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Short title: Constitutive modeling of inherent anisotropy for granular materials

Constitutive modeling of inherent anisotropy in a strain space multiple mechanism
model for granular materials

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SUMMARY

Consideration of fabric anisotropy is crucial to gaining an improved understanding of the behavior of granular materials. This paper presents a constitutive model to describe the sand behavior associated with fabric anisotropy within a framework of a strain space multiple mechanism model. In the proposed model, a second-order fabric tensor is extended by incorporating a new function that represents the effect of inherent (or initial fabric) anisotropy, along with three additional parameters: two of them, a_1 and a_2 , control the degree of anisotropy, and the second mode of inherent anisotropy can be expressed by introducing the parameter a_2 as well as the first mode by the parameter a_1 . The third parameter, θ_0 , expresses the principal direction of inherent anisotropy (e.g., the normal vector direction of bedding planes relative to horizontal axis). The formulation of the dilative component of dilatancy (i.e., positive dilatancy) is also extended to consider the effect of inherent anisotropy based on the interlocking mechanism. Experimental data on the complex anisotropic responses of Fraser River sand and Toyoura sand under monotonic loading is used to validate this model. The proposed model is shown to successfully capture anisotropic responses, which become contractive or dilative depending on different principal-stress directions, with a single set of anisotropy parameters; thus, the model is considered to possess the capability to simulate the anisotropic behaviors of granular materials. In addition to different loadings on the same fabric, the effects of different fabric anisotropies upon the sand behavior under the same loadings are also investigated.

Keywords: inherent anisotropy, constitutive model, granular material, fabric tensor,

dilatancy

1. INTRODUCTION

It is crucial to consider inherent (or initial fabric) anisotropy if one is to better understand the strength and deformation characteristics of granular materials as well as stress-induced anisotropy. While stress-induced anisotropy occurs during non-elastic deformation in response to applied stresses arising from changes in the spatial arrangement of granular material particles and voids [1, 2], inherent anisotropy is thought to arise from the preferred particle arrangement and contact orientation of granular particles during sedimentation [3, 4].

Fabric anisotropy of granular materials has been a subject of extensive investigation for almost 40 years. Arthur and Menzies [3], Oda [4], Arthur and Philips [5], and Oda and Koishikawa [6] have reported inherent anisotropy's effects on cohesionless soil strength and stress-strain relations using conventional triaxial tests. Oda et al. [7] found that anisotropic shear strength due to anisotropic parallel-particle alignment is more clearly exhibited under the plane-strain condition than using triaxial-compression tests. In addition to these tests, the hollow-cylinder torsional-shear apparatus has also been widely used to study soil behavior through inherent anisotropy since the 1990s [8-11]. By controlling the direction of principal stresses relative to the orientation of soil deposition, all tests demonstrated that drained and undrained monotonic behavior both depend upon the major principal-stress direction's orientation relative to bedding planes during shearing. Furthermore, principal stress rotation's influence upon responses under monotonic and cyclic loading has also been investigated [12-15].

In addition to laboratory tests, the discrete element method (DEM) has recently been employed to numerically study the effect of fabric anisotropy upon the behaviors of

granular materials [16-20]: Li and Li [16] proposed a new anisotropic fabric tensor based on a spatial tessellation of the domain and examined the correlation between the anisotropic fabric tensor and the macroscopic behavior of granular materials through two-dimensional DEM. Fu and Dafalias [17] conducted direct shear tests and biaxial compression tests using two-dimensional DEM models consisting of ellipse-shaped particles to investigate shear strength of granular materials with inherent anisotropy; similar biaxial compression tests were simulated by using convex polygon-shaped particles [18]. Additionally, fabric anisotropy at the critical state was examined as well as the evolution of inherent anisotropy based on two- or three-dimensional DEM simulations [19, 20]. These experimental and numerical outcomes can be used to validate and verify constitutive models, which consider inherent anisotropy effects based on continuum mechanics (e.g., [21-23]).

Among the various continuum-constitutive models proposed for granular materials, a strain space multiple mechanism model is one of the promising examples for describing their specific structure [24-28]. The model consists of a multitude of simple shear mechanisms, each oriented in an arbitrary direction. These mechanisms can idealize the internal (or micromechanical) structure of granular materials with an induced fabric anisotropy, as well as the evolution of this structure under monotonic and cyclic loadings and complicated loading paths, including the effect of rotation of the principal stress axes [29, 30]. The model has been implemented in a finite-element program and used to analyze numerous practical problems involved in evaluating the seismic performances of geotechnical works, such as level ground, retaining structures, embankments, and underground structures [31-38]. However, the influence of inherent anisotropy has not been incorporated into the model in an explicit form for describing

the textures of granular materials. Although the formulation without inherent anisotropy works relatively well in engineering practice, the dependence of soil behavior upon inherent anisotropy due to texture should be appropriately considered to gain a more accurate description of granular-material behavior.

This study aims to develop an anisotropic constitutive model for granular materials considering inherent anisotropy within a strain space multiple mechanism framework [28]. As described in subsequent sections, a strain space multiple mechanism model with fabric anisotropy is proposed by incorporating a new function that represents the effect of inherent anisotropy into a second-order fabric tensor. The general form of this tensor can be defined in a two-dimensional plane as

$$F = \int_0^{2\pi} a(\theta) \mathbf{b}(\theta) \otimes \mathbf{c}(\theta) d\theta, \quad (1)$$

where $a(\theta)$ is the orientation distribution of a quantity in a direction θ measured from a specified coordinate axis in a two-dimensional plane, and $\mathbf{b}(\theta)$ and $\mathbf{c}(\theta)$ are the direction vectors (e.g., the contact normal orientation vector, the particle orientation vector, and the void direction vector [20]) associated with θ . The second-order fabric tensor can characterize the internal structure of an assembly of particles, including inherent anisotropy in granular materials as well as stress-induced anisotropy with proper consideration of the fabric evolution [39]. The applicability of the proposed model is investigated through a comparison between simulation and laboratory tests. The discussion in this paper is limited to two dimensions, with a three-dimensional formulation being left for future work.

2. INTEGRATED FORM OF THE STRAIN SPACE MULTIPLE MECHANISM

MODEL WITH INHERENT ANISOTROPY

As defined for a continuum, the macroscopic effective stress in granular materials is given by a certain average of the contact forces between particles. In an assembly of plane-circular particles, contact force $\mathbf{P}$ can be divided into the directions of contact-normal (or along the branch connecting particle centers when circular or spherical particles are of concern), $\mathbf{n}$, and tangential, $\mathbf{t}$, as (Figure 1)

$$\mathbf{P} = f_n \mathbf{n} + f_t \mathbf{t}, \quad (2)$$

where

$$\begin{aligned} \mathbf{n}^T &= [\cos \theta \quad \sin \theta] \\ \mathbf{t}^T &= [\sin \theta \quad -\cos \theta] \end{aligned} \quad (3)$$

Angle θ is measured relative to the x axis. Macroscopic effective stress is given by taking an average over the contact forces within a representative element having volume V as [40, 41]

$$\boldsymbol{\sigma}' = \frac{1}{V} \sum l \mathbf{P} \otimes \mathbf{n}, \quad (4)$$

where l denotes the length of the branch. As shown in Equation (4), contact normal-based fabric tensors are used in this study to quantify the internal structure of granular materials, although other fabric descriptors such as the particle orientation fabric tensor or void vector fabric tensor have been recently proposed for better describing the micromechanical structure [20]. This is because the contact normal fabric tensor is more suitable for continuum modeling than the other fabric tensors and a strong correlation with the void vector fabric tensor is identified for non-elongated particles [42]. The strain space multiple mechanism model takes into account the fabric

due to touching particles as well as that due to non-touching particles through the concept of Dirichlet tessellation and Delaunay network [43]. Thus, contribution of void vector fabric due to non-touching particles may be indirectly incorporated in this model. The primary contributions of non-touching particles are generalized as energy-less strain and incorporated as an essential element of modelling of material anisotropy as discussed in Chapter 4 of this paper.

Following Rothenburg and Bathurst [44], Equation (4) can be rewritten in terms of contact density (i.e., the total number of assembly contacts per unit volume) m_v , the average branch length $\bar{l}_0$, and the contact-distribution function $E(\theta)$ as

$$\boldsymbol{\sigma}' = m_v \bar{l}_0 \left[\int_0^{2\pi} F^*(\theta - \theta_0) f_n(\theta) E(\theta) \mathbf{n} \otimes \mathbf{n} d\theta + \int_0^{2\pi} F^*(\theta - \theta_0) f_t(\theta) E(\theta) \mathbf{t} \otimes \mathbf{n} d\theta \right], \quad (5)$$

where

$$\int_0^{2\pi} E(\theta) d\theta = 1, \quad (6)$$

$$F^*(\theta) = (1 + a_1 \cos 2\theta + a_2 \cos 4\theta). \quad (7)$$

The symbols indicating the components of contact forces, f_n and f_t , are redefined in terms of the average over the contacts pointing in the direction θ . $F^*(\theta)$, which approximates the effect of inherent anisotropy to second order, is newly incorporated into the second-order fabric tensor as shown in Equation (5), which describes the textures of granular materials. The effect of inherent anisotropy can be characterized by three parameters, a_1 , a_2 , and θ_0 in Equation (7): a_1 and a_2 control the degree of anisotropy and θ_0 expresses the principal direction of inherent anisotropy, such as the normal vector direction of the bedding plane relative to the x axis in geotechnical

laboratory tests. Figure 2 illustrates a distribution of the function $F^*(\theta)$ depending on the anisotropic parameters a_1 and a_2 . In addition to the first (or fundamental) mode in Figure 2(a) that can be controlled by the parameter a_1 , the second (or a higher) mode can also be expressed by introducing the parameter a_2 as shown in Figure 2(b). To the best of the authors' knowledge, existing constitutive models accounting for inherent anisotropy (e.g., [21-23]) take into consideration only the first mode but not the second mode. By combining the anisotropic parameters a_1 and a_2 , the proposed model is believed to more precisely reproduce complex behavior associated with inherent anisotropy as shown in the model validation for laboratory tests under general-direction shear loading (see subsection 6.3). Furthermore, the necessity and advantages of introducing the second mode with the parameter a_2 in the expression are examined in Appendix A through comparison with a DEM simulation. If the anisotropic parameters are ignored and thus the function $F^*(\theta)$ is independent of direction θ (i.e., $F^*(\theta)=1$) as shown in the circle in Figure 2, the second-order fabric tensor in Equation (5) is reduced to its original form without any inherent-anisotropy effects [28-30].

