

Periodic solutions of a singular Hamiltonian system of 2-body type

Kazunaga Tanaka (田中和永)

Department of Mathematics, School of Science, Nagoya University
Chikusa-ku, Nagoya 464, JAPAN

0. Introduction and results

In this short note, we study the existence of periodic solutions of a Hamiltonian system

$$\ddot{q} + \nabla V(q, t) = 0, \quad (HS)$$

where $q = (q_1, \dots, q_N) \in \mathbf{R}^N$ ($N \geq 3$) and $V(q, t) : \mathbf{R}^N \times \mathbf{R} \rightarrow \mathbf{R}$ is a given potential. We deal with the case where a potential has a singularity and is related to 2-body problem.

More precisely, we assume $V(q, t)$ satisfies

- (V1) $V(q, t) \in C^2((\mathbf{R}^N \setminus \{0\}) \times \mathbf{R}, \mathbf{R})$ is T -periodic in t ;
- (V2) $V(q, t) < 0$ and $V(q, t), \nabla V(q, t) \rightarrow 0$ as $|q| \rightarrow \infty$ uniformly in t ;
- (V3) $V(q, t)$ is of a form:

$$V(q, t) = -\frac{1}{|q|^\alpha} + W(q, t),$$

where $\alpha > 0$ and $W(q, t) \in C^2((\mathbf{R}^N \setminus \{0\}) \times \mathbf{R}, \mathbf{R})$ satisfies

$$\begin{aligned} &|q|^\alpha W(q, t), |q|^{\alpha+1} \nabla W(q, t), |q|^{\alpha+2} \nabla^2 W(q, t), \\ &|q|^\alpha W_t(q, t) \rightarrow 0 \quad \text{as } q \rightarrow 0 \text{ uniformly in } t. \end{aligned}$$

We consider the following two problems:

- (i) *Prescribed Period Problem (PP)*: For a given $T > 0$, we study the existence of T -periodic solutions of (HS), i.e., solutions of (HS) such that

$$q(t+T) = q(t) \quad \text{for all } t. \quad (HS.P)$$

- (ii) *Prescribed Energy Problem (PE)*: Assume V is independent of t . For a given $H \in \mathbf{R}$, we study the existence of periodic solutions of (HS) such that

$$\frac{1}{2} |\dot{q}(t)|^2 + V(q(t)) = H \quad \text{for all } t. \quad (HS.E)$$

(Here we do not fix the period of $q(t)$.)

We study via variational methods these problems. Recently it is observed that the order α of the singularity of $V(q, t)$ at $q = 0$ plays an important role for the existence of

periodic solutions for both of problems. We consider the following cases separately; (i) the *strong force* case $\alpha \geq 2$ for (PP) and $\alpha > 2$ for (PE) (ii) the *weak force* case $\alpha \in (0, 2)$.

For the Prescribed Period problem (PP), we use the following variational formulation. Let $E = \{q \in H_{loc}^1(\mathbf{R}, \mathbf{R}^N); q(t) \text{ is } T\text{-periodic in } t\}$ is a space of T -periodic functions with norm $\|q\|_E^2 = \int_0^T [|\dot{q}(t)|^2 + |q(t)|^2] dt$ and set

$$\Lambda = \{q \in E; q(t) \neq 0 \text{ for all } t\}.$$

We define the functional $I(q) : \Lambda \rightarrow \mathbf{R}$ by

$$I(q) = \int_0^T \left[\frac{1}{2} |\dot{q}|^2 - V(q(t), t) \right] dt.$$

Then there is one-to-one correspondence between critical points $q \in \Lambda$ of $I(q)$ and T -periodic solutions of (HS), (HS.P). Therefore we try to find critical points of $I(q)$.

