

A quantization of Iwasawa theory and cyclotomic extensions of Kummer fields

Takashi ICHIKAWA

Department of Mathematics
Faculty of Science and Engineering
Saga University, Saga 840, Japan

Introduction

The aim of this paper is to study systematically the Iwasawa theory of Kummer p -extensions of $\mathbf{Q}$, i.e., we shall study the structure of

$$X = \text{Gal}(L_\infty/\mathbf{Q}(\zeta_{p^\infty}, a^{1/p^\infty}))$$

as a $\mathbf{Z}_p[[\text{Gal}(\mathbf{Q}(\zeta_{p^\infty}, a^{1/p^\infty})/\mathbf{Q}(\zeta_{p'}))]]$ -module, where a is a rational number prime to p , L_∞ is the maximal unramified abelian p -extension of $\mathbf{Q}(\zeta_{p^\infty}, a^{1/p^\infty})$, and $p' = p$ (if $p > 2$) = 4 (if $p = 2$). Let q be a topological generator of $\text{Gal}(\mathbf{Q}(\zeta_{p^\infty}, a^{1/p^\infty})/\mathbf{Q}(\zeta_{p^\infty}))$. Then the non-commutative ring

$$\Lambda_q = \mathbf{Z}_p[[\text{Gal}(\mathbf{Q}(\zeta_{p^\infty}, a^{1/p^\infty})/\mathbf{Q}(\zeta_{p'}))]]$$

becomes to the Iwasawa algebra $\Lambda = \mathbf{Z}_p[[\text{Gal}(\mathbf{Q}(\zeta_{p^\infty})/\mathbf{Q}(\zeta_{p'}))]]$ under $q \rightarrow 1$, and $X/(q-1)X$ is related to $X_0 = \text{Gal}(L_0/\mathbf{Q}(\zeta_{p^\infty}))$, where L_0 is the maximal unramified abelian p -extension of $\mathbf{Q}(\zeta_{p^\infty})$. Therefore, our aim can be stated as to quantize the Iwasawa theory of cyclotomic fields. Our results follow from the Iwasawa theory and the commutation relation between $q-1$ and elements of Λ_q .

First we treat general cases. We show that X is a finitely generated Λ_q -module, and that for each $n \in \mathbf{N}$, $X/(q-1)^n X$ is a finitely generated and Λ -torsion Λ -module of μ -invariant 0 whose λ -invariant satisfies asymptotically $\alpha n + \beta$ for certain integers $\alpha \geq 0$ and $\beta \geq 0$. This deduces that the cyclotomic $\mathbf{Z}_p$ -extension of $\mathbf{Q}(a^{1/p^n})$ is of μ -invariant 0, which was already known by results of Ferrero-Washington [1] and Iwasawa [2].

Next we treat special cases where Vandiver's Conjecture holds for $p \neq 2$ and $X/(q-1)X = X_0$. Then it can be shown that there exist

$$F \in \Lambda'_q = \mathbf{Z}_p[[\text{Gal}(\mathbf{Q}(\zeta_{p^\infty}, a^{1/p^\infty})/\mathbf{Q})]]$$

and $x \in X$ such that

$$F|_{q=1, \gamma=(1+p)^s} = \sum_{i=0,1,2,4,\dots,p-3} \varepsilon_i + \sum_{i=3,5,\dots,p-2} \varepsilon_i L_p(s, \omega^{1-i}),$$

and that $\Lambda'_q \ni \alpha \mapsto \alpha x \in X$ induces a surjective Λ'_q -homomorphism $\Lambda'_q/(\Lambda'_q \cdot F) \rightarrow X$, where $\gamma \in \text{Gal}(\mathbf{Q}(\zeta_{p^\infty})/\mathbf{Q})$ is defined by $\gamma(\zeta_{p^n}) = \zeta_{p^n}^{1+p}$ ($n \in \mathbf{N}$), ω denotes the Teichmüller character, ε_i denotes the idempotent for ω^i , and $L_p(s, \omega^{1-i})$ denotes the p -adic L -function for ω^{1-i} . From this, we deduce the following inequality for the λ -invariant of the cyclotomic $\mathbf{Z}_p$ -extension of $\mathbf{Q}(\zeta_p, a^{1/p^n})$:

$$\text{rank}_{\mathbf{Z}_p}(X_n) \leq p^n \cdot \text{rank}_{\mathbf{Z}_p}(X_0),$$

where $L_n/\mathbf{Q}(\zeta_{p^\infty}, a^{1/p^n})$ is the maximal unramified abelian p -extension with Galois group X_n . As an analogy of the Iwasawa theory of cyclotomic fields, it seems to be interesting if there are some relations between F and the q -analogue of $L_p(s, \omega^{1-i})$ constructed by Koblitz [3].