Because of the equilibrium of momentum (i.e., $\int_0^{2\pi} f_t(\theta)E(\theta)(\mathbf{t} \otimes \mathbf{n} - \mathbf{n} \otimes \mathbf{t})d\theta = 0$), Equation (5) can be expressed as a combination of the isotropic and deviator (i.e., biaxial and simple shear) components as follows [41, 45]:

$$\begin{aligned}\boldsymbol{\sigma}' = & \frac{1}{2} m_v \bar{l}_0 \int_0^{2\pi} F^*(\theta - \theta_0) f_n(\theta) E(\theta) |\mathbf{n} \otimes \mathbf{n}| d\theta \\ & + \frac{1}{2} m_v \bar{l}_0 \int_0^{2\pi} F^*(\theta - \theta_0) f_n(\theta) E(\theta) \langle \mathbf{n} \otimes \mathbf{n} \rangle d\theta, \\ & + \frac{1}{2} m_v \bar{l}_0 \int_0^{2\pi} F^*(\theta - \theta_0) f_t(\theta) E(\theta) \langle \mathbf{t} \otimes \mathbf{n} \rangle d\theta\end{aligned}\quad (8)$$

where

$$\begin{aligned} |\mathbf{n} \otimes \mathbf{n}| &= \mathbf{n} \otimes \mathbf{n} + \mathbf{t} \otimes \mathbf{t} \\ \langle \mathbf{n} \otimes \mathbf{n} \rangle &= \mathbf{n} \otimes \mathbf{n} - \mathbf{t} \otimes \mathbf{t} . \\ \langle \mathbf{t} \otimes \mathbf{n} \rangle &= \mathbf{t} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{t} \end{aligned} \quad (9)$$

The first term in Equation (8), which represents the isotropic component, is rewritten as

$$\begin{aligned} & \frac{1}{2} m_v \bar{l}_0 \int_0^{2\pi} F^*(\theta - \theta_0) f_n(\theta) E(\theta) |\mathbf{n} \otimes \mathbf{n}| d\theta \\ &= \frac{1}{2} m_v \bar{l}_0 \int_0^{2\pi} f_n(\theta) E(\theta) |\mathbf{n} \otimes \mathbf{n}| d\theta = \frac{1}{2} m_v \bar{l}_0 \bar{f}_0 \mathbf{I} , \end{aligned} \quad (10)$$

where the average normal contact force, $\bar{f}_0$, is given by

$$\bar{f}_0 = \int_0^{2\pi} f_n(\theta) E(\theta) d\theta . \quad (11)$$

The second and third terms in Equation (8) represent the biaxial shear $\langle \mathbf{n} \otimes \mathbf{n} \rangle$ and simple shear $\langle \mathbf{t} \otimes \mathbf{n} \rangle$, respectively. However, if these shears are idealized in terms of the second-order tensors, they become indistinguishable except for an orientation-angle difference of $\pi/4$ (Figure 3). Thus, Equation (8) can be reduced to a simpler form, with $\omega = 2\theta$ as

$$\boldsymbol{\sigma}' = -p' \mathbf{I} + \frac{1}{2} \int_0^{2\pi} q \langle \mathbf{t} \otimes \mathbf{n} \rangle d\omega , \quad (12)$$

$$\langle \mathbf{t} \otimes \mathbf{n} \rangle = \begin{bmatrix} \cos \omega & \sin \omega \\ \sin \omega & -\cos \omega \end{bmatrix} , \quad (13)$$

$$p' = -\frac{1}{2} m_v \bar{l}_0 \bar{f}_0 , \quad (14)$$

and

$$q = m_v \bar{l}_0 \left[\left\{ F^* \left(\frac{\omega}{2} - \frac{\omega_0}{2} \right) f_n \left(\frac{\omega}{2} \right) E \left(\frac{\omega}{2} \right) - \bar{f}_0 \bar{E}_0 \right\} + F^* \left(\frac{\omega}{2} - \frac{\omega_0}{2} \right) f_t \left(\frac{\omega}{2} + \frac{\pi}{4} \right) E \left(\frac{\omega}{2} + \frac{\pi}{4} \right) \right] , \quad (15)$$

where the average contact distribution, $\bar{E}_0$, is given by

$$\bar{E}_0 = \frac{1}{2\pi}. \quad (16)$$

As shown in Equation (15), the virtual simple shear stress, q , in a multitude of virtual simple shear mechanisms oriented in arbitrary directions is defined as a quantity that depends on the contact distribution function $E(\theta)$ as well as the normal and tangential components of inter-particle contact forces, f_n and f_t , which evolve independently during a loading process [30]. By introducing the anisotropic function $F^*(\theta)$ in the expression, the evolution of induced fabric in inherently anisotropic granular materials changes from the fabric evolution in isotropic granular materials without inherent anisotropy, depending on the newly introduced parameters a_1 , a_2 , and θ_0 . To better understand the performance of the strain space multiple mechanism model with inherent anisotropy, the effect of the anisotropic parameters on the evolution of induced fabric, due to changes in the macroscopic stress and strain during a loading process, is to be further examined by future work, which is out of the scope of the current study.

The virtual simple shear stress, q , in Equation (15) can be written as a combination of the same term for isotropic-fabric granular materials, q_{Iso} , and an additional term due to inherent anisotropy, q_{Aniso} , as

$$q = F^* \left(\frac{\omega}{2} - \frac{\omega_0}{2} \right) q_{\text{Iso}}(\omega) + q_{\text{Aniso}}(\omega), \quad (17)$$

where

$$q_{\text{Iso}}(\omega) = m_v \bar{l}_0 \left[\left\{ f_n \left(\frac{\omega}{2} \right) E \left(\frac{\omega}{2} \right) - \bar{f}_0 \bar{E}_0 \right\} + f_t \left(\frac{\omega}{2} + \frac{\pi}{4} \right) E \left(\frac{\omega}{2} + \frac{\pi}{4} \right) \right], \quad (18)$$

$$q_{\text{Aniso}}(\omega) = \frac{1}{2\pi} \left\{ F^* \left(\frac{\omega}{2} - \frac{\omega_0}{2} \right) - 1 \right\} m_v \bar{l}_0 \bar{f}_0. \quad (19)$$

The additional term due to inherent anisotropy in Equation (19) is rewritten by substituting Equations (7) and (14) as

$$q_{\text{Aniso}}(\omega) = -\frac{1}{\pi} \{ a_1 \cos(\omega - \omega_0) + a_2 \cos 2(\omega - \omega_0) \} p'. \quad (20)$$

Thus, q_{Aniso} is proportional to the effective confining pressure, p' : this may reflect the fact that the evolution of coordination number, which is considered one of causes for anisotropic behavior of granular materials, has a strong correlation with a change in the effective confining pressure [19, 20].

The integrated form (i.e., the direct stress–strain relationship) of the strain space multiple mechanism model is given by relating the macroscopic-strain tensor, $\boldsymbol{\varepsilon}$, to macroscopic-effective-stress tensor, $\boldsymbol{\sigma}'$, using the structure defined by Equation (12). The first step for deriving the explicit form of this relationship is to define the volumetric strain, ε (extension positive), and the virtual simple shear strain, γ , as a projection of the macroscopic strain field to second-order tensors $\mathbf{I}$ and $\langle \mathbf{t} \otimes \mathbf{n} \rangle$, respectively, as follows:

$$\varepsilon = \mathbf{I} : \boldsymbol{\varepsilon}; \quad (21)$$

$$\gamma = \langle \mathbf{t} \otimes \mathbf{n} \rangle : \boldsymbol{\varepsilon}. \quad (22)$$

In addition, the effective volumetric strain, ε' , and the virtual effective volumetric strain, ε'' , are introduced below:

$$\varepsilon' = \varepsilon - \varepsilon_d; \quad (23)$$

$$\varepsilon'' = \varepsilon - \varepsilon_d^c, \quad (24)$$

where the dilatancy, ε_d , in Equation (23) is decomposed into a contractive component, ε_d^c , and a dilative component, ε_d^d , as

$$\varepsilon_d = \varepsilon_d^c + \varepsilon_d^d. \quad (25)$$

The scalar variables defined in Equations (23) and (22) as projections of the macroscopic strain field are used to define the isotropic stress, p' , in Equation (14) and the isotropic component of the virtual simple shear stress, q_{Iso} , in Equation (18), respectively, through path-dependent functions as follows:

$$p' = p'(\varepsilon'), \quad (26)$$

$$q_{\text{Iso}} = q_{\text{Iso}}(\gamma(\omega)). \quad (27)$$

In the strain space multiple mechanism model, the virtual simple shear mechanism is formulated as a nonlinear hysteretic function, where the backbone curve is given by a hyperbolic function [46] as

$$q_{\text{Iso}}(\gamma(\omega)) = \frac{\gamma(\omega) / \gamma_v}{1 + |\gamma(\omega) / \gamma_v|} q_v. \quad (28)$$

Here, the parameters q_v and γ_v are the shear strength and the reference strain of the virtual simple shear mechanism, respectively [25, 28]. As will be described in Section 5, these parameters can be expressed by macroscopic shear strength and elastic shear modulus considering the influence of inherent anisotropy. The transformation process in Equation (28) is decomposed into the following three steps: (a) $\gamma \mapsto \gamma / \gamma_v$, (b) $\gamma / \gamma_v \mapsto q_{\text{Iso}} / q_v$, and (c) $q_{\text{Iso}} / q_v \mapsto q_{\text{Iso}}$. The effect of inherent anisotropy comes into effect in steps (a) and (c), but such anisotropy does not affect the transformation process in step (b), which is essential for describing the non-linear evolution of stress-space

fabric (i.e., stress-induced anisotropy) [29, 30].

By substituting Equation (13) into the definition of the virtual simple shear strain in Equation (22), the following equality describing the shear strain property holds:

$$\gamma(\omega + \pi) = -\gamma(\omega). \quad (29)$$

In addition, substitution of the above equality into Equation (28) yields the equation governing virtual simple shear strain,

$$q_{\text{Iso}}(\omega + \pi) / q_v = -q_{\text{Iso}}(\omega) / q_v. \quad (30)$$

By considering Equations (29) and (30), Equation (12) can be rewritten as follows:

$$\boldsymbol{\sigma}' = -p' \mathbf{I} + \int_0^\pi q \langle \mathbf{t} \otimes \mathbf{n} \rangle d\omega; \quad (31)$$

here, the virtual simple shear stress, q , originally given by Equation (17), is redefined by

$$q = \frac{1}{2} \left\{ F^* \left(\frac{\omega}{2} - \frac{\omega_0}{2} \right) + F^* \left(\frac{\omega + \pi}{2} - \frac{\omega_0}{2} \right) \right\} q_{\text{Iso}}(\omega) + \frac{1}{2} \{ q_{\text{Aniso}}(\omega) - q_{\text{Aniso}}(\omega + \pi) \}. \quad (32)$$

Substitution of Equations (7) and (20) into Equation (32) yields

$$q = F(\omega - \omega_0) q_{\text{Iso}}(\omega) + q_{\text{Aniso}}(\omega), \quad (33)$$

where the function F representing the effect of inherent anisotropy, which differs from Equation (7), is given by

$$F(\omega) = 1 + a_2 \cos 2\omega, \quad (34)$$

and the additional term, q_{Aniso} , due to inherent anisotropy in Equation (20) can be redefined as

$$q_{\text{Aniso}}(\omega) = -\frac{1}{\pi} a_1 \cos(\omega - \omega_0) p'. \quad (35)$$

As shown in Equations (34) and (35), the parameter a_1 has no influence upon the

function F acting on the isotropic fabric term q_{Iso} as an anisotropic-scaling factor, while the parameter a_2 does not affect the additional term q_{Aniso} due to inherent anisotropy. If the anisotropy parameters (i.e., a_1 , a_2 , and ω_0) are ignored in Equation (31), the integrated form of the anisotropic strain space multiple mechanism model is reduced to the original form with no effect upon inherent anisotropy as mentioned earlier [25, 28].

Assuming Equation (26) as being the same, Equation (27) is generalized to take into account the effects of effective stress and state of liquefaction by introducing additional variables as

$$q_{\text{Iso}} = q_{\text{Iso}}(\gamma, \varepsilon', \varepsilon''). \quad (36)$$

where ε' and ε'' denote the effective volumetric strain in Equation (23) and the virtual effective volumetric strain in Equation (24), respectively.

