

A decision method for a set of
first order classical formulas and its application
to decision problems for non-classical propositional logics.

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I. Main Theorem

Let LN be the first order classical predicate logic without equality which has a fixed binary predicate symbol R , unary predicate symbols $P_1, \dots, P_N$ and no other non-logical constant symbols. Suppose that X is a set of sentences in LN. Then a decision method for X is a method by which, given a sentence A in X , we can decide in a finite number of steps whether or not it has a model. X is said to be decidable if there is a decision method for X . It is well-known that the set of all the R -free sentences (sentences in LN which have no occurrences of R) is decidable, but the set of all the sentences in LN is not. R -formulas are formulas belonging to the least set ^(X) such that: (i) ^{All} R -free formulas belong to X , (ii) X is closed under $\neg, \wedge, \vee, \supset$, (iii) If $A(x)$ belongs to X , then $\exists v A(v), \exists v (R(x, v) \wedge A(v)), \exists v (R(v, x) \wedge A(v))$ belong to X . R -positive formulas are formulas which have no negative occurrences of R . Also, Tr is the sentence $\forall u \forall v \forall w (R(u, v) \wedge R(v, w) \supset R(u, w))$ and Sym is the sentence $\forall u \forall v (R(u, v) \supset R(v, u))$. Let FN be the set of finite conjunctions of sentences: R -sentences, R -positive

sentences, Tr and Sym. Then, our main theorem is:

MAIN THEOREM. FN is decidable.

In fact, we show that for each sentence A in FN, we can calculate a natural number $n(A)$. This fact clearly implies our main theorem.

(such that if A has a model, then A has a model whose cardinality is at most $n(A)$)

II. Applications.

Suppose that L is a formal logic. Then a decision method for L is a method by which, given a formula of L , we can decide in a finite number of steps whether or not it is provable in L .

1) Intuitionistic propositional logic.

Let IPL be the intuitionistic propositional logic whose propositional variables are $p_1, \dots, p_N$. For each formula A in IPL, and each free variable x in LN, let (A, x) be the formula in LN defined by:

(p_i, x) is $P_i(x)$, $(\neg A, x)$ is $\forall v(R(x, v) \supset \neg (A, v))$, $(A \wedge B, x)$ is $(A, x) \wedge (B, x)$, $(A \vee B, x)$ is $(A, x) \vee (B, x)$, and, $(A \supset B, x)$ is $\forall v(R(x, v) \supset ((A, v) \supset (B, v)))$.

Then, ^{by} Kripke's completeness theorem for IPL, we have:

Completeness Theorem for IPL. For each formula A in IPL, A is provable in IPL iff the sentence: $\text{Tr} \bigwedge_{i=1}^N \text{Tr}(P_i) \wedge \exists v \neg (A, v)$ has no models, where $\text{Tr}(P_i)$ is the R-sentence $\forall u(P_i(u) \supset \forall v(R(u, v) \supset P_i(v)))$.

Since $\text{Tr} \wedge \bigwedge_{i=1}^N \text{Tr}(P_i) \wedge \exists v \neg(A, v)$ belongs to FN, our main theorem clearly implies that the logic IPL is decidable.

2) Modal propositional logics.

Let MPL be the modal propositional language whose logical constants are $\neg, \wedge, \vee, \supset$ and $\Box$, and whose propositional variables are $p_1, \dots, p_N$. For each formula A in MPL and each free variable x in LN, let $\langle A, x \rangle$ be the formula in LN defined by: $\langle p_i, x \rangle$ is $P_i(x)$, $\langle \neg A, x \rangle$ is $\neg \langle A, x \rangle$, $\langle A \wedge B, x \rangle$ is $\langle A, x \rangle \wedge \langle B, x \rangle$, $\langle A \vee B, x \rangle$ is $\langle A, x \rangle \vee \langle B, x \rangle$, $\langle A \supset B, x \rangle$ is $\langle A, x \rangle \supset \langle B, x \rangle$ and $\langle \Box A, x \rangle$ is $\forall v (R(x, v) \supset \langle A, v \rangle)$.

Let M, S4, B, S5 be modal propositional logics in Kripke [], whose language is MPL. Then, by Kripke's completeness theorem for modal logics, we have:

Completeness Theorem for modal logics. For any formula A in MPL,

- (i) A is provable in M iff $\forall u R(u, u) \wedge \exists v \neg \langle A, v \rangle$ has no models,
- (ii) A is provable in S4 iff $\forall u R(u, u) \wedge \text{Tr} \wedge \exists v \neg \langle A, v \rangle$ has no models,
- (iii) A is provable in B iff $\forall u R(u, u) \wedge \text{Sym} \wedge \exists v \neg \langle A, v \rangle$ has no models,
- (iv) A is provable in S5 iff $\forall u R(u, u) \wedge \text{Tr} \wedge \text{Sym} \wedge \exists v \neg \langle A, v \rangle$ has no models.

