

Large-time Behaviour of Solutions to Phase-Separation Models in One-Dimensional case

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1. Introduction

Let us consider a one-dimensional model for phase separation, which is described as the following system, noted by (P):

$$\frac{u_t}{u^2} + ww_t - u_{xx} = f \quad \text{in } Q := (0, +\infty) \times (-1, 1), \quad (1.1)$$

$$w_t - \{-\kappa w_{xx} + \xi + w^3 - (1+u)w\}_{xx} = 0 \quad \text{in } Q, \quad (1.2)$$

$$\xi \in \partial I_{[-0.5, 0.5]}(w) \quad \text{in } Q, \quad (1.3)$$

$$\pm u_x(t, \pm 1) + u(t, \pm 1) = 0 \quad \text{for } t > 0, \quad (1.4)$$

$$w_x(t, \pm 1) \quad \text{for } t > 0, \quad (1.5)$$

$$[-\kappa w_{xx}(t, \cdot) + (w(t, \cdot))^3 - (1+u(t, \cdot))w(t, \cdot)]_x|_{x=\pm 1} = 0 \quad \text{for } t > 0, \quad (1.6)$$

$$u(0, x) = u_0(x), \quad w(0, x) = w_0(x) \quad \text{for } x \in (-1, 1). \quad (1.7)$$

Here, κ is a positive constant; $\partial I_{[-0.5, 0.5]}$ is the subdifferential of the indicator function $I_{[-0.5, 0.5]}$ of the interval $[-0.5, 0.5]$; f , $h_{\pm}$, u_0 and w_0 are given data.

This system arises in the phase separation of a binary mixture with components A and B.

In this paper, $\theta := -\frac{1}{u}$ represents the absolute temperature and w_A the order parameter which is the local concentration of the component A; you note that $-0.5 \leq w(t, x) := w_A(t, x) - 0.5 \leq 0.5$, and $w(t, x) = 0.5$ (resp. $w(t, x) = -0.5$) means that the physical situation of the system at (t, x) is of pure A (resp. pure B), while $-0.5 < w(t, x) < 0.5$ means that the physical situation is mixture.

About this problem, by N. Kenmochi & M. Niezgodka [6] and [7], we know that (P) has a global and unique solution and under some assumptions on the convergences of the data

$f(t) \rightarrow 0$ and $h_{\pm}(t) \rightarrow h^{\infty}$ as $t \rightarrow +\infty$ in some senses, $u(t) \rightarrow u^{\infty}(=h^{\infty})$ as $t \rightarrow +\infty$ and any ω -limit function w^{∞} of the order parameter $w(t)$ is a solution of the following steady-state problem, noted by $(P)^{\infty}$:

$$-\kappa w_{xx}^{\infty} + \xi^{\infty} + (w^{\infty})^3 - (1 + u^{\infty})w^{\infty} = \sigma \quad \text{in } (-1, 1), \quad (1.8)$$

$$\xi^{\infty} \in \partial I_{[-0.5, 0.5]}(w^{\infty}) \quad \text{in } (-1, 1), \quad (1.9)$$

$$\xi^{\infty} \in L^2(-1, 1), \quad (1.10)$$

$$w_x^{\infty}(\pm 1) = 0, \quad (1.11)$$

$$\frac{1}{2} \int_{-1}^1 w^{\infty}(x) dx = m_0, \quad (1.12)$$

where $m_0 = \frac{1}{2} \int_{-1}^1 w_0(x) dx$.

Here, from (1.8) and (1.10), we note that

$$\sigma = \frac{1}{2} \int_{-1}^1 \{ \xi^{\infty} + (w^{\infty}(x))^3 - (1 + u^{\infty})w^{\infty}(x) \} dx.$$

In this paper, we consider the structure of the ω -limit set of the order parameter w , which is defined by

$$\omega(u_0, w_0) := \{ z \in H^1(-1, 1); w(t_n) \rightarrow z \text{ in } H^1(-1, 1) \text{ for some } t_n \uparrow +\infty \text{ as } n \rightarrow +\infty \}.$$

Notations. For simplicity, we use the following notations:

$H^1(-1, 1)$: the usual Sobolev space with norm $|\cdot|_{H^1(-1, 1)}$ given by

$$|z|_{H^1(-1, 1)} := (|z_x|_{L^2(-1, 1)}^2 + |z(-1)|^2 + |z(1)|^2)^{\frac{1}{2}};$$

$H^1(-1, 1)^*$: the dual space of $H^1(-1, 1)$;

$(\cdot, \cdot)$: the standard inner product in $L^2(-1, 1)$;

$\langle \cdot, \cdot \rangle$: the duality pairing between $H^1(-1, 1)^*$ and $H^1(-1, 1)$;

$$a(v, z) := \int_{-1}^1 v_x(x) z_x(x) dx \quad \text{for } v, z \in H^1(-1, 1).$$

2. Assumptions and known results

Problems (P) and $(P)^{\infty}$ are discussed under the following assumptions:

(A1) κ is a positive constant.

(A2) $f \in W_{loc}^{1,2}(0, +\infty; L^2(-1, 1)) \cap L^2(0, +\infty; L^2(-1, 1))$ such that

$$\sup_{t \geq 0} |f|_{W^{1,2}(t, t+1; L^2(-1, 1))} < +\infty.$$

(A3) $h_{\pm} \in W_{loc}^{1,2}(0, +\infty)$ such that

$$\sup_{t \geq 0} \{|h_+|_{W^{1,2}(t,t+1)} + |h_-|_{W^{1,2}(t,t+1)}\} < +\infty,$$

and for some constant $h^\infty \in (-\infty, 0)$

$$h_{\pm} - h^\infty \in L^2(0, +\infty).$$

(A4) $h_{\pm}(t) \in (-\infty, 0]$ for all $t \geq 0$ and there exist positive constants A_1 and A_2 such that

$$\frac{h_{\pm}(t)}{r} - 1 \geq -A_1|r| - A_2 \quad \text{for all } r \in (-\infty, 0) \text{ and all } t \geq 0.$$

(A5) $u_0 \in H^1(-1, 1)$ and $w_0 \in H^1(-1, 1)$ such that

$$-\frac{1}{u_0} \in L^2(-1, 1),$$

$$w_{0x}(\pm 1) = 0, \quad -0.5 \leq w_0 \leq 0.5 \text{ on } [-1, 1]$$

$$-0.5 < m_0 := \frac{1}{2} \int_{-1}^1 w_0(x) dx < 0.5$$

and there exists $\xi_0 \in L^2(-1, 1)$ satisfying

$$\xi_0 \in \partial I_{[-0.5, 0.5]}(w_0) \text{ a.e. in } (-1, 1), \quad -\kappa w_{0xx} + \xi_0 \in H^1(-1, 1).$$

Next, we give a weak variational formulation for (P).

Definition 2.1. For $0 < T < +\infty$ a coupled $\{u, w\}$ of functions $u : [0, T] \longrightarrow H^1(-1, 1)$ and $w : [0, T] \longrightarrow H^1(-1, 1)$ is called a (weak) solution of (P) on $[0, T]$, if the following conditions (w1)-(w4) are fulfilled:

(w1) $u \in L^\infty(0, T; H^1(-1, 1))$,

$-\frac{1}{u}$ is weakly continuous from $[0, T]$ into $L^2(-1, 1)$ with

$$\frac{u_t}{u^2} \in L^1(0, T; H^1(-1, 1)^*),$$

$$w \in L^\infty(0, T; H^1(-1, 1)) \cap L^2(0, T; H^2(-1, 1)), \quad w_t \in L^2(0, T; H^1(-1, 1)^*),$$

$$ww_t \in L^1(0, T; H^1(-1, 1)^*).$$

(w2) $u(0) = u_0$ and $w(0) = w_0$.

(w3) (1.1) holds in the standard variational sense, that is,

$$\begin{aligned} & \frac{d}{dt} \left(-\frac{1}{u(t)} + \frac{1}{2} w^2(t), z \right) + a(u(t), z) \\ & + (u(t, -1) - h_-(t))z(-1) + (u(t, 1) - h_+(t))z(1) = (f(t), z) \end{aligned} \quad (2.1)$$

for a.e. $t \in [0, T]$ and all $z \in H^1(-1, 1)$.