3. INCREMENTAL FORM OF THE STRAIN SPACE MULTIPLE MECHANISM MODEL WITH INHERENT ANISOTROPY

The incremental form of the constitutive model is derived by taking the time derivative of both sides of Equation (31) as

$$\dot{\boldsymbol{\sigma}}' = -\dot{p}' \mathbf{I} + \int_0^\pi \left\{ F(\omega - \omega_0) \dot{q}_{\text{Iso}} + \dot{q}_{\text{Aniso}} \right\} \langle \mathbf{t} \otimes \mathbf{n} \rangle d\omega, \quad (37)$$

where

$$\dot{p}' = \frac{dp'}{d\varepsilon'} \dot{\varepsilon}'; \quad (38)$$

$$\dot{q}_{\text{Iso}} = \frac{\partial q_{\text{Iso}}}{\partial \gamma} \dot{\gamma} + \frac{\partial q_{\text{Iso}}}{\partial \varepsilon'} \dot{\varepsilon}' + \frac{\partial q_{\text{Iso}}}{\partial \varepsilon''} \dot{\varepsilon}''; \quad (39)$$

$$\dot{q}_{\text{Aniso}} = \frac{dq_{\text{Aniso}}}{dp} \dot{p}' = \frac{dq_{\text{Aniso}}}{dp'} \frac{dp'}{d\varepsilon'} \dot{\varepsilon}', \quad (40)$$

A volumetric strain rate due to contractive dilatancy is formulated using a parameter, M_v^* , determined by the phase transformation angle through a second-order fabric tensor, $\mathbf{I}_d^c$, as follows [28]:

$$\dot{\varepsilon}_d^c = \mathbf{I}_d^c : \dot{\varepsilon}, \quad (41)$$

$$\mathbf{I}_d^c = -\int_0^\pi M_v^* |\langle \mathbf{t} \otimes \mathbf{n} \rangle|^* d\omega, \quad (42)$$

where

$$|\langle \mathbf{t} \otimes \mathbf{n} \rangle|^* = \begin{cases} \langle \mathbf{t} \otimes \mathbf{n} \rangle & \text{for } \langle \mathbf{t} \otimes \mathbf{n} \rangle : \dot{\varepsilon} \geq 0 \\ -\langle \mathbf{t} \otimes \mathbf{n} \rangle & \text{for } \langle \mathbf{t} \otimes \mathbf{n} \rangle : \dot{\varepsilon} < 0 \end{cases}. \quad (43)$$

The contractive component of dilatancy is assumed not to be directly affected by inherent anisotropy; thus refer to Iai et al. [28] for further details. On the other hand, a volumetric strain rate due to the dilative dilatancy is obtained through a derivation described in Section 4.2 as

$$\dot{\varepsilon}_d^d = \mathbf{I}_d^d : \dot{\varepsilon}. \quad (44)$$

Consequently, the strain rates appearing on the right-hand side of Equations (39) and (40) are obtained by taking the time derivatives of Equations (22) through (24) with Equations (21), (44) and (41) as follows:

$$\dot{\gamma} = \langle \mathbf{t} \otimes \mathbf{n} \rangle : \dot{\varepsilon}; \quad (45)$$

$$\dot{\varepsilon}' = (\mathbf{I} - \mathbf{I}_d) : \dot{\varepsilon}; \quad (46)$$

$$\dot{\varepsilon}'' = (\mathbf{I} - \mathbf{I}_d^c) : \dot{\varepsilon}. \quad (47)$$

Substitution of Equations (45) through (47) into Equations (38) through (40) yields

$$\dot{p}' = \frac{dp'}{d\varepsilon'} (\mathbf{I} - \mathbf{I}_d) : \dot{\boldsymbol{\varepsilon}}, \quad (48)$$

$$\dot{q}_{\text{Iso}} = \frac{\partial q_{\text{Iso}}}{\partial \gamma} \langle \mathbf{t} \otimes \mathbf{n} \rangle : \dot{\boldsymbol{\varepsilon}} + \frac{\partial q_{\text{Iso}}}{\partial \varepsilon'} (\mathbf{I} - \mathbf{I}_d) : \dot{\boldsymbol{\varepsilon}} + \frac{\partial q_{\text{Iso}}}{\partial \varepsilon''} (\mathbf{I} - \mathbf{I}_d^c) : \dot{\boldsymbol{\varepsilon}}, \quad (49)$$

and

$$\dot{q}_{\text{Aniso}} = \frac{dq_{\text{Aniso}}}{dp'} \frac{dp'}{d\varepsilon'} (\mathbf{I} - \mathbf{I}_d) : \dot{\boldsymbol{\varepsilon}}. \quad (50)$$

By substituting these equations into Equation (37), the incremental form of the constitutive model is given as

$$\dot{\boldsymbol{\sigma}}' = \mathbb{C} : \dot{\boldsymbol{\varepsilon}}, \quad (51)$$

where

$$\begin{aligned} \mathbb{C} = & K_{\text{LU}} \mathbf{I} \otimes \mathbf{I} + \int_0^\pi F(\omega - \omega_0) G_{\text{LU}} \langle \mathbf{t} \otimes \mathbf{n} \rangle \otimes \langle \mathbf{t} \otimes \mathbf{n} \rangle d\omega - K_{\text{LU}} \mathbf{I} \otimes \mathbf{I}_d \\ & + \int_0^\pi H \langle \mathbf{t} \otimes \mathbf{n} \rangle \otimes (\mathbf{I} - \mathbf{I}_d) d\omega + \int_0^\pi F(\omega - \omega_0) L \langle \mathbf{t} \otimes \mathbf{n} \rangle \otimes (\mathbf{I} - \mathbf{I}_d^c) d\omega, \end{aligned} \quad (52)$$

$$K_{\text{LU}} = -\frac{dp'}{d\varepsilon'}, \quad (53)$$

$$G_{\text{LU}} = \frac{\partial q_{\text{Iso}}}{\partial \gamma}, \quad (54)$$

$$H = F(\omega - \omega_0) H_{\text{Iso}} + H_{\text{Aniso}}, \quad (55)$$

$$H_{\text{Iso}} = \frac{\partial q_{\text{Iso}}}{\partial \varepsilon'}, \quad (56)$$

$$H_{\text{Aniso}} = \frac{\partial q_{\text{Aniso}}}{\partial p'} \frac{dp'}{d\varepsilon'}, \quad (57)$$

and

$$L = \frac{\partial q_{\text{Iso}}}{\partial \varepsilon''}. \quad (58)$$

Thus, the incremental form of the anisotropic strain space multiple mechanism model is

given through a fourth-order fabric tensor $\mathbb{C}$, taking the influence of both inherent and stress-induced anisotropy into account. The first and second terms of Equation (52) represent the volumetric and deviator relationships, being major-symmetric. The third term represents the effect of dilatancy, the fourth the dependency of the shear mechanism upon effective stress, and the fifth the state of liquefaction. The third through fifth terms represent coupling between the volumetric and deviator mechanisms, being major-asymmetric. The second, fourth, and fifth terms, which include the distribution function expressing inherent anisotropy (i.e., $F(\omega)$), are directly influenced by the existence of such anisotropy.

By ignoring the terms associated with inherent anisotropy in Equation (52), the fourth-order fabric tensor, $\mathbb{C}$, is reduced to the following expression, which has already been established for isotropic granular materials without inherent anisotropy [28]:

$$\begin{aligned} \mathbb{C} = & K_{LU} \mathbf{I} \otimes \mathbf{I} + \int_0^{2\pi} G_{LU} \langle \mathbf{n} \otimes \mathbf{n} \rangle \otimes \langle \mathbf{n} \otimes \mathbf{n} \rangle d\omega - K_{LU} \mathbf{I} \otimes \mathbf{I}_d \\ & + \int_0^{2\pi} H_{iso} \langle \mathbf{n} \otimes \mathbf{n} \rangle \otimes (\mathbf{I} - \mathbf{I}_d) d\omega + \int_0^{2\pi} L \langle \mathbf{n} \otimes \mathbf{n} \rangle \otimes (\mathbf{I} - \mathbf{I}_d^c) d\omega \end{aligned} \quad (59)$$

4. STRAIN ENERGY AND THE DILATIVE COMPONENT OF DILATANCY

4.1. Micro- and macroscopic strain energy

As a prelude to the formulation of dilatancy in the anisotropic strain space multiple mechanism model, the relationship between microscopic and macroscopic strain energies is identified following Iai et al. [28]. First, the rate of strain energy, $\dot{W}$, in the constitutive model is computed through Equations (31), (21), and (22) as

$$\dot{W} = \boldsymbol{\sigma}':\dot{\boldsymbol{\varepsilon}} = \dot{W}_p + \int_0^\pi \left\{ F(\omega - \omega_0) \dot{W}_{q_{iso}} + \dot{W}_{q_{Aniso}} \right\} d\omega, \quad (60)$$

where

$$\dot{W}_p = -p' \dot{\varepsilon}, \quad (61)$$

$$\dot{W}_{q_{\text{Iso}}} = q_{\text{Iso}} \dot{\gamma}, \quad (62)$$

$$\dot{W}_{q_{\text{Aniso}}} = q_{\text{Aniso}} \dot{\gamma} = -\frac{1}{\pi} a_1 \cos(\omega - \omega_0) p' \dot{\gamma}. \quad (63)$$

Equation (60) defines the fundamental relationship between the macroscopic strain-energy rate $\dot{W}$ and the microscopic strain-energy rates, $\dot{W}_{q_{\text{Iso}}}$ and $\dot{W}_{q_{\text{Aniso}}}$ given by the individual virtual simple shear mechanism that constitutes the anisotropic strain space multiple mechanism model. This relationship will play an important role in formulating dilatancy as described later.

Based on Equations (23) and (25), the volumetric-strain rate is divided into three components (i.e., the isotropic stress change and the contractive and dilative components of dilatancy) as follows:

$$\dot{\varepsilon} = \dot{\varepsilon}' + \dot{\varepsilon}_d^c + \dot{\varepsilon}_d^d. \quad (64)$$

By substituting Equation (64) into Equation (60), the macroscopic-strain-energy rate can be given as

$$\dot{W} = \dot{W}'_p + \dot{W}_p^{\text{dc}} + \dot{W}_p^{\text{dd}} + \int_0^\pi \left\{ F(\omega - \omega_0) \dot{W}_{q_{\text{Iso}}} + \dot{W}_{q_{\text{Aniso}}} \right\} d\omega, \quad (65)$$

where

$$\dot{W}'_p = -p' \dot{\varepsilon}', \quad (66)$$

$$\dot{W}_p^{\text{dc}} = -p' \dot{\varepsilon}_d^c, \quad (67)$$

$$\dot{W}_p^{\text{dd}} = -p' \dot{\varepsilon}_d^d. \quad (68)$$

4.2. Dilative component of dilatancy

In the 1940s, Taylor [47] documented the importance of the energy-less component of deformation and used the term “interlocking” to describe this mechanism. According to Iai and Ueda [43], the volumetric-strain component of the energy-less strain (i.e., the strain due to non-touching particles) is defined as the dilative component of dilatancy in granular materials. Following their finding, a fraction of the shear-strain energy is assumed to be consumed in the interlocking mechanism as

$$\dot{W}_p^{dd} + \int_0^\pi \left\{ F(\omega - \omega_0) \dot{W}_{q_{Iso}}^{dd} + \dot{W}_{q_{Aniso}}^{dd} \right\} d\omega = 0, \quad (69)$$

where

$$\dot{W}_{q_{Iso}}^{dd} = r_{\varepsilon_d}^{Iso} \dot{W}_{q_{Iso}} = r_{\varepsilon_d}^{Iso} q_{Iso} \dot{\gamma}, \quad (70)$$

$$\dot{W}_{q_{Aniso}}^{dd} = r_{\varepsilon_d}^{Aniso} \dot{W}_{q_{Aniso}} = r_{\varepsilon_d}^{Aniso} q_{Aniso} \dot{\gamma}. \quad (71)$$

In this study, it is also hypothesized that the shear stress contribution, q_{Iso} , to the interlocking in Equation (70) follows the backbone curve defined by Equation (28). This hypothesis is formulated to avoid accumulation of the dilative component of dilatancy during cyclic loading, thus simplifying the formulation of dilatancy. In addition, the parameter $r_{\varepsilon_d}^{Aniso}$ in Equation (71) is assumed to be 1.0 in this study for simplicity, and the parameter $r_{\varepsilon_d}^{Iso}$ in Equation (70) is hereafter rewritten as r_{ε_d} .