If (V1)–(V3) holds and $\alpha \geq 2$, more generally, under the conditions of (V2)–(V3) and (V1') $V(q, t) \in C^1((\mathbf{R}^N \setminus \{0\}) \times \mathbf{R}, \mathbf{R})$ is T -periodic in t and the following strong force condition (SF) of Gordon [Go]:

(SF) there is a neighborhood Ω of 0 and $U(q) \in C^1(\Omega \setminus \{0\}, \mathbf{R})$ such that

$$\begin{aligned} U(q) &\rightarrow \infty, & q &\rightarrow 0, \\ -V(q, t) &\geq |\nabla U(q)|^2 & \text{for all } q \in \Omega \setminus \{0\} \text{ and } t, \end{aligned}$$

we can see the functional $I(q)$ satisfies the Palais-Smale condition and we can apply min-max methods to obtain critical points of $I(q)$. We refer to [BR, Gr, AC1]. Our main purpose is to study the weak force case $\alpha \in (0, 2)$. We remark that the Palais-Smale condition does not hold in this case. Our result is as follows:

Theorem 0.1 ([T2]). Suppose (V1)–(V3) and $\alpha \in (1, 2)$. Then the prescribed period problem (HS), (HS.P) possesses at least one periodic solution.

For the Prescribed energy problem (PE), we can expect the existence of periodic solutions only under the situations

- (i) $H > 0$ if $\alpha > 2$, or
- (i) $H < 0$ if $\alpha \in (0, 2)$.

Actually, if $V(q) = -\frac{1}{|q|^\alpha}$, we can easily see that periodic solutions of (HS), (HS.E) exist if and only if (i) or (ii) holds. In the strong force case $\alpha > 2$, the Palais-Smale condition holds under additional assumptions and we refer to Ambrosetti and Coti Zelati [AC2] for the existence result. We study the case $\alpha \in (0, 2)$. Here we assume

(V4) there is $\bar{\alpha} \in (0, \alpha]$ such that

$$\nabla V(q)q \geq -\bar{\alpha}V(q) \quad \text{for all } q \in \mathbf{R}^N \setminus \{0\}$$

in addition to (V1)–(V3).

Theorem 0.2 ([T3]). Suppose V is independent of t , $H < 0$ and (V1)–(V4). Moreover assume $\alpha \in (1, 2)$ if $N \geq 4$ and $\alpha \in (4/3, 2)$ if $N = 3$. Then the prescribed energy problem (HS), (HS.E) possesses at least one periodic solution.

We remark that in case of weak force the existence of *generalized solutions*, which may enter the singularity 0, is obtained by [BR] for the prescribed period problem (PP) and by [AC2] for the prescribed energy problem (PE). We also remark the result very closely related to Theorem 0.1 is obtained by Coti Zelati and Serra [CS] independently.

In what follows, we sketch outline of the proof of Theorem 0.1. The proof of Theorem 0.2 is done essentially in a same way (but more complicated) to Theorem 0.1 and we refer to [T3].

1. Perturbed functionals

We take the following approach, which is used by [BR] first time.

1° First we introduce a perturbed potential $V_\epsilon(q, t) = V(q, t) - \frac{\epsilon}{|q|^2}$. The corresponding functional

$$\begin{aligned} I_\epsilon(q) &= \int_0^T \left[\frac{1}{2} |q|^2 - V_\epsilon(q, t) \right] dt \\ &= \int_0^T \left[\frac{1}{2} |q|^2 - V(q, t) + \frac{\epsilon}{|q|^2} \right] dt \end{aligned}$$

satisfies a variant of the Palais-Smale condition and we can apply a minimax method of [BR] to get approximate solution $q_\epsilon(t)$ for each $\epsilon \in (0, 1]$.

2° Second we try to pass to the limit as $\epsilon \rightarrow 0$ and we try to obtain a solution as a limit of $q_\epsilon(t)$

More precisely, we use the following minimax method; we set

$$\Gamma = \{\gamma \in C(S^{N-2}, \Lambda); \deg \tilde{\gamma} \neq 0\} \quad (1.1)$$

where $\tilde{\gamma} : S^1 \times S^{N-2} \simeq ([0, T]/\{0, T\}) \times S^{N-2} \rightarrow S^{N-1}$ is defined by

$$\tilde{\gamma}(t, x) = \frac{\gamma(x)(t)}{|\gamma(x)(t)|}$$

and $\deg \tilde{\gamma}$ denote the Brower degree of $\tilde{\gamma}$. We define

$$b_\epsilon = \inf_{\gamma \in \Gamma} \max_{x \in S^{N-2}} I_\epsilon(\gamma(x)). \quad (1.2)$$

