

On the fixed point set of a unipotent
transformation on generalized flag varieties

Naohisa Shimomura
Hiroshima University

Introduction

Let $G = GL_n$ be the general linear group defined over a field K . Let P be a parabolic subgroup of G . For a unipotent element u of G , put

$$(G/P)_u = \{gP \in G/P \mid u \cdot gP = gP\},$$

the fixed point subvariety of u in a generalized flag variety G/P . The author [2] obtained a locally closed partition of $(G/P)_u$ into affine spaces. This is a generalization of a result of N. Spaltenstein [3]. The purpose of this report is to give an alternate proof to the result of [2]. The proof in this report is simpler than that of [2] and, it seems, applicable for other groups. Some applications (in particular, on the character theory of the finite general linear groups) of this paper are described in [1] with other results on the Springer representations of Weyl groups for reductive groups.

Notations. Let V be a vector space over a field K . If $\{x_v \mid v \in I\}$ is a subset of V , then we denote by $\langle x_v \mid v \in I \rangle$ the subspace spanned by $\{x_v\}$. We denote by N the set of all natural numbers. For $n \in N$, let A^n be the n -dimensional affine space over K . If $\{X_v\}$ is a family of subsets of a set X , then $X = \coprod_v X_v$ means the direct sum decomposition of X . A partition λ of n means a sequence $\lambda = (n_1, n_2, \dots, n_r)$ such that $n_i \in N$ ($i=1, \dots, r$), $n_1 + n_2 + \dots + n_r = n$ and $n_1 \geq n_2 \geq \dots \geq n_r > 0$.

§1. Preliminaries

Let G, P and u be as in the introduction. There exists $\mu = (\mu_1, \dots, \mu_r)$ (resp. $\lambda = (\lambda_1, \dots, \lambda_s)$), a partition of n , such that P (resp. u) is conjugate to P_μ (resp. u_λ), where P_μ is a parabolic subgroup of G whose Levi subgroup is isomorphic to $\prod_{i=1}^r GL_{\mu_i}$ (resp. the unipotent element of Jordan type diag $(J_1, \dots, J_s)$, $J_i = \left(\begin{smallmatrix} 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & \\ & & & 1 \end{smallmatrix} \right)_{\lambda_i}$). Then $(G/P)_u$ is isomorphic to $(G/P_\mu)_{u_\lambda}$.

For λ , a partition of n , we can associate the Young diagram of type λ , in the usual way.

Definition 1. Let λ and μ be partitions of n . Put $\mu = (\mu_1, \dots, \mu_r)$

(1) A μ -tableau of type λ is a Young diagram of type λ whose nodes are numbered with the figures from 1 to r such that the cardinality of the nodes with figure i is μ_i .

(2) A μ -tableau is said to be semi-standard if, in each row,

the sequence of the figures on the nodes increases (may be stationary).

Example. If $\lambda = (3, 2, 1)$ and $\mu = (2, 2, 1, 1)$, then

$$(1) \begin{array}{|c|c|c|} \hline 1 & 3 & 1 \\ \hline 4 & 2 & \\ \hline 2 & & \\ \hline \end{array} \quad (2) \begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 2 & 2 & \\ \hline 4 & & \\ \hline \end{array}$$

are μ -tableaus of type λ ((2) is semi-standard). If $\mu = (\mu_1, \mu_2)$, then, for simplicity, we may write $\boxed{1}$ as $\boxed{\text{diagonal lines}}$ and $\boxed{2}$ as $\boxed{\text{empty}}$.

Example. If $\lambda = (3, 2, 1)$ and $\mu = (4, 2)$, then

$$\begin{array}{|c|c|c|} \hline 2 & 1 & 1 \\ \hline 1 & 2 & \\ \hline 1 & & \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline & \boxed{\text{diagonal lines}} & \boxed{\text{diagonal lines}} \\ \hline \boxed{\text{diagonal lines}} & & \\ \hline \boxed{\text{diagonal lines}} & & \\ \hline \end{array}.$$

Let $\widetilde{L}_\mu(\lambda)$ (resp. $L_\mu(\lambda)$) be the set of all μ -tableaus of type λ (resp. the set of all semi-standard μ -tableaus of type λ).