Based on these postulates, together with Equations (68), (28), and (35), the time derivative of the dilative component is expressed as

$$\dot{\varepsilon}_d^d = r_{\varepsilon_d} \int_0^\pi F(\omega - \omega_0) \frac{\gamma / \gamma_v}{1 + |\gamma / \gamma_v|} M_{fv} \dot{\gamma} d\omega - \frac{1}{\pi} a_1 \int_0^\pi \cos(\omega - \omega_0) \dot{\gamma} d\omega, \quad (72)$$

where

$$M_{fv} = \frac{q_v}{p'}. \quad (73)$$

Equation (73) is a slope of the shear failure line defined for the virtual simple shear mechanism. By integrating Equation (72) with respect to γ under the condition that $\varepsilon_d^d = 0$ for $\gamma = 0$, an integrated form of the dilative component is derived as

$$\varepsilon_d^d = r_{\varepsilon_d} \int_0^\pi F(\omega - \omega_0) \left[\left| \frac{\gamma}{\gamma_v} \right| - \ln \left(1 + \left| \frac{\gamma}{\gamma_v} \right| \right) \right] \gamma_v M_{fv} d\omega - \frac{1}{2} a_1 \langle \mathbf{t} \otimes \mathbf{n} \rangle_{\omega_0} : \boldsymbol{\varepsilon}, \quad (74)$$

where

$$\langle \mathbf{t} \otimes \mathbf{n} \rangle_{\omega_0} = \begin{bmatrix} \cos \omega_0 & \sin \omega_0 \\ \sin \omega_0 & -\cos \omega_0 \end{bmatrix}. \quad (75)$$

By ignoring the anisotropy parameters (i.e., a_1 , a_2 , and ω_0), Equation (74) is reduced to the dilative component of dilatancy for granular materials without inherent anisotropy [28].

By substituting the time derivative of Equation (22) into Equation (72), the incremental relationship between the dilative component of dilatancy and the macroscopic strain is given by a second-order fabric tensor, $\mathbf{I}_d^d$, as follows:

$$\dot{\varepsilon}_d^d = \mathbf{I}_d^d : \dot{\boldsymbol{\varepsilon}}. \quad (76)$$

Here,

$$\mathbf{I}_d^d = r_{\varepsilon_d} \int_0^\pi F(\omega - \omega_0) \frac{\gamma / \gamma_v}{1 + |\gamma / \gamma_v|} M_{fv} \langle \mathbf{t} \otimes \mathbf{n} \rangle d\omega - \frac{1}{2} a_1 \langle \mathbf{t} \otimes \mathbf{n} \rangle_{\omega_0}. \quad (77)$$

5. MODEL PARAMETERS

5.1. Shear-mechanism parameters under simple-shear loading

In this section, the model parameters and functions required to complete the description of the shear mechanism in the proposed constitutive model (see Table 1) are described by relating the macroscopic-elastic-shear modulus, G_m^{Aniso} , and the macroscopic-shear strength, τ_m^{Aniso} , considering inherent anisotropy to their microscopic counterparts in the virtual simple shear mechanism.

First, let us consider a strain-controlled monotonic simple-shear test (Figure 4(a)), in which only non-zero shear strain is the (engineering) shear strain, γ_{xy} , under drained-loading conditions with an initially isotropic stress state. The virtual simple shear strain in Equation (22) is reduced to

$$\gamma = \gamma_{xy} \sin \omega. \quad (78)$$

The macroscopic-simple-shear stress, τ_{xy} , is given from Equation (31) as follows:

$$\tau_{xy} = \int_0^\pi \left[\left\{ 1 + a_2 \cos 2(\omega - \omega_0) \right\} q_{\text{Iso}}(\gamma(\omega)) - \frac{1}{\pi} a_1 \cos(\omega - \omega_0) p' \right] \sin \omega d\omega. \quad (79)$$

The derivative of Equations (79) with respect to the simple-shear strain, γ_{xy} , leads to the macroscopic-elastic-shear modulus as

$$G_m^{\text{Aniso}} = \left. \frac{\partial \tau_{xy}}{\partial \gamma_{xy}} \right|_{\gamma_{xy}=0} = \int_0^\pi \left\{ 1 + a_2 \cos 2(\omega - \omega_0) \right\} \left. \frac{\partial q_{\text{Iso}}}{\partial \gamma_{xy}} \right|_{\gamma_{xy}=0} \sin \omega d\omega. \quad (80)$$

Here, the derivative of the isotropic virtual simple shear stress, q_{Iso} , with respect to γ_{xy} is rewritten using Equations (28) and (78) as follows:

$$\left. \frac{\partial q_{\text{Iso}}}{\partial \gamma_{xy}} \right|_{\gamma_{xy}=0} = \left. \frac{\partial q_{\text{Iso}}}{\partial \gamma} \right|_{\gamma=0} \frac{\partial \gamma}{\partial \gamma_{xy}} = \frac{q_v}{\gamma_v} \sin \omega. \quad (81)$$

Thus, the macroscopic shear modulus at a small strain level considering inherent anisotropy, G_m^{Aniso} , is given as

$$G_m^{\text{Aniso}} = \int_0^\pi \left\{ 1 + a_2 \cos 2(\omega - \omega_0) \right\} \frac{q_v}{\gamma_v} \sin^2 \omega d\omega = G_m^{\text{Iso}} \left(1 - \frac{1}{2} a_2 \cos 2\omega_0 \right), \quad (82)$$

where G_m^{Iso} is the macroscopic elastic shear modulus without inherent anisotropy given as [25, 28]

$$G_m^{\text{Iso}} = \frac{q_v}{\gamma_v} \int_0^\pi \sin^2 \omega d\omega = \frac{\pi}{2} \frac{q_v}{\gamma_v}. \quad (83)$$

Equation (82) suggests that the anisotropy parameter, a_2 , can be determined from laboratory-test results if the anisotropic elastic shear modulus, G_m^{Aniso} , corresponding to two different deposition angles of bedding planes (i.e., $\theta_0 = \omega_0 / 2$) is obtained. However, it is much harder to alter the deposition angles in laboratory tests (particularly torsional shear tests in a hollow cylinder apparatus) than it is to change the principal stress direction with the same fabric.

When the shear strain is very large such that $\gamma_{xy} = \infty$, Equation (79) with Equation (28) leads to the macroscopic shear strength considering inherent anisotropy, τ_m^{Aniso} , as follows:

$$\tau_m^{\text{Aniso}} = \tau_{xy} \Big|_{\gamma_{xy}=\infty} = \int_0^\pi \left[\left\{ 1 + a_2 \cos 2(\omega - \omega_0) \right\} q_v - \frac{1}{\pi} a_1 \cos(\omega - \omega_0) p' \right] \sin \omega d\omega. \quad (84)$$

Here, the macroscopic shear strength without inherent anisotropy is given as [25, 28]

$$\tau_m^{\text{Iso}} = q_v \int_0^\pi \sin \omega d\omega = 2q_v. \quad (85)$$

Substitution of Equation (85) into Equation (84) yields

$$\tau_m^{\text{Aniso}} = \left(1 - \frac{1}{3} a_2 \cos 2\omega_0 \right) \tau_m^{\text{Iso}} - \frac{1}{2} a_1 p' \sin \omega_0; \quad (86)$$

by normalizing Equation (86) with the isotropic stress p' , the internal friction angle

with inherent anisotropy, ϕ_f^{Aniso} , can be related to that without inherent anisotropy, ϕ_f^{Iso} , using the anisotropy parameters (i.e., a_1 , a_2 , and ω_0) as follows:

$$\sin \phi_f^{\text{Aniso}} = \left(1 - \frac{1}{3} a_2 \cos 2\omega_0 \right) \sin \phi_f^{\text{Iso}} - \frac{1}{2} a_1 \sin \omega_0. \quad (87)$$

This equation suggests that once the parameter a_2 is estimated, the another parameter a_1 can be determined from the ϕ_f^{Aniso} values corresponding to two different deposition angles of bedding planes (i.e., $\theta_0 = \omega_0 / 2$) in laboratory tests.

The microscopic simple shear strength, q_v , and the microscopic reference strain, γ_v , in the virtual simple shear mechanism (i.e., Equation (28)) are related to their macroscopic counterparts from Equations (83) and (85) as follows [25, 28]:

$$q_v = \frac{\tau_m^{\text{Iso}}}{2}; \quad (88)$$

$$\gamma_v = \frac{\pi}{4} \frac{\tau_m^{\text{Iso}}}{G_m^{\text{Iso}}} = \frac{\pi}{4} \gamma_m^{\text{Iso}}; \quad (89)$$

here, γ_m^{Iso} is the macroscopic reference (shear) strain.