Then we have

Proposition 1.1 ([BR]). For any $\epsilon \in (0, 1]$, b_ϵ is a critical value of $I_\epsilon(q)$. That is, there is a critical point $q_\epsilon(t)$ of $I_\epsilon(q)$ such that $I_\epsilon(q_\epsilon) = b_\epsilon$. Moreover, there are constants $M, m > 0$ independent of $\epsilon \in (0, 1]$ such that

$$m \leq b_\epsilon \leq M \quad \text{for all } \epsilon \in (0, 1]. \quad (1.3)$$

Using the uniform estimate (1.3), we can get

Proposition 1.2 ([BR]). There is a constant $C > 0$ independent of $\epsilon \in (0, 1]$ such that

$$\|q_\epsilon\|_E \leq C \quad \text{for all } \epsilon \in (0, 1].$$

Therefore we can choose a subsequence — still we denote by $\epsilon \rightarrow 0$ — such that $q_\epsilon \rightarrow q_0 \in E$ weakly in E and strongly in L^∞ . If $q_0(t) \neq 0$ for all t , in other words, if $q_0 \in \Lambda$, we can easily see $q_0(t)$ is a periodic solution of (HS), (HS.P). The difficulty is to prove $q_0 \in \Lambda$.

Even if $q_0 \notin \Lambda$, we can see

- (i) Set $D = \{t; q_0(t) = 0\}$. Then $\text{meas } D = 0$;
- (ii) $q_0(t) \in C^2(\mathbf{R} \setminus D, \mathbf{R}^N) \cap C(\mathbf{R}, \mathbf{R}^N)$;
- (iii) $q_0(t)$ satisfies (HS) in $\mathbf{R} \setminus D$.

Bahri and Rabinowitz [BR] called such a limit function $q_0(t)$ *generalized solution* of (HS), (HS.P). They constructed generalized solutions under the conditions (V1'), (V2) and

(V3') $V(q, t) \rightarrow -\infty$ as $q \rightarrow 0$ uniformly in t .

To prove $q_0(t)$ does not enter the singularity 0, we use a combination of a re-scaling argument and an estimate of Morse indices.

2. Re-scaling argument

Suppose $q_0(t)$ enters the singularity 0 at $t_0 \in (0, T]$, i.e., $q_0(t_0) = 0$. Then there is a sequence $t_\epsilon \in (0, T]$ such that

1° $t_\epsilon \rightarrow t_0$;

2° $|q_\epsilon(t)|$ takes its local minimum at t_ϵ .

Case 1: First we study the behavior of $q_\epsilon(t)$ near the singularity 0 more precisely via a re-scaling argument. We set

$$\begin{aligned} \delta_\epsilon &= |q_\epsilon(t_\epsilon)|, \\ x_\epsilon(s) &= \delta_\epsilon^{-1} q_\epsilon(\delta_\epsilon^{(\alpha+2)/2} s + t_\epsilon). \end{aligned}$$

Then $x_\epsilon(s)$ satisfies $|x_\epsilon(0)| = 1$ and

$$\ddot{x}_\epsilon + \frac{\alpha x_\epsilon}{|x_\epsilon|^{\alpha+2}} + \delta_\epsilon^{\alpha+1} \nabla W(\delta_\epsilon x_\epsilon(s), \delta_\epsilon^{(\alpha+2)/2} s + t_\epsilon) + \frac{2\epsilon}{\delta_\epsilon^{2-\alpha}} \frac{x_\epsilon}{|x_\epsilon|^4} = 0.$$

We study the behavior of $x_\epsilon(s)$ instead of $q_\epsilon(t)$.

After taking a suitable subsequence — still we denote by ϵ —, we may assume that

$$d = \lim_{\epsilon \rightarrow 0} \frac{\epsilon}{\delta_\epsilon^{2-\alpha}} \in [0, \infty] \quad (2.1)$$

exists. We consider the following two cases separately.

Case 1: $d < \infty$;

Case 2: $d = \infty$.

Case 1: First we deal with Case 1.