§2. The Grassmann manifold

Let $V = \langle v_1, \dots, v_n \rangle$ be an n -dimensional vector space over a field K with basis $\{v_1, \dots, v_n\}$. We denote by $G_k(V)$ the Grassmann manifold defined by the set of all k -dimensional subspaces of V . Put $L_k = \{(s_1, \dots, s_k) \in \mathbb{N}^k \mid 1 \leq s_1 < s_2 < \dots < s_k \leq n\}$, a set of increasing sequence of natural numbers. For $s = (s_1, \dots, s_k) \in L_k$, let S_s be the set of vector subspaces defined by

$$\{ \langle v_{s_m} + \sum_{i>s_m} a_{mi} v_i \mid 1 \leq m \leq k \rangle \mid a_{mi} \in K \}.$$

We remark that we can associate to S_s the following tableau :

$$\begin{array}{|c|c|c|c|c|c|c|c|} \hline v_1 & v_2 & \cdots & v_{s_1} & \cdots & v_{s_m} & \cdots & v_{s_k} & \cdots & v_n \\ \hline \end{array}.$$

The next lemma gives a well-known cellular decomposition of the Grassmann manifold.

Lemma 1. (1) $G_k(V) = \bigsqcup_{s \in L_k} S_s,$

$$(2) \quad \langle v_{s_m} + \sum_{i > s_m} a_{mi} v_i \mid 1 \leq m \leq k \rangle = \langle v_{s_m} + \sum_{I_m} a'_{mi} v_i \mid 1 \leq m \leq k \rangle,$$

where I_m is a condition: $i > s_m, i \neq s_{m+1}, \dots, s_k,$

(3) put $e(s) = \sum_{m=1}^k \{(n-s_m) - (k-m)\},$ then by (2), we have an isomorphism $A^{e(s)} \xrightarrow{\sim} S_s$ under a mapping: $(\dots, a_{mi}, \dots) \mapsto \langle v_{s_m} + \sum_{I_m} a_{mi} v_i \mid 1 \leq m \leq k \rangle,$

(4) S_s is a locally closed subset of $G_k(V)$ in the K-Zariski topology.

Let N be a nilpotent transformation of V . We take a Jordan basis $\{w_{ij_i} \mid 1 \leq j_i \leq l_i\}$ of V satisfying the following requirement:

$$l_1 \leq l_2 \leq \dots \leq l_d, \quad Nw_{ij} = w_{i+1j} \quad \text{and} \quad Nw_{dj} = 0.$$

We remark that this basis forms a Young diagram of degree n and of type $\lambda = \lambda(N) = (\underbrace{d, \dots, d}_{l_1}, \dots, \underbrace{1, \dots, 1}_{l_d - l_{d-1}}).$

Example. Let $\dim V = 8$. If N has two Jordan blocks of dimension 3 and one Jordan block of dimension 2, then

$$\begin{array}{|c|c|c|} \hline w_{31} & w_{21} & w_{11} \\ \hline w_{32} & w_{22} & w_{12} \\ \hline w_{33} & w_{23} & \\ \hline \end{array}.$$

Put $u_\lambda = I_n + N$, I_n is the identity matrix of size n , then u_λ is a unipotent element of $GL_n = GL(V)$ of Jordan type λ . We place w_{ij} in the following way:

$$v_1 = w_{1\ell_1}, \dots, v_{\ell_1} = w_{11}, v_{\ell_1+1} = w_{2\ell_2}, \dots, v_{\ell_1+\ell_2} = w_{21}, \dots, v_n = w_{d1}.$$

For $\bar{k} = (k, n-k)$ and $\lambda = \lambda(N)$, put

$$\widetilde{L_{\bar{k}}}(\lambda) = \{ \bar{k}\text{-tableaus of type } \lambda \} = \left\{ \begin{array}{c} \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array} & \left| \begin{array}{l} \text{the number of} \\ \square \text{ is } k \end{array} \right. \end{array} \right\}.$$

We have a bijective correspondence between $\widetilde{L_{\bar{k}}}(\lambda)$ and L_k by making a sequence $(s_1, \dots, s_k) \in L_k$ if v_{s_1} is in a node $\square$.