5.2. Shear-mechanism parameters under biaxial-shear loading

In this subsection, we consider a strain-controlled monotonic biaxial-shear test (Figure 4(b)) in which the strain difference, $\varepsilon_x - \varepsilon_y$, is non-zero, under drained loading conditions with an initially isotropic stress state. The virtual simple shear strain in Equation (22) is reduced to

$$\gamma = (\varepsilon_x - \varepsilon_y) \cos \omega. \quad (90)$$

The macroscopic biaxial shear stress, $(\sigma_x' - \sigma_y')/2$, is given from Equation (31) as

follows:

$$\frac{\sigma_x' - \sigma_y'}{2} = \int_0^\pi \left[\{1 + a_2 \cos 2(\omega - \omega_0)\} q_{\text{iso}}(\gamma(\omega)) - \frac{1}{\pi} a_1 \cos(\omega - \omega_0) p' \right] \cos \omega d\omega. \quad (91)$$

The derivative of Equations (91) with respect to the biaxial shear strain, $\varepsilon_x - \varepsilon_y$, leads to the macroscopic elastic shear modulus as

$$\begin{aligned} G_m^{\text{Aniso}} &= \frac{\partial \{(\sigma_x' - \sigma_y')/2\}}{\partial (\varepsilon_x - \varepsilon_y)} \bigg|_{\varepsilon_x - \varepsilon_y = 0} \\ &= \int_0^\pi \{1 + a_2 \cos 2(\omega - \omega_0)\} \frac{\partial q_{\text{iso}}}{\partial (\varepsilon_x - \varepsilon_y)} \bigg|_{\varepsilon_x - \varepsilon_y = 0} \cos \omega d\omega. \end{aligned} \quad (92)$$

Here, the derivative of the isotropic virtual simple shear stress, q_{iso} , with respect to $\varepsilon_x - \varepsilon_y$ is rewritten using Equations (28) and (90) as follows:

$$\frac{\partial q_{\text{iso}}}{\partial (\varepsilon_x - \varepsilon_y)} \bigg|_{\varepsilon_x - \varepsilon_y = 0} = \frac{\partial q_{\text{iso}}}{\partial \gamma} \bigg|_{\gamma=0} \frac{\partial \gamma}{\partial (\varepsilon_x - \varepsilon_y)} = \frac{q_v}{\gamma_v} \cos \omega. \quad (93)$$

Thus, the macroscopic shear modulus at a small strain level considering inherent anisotropy, G_m^{Aniso} , is given as

$$G_m^{\text{Aniso}} = \int_0^\pi \{1 + a_2 \cos 2(\omega - \omega_0)\} \frac{q_v}{\gamma_v} \cos^2 \omega d\omega = G_m^{\text{Iso}} \left(1 + \frac{1}{2} a_2 \cos 2\omega_0 \right). \quad (94)$$

Here, G_m^{Iso} is the macroscopic elastic shear modulus without inherent anisotropy given as [25, 28]

$$G_m^{\text{Iso}} = \frac{q_v}{\gamma_v} \int_0^\pi \cos^2 \omega d\omega = \frac{\pi}{2} \frac{q_v}{\gamma_v}, \quad (95)$$

which is equivalent to Equation (83). As is the case with simple shear, the anisotropy parameter, a_2 , can be determined from Equation (94) if the deposition angles of the

bedding planes are alterable in laboratory biaxial-shear tests.

By comparing Equations (82) and (94), we can obtain the relationship between the anisotropic elastic-shear modulus under simple and under biaxial shear as follows:

$$\left(G_m^{\text{Aniso}}\right)_{\text{biaxial}} = \left(G_m^{\text{Aniso}}\right)_{\text{simple}} + (a_2 \cos 2\omega_0) G_m^{\text{Iso}}. \quad (96)$$

When the shear strain, $\varepsilon_x - \varepsilon_y$, is very large, Equation (91) with Equation (28) leads to the macroscopic shear strength considering inherent anisotropy, τ_m^{Aniso} , as follows:

$$\begin{aligned} \tau_m^{\text{Aniso}} &= \left(\frac{\sigma_x' - \sigma_y'}{2} \right) \bigg|_{\varepsilon_x - \varepsilon_y = \infty} \\ &= \int_0^\pi \left[\left\{ 1 + a_2 \cos 2(\omega - \omega_0) \right\} q_v - \frac{1}{\pi} a_1 \cos(\omega - \omega_0) p' \right] \text{sgn}(\gamma) \cos \omega d\omega \end{aligned} \quad (97)$$

Here, the macroscopic shear strength without inherent anisotropy is given as [25, 28]

$$\tau_m^{\text{Iso}} = q_v \int_0^\pi \text{sgn}(\gamma) \cos \omega d\omega = 2q_v, \quad (98)$$

and becomes equivalent to Equation (85). Substitution of Equation (98) into Equation (97) yields

$$\tau_m^{\text{Aniso}} = \left(1 + \frac{1}{3} a_2 \cos 2\omega_0 \right) \tau_m^{\text{Iso}} - \frac{1}{2} a_1 p' \cos \omega_0. \quad (99)$$

By normalizing Equation (99) with the isotropic stress p' , the internal friction angle with inherent anisotropy, ϕ_f^{Aniso} , can be related to that without inherent anisotropy, ϕ_f^{Iso} , using the anisotropy parameters (i.e., a_1 , a_2 , and ω_0) as follows:

$$\sin \phi_f^{\text{Aniso}} = \left(1 + \frac{1}{3} a_2 \cos 2\omega_0 \right) \sin \phi_f^{\text{Iso}} - \frac{1}{2} a_1 \cos \omega_0. \quad (100)$$

Comparing Equations (87) and (100), we obtain the relationship between the internal friction angle under simple and under biaxial shear as follows:

$$\begin{aligned} \left(\sin \phi_f^{\text{Aniso}} \right)_{\text{biaxial}} &= \left(\sin \phi_f^{\text{Aniso}} \right)_{\text{simple}} \\ &+ \left(\frac{2}{3} a_2 \cos 2\omega_0 \right) \sin \phi_f^{\text{Iso}} + \frac{1}{2} a_1 (\sin \omega_0 - \cos \omega_0) \end{aligned} \quad (101)$$

5.3. Shear-mechanism parameters under general-directional-shear loading

In this subsection, a strain-controlled monotonic general-directional-shear test (Figure 4(c)), in which the shear strain is given as a combination of the simple shear strain, γ_{xy} , and the strain difference, $\varepsilon_x - \varepsilon_y$, under drained loading conditions, is considered with an initially isotropic stress state. As shown in Figure 4(c), an angle between the normal vector of the bedding plane, θ_0 ($= \pi/2$ for horizontal bedding planes), and the major principal-strain (i.e., ε_1) direction is given as $\theta_0 - \delta_\varepsilon = \theta_0 + \alpha_\varepsilon - \pi/2$. By contrast, the angle is defined as θ_0 under biaxial-shear loading in Figure 4(b). Thus, the macroscopic-elastic-shear modulus, G_m^{Aniso} , and the macroscopic-shear strength, τ_m^{Aniso} , under general-directional-shear loading can be derived by changing ω_0 in Equations (94) and (99) under biaxial-shear loading into $\omega_0 + 2\alpha_\varepsilon - \pi$ as follows:

$$G_m^{\text{Aniso}} = G_m^{\text{Iso}} \left\{ 1 + \frac{1}{2} a_2 \cos 2(\omega_0 + 2\alpha_\varepsilon) \right\}, \quad (102)$$

$$\tau_m^{\text{Aniso}} = \left\{ 1 + \frac{1}{3} a_2 \cos 2(\omega_0 + 2\alpha_\varepsilon) \right\} \tau_m^{\text{Iso}} + \frac{1}{2} a_1 p' \cos(\omega_0 + 2\alpha_\varepsilon), \quad (103)$$

where α_ε denotes the major principal strain direction relative to the vertical. By normalizing Equation (103) with the isotropic stress p' , the internal friction angle considering inherent anisotropy, ϕ_f^{Aniso} , under general-directional-shear loading is given as

$$\sin \phi_f^{\text{Aniso}} = \left\{ 1 + \frac{1}{3} a_2 \cos 2(\omega_0 + 2\alpha_\epsilon) \right\} \sin \phi_f^{\text{Iso}} + \frac{1}{2} a_1 \cos(\omega_0 + 2\alpha_\epsilon). \quad (104)$$

Equations (102) and (103) are reduced to Equations (82) and (86), respectively, by substituting $\alpha_\epsilon = \pi/4$ under simple-shear loading, and to Equations (94) and (99), respectively, by substituting $\alpha_\epsilon = \pi/2$ under biaxial-shear loading.

Equation (102) suggests that the anisotropy parameter a_2 can be determined from laboratory tests for a constant deposition angle of the bedding planes, normally $\theta_0 = \omega_0/2 = 90^\circ$ in a hollow-cylinder apparatus, if the anisotropic elastic-shear modulus G_m^{Aniso} corresponding to two different principal directions of the strain tensors is obtained. Once a_2 is estimated, another parameter, a_1 , can be determined from Equation (103) if the shear strength corresponding to the major principal strain direction, α_ϵ , is well-evaluated.

6. EXAMPLES

6.1. Simulations for undrained behavior under simple shear

The strain space multiple mechanism model's performance incorporating the effects of inherent anisotropy, which has three anisotropy parameters (i.e., a_1 , a_2 , and θ_0) in addition to 15 existing parameters [28] (as shown in Table 1), is investigated in this section. First, the effects of different fabric anisotropies (arising from different deposition processes or sample-preparation methods) upon the undrained response under the same loadings are examined. Note that the proposed model may not be suitable for simulating drained soil behaviors accompanied by a massive change in

dilatancy properties as well as a density change, because the current form is not state dependent; however, the dependency can be readily taken into account in future by redefining some of the model parameters as state dependent by introducing void ratio as an additional variable in the formulation. In this subsection, the undrained behavior under monotonic simple shear loading is simulated by changing the normal-vector direction of the bedding planes relative to the x axis (i.e., $\theta_0 = \omega_0 / 2$), as shown in Figure 4(a).

As an example of model performance, the undrained monotonic behavior under strain-controlled simple-shear loading with the anisotropy parameters $a_1 = 0.2$ and $a_2 = 0.0$ is shown in Figure 5. The rest of the model parameters in Table 1 are specified as shown in Table 2, and the undrained shear strength, q_{us} , for steady-state analysis [28] is not used in this study because clean sands such as Toyoura sand have very high values of this parameter at the steady state. Figures 5(a) and (b) indicate that the deposition angle, together with a_1 , greatly influences both the stress–strain relationship and the stress path. The soil behavior is contractive for $\theta_0 = 30^\circ$ through 60° , but becomes dilative as the angle approaches 0° or 90° . Figure 5(c) compares the dependency of the internal friction angle on θ_0 (or the major-principal-strain direction relative to bedding plane δ_e in Figure 4(a)) between simulation and theory (Equation (87)). The simulated internal friction angles, which differ from the input value $\phi_f = 39.7^\circ$ due to the effects of inherent anisotropy, accurately capture the theoretical solutions as a code-verification process.

Figure 6 shows a similar example of the model performance under strain-controlled

simple-shear loading with anisotropy parameters $a_1 = 0.0$ and $a_2 = 0.2$. Unlike the case with $a_1 = 0.2$ and $a_2 = 0.0$ in Figures 5(a) and (b), $\theta_0 = 45^\circ$ yields the most dilative behavior in Figures 6(a) and (b). Although the soil response becomes less dilative as θ_0 approaches 0° or 90° , contractive behavior is not obtained regardless of the deposition angle. This suggests that the impact of parameter a_2 on the undrained response under monotonic simple shear loading may not be as pronounced as that of a_1 . However, the dependency of internal friction angle upon the bedding-plane-direction angle θ_0 , which agrees well with Equation (87), is affected by a_2 in Figure 6(c) to the same degree as it is affected by a_1 in Figure 5(c), even though the internal friction angles are out of phase by 45 degrees in terms of the bedding-plane direction between Figures 5(c) and 6(c).

Simulation results with the anisotropy parameters $a_1 = 0.2$ and $a_2 = 0.2$ are depicted in Figure 7. The stress–strain relationship (Figure 7(a)) and the stress path (Figure 7(b)) are generally comparable to those with $a_1 = 0.2$ and $a_2 = 0.0$ in Figures 5(a) and (b), in spite of the difference in the parameter a_2 . However, the fluctuation range of the internal friction angle depending on the bedding-plane direction in Figure 7(c) is relatively small compared to the cases in which either of the parameters a_1 or a_2 are considered.

Figures 5 through 7 suggest that the inherently anisotropic soil, for which the normal direction of the bedding plane is specified as $\pi/2 - \theta_0$, shows the same responses (e.g., stress–strain relationship, stress path including internal friction angle) as in the case of θ_0 under simple shear loading, regardless of the combination of the anisotropy

parameters a_1 and a_2 .