Proposition 2.1. Suppose $d < \infty$. After taking a subsequence — still denoted by ϵ —, $x_\epsilon(s)$ converges to a function $y_{\alpha,d}(s)$ in $C_{loc}^2(\mathbf{R}, \mathbf{R}^N)$, where $y_{\alpha,d}(s)$ is the solution of

$$\begin{aligned} \ddot{y} + \frac{\alpha y}{|y|^{\alpha+2}} + \frac{2dy}{|y|^4} &= 0, \quad \text{in } \mathbf{R}, \\ y(0) &= e_1, \\ \dot{y}(0) &= \sqrt{2(1+d)}e_2. \end{aligned} \quad (2.2)$$

Here, $e_1, e_2, \dots, e_N \in \mathbf{R}^N$ are vectors satisfying $e_i \cdot e_j = \delta_{ij}$. ■

Case 2: In this case, we introduce another re-scaled function

$$z_\epsilon(s) = \delta_\epsilon^{-1} q_\epsilon(\epsilon^{-1/2} \delta_\epsilon^2 s + t_\epsilon).$$

Then $z_\epsilon(s)$ satisfies

$$\ddot{z}_\epsilon + \frac{\alpha \delta_\epsilon^{2-\alpha}}{\epsilon} \frac{z_\epsilon}{|z_\epsilon|^{\alpha+2}} + \frac{\alpha \delta_\epsilon^{2-\alpha}}{\epsilon} \delta_\epsilon^{\alpha+1} \nabla W(\delta_\epsilon z_\epsilon, \epsilon^{-1/2} \delta_\epsilon^2 s + t_\epsilon) + \frac{2z_\epsilon}{|z_\epsilon|^4} = 0.$$

We have

Proposition 2.2. Suppose $d = \infty$. Then, after taking a subsequence — still denoted by ϵ —, we have

$$z_\epsilon(s) \rightarrow z_0(s) = e_1 \cos \sqrt{2}s + e_1 \sin \sqrt{2}s \quad \text{in } C_{loc}^2(\mathbf{R}, \mathbf{R}^N).$$

Here, $e_1, e_2, \dots, e_N \in \mathbf{R}^N$ are vectors satisfying $e_i \cdot e_j = \delta_{ij}$. ■

We remark $z_0(s)$ is a solution of

$$\begin{aligned} \ddot{z} + \frac{2z}{|z|^4} &= 0, \quad \text{in } \mathbf{R}, \\ z(0) &= e_1, \\ \dot{z}(0) &= \sqrt{2}e_2. \end{aligned}$$

3. Estimates of Morse index

Using the propositions 2.1 and 2.2, we have the following estimate of Morse indices.

Proposition 3.1. *Suppose $q_0(t)$ enters the singularity 0 and set*

$$\nu = \#\{t \in (0, t]; q_0(t) = 0\}.$$

Then

$$\liminf_{\epsilon \rightarrow 0} \text{index } I''_{\epsilon}(q_{\epsilon}) \geq (N-2)i(\alpha)\nu \quad (3.1)$$

where

$$i(\alpha) = \max\{k \in \mathbf{N}; k < \frac{2}{2-\alpha}\}. \quad \text{■}$$

Before we sketch the proof of Proposition 3.1, we give a proof of Theorem 0.1.

Proof of Theorem 0.1. First we remark that the following estimate of Morse index follows from the minimax characterization (1.1)–(1.2) of b_{ϵ} .

Proposition 3.2 (c.f.[BL, LS, T1]). *$q_{\epsilon}(t) \in \Lambda$ satisfies*

$$\text{index } I''_{\epsilon}(q_{\epsilon}) \leq N-2 \quad \text{for all } \epsilon \in (0, 1]. \quad (3.2)$$

■

Comparing (3.1) and (3.2), we have

$$i(\alpha)\nu \leq 1. \quad (3.3)$$

Since $i(\alpha) \geq 2$ for $\alpha \in (1, 2)$ and $i(\alpha) = 1$ for $\alpha \in (0, 1]$, we find

$$\begin{aligned} \nu &= 0, & \text{if } \alpha \in (1, 2), \\ \nu &\leq 1, & \text{if } \alpha \in (0, 1]. \end{aligned}$$

Therefore in case $\alpha \in (1, 2)$, we obtain $q_0(t) \neq 0$ for all t and it is a classical solution. $\blacksquare$

Sketch of the proof of Proposition 3.1. Suppose $q_0(t_0) = 0$ and choose $t_\epsilon \in (0, T]$ as above. We deal with only the Case 1: $d < \infty$. The Case 2: $d = 0$ can be treated similarly. For $L > 0$, $\varphi(s) \in H_0^1(-L, L; \mathbf{R})$ and $j = 1, 2, \dots, N$, we set

$$h_{\epsilon,j}(t) = \delta_\epsilon \varphi(\delta_\epsilon^{-(\alpha+2)/2}(t - t_\epsilon))e_j.$$