Since finite conjunctions of sentences $\forall u R(u, u)$, Tr , Sym , and $\exists v \neg \langle A, v \rangle$ belong to FN, our main theorem clearly implies that four logics M, S4, B, S5 are all decidable.

III. A proof.

1) R-degree. For each R-formula A, let $R\text{-deg}(A)$ be the non-negative integer, called the R-degree of A, defined by: $R\text{-deg}(A) = 0$ if A is R-free, $R\text{-deg}(\neg A) = R\text{-deg}(A)$, $R\text{-deg}(A \wedge B) = R\text{-deg}(A \vee B) = R\text{-deg}(A \supset B) = \max \{ R\text{-deg}(A), R\text{-deg}(B) \}$, $R\text{-deg}(\exists v A(v)) = R\text{-deg}(A(x))$, and $R\text{-deg}(\exists v (R(x, v) \wedge A(v))) = R\text{-deg}(\exists v (R(v, x) \wedge A(v))) = R\text{-deg}(A(x)) + 1$.

2) R-basic sentences.

Define $\Sigma_n (n = 0, 1, 2, \dots)$ and Σ by: $\Sigma_0 = \text{Pow}(\{1, \dots, N\})$

$\Sigma_{n+1} = \Sigma_n \times \text{Pow}(\Sigma_n) \times \text{Pow}(\Sigma_n)$, $(n = 0, 1, 2, \dots)$ and $\Sigma = \bigcup_{n < \omega} \Sigma_n$,

where $\text{Pow}(Z)$ is the power set of Z.

For each σ in Σ , let $A(\sigma, x)$ be the unary formula defined by:

$A(\sigma, x)$ is $\bigwedge_{i \in \sigma} P_i(x) \wedge \bigwedge_{i \notin \sigma} \neg P_i(x)$ if $\sigma \in \Sigma_0$ and

$A(\sigma, x)$ is $A(\nu, x) \wedge \bigwedge_{\alpha \in 1} \exists v (R(v, x) \wedge A(\alpha, v)) \wedge \bigwedge_{\alpha \notin 1} \neg \exists v (R(v, x) \wedge A(\alpha, v)) \wedge$

$\bigwedge_{\alpha \in r} \exists v (R(x, v) \wedge A(\alpha, v)) \wedge \bigwedge_{\alpha \notin r} \neg \exists v (R(x, v) \wedge A(\alpha, v))$

if $\sigma = \langle \nu, l, r \rangle \in \Sigma_{n+1}$.

Then, $A(\sigma, x)$ is an R-formula whose R-degree is n if $\sigma \in \Sigma_n$.

For each subset X of Σ_n , let AX be the sentence;

$\bigwedge_{\sigma \in X} \exists v A(\sigma, v) \wedge \bigwedge_{\sigma \notin X} \neg \exists v A(\sigma, v)$

$AX (X \subseteq \Sigma_n)$ are called R-basic sentences of R-degree n.

3) Representation theorem.

(1) For each R-formula $A(x, \dots, y)$ of R-degree n , whose free variables are among $x, \dots, y$, we can concretely construct a Boolean combination $B(x, \dots, y)$ of formulas of the forms: $\exists v A(\sigma, v), A(\sigma, x), \dots, A(\sigma, y)$, where $\sigma \in \Sigma_n$ such that A and B are equivalent in LN .

(2) For each R-sentence A of R-degree n , we can concretely obtain finite subsets $X_1, \dots, X_n$ of Σ_n such that $A \equiv \bigvee_{x_1 \in X_1} \dots \bigvee_{x_n \in X_n} A$ and A are equivalent in LN .

4) Reduction lemmas.

Let GN (HN) be the set of sentences in FN which are finite conjunctions of the sentences: R-basic sentence (R-basic sentences of R-degree 1), R-positive sentences, Tr and Sym . Then $HN \subseteq GN \subseteq FN$.

Reduction Lemma 1. If GN is decidable, then FN is decidable.

Reduction Lemma 2. If HN ($N = 1, 2, \dots$) are all decidable, then GN ($N = 1, 2, \dots$) are all decidable.

5) Main Lemma. For each sentence A in HN , if A has a model, then A has a model of cardinality no more than $2^{\aleph_0} \times 2^{\aleph_0} \times 2^{\aleph_0}$.

Clearly, Reduction Lemma 1, Reduction lemma 2 and Main Lemma imply our main theorem.