6.2. Simulations for undrained behavior under biaxial shear

As shown in Figure 4(b), the undrained monotonic behavior under strain-controlled biaxial-shear loading is simulated by changing the bedding-plane direction, θ_0 , in this subsection. At first, the simulation results with the anisotropy parameters $a_1 = 0.2$ and $a_2 = 0.0$ are shown in Figure 8. Unlike simple shear loading with the same anisotropy parameters in Figures 5(a) and (b), the soil behavior changes from contractive to dilative as the θ_0 increases, as shown in Figures 8(a) and (b). The simulated internal friction angle in Figure 8(c), which agrees well with the theoretical solution (Equation (100)), monotonically increases between $\theta_0 = 0^\circ$ and 90° .

Figure 9 shows a similar example of the model performance under strain-controlled biaxial-shear loading with $a_1 = 0.0$ and $a_2 = 0.2$. The undrained responses in Figures 9(a) and (b) look similar to those with the same anisotropy parameters under simple shear loading in Figures 6(a) and (b), but $\theta_0 = 0^\circ$ and 90° yield the most dilative behavior under biaxial-shear loading. The presence of dilative behavior regardless of the bedding-plane direction suggests that a_2 may not significantly impact undrained behavior under monotonic-biaxial-shear loading or simple shear loading. However, the internal friction angle, which depends on θ_0 , is heavily affected, as shown in Figure 9(c). The friction angle assumes its lowest value at $\theta_0 = 45^\circ$ and shows a symmetric shape with respect to the bedding-plane angle.

Simulation results with anisotropy parameters $a_1 = 0.2$ and $a_2 = 0.2$ are shown in

Figure 10. The stress-strain relationship (Figure 10(a)) and the stress path (Figure 10(b)) are almost the same as those with $a_1 = 0.2$ and $a_2 = 0.0$ in Figures 8(a) and (b), regardless of a_2 . As shown in Figure 10(c), both the simulated internal friction angle and the theoretical solution take their lowest value at $\theta_0 = 30^\circ$, beyond which the friction angle drastically increases. This non-symmetric and non-monotonic trend between $\theta_0 = 0^\circ$ and 90° arises from a combination of the non-zero anisotropy parameters a_1 and a_2 . Note that, even if the same parameters are used, the dependency of internal friction angle upon the bedding-plane angle becomes totally different depending on the types of shear loading (i.e., simple or biaxial), as shown in Figures 7(c) and 10(c).

6.3. Simulations of undrained behavior under general-direction shear

The effects of different loadings on the same fabric are investigated in this subsection to verify the capability of the proposed model. The typical behavior of sand with an intermediate principal-stress coefficient, b , of 0.5, as has been identified through strain-controlled laboratory tests for water-pluviated Fraser River sand [9] and dry-pluviated Toyoura sand [8, 10], is the target of this simulation. These data can demonstrate the effects of general-directional-shear loading in the two-dimensional plane for the same fabric (i.e., $\theta_0 = 90^\circ$, which corresponds to the laboratory tests) upon the undrained responses of granular materials.

As mentioned in subsection 5.3, the anisotropy parameters a_1 and a_2 can be determined from laboratory test results by applying Equations (102) and (103) if the major-principal-strain direction is known experimentally. However, it is difficult to

obtain accurate readings of the anisotropic-elastic-shear modulus and anisotropic-shear strength from laboratory data; thus, the anisotropy parameters were determined based on trial and error, as well as the dilatancy parameters in Table 1, to achieve comparable responses to the overall trend of the experimental results. The model parameters used for each simulation are listed in Tables 3 through 5, except for the parameters unchanged from Table 2.

Note that the major principal-stress direction relative to the vertical, α_σ , is hereafter used as an anisotropy index [8-10, 21] instead of the major principal strain direction, α_ϵ , even though the laboratory tests and their simulations are performed under strain control. In general, granular materials with fabric anisotropy (i.e., inherent anisotropy and its evolution) show the noncoaxiality between the principal directions of the stress and strain tensors under complex-loading conditions (e.g., reverse loading and principal-stress rotation) [48, 49], but α_σ is proven to be approximately equal to α_ϵ with an angle difference of only two or three degrees under monotonic loading in Appendix B. In addition, tensile stresses and strains are taken to be positive in this study, thus a 90-degree phase shift is corrected for the experimental major principal-stress (or -strain) direction relative to the vertical in the laboratory tests [8-10], in which compressive stresses and strains have a positive sign.

Figure 11 presents the undrained-torsional-shear-test results with $b = 0.5$ for water-pluviated Fraser River sand with a relative density of 30% [9], as well as the corresponding model simulations. By employing the anisotropy parameters in Table 3, the model well-captures the experimental trend whereby larger values of α_σ relative to the vertical result in harder (i.e., larger shear stress at the same shear strain) and more

dilative responses. In particular, the most contractive behavior under $\alpha_\sigma = 30^\circ$ is reasonably simulated. Thus, the proposed model demonstrates the capability to capture the essential features of laboratory-test data for monotonic general-direction loading. As a supplement, the dependency of the internal friction angle upon the major-principal-stress direction, which is assumed to be equal to α_ϵ , is depicted in Figure 12; the consistency between the simulated results and the theoretical solution obtained by Equation (104) verifies the software code that incorporates the proposed model.

A comparison of the model simulations with the undrained torsional-shear-test results with $b = 0.5$ for dry-pluviated Toyoura sand with a relative density of 40% [10] is presented in Figure 13. The simulations agree well with the test data; both the dilative and contractive behavior of sand associated with different principal-stress directions can be well-captured with a single set of model parameters in Table 4. Figure 13(c) suggests that the steady state of clean sands is different depending on the principal stress direction under 15% shear strain; the tendency is also properly simulated in Figure 13(d). This fact is contradictory to some of the previous studies suggesting that a unique steady state is achieved for granular materials with inherent anisotropy. Figure 14 compares the simulated internal friction angle with the theoretical solution (Equation (104)). The model reasonably captures the experimental trend whereby larger values of the principal-stress direction α_σ lead to larger values of the friction angle as estimated from the stress path in Figure 13(c).

Figures 15 through 16 present similar comparisons of the stress-strain relationship and stress path for dry-pluviated Toyoura sand with three different relative densities (i.e., 90,

60, and 30%) [8]. The variations in sand responses depending on different principal-stress directions are properly simulated for each relative density. Variation in the internal friction angle for the test data with a relative density of 30% in Figure 17(c) is hard to accurately estimate, but the experimental variation trend for the other two relative densities in Figures 15(c) and 16(c) is well-reflected by the simulated dependency of the internal friction angle upon the principal-stress direction, as shown in Figures 18(a) and (b). These results demonstrate the model's capability to capture the complex behavior associated with inherent anisotropy, which may arise from different deposition processes or sample-preparation methods, with a single set of anisotropy parameters.

7. CONCLUSIONS

In this study, a constitutive model for describing the anisotropic behavior of granular materials with fabric anisotropy was discussed within the framework of a strain space multiple mechanism model [25, 28] in a two-dimensional plane. This model idealizes the behavior of granular materials based on a multitude of virtual simple shear mechanisms oriented in arbitrary directions. These mechanisms reflect the internal (or micromechanical) structure of granular materials with an induced anisotropy and form a second-order fabric tensor. To describe the anisotropic behavior, the second-order fabric tensor was extended by incorporating a new function to represent the effect of inherent anisotropy, with three additional parameters: two, a_1 and a_2 , to control the degree of anisotropy and the other, θ_0 , to express the principal direction of inherent anisotropy (e.g., the normal vector direction of the bedding plane relative to horizontal axis in

geotechnical laboratory tests). The second (or a higher) mode of inherent anisotropy can be expressed by introducing the parameter a_2 as well as the first (or fundamental) mode by the parameter a_1 . The formulation of the dilative component of dilatancy was also extended to consider the effect of inherent anisotropy based on the interlocking mechanism, which is defined as the energy-less component of macroscopic strain [43].

In addition to the effects of different fabric anisotropies upon the undrained behavior under the same loading conditions (i.e., simple or biaxial shear), the effects of different loadings for the same fabric were examined to investigate the capability of the proposed model. Complex anisotropic sand behavior under undrained-monotonic loading with an intermediate principal-stress coefficient of 0.5 (which has been identified through laboratory tests of Fraser River sand [9] and Toyoura sand [8, 10]) was used to validate this model. The proposed model with a single set of anisotropy parameters was shown to successfully capture the anisotropic stress-strain relationships and stress paths, which become contractive or dilative depending on the principal-stress direction. These results demonstrate that the strain space multiple mechanism model that incorporates an anisotropic function should be able to simulate the anisotropic behavior of granular materials.

It should be noted that the behavior of granular materials under more complex-loading conditions (e.g., reverse loading and principal-stress rotation) has not been studied in this paper. Future work should be focused on the issue, as well as developing a three-dimensional formulation of a strain space multiple mechanism model with inherent anisotropy.

APPENDIX A

The necessity and advantages of introducing the second-mode fabric anisotropy with the parameter a_2 in Equation (7) are examined through comparison with a DEM simulation by Fu and Dafalias [17]. Based on direct shear tests using DEM models consisting of ellipse-shaped particles, they have determined a shear failure criterion as a function of the inclination angle of the bedding plane with respect to the shear plane. Circle symbols in Figure A.1 show the change of peak shear strength as the inclination angle ψ increases from 0° to 360° in polar coordinate systems. The friction angle is dependent on the inclination angle: the peak friction angle first slightly decreases toward its minimum value at $\psi = 60^\circ$, and then increases as the inclination angle increases beyond 60° and reaches its maximum value at $\psi = 115^\circ$. The most remarkable finding may be that a phase shifting between the maximum and minimum strengths is 55° , less than 90° , with respect to the inclination angle.

As a comparison, a change of peak shear strength calculated based on Equation (104) is depicted in Figure A.1. The parameters used are as follows: $a_1 = 0.40$, $a_2 = 0.35$, $\sin \phi_f^{\text{iso}} = 0.75$ and $\alpha_\varepsilon = 25^\circ$ with the relationship $\psi = 90^\circ - \theta_0$ as shown in Figure 4(c). The reason why the major principal strain direction differs from 45° , which corresponds to simple-shear loading shown in Figure 4(a), is that the normal stress acting on the vertical plane was different from that on the horizontal (or shear) plane in the direct shear tests and thus biaxial-shear loading should be also considered. The solid and broken lines in Figure A.1 indicate the peak shear strength with and without considering the second-mode fabric anisotropy, respectively. The calculated strength distribution without the parameter a_2 is ellipse- or peanut-shaped depending on the

parameter a_1 , and a phase shifting between the maximum and minimum strengths is always 90° . Thus, the characteristics of the strength distribution obtained from the DEM simulations cannot be properly simulated. On the other hand, the introduction of the parameter a_2 for expressing a second-mode fabric anisotropy makes the calculated strength distribution shape much closer to the DEM simulation results, including a phase shifting between the maximum and minimum strengths: the calculated phase shifting (i.e., 62°) from Equation (104) with the parameters $a_1 = 0.40$ and $a_2 = 0.35$ is comparable with the value obtained from the DEM simulations.

