}^N$ be a **Fermat** hypersurf of deg $ep + 1$ ($e \in \mathbb{N}$). Then, (1) $N \geq 2ep + 1 \Rightarrow X$ has a **free** rat curve.
(2) $N > e(p + 1) \Rightarrow X$ has **no minimal free** rat curve.

Thus a Fermat hypersurf in $\mathbb{P}^N$ of deg $ep + 1$ with $N \geq 2ep + 1$ gives a **counter-example** for Theorem A in **each** $p > 0$.

References. [1] S. Fukasawa, K. Furukawa, H. Kaji, Projective varieties admitting an embedding with Gauss map of rank zero, Adv. Math. **224** (2010) 2645–2661.

[2] K. Furukawa, preprint. [3] J. Kollár, Rational curves on algebraic varieties, Ergebnisse Der Mathematik Und Ihrer Grenzgebiete, 3. Folge, Bd. **32**, Springer, Berlin, 1996.