After the change of variable, we take a limit as $\epsilon \rightarrow 0$ and obtain

$$\begin{aligned} & \delta_\epsilon^{-(2-\alpha)/2} I_\epsilon''(q_\epsilon)(h_{\epsilon,j}, h_{\epsilon,j}) \\ & \rightarrow \int_{-L}^L \left[|\dot{\varphi}|^2 - \frac{\alpha |\varphi|^2}{|y_{\alpha,d}|^{\alpha+2}} + \frac{\alpha(\alpha+2)(y_{\alpha,d}, e_j)^2 |\varphi|^2}{|y_{\alpha,d}|^{\alpha+4}} \right. \\ & \quad \left. - \frac{2d |\varphi|^2}{|y_{\alpha,d}|^4} + \frac{8d(y_{\alpha,d}, e_j)^2 |\varphi|^2}{|y_{\alpha,d}|^6} \right] ds. \end{aligned}$$

Recalling $y_{\alpha,d}(s) \in \text{span}\{e_1, e_2\}$ for all s , we can see

$$\liminf_{\epsilon \rightarrow 0} \text{index } I_\epsilon''(q_\epsilon) \geq (N-2)i(\alpha, d) \quad (3.4)$$

where

$$i(\alpha, d) = \sup_{L>0} \left(\begin{array}{l} \text{the number of negative eigenvalues} \\ \text{of the following eigenvalue problem:} \\ -\ddot{\varphi} - \left(\frac{\alpha}{|y_{\alpha,d}|^{\alpha+2}} + \frac{2d}{|y_{\alpha,d}|^4} \right) \varphi = 0, \\ \varphi(L) = \varphi(-L) = 0. \end{array} \right).$$

We repeat the above argument at all other $t'_0 \in (0, T]$ such that $q_0(t'_0) = 0$ and we find

$$\liminf_{\epsilon \rightarrow 0} \text{index } I_\epsilon''(q_\epsilon) \geq (N-2)i(\alpha)\nu$$

where

$$i(\alpha) = \min_{d \geq 0} i(\alpha, d).$$

Now Proposition 3.1 follows from the following proposition. $\blacksquare$

Proposition 3.3.

$$i(\alpha, d) = \max\{k \in \mathbf{N}; k < \frac{2\sqrt{1+d}}{2-\alpha}\}. \quad (3.5)$$

Thus $i(\alpha) = \max\{k \in \mathbf{N}; k < \frac{2}{2-\alpha}\}$.

Proof. The case $d = 0$ is proved in [T2]. The case $d > 0$ is proved similarly. The key of the proof is the Sturm comparison theorem and the following property of $y_{\alpha,d}(s)$. We use the polar coordinate and write

$$y_{\alpha,d}(s) = r(s)(e_1 \cos \theta(s) + e_2 \sin \theta(s))$$

where $r(s) > 0$ and $\theta(s) \in \mathbf{R}$ with $\theta(0) = 0$. Then we have

- (i) $s \dot{r}(s) > 0$ for all $s \neq 0$ and $r(s) \rightarrow \infty$ as $s \rightarrow \pm\infty$;
- (ii) $\dot{\theta}(s) > 0$ for all s ;
- (iii) $\theta(s) \rightarrow \pm \frac{2\pi\sqrt{1+d}}{2-\alpha}$ as $s \rightarrow \pm\infty$.

■

4. Remarks

In case $\alpha \in (0, 1]$, it seems that the existence of classical periodic solutions is not known. However by (3.3) we can see there is a generalized solution of (HS), (HS.P) that enters at most one time in its period. By (3.4) and (3.5), we also have

$$d \leq (2 - \alpha)^2 - 1 \quad (4.1)$$

where d is defined in (2.1).