1 Quantized Iwasawa algebra

1.1. Let p be a fixed prime number, and put $p' = p$ (if $p > 2$) and $p' = 4$ (if $p = 2$). Let Σ be the pro- p group generated by γ and q with the single relation

$$\gamma \cdot q = q^{p'+1} \cdot \gamma.$$

Let Θ be the closed subgroup of Σ generated by q , and put $\Gamma = \Sigma/\Theta$. Then Θ and Γ are isomorphic to $\mathbf{Z}_p$ with generators q and $\gamma\Theta$ respectively. Let Λ_q denote the completed group ring $\mathbf{Z}_p[[\Sigma]]$. Then by a result of Serre [4], under the correspondence $\gamma \leftrightarrow 1 + T$ and $q \leftrightarrow 1 + S$, Λ_q is isomorphic to, and hence is identified with, the quotient ring of $\mathbf{Z}_p[[T, S]]_{\text{n.c.}}$ (: the non-commutative power series ring over $\mathbf{Z}_p$ with variables T and S) by the single relation

$$(1.1.1) \quad (1 + T)(1 + S) = (1 + S)^{p'+1}(1 + T),$$

which is equivalent to

$$(1.1.2) \quad TS = ST + p'S(1 + T) + \sum_{i=2}^{p'+1} \binom{p'+1}{i} S^i(1 + T).$$

The algebra Λ_q is a complete local ring with maximal ideal (p, T, S) . Let Λ denote the Iwasawa algebra $\mathbf{Z}_p[[\Gamma]] = \mathbf{Z}_p[[T]]$. Then by putting $q = 1$, Λ_q becomes to Λ , so we call Λ_q the *quantized Iwasawa algebra*. By (1.1.2), any $\alpha \in \Lambda_q$ can be uniquely expressed as

$$\alpha = \sum_{n=0}^{\infty} S^n \alpha_n \quad (\alpha_n \in \Lambda).$$

In what follows, Λ_q (resp. Λ)-modules mean topological additive groups on which Λ_q (resp. Λ) acts continuously. Since Λ_q contains Λ naturally, any Λ_q -module can be regarded as a Λ -module.

1.2. Lemma. *If M is a compact left Λ_q -module such that $v_1, \dots, v_n \in M$ generate $M/(p, T, S)M$ over $\mathbf{F}_p$, then they generate M over Λ_q . In particular,*

$$M/(p, T, S)M = \{0\} \iff M = \{0\}.$$

Proof. One can prove this in the same way as for Lemma 13.16 of [4].

1.3. Lemma. *For any $\alpha \in \Lambda_q$, there exists a unique $\alpha' \in \Lambda_q$ such that $\alpha S = S\alpha'$.*

Proof. It follows from (1.1.2).

1.4. Corollary. *For any left Λ_q -module M and $n \in \mathbf{N}$, $S^n M$ is a left sub Λ_q -module of M .*

1.5. Lemma. *For any $\alpha \in \Lambda_q$, there exists a unique $\alpha' \in \Lambda_q$ such that $S\alpha = \alpha'S$. Then*

$$\min\{n \mid p \nmid \alpha_{0,n}\} = \min\{n \mid p \nmid \alpha'_{0,n}\},$$

where $\alpha_0 = \sum_{n=0}^{\infty} \alpha_{0,n} T^n$ ($\alpha_{0,n} \in \mathbf{Z}_p$) and $\alpha'_0 = \sum_{n=0}^{\infty} \alpha'_{0,n} T^n$ ($\alpha'_{0,n} \in \mathbf{Z}_p$).