Then by Lemma 1, (1), we can write $G_k(V) = \sum_{\ell \in \widetilde{L_{\bar{k}}}(\lambda)} S_\ell$. Put

$$G_k(V)^N = \{W \in G_k(V) \mid N(W) \subset W\},$$

$$S_\ell^N = S_\ell \cap G_k(V)^N.$$

Let $L_{\bar{k}}(\lambda)$ be the set of all semi-standard $\bar{k}$ -tableau of type

$$\lambda = \lambda(N), \text{ e.g. } L_{\bar{k}}(\lambda) = \left\{ \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array} \right\}.$$

Lemma 2. Let $\ell \in \widetilde{L_{\bar{k}}}(\lambda)$. In order to have $S_\ell^N \neq \emptyset$, it is necessary and sufficient that $\ell \in L_{\bar{k}}(\lambda)$

Proof. We assume $S_\ell^N \neq \emptyset$. For this $\ell \in \widetilde{L_{\bar{k}}}(\lambda)$, let $v_{k_m} = w_{i_m j_m}$ ($m=1, 2, \dots$; $k_1 < k_2 < \dots$) be the w_{ij} which is in the rightest node in the m -th row from the top in the tableau obtained by extracting the nodes $\square$ from ℓ . For example

$$\lambda = \begin{array}{|c|c|c|c|} \hline \text{shaded} & \text{shaded} & & v_{k_1} \\ \hline \text{shaded} & \text{shaded} & v_{k_2} & \\ \hline v_{k_3} & & & \\ \hline & & & \\ \hline \end{array}.$$

For $W \in S_{\lambda}^N$, there exists $a_{mj} \in K$ such that

$$v_{k_m} + \sum_{j > k_m} a_{mj} v_j \in W \quad (m=1,2,\dots).$$

By $N(W) \subset W$, we have

$$N^{h_m}(v_{k_m} + \sum_{j > k_m} a_{mj} v_j) \in W \quad (0 \leq h_m \leq d - i_m).$$

The set $\{N^{h_m}(v_{k_m} + \sum_{j > k_m} a_{mj} v_j) \mid \begin{matrix} m=1,2,\dots \\ 0 \leq h_m \leq d - i_m \end{matrix}\}$ is linearly

independent and the number of its elements is greater than $k = \dim W$. Hence the set

$$\left\{ N^{h_m}(v_{k_m} + \sum_{j > k_m} a_{mj} v_j) \mid \begin{matrix} m=1,2,\dots \\ 0 \leq h_m \leq d - i_m \end{matrix} \right\}$$

must be a basis of W , which implies, by definition,

$$\lambda \in L_{\bar{K}}(\lambda).$$

Conversely, for a semi-standard $\lambda \in L_{\bar{K}}(\lambda)$, put

$$\begin{array}{|c|c|c|c|} \hline \text{shaded} & \text{shaded} & \text{shaded} & \\ \hline & & & \\ \hline \text{shaded} & & & \\ \hline \text{shaded} & & & \\ \hline \end{array}$$

$W = \langle w_{i,j} \text{ in } \text{shaded} \rangle$. Then $W \in S_{\lambda}^N$. This means that $S_{\lambda}^N \neq \emptyset$. The proof of the lemma is thus completed.

Let $\lambda \in L_{\bar{K}}(\lambda)$. In the tableau λ , let $v_{k_m} = w_{i_m j_m}$ ($m=1,2,\dots$; $k_1 < k_2 < \dots$) be as in the proof of Lemma 2. Put

$$M_\ell = \left\{ N^{h_m} v_{k_m} \mid \begin{array}{l} m=1,2,\dots, \\ 0 \leq h_m \leq d-i_m \end{array} \right\} = \{w_{ij} \text{ in } \boxed{\text{diagonal}}\}.$$

Lemma 3. For $\ell \in L_{\overline{K}}(\lambda)$, we have

$$S_\ell^N = \left\{ \left\langle N^{h_m} (v_{k_m} + \sum_{j > k_m} a_{mj} v_j) \mid \begin{array}{l} m=1,2,\dots, \\ 0 \leq h_m \leq d-i_m \end{array} \right\rangle \mid \begin{array}{l} a_{mi} \in K, \\ a_{mi}=0 \text{ if } v_i \in M_\ell \end{array} \right\}.$$

Proof. It is obvious that S_ℓ^N contains the right-hand side. Apply Lemma 1 (2) to elements of S_ℓ . Then the proof of this lemma is similar to that of Lemma 2. Thus the lemma.