We get the following additional information under slightly stronger conditions: (V1), (V2) and

(V3'') $V(q, t)$ is of a form:

$$V(q, t) = -\frac{1}{|q|^\alpha} + W(q, t),$$

where $\alpha > 0$ and $W(q, t) \in C^2((\mathbf{R}^N \setminus \{0\}) \times \mathbf{R}, \mathbf{R})$ satisfies

$$\begin{aligned} & |q|^{\alpha-\rho} W(q, t), |q|^{\alpha-\rho+1} \nabla W(q, t), |q|^{\alpha-\rho+2} \nabla^2 W(q, t), \\ & |q|^{\alpha-\rho} W_t(q, t) \rightarrow 0 \quad \text{as } q \rightarrow 0 \text{ uniformly in } t \end{aligned}$$

for some $\rho \in (0, \alpha)$.

We assume $q_0(t)$ is a generalized solution such that $q_0(t_0) = 0$. Beaulieu [B] proved that the limits

$$a_{\pm} = \lim_{t \rightarrow t_0 \pm 0} \frac{q_0(t)}{|q_0(t)|} \in S^{N-1}$$

exist. We have

Theorem 4.1 ([T4]). Assume (V1), (V2), (V3'') and let $q_{\epsilon}(t)$ be a critical point of $I_{\epsilon}(q)$ which is obtained through a minimax method (1.1)–(1.2). Suppose $q_0(t) = \lim_{\epsilon \rightarrow 0} q_{\epsilon}(t)$ is a generalized solution such that $q_0(t_0) = 0$ and let $a_{\pm} = \lim_{t \rightarrow t_0 \pm 0} \frac{q_0(t)}{|q_0(t)|} \in S^{N-1}$. Then we have

$$\text{the angle between } a_+ \text{ and } a_- = \frac{2\pi\sqrt{1+d}}{2-\alpha} \text{ modulo } 2\pi$$

where $d \in [0, (2 - \alpha)^2 - 1]$ is defined in (2.1).

■

In particular, in case $\alpha = 1$ we have $d = 0$ and $a_+ = a_-$.

References

- [AC1] A. Ambrosetti and V. Coti Zelati, Critical points with lack of compactness and applications to singular dynamical system, *Annali Mat. Pura Appl.* **149** (1987), 237–259.
- [AC2] — and —, Closed orbits of fixed energy for singular Hamiltonian systems, *Arch. Rat. Mech. Anal.* **112** (1990), 339–362.
- [B] A. Beaulieu, Etude de solutions generalisees pour un systeme hamiltonien avec potentiel singulier, *preprint*.
- [BL] A. Bahri and P.L. Lions, Morse index of some min-max critical points. I. Application to multiplicity results, *Comm. Pure Appl. Math.* **41** (1988), 1027–1037.
- [BR] A. Bahri and P.H. Rabinowitz, A minimax method for a class of Hamiltonian systems with singular potentials, *J. Funct. Anal.* **82** (1989), 412–428.
- [C] V. Coti Zelati, Periodic solutions for a class of planar, singular dynamical systems, *J. Math. Pure Appl.* **68** (1989), 109–119.
- [CS1] V. Coti Zelati and E. Serra, Collisions and non-collisions solutions for a class of Keplerian-like dynamical systems, *preprint*.
- [CS2] — and —, Multiple brake orbits for some classes of Hamiltonian systems, *preprint*.
- [Go] W.B. Gordon, Conservative dynamical systems involving strong forces, *Trans. Amer. Math. Soc.* **204** (1975), 113–135.
- [Gr] C. Greco, Periodic solutions of a class of singular Hamiltonian systems, *Nonlinear Analysis; T.M.A.* **12** (1988), 259–269.
- [LS] A. C. Lazer and S. Solimini, Nontrivial solutions of operator equations and Morse indices of critical points of min-max type, *Nonlinear Analysis: T. M. A.* **12** (1988), 761–775.
- [R] P. H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, CBMS Regional Conf. Ser. in Math. **65** Amer. Math. Soc., Providence 1986.
- [ST] E. Serra and S. Terracini, Noncollision solutions to some singular minimization problems with Keplerian-like potentials, *preprint*.
- [T1] K. Tanaka, Morse indices at critical points related to the symmetric mountain pass theorem and applications, *Comm. Partial Diff. Eq.* **14** (1989), 99–128.
- [T2] —, Non-collision solutions for a second order singular Hamiltonian system with weak force, *Ann. Inst. Henri Poincaré, Analyse nonlinéaire*, (to appear).
- [T3] —, A prescribed energy problem for a singular Hamiltonian system with a weak force, *J. Funct. Anal.*, (to appear).
- [T4] —, *preprint*.