Proof. By (1.1.2),

$$(1 + p' + \sum_{i=2}^{p'+1} \binom{p'+1}{i} S^{i-1}) ST = (T - p' - \sum_{i=2}^{p'+1} \binom{p'+1}{i} S^{i-1}) S.$$

Therefore, if $\alpha = T$, then

$$\alpha' = (1 + p' + \sum_{i=2}^{p'+1} \binom{p'+1}{i} S^{i-1})^{-1} (T - p' - \sum_{i=2}^{p'+1} \binom{p'+1}{i} S^{i-1}),$$

and hence

$$\alpha'_0 = (1 + p')^{-1} (T - p') \equiv T \pmod{p\Lambda}.$$

1.6. Lemma. For each integer $n \geq 0$, put $\sigma_n = (1 + S)^{p^n} - 1$. Then for any $\alpha \in \Lambda_q$, there exists a unique $\alpha' \in \Lambda_q$ such that $\alpha\sigma_n = \sigma_n\alpha'$.

Proof. It follows from

$$T\sigma_n = \sigma_n \left\{ \left(\sum_{i=0}^{p'-1} (\sigma_n + 1)^i \right) (1 + T) - 1 \right\}.$$

1.7. Corollary. For any left Λ_q -module M and $n \geq 0$, $\sigma_n M$ is a left sub Λ_q -module of M .

2 General Case

2.1. Let $\{\zeta_{p^n}\}_{n \in \mathbb{N}}$ be a set of primitive p^n -th roots ζ_{p^n} of 1 such that $\zeta_{p^{n+1}}^p = \zeta_{p^n}$ ($n \in \mathbb{N}$), and put

$$\mathbb{Q}(\zeta_{p^\infty}) = \bigcup_{n \in \mathbb{N}} \mathbb{Q}(\zeta_{p^n}).$$

Let $l_i \neq p$ be prime numbers, m_i positive integers ($i = 1, \dots, k$) prime to p , and put $a = \prod_{i=1}^k l_i^{m_i}$. Let $\{a^{1/p^n}\}_{n \in \mathbb{N}}$ be a set of p^n -th roots a^{1/p^n} of a such that $(a^{1/p^{n+1}})^p = a^{1/p^n}$ ($n \in \mathbb{N}$), and put

$$\mathbb{Q}(\zeta_{p^\infty}, a^{1/p^\infty}) = \bigcup_{n \in \mathbb{N}} \mathbb{Q}(\zeta_{p^n}, a^{1/p^n}).$$

Let γ and q be elements of $\text{Gal}(\mathbb{Q}(\zeta_{p^\infty}, a^{1/p^\infty})/\mathbb{Q}(\zeta_{p'}))$ defined by

$$\gamma(\zeta_{p^n}) = \zeta_{p^n}^{1+p'}, \quad \gamma(a^{1/p^n}) = a^{1/p^n} \quad (n \in \mathbb{N})$$

and

$$q(\zeta_{p^n}) = \zeta_{p^n}, \quad q(a^{1/p^n}) = \zeta_{p^n} \cdot a^{1/p^n} \quad (n \in \mathbb{N})$$

respectively. Then $\text{Gal}(\mathbb{Q}(\zeta_{p^\infty}, a^{1/p^\infty})/\mathbb{Q}(\zeta_{p'}))$ is isomorphic to, and hence is identified with, the group Σ defined in 1.1, and via this identification, $\Theta = \text{Gal}(\mathbb{Q}(\zeta_{p^\infty}, a^{1/p^\infty})/\mathbb{Q}(\zeta_{p^\infty}))$ and $\Gamma = \text{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q}(\zeta_{p'}))$. Let L_∞

be the maximal unramified abelian p -extension of $\mathbf{Q}(\zeta_{p^\infty}, a^{1/p^\infty})$, and put $X = \text{Gal}(L_\infty/\mathbf{Q}(\zeta_{p^\infty}, a^{1/p^\infty}))$. Let $\sigma \in \Sigma = \text{Gal}(\mathbf{Q}(\zeta_{p^\infty}, a^{1/p^\infty})/\mathbf{Q}(\zeta_{p'}))$ act on X as

$$\sigma \cdot x = \tilde{\sigma} x \tilde{\sigma}^{-1} \quad (x \in X),$$

where $\tilde{\sigma} \in \text{Gal}(L_\infty/\mathbf{Q}(\zeta_{p'}))$ is a lifting of σ . Then this action is well-defined, and hence we can regard X as a left $\Lambda_q (= \mathbf{Z}_p[[\Sigma]])$ -module.