Definition 2. Let $\ell \in L_{\overline{K}}(\lambda)$. For $v_{k_m} = w_{i_m j_m}$ ($m=1,2,\dots$), let $n(\ell)_m$ be the number of $\boxed{\text{diagonal}}$ in ℓ which lies in the left-hand side of the column on which v_{k_m} lies, or in the upper position than that of $\boxed{v_{k_m}}$ in the column on which v_{k_m} lies. Put

$$n(\ell) = n(\ell)_1 + n(\ell)_2 + \dots.$$

We remark that $n(\ell) \leq e(\ell)$, where $e(\ell)$ is defined in Lemma 1 (3).

Example.

For $\ell =$

v_{k_3}	$\circ$	$\circ$	
$\boxed{\text{diagonal}}$	$\boxed{\text{diagonal}}$	v_{k_2}	
$\boxed{\text{diagonal}}$	$\boxed{\text{diagonal}}$	v_{k_1}	
$\circ$	$\circ$		

, $n(\ell)_2$ is the number of $\boxed{\circ}$ in ℓ .

In this case $n(\ell)_2 = n(\ell)_1 = 4$, $n(\ell)_3 = 0$ and

$$n(\ell) = n(\ell)_1 + n(\ell)_2 + n(\ell)_3 = 4 + 4 + 0 = 8.$$

In view of Lemma 1, (3), we have :

Corollary. For $\ell \in L_{\overline{K}}(\lambda)$, we have

$$S_{\ell}^N \simeq \mathbb{A}^{n(\ell)}.$$

Put $T_{\ell} = S_{\ell}^N$. By Lemma 3, T_{ℓ} is a closed subset (linear subvariety) of S_{ℓ} . Summing up the above statements, we have :

Theorem 1. Let the notations be as above. We have

$$G_K(V)^N = \bigsqcup_{\ell \in L_{\overline{K}}(\lambda)} T_{\ell},$$

where T_{ℓ} is a locally closed subset of $G_K(V)^N$ and isomorphic to an $n(\ell)$ -dimensional affine space $\mathbb{A}^{n(\ell)}$.

§3. The flag manifold

Let $\mu = (\mu_1, \dots, \mu_p, \mu_{p+1})$ be a partition of n . Put $k_j = \mu_1 + \dots + \mu_j$ ($j=1, 2, \dots, p, p+1$). Then $1 \leq k_1 < k_2 < \dots < k_p < k_{p+1} = n$. For $j=1, 2, \dots, p$, we denote by $\mathcal{F}_j$ the flag manifold of type $(k_1, \dots, k_j)$ defined by

$$\{(W_1, \dots, W_j) \in G_{k_1}(V) \times \dots \times G_{k_j}(V) \mid W_i \subset W_{i+1} \ (1 \leq i \leq j-1)\}.$$

Then, $\mathcal{F}_j$ is isomorphic to $GL_{k_{j+1}}/P(\mu_1, \dots, \mu_{j+1})$, where

$P(\mu_1, \dots, \mu_{j+1})$ is a parabolic subgroup of $GL_{k_{j+1}}$ whose Levi subgroup is isomorphic to $\prod_{i=1}^{j+1} GL_{\mu_i}$. In particular, if $j=p$, then

$\mathcal{F}_p \simeq GL_n/P_{\mu}$. For a nilpotent transformation N of V , put

$$\mathcal{Y}_j^N = \{(W_i) \in \mathcal{Y}_j \mid N(W_i) \subset W_i \quad (1 \leq i \leq j)\}.$$

If $u_\lambda = 1_n + N$ is the corresponding unipotent element of GL_n , then $\mathcal{Y}_p^N \simeq (GL_n/P_\mu)_{u_\lambda}$.