APPENDIX B

The noncoaxiality between the principal directions of the stress and strain tensors under monotonic loading is investigated through strain-controlled simulations for Fraser River sand [9] and Toyoura sand [8, 10]. Strain paths shown in Figure B.1 were applied in the simulations with the model parameters listed in Tables 3 through 5. Figure B.2 illustrates stress paths obtained from the simulation for the undrained torsional-shear tests of Fraser River sand: the major principal stress direction, α_σ , which can be calculated from the stress paths, is identical to α_ϵ for $\alpha_\epsilon = 0^\circ$ and 90° whereas α_σ slightly differs from α_ϵ for $\alpha_\epsilon = 45^\circ$, 60° , and 75° . However, the angle difference is only two or three degrees, thus the principal directions of the stress and strain tensors can be assumed to be nearly coaxial within the strain level (i.e., $\pm 20\%$). Stress paths obtained for Toyoura sand shown in Figure B.3 indicate a similar tendency (i.e., the coaxiality between the stress and strain tensors may work), and α_σ tends to get closer

to α_ε with the increase in a relative density.

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Table 1. Model parameters of the strain space multiple mechanism model.

Symbol	Mechanism	Parameter designation
$K_{L/Ua}$	Volumetric	Bulk modulus under the confining pressure p_a
r_K	Volumetric	Reduction factor of bulk modulus for liquefaction analysis
l_K	Volumetric	Power index of bulk modulus for liquefaction analysis
G_{ma}^{iso}	Shear	Isotropic shear modulus under the confining pressure p_a
ϕ_f^{iso}	Shear	Isotropic internal friction angle
h_{max}	Shear	Upper bound for hysteretic damping factor
ϕ_p	Dilatancy	Phase transformation angle
r_{ε_d}	Dilatancy	Parameter controlling dilative and contractive components
$r_{\varepsilon_d^c}$	Dilatancy	Parameter controlling contractive component
q_1	Dilatancy	Parameter controlling initial phase of contractive component
q_2	Dilatancy	Parameter controlling final phase of contractive component
ε_d^{cm}	Dilatancy	Limit of contractive component
S_1	Dilatancy	Small positive number to avoid zero confining pressure
c_1	Dilatancy	Parameter controlling elastic range for contractive component
q_{us}	Dilatancy	Undrained shear strength (for steady-state analysis)
a_1	Fabric anisotropy	Parameter controlling the degree of anisotropy
a_2	Fabric anisotropy	Parameter controlling the degree of anisotropy
θ_0	Fabric anisotropy	Parameter controlling the direction of anisotropy

Table 2. Model parameters used as the standard set for simulation.

$K_{L/Ua}$ (kPa)	r_K	l_K	G_{ma} (kPa)	ϕ_f ($^\circ$)	h_{max}	ϕ_p ($^\circ$)	r_{ε_d}	$r_{\varepsilon_d^c}$	q_1	q_2	ε_d^{cm}	S_1	c_1
220300	0.5	2	84490	39.7	0.24	28.0	0.1	1.5	1.0	1.0	0.2	0.005	1.0

($p_a = 98$ kPa)

Table 3. Model parameters used for water-pluviated Fraser River sand with a relative density of 30%.

$K_{L/Ua}$ (kPa)	G_{ma} (kPa)	ϕ_f ($^\circ$)	r_{ε_d}	$r_{\varepsilon_d^c}$	q_1	q_2	ε_d^{cm}	a_1	a_2	θ_0 ($^\circ$)
123048	47184	38.0	0.2	3.0	2.5	0.55	0.2	0.08	0.12	90

($p_a = 150$ kPa)

Table 4. Model parameters used for dry-pluviated Toyoura sand with a relative density

of 40%.

$K_{L/Ua}$ (kPa)	G_{ma} (kPa)	ϕ_f ($^\circ$)	r_{ε_d}	$r_{\varepsilon_d^c}$	q_1	q_2	ε_d^{cm}	a_1	a_2	θ_0 ($^\circ$)
171025	65581	41.0	0.2	3.0	2.5	0.75	0.2	0.12	0.20	90

($p_a = 98$ kPa)

Table 5. Model parameters used for dry-pluviated Toyoura sand with a relative density of: (a) 90%, (b) 60%, (c) 30%.

(a)

$K_{L/Ua}$ (kPa)	G_{ma} (kPa)	ϕ_f ($^\circ$)	r_{ε_d}	$r_{\varepsilon_d^c}$	q_1	q_2	ε_d^{cm}	a_1	a_2	θ_0 ($^\circ$)
239162	91709	45.0	0.2	2.0	2.5	1.2	0.1	0.08	0.18	90

($p_a = 98$ kPa)

(b)

$K_{L/Ua}$ (kPa)	G_{ma} (kPa)	ϕ_f ($^\circ$)	r_{ε_d}	$r_{\varepsilon_d^c}$	q_1	q_2	ε_d^{cm}	a_1	a_2	θ_0 ($^\circ$)
196033	75170	42.0	0.2	2.5	2.5	0.75	0.15	0.08	0.18	90

($p_a = 98$ kPa)

(c)

$K_{L/Ua}$ (kPa)	G_{ma} (kPa)	ϕ_f ($^\circ$)	r_{ε_d}	$r_{\varepsilon_d^c}$	q_1	q_2	ε_d^{cm}	a_1	a_2	θ_0 ($^\circ$)
159523	61170	40.0	0.2	3.5	3.5	0.35	0.2	0.15	0.05	90

($p_a = 98$ kPa)

Figure 1. Contact normal ($\mathbf{n}$) and tangential ($\mathbf{t}$) directions and contact force $\mathbf{P}$.

Figure 2. Distribution of the function $F^*(\theta)$: (a) effects of the anisotropic parameter a_1 with $a_2 = 0.0$ (the first mode); (b) effects of the anisotropic parameter a_2 with $a_1 = 0.0$ (the second mode).

Figure 3. Stress component in the direction of θ to define the two-dimensional mechanism: (a) biaxial shear $\langle \mathbf{n} \otimes \mathbf{n} \rangle$; (b) simple shear $\langle \mathbf{t} \otimes \mathbf{n} \rangle$.

Figure 4. Schematic image of numerical simulation for strain-controlled laboratory tests under plane-strain condition: (a) simple shear with different bedding-plane angles; (b) biaxial shear with different bedding-plane angles; (c) multi-directional shear (in arbitrary directions) with horizontal bedding planes.

Figure 5. Simulations for undrained behavior under simple shear with $a_1 = 0.2$ and $a_2 = 0.0$: (a) stress–strain relationship; (b) stress path; (c) internal friction angle.

Figure 6. Simulations for undrained behavior under simple shear with $a_1 = 0.0$ and $a_2 = 0.2$: (a) stress–strain relationship; (b) stress path; (c) internal friction angle.

Figure 7. Simulations for undrained behavior under simple shear with $a_1 = 0.2$ and $a_2 = 0.2$: (a) stress–strain relationship; (b) stress path; (c) internal friction angle.

Figure 8. Simulations for undrained behavior under biaxial shear with $a_1 = 0.2$ and $a_2 = 0.0$: (a) stress–strain relationship; (b) stress path; (c) internal friction angle.

Figure 9. Simulations for undrained behavior under biaxial shear with $a_1 = 0.0$ and $a_2 = 0.2$: (a) stress–strain relationship; (b) stress path; (c) internal friction angle.

Figure 10. Simulations for undrained behavior under biaxial shear with $a_1 = 0.2$ and $a_2 = 0.2$: (a) stress–strain relationship; (b) stress path; (c) internal friction angle.

Figure 11. Experimental results and model simulations for the undrained torsional-shear tests of Fraser River sand with a relative density of 30% (data from Uthayakumar and Vaid [9]): (a) measured stress–strain relationship; (b) computed stress–strain relationship; (c) measured stress path; (d) computed stress path.

Figure 12. Comparison of simulated internal friction angles with theoretical solutions for Fraser River sand with a relative density of 30%.

Figure 13. Experimental results and model simulations for undrained torsional-shear tests of Toyoura sand with a relative density of 40% (data from Yoshimine et al. [10]): (a) measured stress–strain relationship; (b) computed stress–strain relationship; (c)

measured stress path; (d) computed stress path.

Figure 14. Comparison of simulated internal friction angles with theoretical solutions for Toyoura sand with a relative density of 40%.

Figure 15. Experimental results and model simulations for undrained torsional-shear tests of Toyoura sand with a relative density of 90% (data from Nakata et al. [8]): (a) measured stress–strain relationship; (b) computed stress–strain relationship; (c) measured stress path; (d) computed stress path.

Figure 16. Experimental results and model simulations for undrained torsional-shear tests of Toyoura sand with a relative density of 60% (data from Nakata et al. [8]): (a) measured stress–strain relationship; (b) computed stress–strain relationship; (c) measured stress path; (d) computed stress path.

Figure 17. Experimental results and model simulations for the undrained torsional-shear tests of Toyoura sand with a relative density of 30% (data from Nakata et al. [8]): (a) measured stress–strain relationship; (b) computed stress–strain relationship; (c) measured stress path; (d) computed stress path.

Figure 18. Comparison between simulated internal friction angles and theoretical results for Toyoura sand with a relative density of: (a) 90%; (b) 60%; (c) 30%.

Figure A.1. Variation of peak shear strength with respect to inclination angles (DEM data from Fu and Dafalias [17]).

Figure B.1. Strain path applied in simulations of undrained monotonic behavior under general-directional-shear loading.

Figure B.2. Stress path obtained from the simulation of Fraser River sand with a relative density of 30%.

Figure B.3. Stress path obtained from the simulation of Toyoura sand with a relative density of: (a) 90%; (b) 60%; (c) 40%; (d) 30%.