2.2. Lemma. *Let $l \neq p$ be a rational prime. Then the set of primes of $\mathbf{Q}(\zeta_{p^\infty})$ lying above l is a finite set whose cardinality is equal to the index of $\langle l \rangle$ in $\mathbf{Z}_p^\times$, where $\langle l \rangle$ denotes the closed subgroup of $\mathbf{Z}_p^\times$ generated by l .*

2.3. Let $\{l_i\}$ be as above. Then by Lemma 2.2, there exist finitely many primes of $\mathbf{Q}(\zeta_{p^\infty})$ lying above $l_1, \dots, l_k$, which we denote by $\lambda_1, \dots, \lambda_m$. For each $j = 1, \dots, m$, let $\tilde{\lambda}_j$ be a prime of L_∞ lying above λ_j , and $I_j \subset \text{Gal}(L_\infty/\mathbf{Q}(\zeta_{p^\infty}))$ the inertia group of $\tilde{\lambda}_j/\lambda_j$. Since $L_\infty/\mathbf{Q}(\zeta_{p^\infty}, a^{1/p^\infty})$ is unramified and $\mathbf{Q}(\zeta_{p^\infty}, a^{1/p^\infty})/\mathbf{Q}(\zeta_{p^\infty})$ is totally ramified at λ_j , the inclusion $I_j \hookrightarrow \text{Gal}(L_\infty/\mathbf{Q}(\zeta_{p^\infty}))$ induces a bijection

$$I_j \xrightarrow{\sim} \text{Gal}(L_\infty/\mathbf{Q}(\zeta_{p^\infty}))/X \cong \Theta,$$

and hence

$$\text{Gal}(L_\infty/\mathbf{Q}(\zeta_{p^\infty})) = XI_j \quad (j = 1, \dots, m).$$

Let $\sigma_j \in I_j$ maps to q . Then σ_j is a topological generator of I_j and there exists a unique $x_j \in X$ such that $\sigma_j = x_j \sigma_1$.

2.4. Proposition. *Let L_n be the maximal unramified abelian p -extension of $\mathbf{Q}(\zeta_{p^\infty}, a^{1/p^n})$, and put*

$$X' = SX + \sum_{j=2}^m \mathbf{Z}_p x_j, \quad X_n = X/(\sigma_n/S)X'.$$

Then

$$\text{Gal}(L_n/\mathbf{Q}(\zeta_{p^\infty}, a^{1/p^n})) \cong X_n.$$

Proof. One can prove this in the same way as for Lemma 13.15 of [5].

2.5. Proposition. X/SX is a finitely generated and Λ -torsion Λ -module with $\mu = 0$.

Proof. In Proposition 2.4, let $n = 0$. Then by the Iwasawa theory of $\mathbf{Z}_p$ -extensions and a result of [1], $\text{Gal}(L_0/\mathbf{Q}(\zeta_{p^\infty})) \cong X/X'$ is a finitely generated and Λ -torsion Λ -module with $\mu = 0$. On the other hand,

$$X'/SX \cong N/(N \cap SX) \quad (N := \sum_{j=2}^m \mathbf{Z}_p x_j)$$

is isomorphic modulo a finite group to a free $\mathbf{Z}_p$ -module of finite rank, and hence this μ -invariant is equal to 0. Therefore, X/SX is a finitely generated and Λ -torsion Λ -module with $\mu = 0$.

2.6. Theorem. X is a finitely generated left Λ_q -module.

Proof. It follows from Lemma 1.2 and Proposition 2.5.

2.7. Lemma. Let M be a left Λ_q -module such that there exists a Λ -homomorphism with finite cokernel

$$\varphi : \bigoplus_{i=1}^n (\Lambda/(g_i)) \longrightarrow M/SM$$

for some $g_i \in \Lambda$ ($i = 1, \dots, n$). Then there exists a Λ -homomorphism with finite cokernel

$$\psi : \bigoplus_{i=1}^n (\Lambda/(h_i)) \longrightarrow SM/S^2M,$$

where $h_i \in \Lambda$ ($i = 1, \dots, n$) such that $Sg_i \equiv h_i S \pmod{S^2\Lambda_q}$.