We preserve the notations in §2. For $\ell \in L_{\bar{k}_p}(\lambda)$ ($\bar{k}_p = (k_p, \mu_{p+1})$), put $V_\ell = \langle w_{ij} \text{ in } \mathbb{Z} \rangle$. We remark that V_ℓ is a element of $T_\ell = S_\ell^N$. If $W \in T_\ell$, then the projection $f: V \rightarrow V_\ell$ induces an N -module isomorphism $f_W: W \xrightarrow{\sim} V_\ell$. By the projection

$$\pi_p: \mathcal{Y}_p \longrightarrow G_{k_p}(V) \quad ((W_1, \dots, W_p) \mapsto W_p),$$

we have the following trivialization:

$$\pi_p^{-1}(T_\ell) \xrightarrow{\sim} \mathcal{Y}_{p-1} \times T_\ell \quad ((W_i) \mapsto (f_{W_p}(W_1), \dots, f_{W_p}(W_{p-1})), W_p)).$$

Under this trivialization, we have

$$\pi_p^{-1}(T_\ell) \cap \mathcal{Y}_p^N \xrightarrow{\sim} \mathcal{Y}_{p-1}^N \times T_\ell,$$

and therefore $\mathcal{Y}_p^N \xrightarrow{\sim} \frac{1 \mid 1}{\ell \in L_{\bar{k}_p}(\lambda)} \mathcal{Y}_{p-1}^N \times T_\ell$. By induction, we have

$$\mathcal{Y}_p^N \xrightarrow{\sim} \frac{1 \mid 1}{\substack{\ell_j \in L_{\bar{k}_j}(\lambda_j) \\ j=1, \dots, p}} T_{\ell_1} \times T_{\ell_2} \times \dots \times T_{\ell_p},$$

where λ_j is the Young tableau obtained by extracting the nodes with figure $j+2, \dots, p+1$. Therefore, we can write

$$\mathcal{Y}_p^N = \frac{1 \mid 1}{\ell \in L_\mu(\lambda)} T_\ell,$$

where $L_\mu(\lambda)$ is the set of all semi-standard μ -tableaus of type λ and T_ℓ is isomorphic to some $T_{\ell_1} \times \dots \times T_{\ell_p}$ ($\ell_j \in L_{\bar{k}_j}(\lambda_j)$, $j=1, \dots, p$).

Remark. Similarly, we can prove that

$$\tilde{\mathcal{H}}_p = \frac{1}{|\widetilde{L_\mu(\lambda)}|} S_\ell,$$

where $\widetilde{L_\mu(\lambda)}$ is the set of all μ -tableaus of type λ . About this decomposition, we note that $T_\ell = S_\ell^N$ and $S_\ell^N \neq \emptyset$ if and only if $\ell \in L_\mu(\lambda)$.

Definition 3. For $\ell \in L_\mu(\lambda)$, let $n(\ell)$ be a non-negative integer defined by the following recurrence rule :

(1) If $\mu = (\mu_1, \mu_2)$ or (n) , then $n(\ell)$ is defined in

Definition 2.

(2) For $\mu = (\mu_1, \dots, \mu_p, \mu_{p+1})$, put $\mu' = (\mu_1, \dots, \mu_p)$ and $k_p = \mu_1 + \dots + \mu_p$. Let $\ell_1 \in L_{\bar{k}_p}(\lambda)$ ($\bar{k}_p = (k_p, \mu_{p+1})$) be the semi-standard $\bar{k}_p$ -tableau obtained from ℓ by changing the figures $p+1$ into 2 (or $\square$) and figures i ($1 \leq i \leq p$) into 1 (or $\boxplus$). Let ℓ_2 be the μ' -tableau obtained by extracting the nodes with figure $p+1$ from ℓ and by rearranging the rows in the appropriate order. Thus $\ell_2 \in L_{\mu'}(\lambda_{p-1})$ for some partition λ_{p-1} of k_p . Then we defines

$$n(\ell) = n(\ell_1) + n(\ell_2).$$

Theorem 2. Let λ and μ be a partition of n . The variety $(GL_n/P_\mu)_{u_\lambda}$ has a partition

$$(GL_n/P_\mu)_{u_\lambda} = \frac{1}{|\widetilde{L_\mu(\lambda)}|} T_\ell,$$

where T_ℓ is a locally closed subset of $(GL_n/P_\mu)_{u_\lambda}$ and isomorphic to an $n(\ell)$ -dimensional affine space $\mathbb{A}^{n(\ell)}$ and this

partition is defined over K .

References

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