(w4) For a.e. $t \in [0, T]$,

$$w_x(t, \pm 1) = 0,$$

and there exists a function $\xi \in L^2(0, T; L^2(-1, 1))$ such that

$$\xi \in \partial I_{[-0.5, 0.5]}(w) \quad \text{for a.e. in } (0, T) \times (-1, 1) \quad (2.2)$$

and

$$\frac{d}{dt}(w(t), \eta) + \kappa(w_{xx}(t), \eta_{xx}) - (\xi(t) + (w(t))^3 - (1 + u(t))w(t), \eta_{xx}) = 0 \quad (2.3)$$

for all $\eta \in H^2(-1, 1)$ with $\eta_x(\pm 1) = 0$ and a.e. $t \in [0, T]$.

As is easily seen from the above definition, for any solution $\{u, w\}$ of (P) on $[0, T]$ it holds that

$$\frac{1}{2} \int_{-1}^1 w(t, x) dx = \frac{1}{2} \int_{-1}^1 w_0(x) dx = m_0$$

and

$$\frac{u_t}{u^2} + ww_t \in L^\infty(0, T; H^1(-1, 1)^*).$$

Also, the inequalities $-0.5 \leq m_0 \leq 0.5$ are necessary in order for (P) to have a solution; if $m_0 = 0.5$ (resp. -0.5), then we see that $w \equiv 0.5$ (resp. -0.5).

We say that a couple $\{u, w\}$ of functions $u : [0, +\infty) \rightarrow H^1(-1, 1)$ and $w : [0, +\infty) \rightarrow H^1(-1, 1)$ is a solution of (P) on $[0, +\infty)$, if it is a solution of (P) on $[0, T]$ for every finite $T > 0$.

We now recall an existence and uniqueness results.

Theorem 2.1. [cf. 7] Assume that (A1)-(A5) hold. Then (P) has one and only one solution $\{u, w\}$ on $[0, +\infty)$, and it satisfies that for every finite $T > 0$

$$\begin{cases} u \in L^2(0, T; H^2(-1, 1)), & u_t \in L^2(0, T; L^2(-1, 1)), \\ w \in L^\infty(0, T; H^2(-1, 1)), & w_t \in L^\infty(0, T; H^1(-1, 1)^*) \cap L^2(0, T; H^1(-1, 1)), \\ \xi \in L^\infty(0, T; L^2(-1, 1)), \end{cases} \quad (2.4)$$

where ξ is the function as in (w4) of Definition 2.1.

As to global estimates for solutions we have the following theorem

Theorem 2.2. [cf. 3] Assume that (A1)-(A5) hold. Let $\{u, w\}$ be the solution of (P) on $[0, +\infty)$. Then,

$$u - u^\infty \in L^2(0, +\infty; H^1(-1, 1)), \quad u \in L^\infty(0, +\infty; H^1(-1, 1)), \quad (2.5)$$

$$\sup_{t \geq 0} |u_t|_{L^2(t, t+1; L^2(-1, 1))} < +\infty, \quad (2.6)$$

$$w \in L^\infty(0, +\infty; H^2(-1, 1)), \quad (2.7)$$

$$w_t \in L^\infty(0, +\infty; H^1(-1, 1)) \cap L^2(0, +\infty; H^1(-1, 1)^*) \quad (2.8)$$

and

$$\sup_{t \geq 0} |w_t|_{L^2(t, t+1; H^1(-1, 1))} < +\infty. \quad (2.9)$$

From this theorem, we have the following corollary.

Corollary 2.1. [cf. 3] *Under the same assumptions as in Theorem 2.2, the following statements hold:*

- (a) $u(t) \longrightarrow u^\infty (= h^\infty)$ weakly in $H^1(-1, 1)$ as $t \rightarrow +\infty$.
- (b) The ω -limit set $\omega(u_0, w_0)$ is non-empty, compact and connected in $H^1(-1, 1)$. Also, $\omega(u_0, w_0)$ is bounded in $H^2(-1, 1)$.
- (c) $\lim_{t \rightarrow +\infty} \left\{ \frac{\kappa}{2} |w_x(t)|_{L^2(-1, 1)}^2 + \int_{-1}^1 \left(\frac{1}{4} (w(t, x))^4 - \frac{1}{2} (1 + u^\infty) (w(t, x))^2 \right) dx \right\}$ exists.
- (d) Any ω -limit function $v \in \omega(u_0, w_0)$ is solution of $(P)^\infty$.