Proof. By the assumption, there exist $\alpha_i \in M$ such that $g_i \alpha_i \in SM$ and that $\sum_{i=1}^n \Lambda \alpha_i + SM$ is finite index in M . Hence there exist $e_j \in M$ ($j = 1, \dots, m$) such that

$$M = \bigcup_{j=1}^m \left(\sum_{i=1}^n \Lambda \alpha_i + SM \right) + e_j.$$

By Propositions 1.3 and 1.5, $S\Lambda \alpha_i \subset \Lambda S\alpha_i + S^2M$. Hence we have

$$SM = \bigcup_{j=1}^m \left(\sum_{i=1}^n \Lambda S\alpha_i + S^2M \right) + Se_j$$

and

$$h_i S\alpha_i \equiv Sg_i \alpha_i \equiv 0 \pmod{S^2M}.$$

This completes the proof.

2.8. Lemma. *Let N be a finitely generated Λ -module. Then N is a Λ -torsion Λ -module with $\mu = 0$ if and only if N/pN is a finite group.*

Proof. It follows from the structure theorem of finitely generated Λ -modules ([5], Theorem 13.12).

2.9. Theorem. *For each $n \in \mathbf{N}$, $X/S^n X$, $X/\sigma_n X$ and X_n are finitely generated and Λ -torsion Λ -modules with $\mu = 0$. Moreover, there exist integers $\alpha \geq 0$ and $\beta \geq 0$ independent of n , and an integer n_0 such that for all $n \geq n_0$,*

$$\text{rank}_{\mathbf{Z}_p}(X/S^n X) = \alpha n + \beta.$$

Proof. By Corollary 1.4 and Lemma 2.7, $X/S^n X$ is a finitely generated and Λ -torsion Λ -module with $\mathbf{Z}_p$ -rank satisfying the above asymptotic behavior. Hence by Lemma 2.8 and that $\sigma_n X + pX = S^n X + pX$, $X/\sigma_n X$ and X_n are finitely generated and Λ -torsion Λ -modules with $\mu = 0$.

3 Special Case

3.1. Let p be an odd prime not dividing the class number of $\mathbf{Q}(\zeta_p + \zeta_p^{-1})$, and let l be a prime congruent modulo p^2 to a topological generator of $\mathbf{Z}_p^\times$ (there exist infinitely many such primes by Dirichlet's theorem on arithmetic progressions). Then by Lemma 2.2, there exists only one prime of $\mathbf{Q}(\zeta_{p^\infty})$ lying above l . Put $a = l$ and let the notation be as in §2. Then $X_n = X/\sigma_n X$. Let Λ'_q (resp. Λ') be the completed group algebra $\mathbf{Z}_p[[\Sigma']]$ (resp. $\mathbf{Z}_p[[\Gamma']]$) of $\Sigma' = \text{Gal}(\mathbf{Q}(\zeta_{p^\infty}, a^{1/p^\infty})/\mathbf{Q})$ (resp. $\Gamma' = \text{Gal}(\mathbf{Q}(\zeta_{p^\infty})/\mathbf{Q})$). Then $\Gamma' = \Sigma'/\Theta$, and hence Λ'_q becomes to Λ by putting $q = 1$. Regard Γ' as a subgroup of Σ' by

$$\gamma(a^{1/p^n}) = a^{1/p^n} \quad (\gamma \in \Gamma', n \in \mathbf{N}).$$

Then Λ'_q contains Λ' naturally. Put $\Delta = \text{Gal}(\mathbf{Q}(\zeta_p)/\mathbf{Q})$, and regard Δ as a subgroup of Γ' by the Teichmüller character $\omega : \mathbf{F}_p^\times \rightarrow \mathbf{Z}_p^\times$ and the identifications $\Delta = \mathbf{F}_p^\times$, $\Gamma' = \mathbf{Z}_p^\times$. For each $i = 0, 1, \dots, p-2$, put

$$\varepsilon_i = \frac{1}{|\Delta|} \sum_{\delta \in \Delta} \omega^{-i}(\delta) \cdot \delta \in \mathbf{Z}_p[\Delta],$$

For each $i = 3, 5, \dots, p-2$, let $L_p(s, \omega^{1-i})$ denote the p -adic L -function with character ω^{1-i} , and f_i the element of Λ such that

$$f_i|_{T=(1+p)^s-1} = L_p(s, \omega^{1-i}).$$

Let $\sigma \in \Sigma'$ act on X as

$$\sigma \cdot x = \tilde{\sigma} x \tilde{\sigma}^{-1} \quad (\sigma \in \Sigma', x \in X),$$

where $\tilde{\sigma} \in \text{Gal}(L_\infty/\mathbf{Q})$ is a lifting of σ . Then this action is an extension of the above action of Σ on X .