From this corollary, the absolute temperature $-\frac{1}{u(t)}$ converges to a constant $-\frac{1}{u^\infty}$. On the other hand, in general the order parameter $w(t)$ does not converge, but any ω -limit function of $w(t)$ is a solution of (P). So, in the next section we consider the structure of the solutions of $(P)^\infty$ and ω -limit set $\omega(u_0, w_0)$.

3. The structure of ω -limit set $\omega(u_0, w_0)$

In this section, we consider the structure of the solution of $(P)^\infty$ and $\omega(u_0, w_0)$. Here, we note that the shape of the function $w^3 - (1 + u^\infty)w$ changes as u^∞ changes. From this results and (a) of Corollary 2.1, we consider $u^\infty (= h^\infty)$ as a controll parameter.

For simplicity, we use the following notations:

$$G(w; u^\infty) := \int_0^w \{v^3 - (1 + u^\infty)v\} dv$$

and

$$H(w; \sigma, u^\infty) := \int_0^w \{v^3 - (1 + u^\infty)v - \sigma\} dv = G(w; u^\infty) - \sigma w.$$

Lemma 3.1. [cf. 3] *Let w^∞ be any solution of $(P)^\infty$ and put $b = H(w^\infty(-1); \sigma, u^\infty)$. Then, $H(w^\infty(x); \sigma, u^\infty) \geq b$ for all $x \in [-1, 1]$.*

Moreover, $w_x^\infty(x) = 0$ if and only if $H(w^\infty(x); \sigma, u^\infty) = b$, hence $H(w^\infty(1); \sigma, u^\infty) = b$.

Proof. Multiplying (1.8) by w_x^∞ and integrating it over $[-1, x]$, from (1.11) we have

$$-\frac{\kappa}{2} |w_x^\infty(x)|^2 + H(w^\infty(x); \sigma, u^\infty) = b \quad \text{for all } x \in [-1, 1].$$

Hence, this lemma holds. $\diamond$.

Next, since there exist two cases of the shape of the function $w^3 - (1 + u^\infty)w$, we consider the two cases one by one.

(i) Case 1: $u^\infty \leq -1$

In this case, $w^3 - (1 + u^\infty)w$ is strictly increasing. So, there exists one and only one solution $\zeta(\sigma)$ of the algebraic equation $w^3 - (1 + u^\infty)w = \sigma$, that is, $H(w; \sigma, u^\infty)$ has the following properties:

$$H(w; \sigma, u^\infty) \text{ is strictly decreasing on } (-\infty, \zeta(\sigma)),$$

$$H(w; \sigma, u^\infty) \text{ is strictly increasing on } (\zeta(\sigma), +\infty)$$

and

$$H(w; \sigma, u^\infty) \geq H(\zeta(\sigma); \sigma, u^\infty).$$

Theorem 3.1. $(P)^\infty$ has no non-constant solution.

Proof. We assume that w^∞ is a non-constant solution of $(P)^\infty$.

Then, from Lemma 3.1 and the properties of $H(w; \sigma, u^\infty)$ we can see that there exist two following cases (α) and (β) for w^∞ .

$$(\alpha) \quad w^\infty(-1) \leq \zeta(\sigma) \text{ and } w^\infty \text{ is decreasing on } [-1, 1].$$

$$(\beta) \quad w^\infty(-1) \geq \zeta(\sigma) \text{ and } w^\infty \text{ is increasing on } [-1, 1].$$

In both cases (α) and (β) we have $w_x^\infty \neq 0$ on $(-1, 1]$ which contradicts the boundary condition $w_x^\infty(1) = 0$. Therefore, we obtain this theorem. $\diamond$

From Theorem 3.1, we can see that the following theorem, easily.

Theorem 3.2. $(P)^\infty$ has a constant solution $v \equiv m_0$ on $[-1, 1]$, only.

Moreover, $\sigma = G(m_0; u^\infty)$ and $b = (1 - m_0)G(m_0; u^\infty)$.