3.2. Proposition. *There exist $F \in \Lambda'_q$ and $x \in X$ such that*

$$F|_{S=0} = \sum_{i=0,1,2,4,\dots,p-3} \varepsilon_i + \sum_{i=3,5,\dots,p-2} \varepsilon_i f_i$$

and that $\Lambda'_q \ni \alpha \mapsto \alpha x \in X$ induces a surjective Λ'_q -homomorphism

$$\Lambda'_q / (\Lambda'_q \cdot F) \longrightarrow X.$$

Proof. Put

$$f = \sum_{i=0,1,2,4,\dots,p-3} \varepsilon_i + \sum_{i=3,5,\dots,p-2} \varepsilon_i f_i.$$

Then it is known (cf. [5], Theorem 10.14 and 10.16) that there exists $x \in X$ such that $\Lambda' \ni a \mapsto ax \bmod(SX) \in X_0$ induces an Λ' -isomorphism $\Lambda'/(f) \xrightarrow{\sim} X_0$. Hence x generates X over Λ'_q (cf. Lemma 1.2) and there exist $F \in \Lambda'_q$ satisfying the above conditions.

3.3. Theorem. $\text{rank}_{\mathbb{Z}_p}(X_n) \leq p^n \cdot \text{rank}_{\mathbb{Z}_p}(X_0).$

Proof. Let $x \in X$ and $F \in \Lambda'_q$ be as in Proposition 3.2. Then $S^k \varepsilon_i x$ ($i = 0, \dots, p-2, k = 0, \dots, p^n-1$) generate X_n over Λ . Since $S^l \varepsilon_j F x = 0$, $\varepsilon_j S \in S\Lambda'_q$, and $S^{p^n} \in \sigma_n \Lambda_q + p\Lambda_q$, for each $j = 0, \dots, p-2$ and $l = 0, \dots, p^n-1$, there exist $a_{ijkl} \in \Lambda$ such that

$$\begin{aligned} a_{ijkl} &= 0 \quad (k < l), \\ a_{ij00} &= \begin{cases} \delta_{ik} & (i = 0, 1, 2, 4, \dots, p-3) \\ \delta_{ik} f_i & (i = 3, 5, \dots, p-2), \end{cases} \\ S a_{ijkk} &\equiv a_{ijk+1k+1} S \bmod(S^2 \Lambda_q), \end{aligned}$$

and

$$\sum_{i=0}^{p-2} \sum_{k=0}^{p^n-1} a_{ijkl} S^k \varepsilon_i x \in \sigma_n X + pX.$$

Let d_i ($i = 3, 5, \dots, p-2$) be the minimal degree of non-zero terms of $f_i \bmod(p)$. Then by Lemma 1.5, the minimal degree of non-zero terms of

$a_{iik} \bmod(p)$ is also d_i , and hence

$$\begin{aligned} \operatorname{rank}_{\mathbf{Z}_p}(X_n) &\leq \operatorname{rank}_{\mathbf{F}_p}(X_n \otimes_{\mathbf{Z}_p} \mathbf{F}_p) \\ &= p^n \cdot \sum_{i=3,5,\dots,p-2} d_i \\ &= p^n \cdot \operatorname{rank}_{\mathbf{Z}_p}(X_0). \end{aligned}$$

References

1. Ferrero, B., Washington, L.: The Iwasawa invariant μ_p vanishes for abelian number fields. *Ann. Math.* **109**, 377-395 (1979)
2. Iwasawa, K.: On the μ -invariants of $\mathbf{Z}_l$ -extensions. In: *Number theory, Algebraic Geometry and Commutative Algebra*, in honor of Yasuo Akizuki, pp.1-11. Tokyo: Kinokuniya 1973
3. Koblitz, N.: On Carlitz's q -Bernoulli numbers. *J. Number Theory.* **14**, 332-339 (1982)
4. Serre, J. P.: *Cohomologie galoisienne*. *Lecture Notes Mathematics*, Vol.5. Berlin Heidelberg New York: Springer 1965
5. Washington, L.: *Introduction to Cyclotomic Fields*. *Graduate Texts in Mathematics* 83. Berlin Heidelberg New York: Springer 1982