Proof. From (1.12), $w^\infty \equiv m_0$ on $[-1, 1]$ must hold. Since $-0.5 < m_0 < 0.5$, $\xi^\infty \equiv 0$ on $[-1, 1]$. So,

$$\begin{aligned} \sigma &= \frac{1}{2} \int_{-1}^1 \{\xi^\infty + m_0^3 - (1 + u^\infty)m_0\} dx \\ &= m_0^3 - (1 + u^\infty)m_0 = G(m_0; u^\infty)0. \end{aligned}$$

Moreover,

$$b = G(m_0; u^\infty) - \sigma m_0 = (1 - m_0)G(m_0; u^\infty). \quad \diamond$$

Remark 3.1. From Corollary 2.1 and 3.2, the order parameter $w(t)$ converges $w^\infty \equiv m_0$ as $t \rightarrow +\infty$. So, there exists one and only one ω -limit set $\omega(u_0, w_0) = \{w^\infty\}$.

Case 2: $-1 < u^\infty < 0$

In this case, $w^3 - (1 + u^\infty)w$ is non-monotone and N-shape. So, we consider the case

when $m_0 = 0$.

Here, We note that there exist two cases for the position of constraints -0.5 and 0.5 .

First case, when $-0.75 \leq u^\infty < 0$, these constraints are outside of zero points of $w^3 - (1 + u^\infty)w$, that is,

$$-0.5 \leq -\sqrt{1 + u^\infty} < 0 < \sqrt{1 + u^\infty} \leq 0.5.$$

Second case, when $-1 < u^\infty < -0.75$, they are inside, that is,

$$-\sqrt{1 + u^\infty} < -0.5 < 0 < 0.5 < \sqrt{1 + u^\infty}.$$

At first, by using the same technique as in Theorem 3.2, we obtain the following theorem about a constant solution.

Theorem 3.3. $(P)^\infty$ has one and only one constant solution $w^\infty \equiv 0$ on $[-1, 1]$. Moreover, in this case $\sigma = b = 0$.

In the rest of this case, we consider non-constant solutions of $(P)^\infty$. To do so, we note that there exist three following cases of the shape of the function $H(w; \sigma, u^\infty)$ by the value of σ .

(a) When $\sigma \geq 2\left(\frac{1 + u^\infty}{3}\right)^{\frac{3}{2}}$, $H(w; \sigma, u^\infty)$ has the following properties:

$H(w; \sigma, u^\infty)$ is strictly decreasing on $(-\infty, \zeta_+(\sigma))$,

$H(w; \sigma, u^\infty)$ is strictly increasing on $(\zeta_+(\sigma), +\infty)$

and

$$H(w; \sigma, u^\infty) \geq H(\zeta_+(\sigma); \sigma, u^\infty),$$

where $\zeta_+(\sigma)$ is a root of the algebraic equation $w^3 - (1 + u^\infty)w = \sigma$ such that

$$\zeta_+(\sigma) > -\left(\frac{1 + u^\infty}{3}\right)^{\frac{1}{2}}.$$

(b) When $\sigma \leq -2\left(\frac{1 + u^\infty}{3}\right)^{\frac{3}{2}}$, $H(w; \sigma, u^\infty)$ has the following properties:

$H(w; \sigma, u^\infty)$ is strictly decreasing on $(-\infty, \zeta_-(\sigma))$,

$H(w; \sigma, u^\infty)$ is strictly increasing on $(\zeta_-(\sigma), +\infty)$

and

$$H(w; \sigma, u^\infty) \geq H(\zeta_-(\sigma); \sigma, u^\infty),$$

where $\zeta_-(\sigma)$ is a root of the algebraic equation $w^3 - (1 + u^\infty)w = \sigma$ such that

$$\zeta_-(\sigma) < \left(\frac{1 + u^\infty}{3}\right)^{\frac{1}{2}}.$$

(c) When $-2\left(\frac{1+u^\infty}{3}\right)^{\frac{3}{2}} < \sigma < 2\left(\frac{1+u^\infty}{3}\right)^{\frac{3}{2}}$, $H(w; \sigma, u^\infty)$ has the following properties:

$H(w; \sigma, u^\infty)$ is strictly decreasing on $(-\infty, \zeta_-(\sigma)) \cup (\zeta(\sigma), \zeta_+(\sigma))$

and

$H(w; \sigma, u^\infty)$ is strictly increasing on $(\zeta_-(\sigma), \zeta(\sigma)) \cup (\zeta_+(\sigma), +\infty)$,

where $\zeta_-(\sigma)$, $\zeta(\sigma)$ and $\zeta_+(\sigma)$ are roots of the algebraic equation $w^3 - (1+u^\infty)w = \sigma$

such that $\zeta_-(\sigma) < -\left(\frac{1+u^\infty}{3}\right)^{\frac{1}{2}} < \zeta(\sigma) < \left(\frac{1+u^\infty}{3}\right)^{\frac{1}{2}} < \zeta_+(\sigma)$.

To the cases (a) and (b), by using the same technique as in Theorem 3.1, we can see that the following theorem holds.

Theorem 3.4. *We assume that $\sigma \leq -2\left(\frac{1+u^\infty}{3}\right)^{\frac{3}{2}}$ or $\sigma \geq 2\left(\frac{1+u^\infty}{3}\right)^{\frac{3}{2}}$. Then, $(P)^\infty$ has no non-constant solution.*

From this theorem, we only consider the case (c). In this case, by the results of A. Ito & N. Kenmochi [6], we know that the following theorem holds.

Theorem 3.5. *Let w^∞ be non-constant solution of $(P)^\infty$. Then,*

(1) $\sigma = 0$.

(2) *If $-0.75 \leq u^\infty < 0$, then all ω -limit set $\omega(u_0, w_0)$ is a singleton, that is, $\omega(u_0, w_0) = \{w^\infty\}$. Moreover, the number of $\omega(u_0, w_0)$ is equal to $2n_1 + 1$, where n_1 is the number of b with $G(-\sqrt{1+u^\infty}; u^\infty) = G(\sqrt{1+u^\infty}; u^\infty) < b < 0$ satisfying the following condition (*):*

(*) *There exist a natural number $N(b)$ such that $N(b)I(b) = 2$,*

where $\pm\eta(b)$ are roots of the algebraic equation $G(w; u^\infty) = b$ such that $-\sqrt{1+u^\infty} < -\eta(b) < 0 < \eta(b) < \sqrt{1+u^\infty}$ and

$$I(b) := \left(\frac{\kappa}{2}\right)^{\frac{1}{2}} \int_{-\eta(b)}^{\eta(b)} \frac{1}{\{G(w; u^\infty) - b\}} dw.$$

(3) *If $-1 < u^\infty < -0.75$, there exist two possibilities (i) and (ii) of the structure of $\omega(u_0, w_0)$:*

(i) $\omega(u_0, w_0)$ is a singleton.

(ii) $\omega(u_0, w_0)$ contains a continuum of the solutions of $(P)^\infty$. Moreover, in this case the following properties hold:

(α) $b = G(-0.5; u^\infty) = G(0.5; u^\infty)$.

(β) $\eta(b) = 0.5$. Hence, in particular boundary values $w^\infty(-1)$ and $w^\infty(1)$ take -0.5 or 0.5 .

(γ) $|J_B| = |J_A|$, where $|J_A|$ and $|J_B|$ are the length of the pure region of the components A and B , respectively.

Moreover, the number of $\omega(u_0, w_0)$ is equal to $2n_1 + 2n_2 + 1$, where n_1 is the number of b with $G(-0.5; u^\infty) = G(0.5; u^\infty) < b < 0$ satisfying (*) and n_2 is the number of the natural number n satisfying the following conditions (**):

$$(**) \quad nI(G(-0.5; u^\infty)) = nI(G(0.5; u^\infty)) \leq 2.$$

From this theorem, we are interested in the case when (ii) of (3).

But, this case is very dependent upon the coefficient κ .

At last, we give the theorem to show that ω -limit set is very dependent upon κ .

Theorem 3.6. *If κ is large enough to satisfy the following condition (**)*

$$2I(G(0.5; u^\infty)) \geq 2.$$

Then, all ω -limit set are singleton, that is, the order parameter $w(t)$ converges to some ω -limit function w^∞ as $t \rightarrow +\infty$.

Proof. It is clear from the above theorem. $\diamond$.

Remark 3.2. We can see that ω -limit set is very dependent upon the length of the interval when κ is fixed.

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