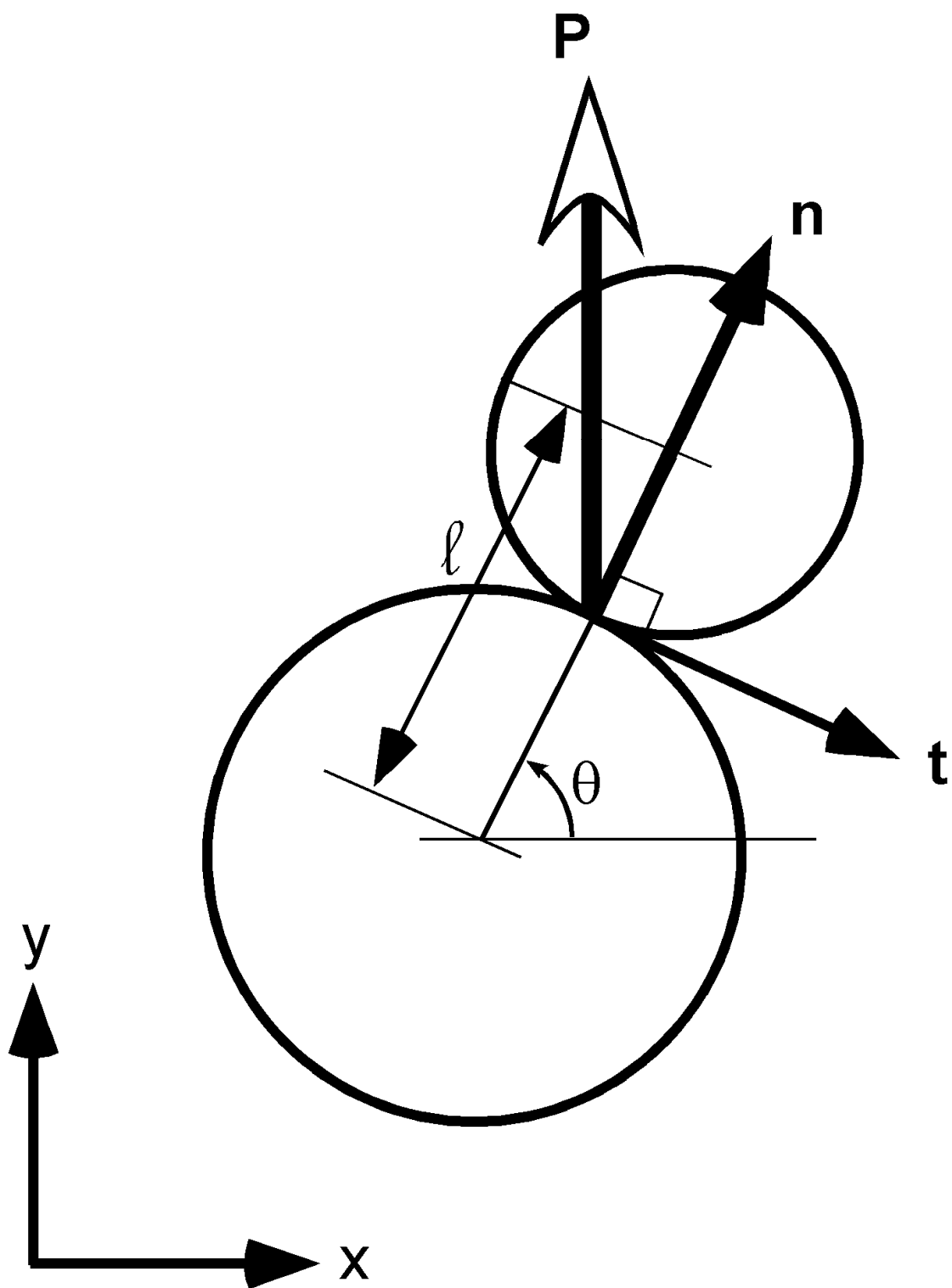
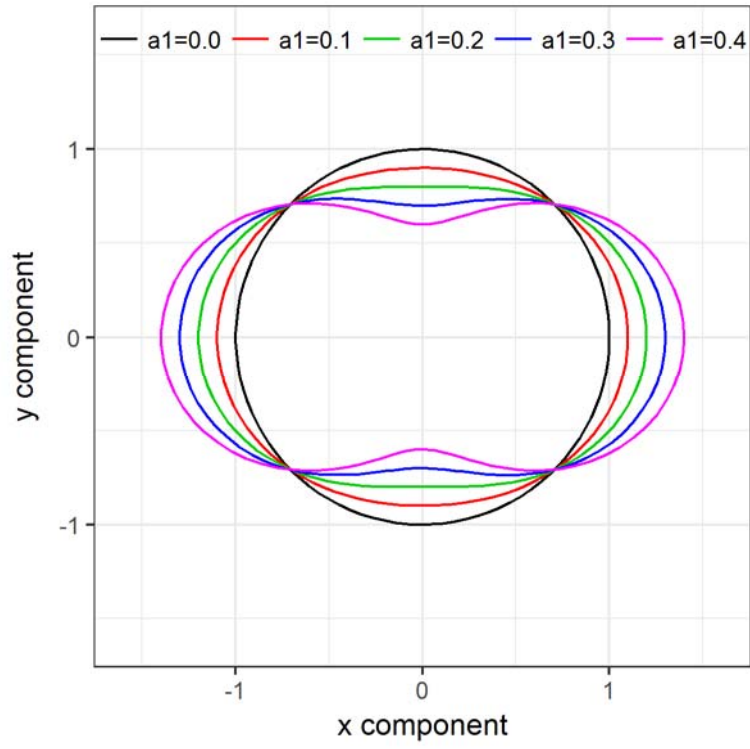
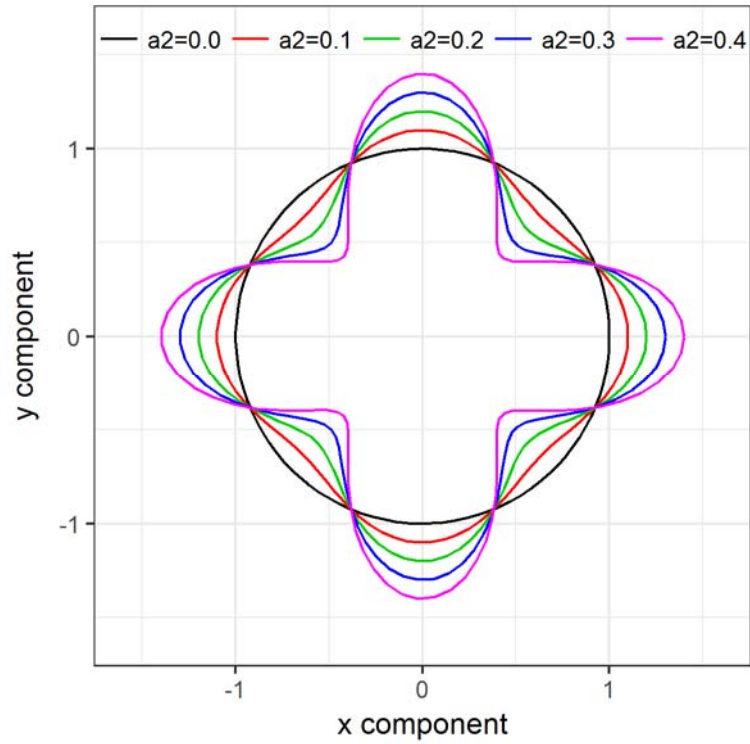


Figure 1. Contact normal ($\mathbf{n}$) and tangential ($\mathbf{t}$) directions and contact force $\mathbf{P}$.



(a)



(b)

Figure 2. Distribution of the function $F^*(\theta)$: (a) effects of the anisotropic parameter a_1 with $a_2 = 0.0$ (the first mode); (b) effects of the anisotropic parameter a_2 with $a_1 = 0.0$ (the second mode).

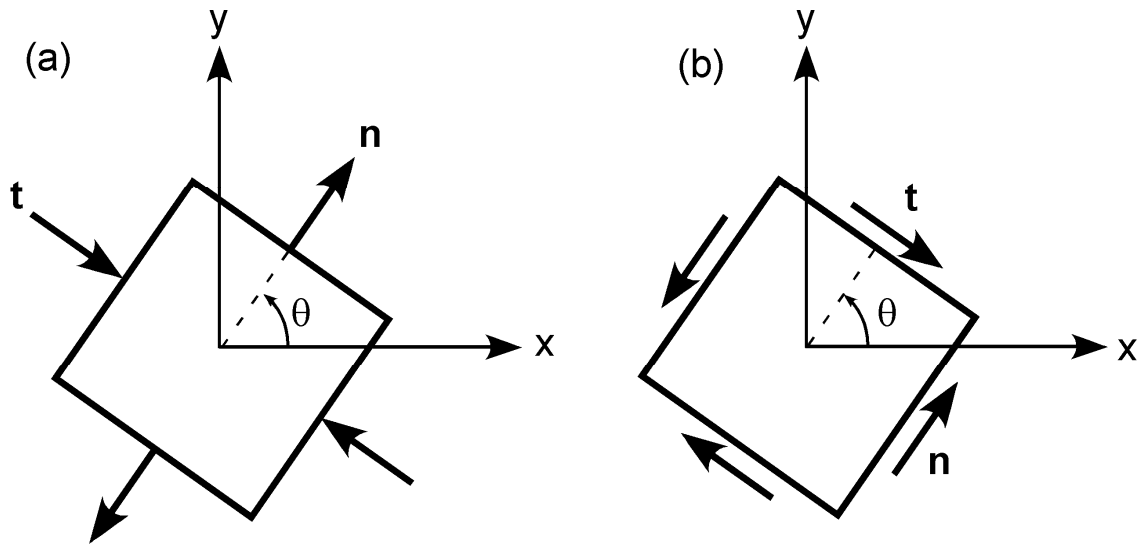
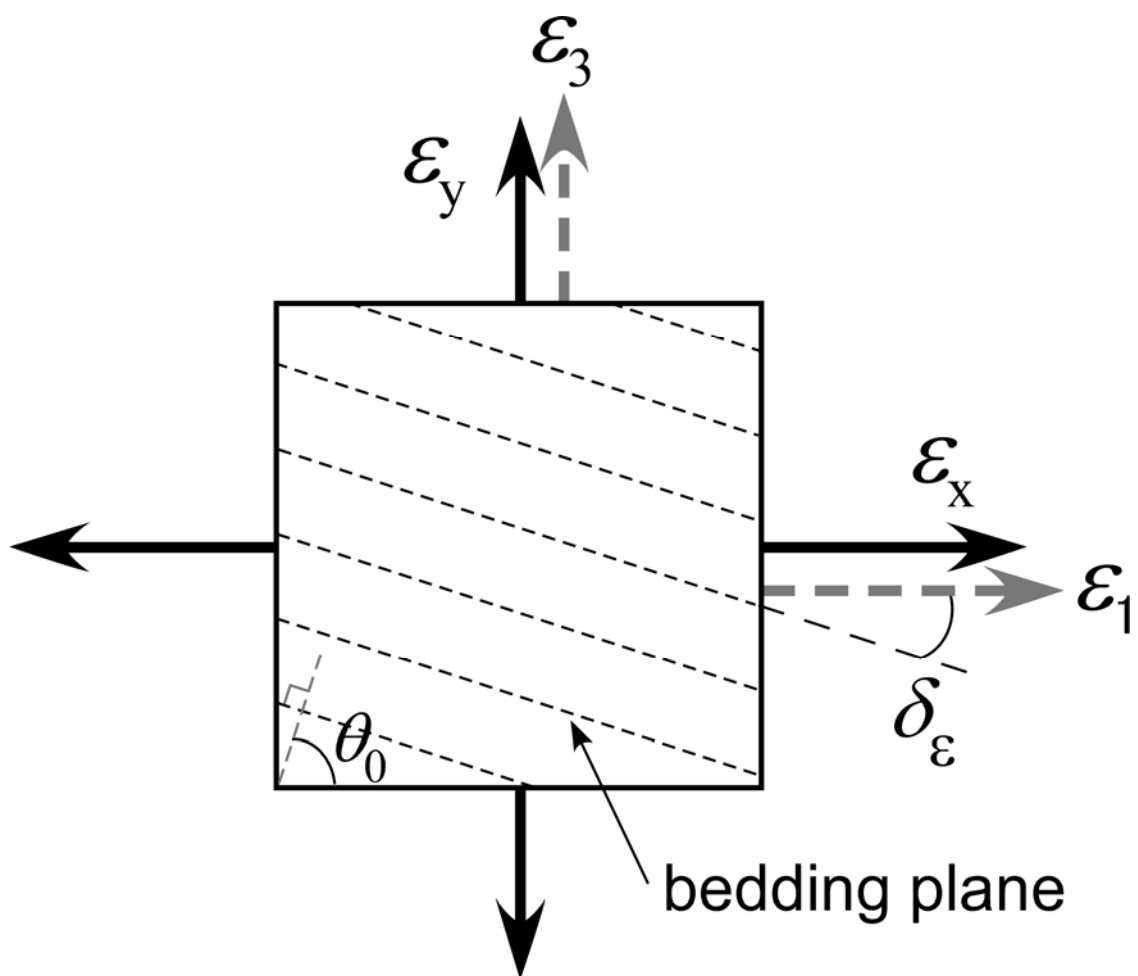
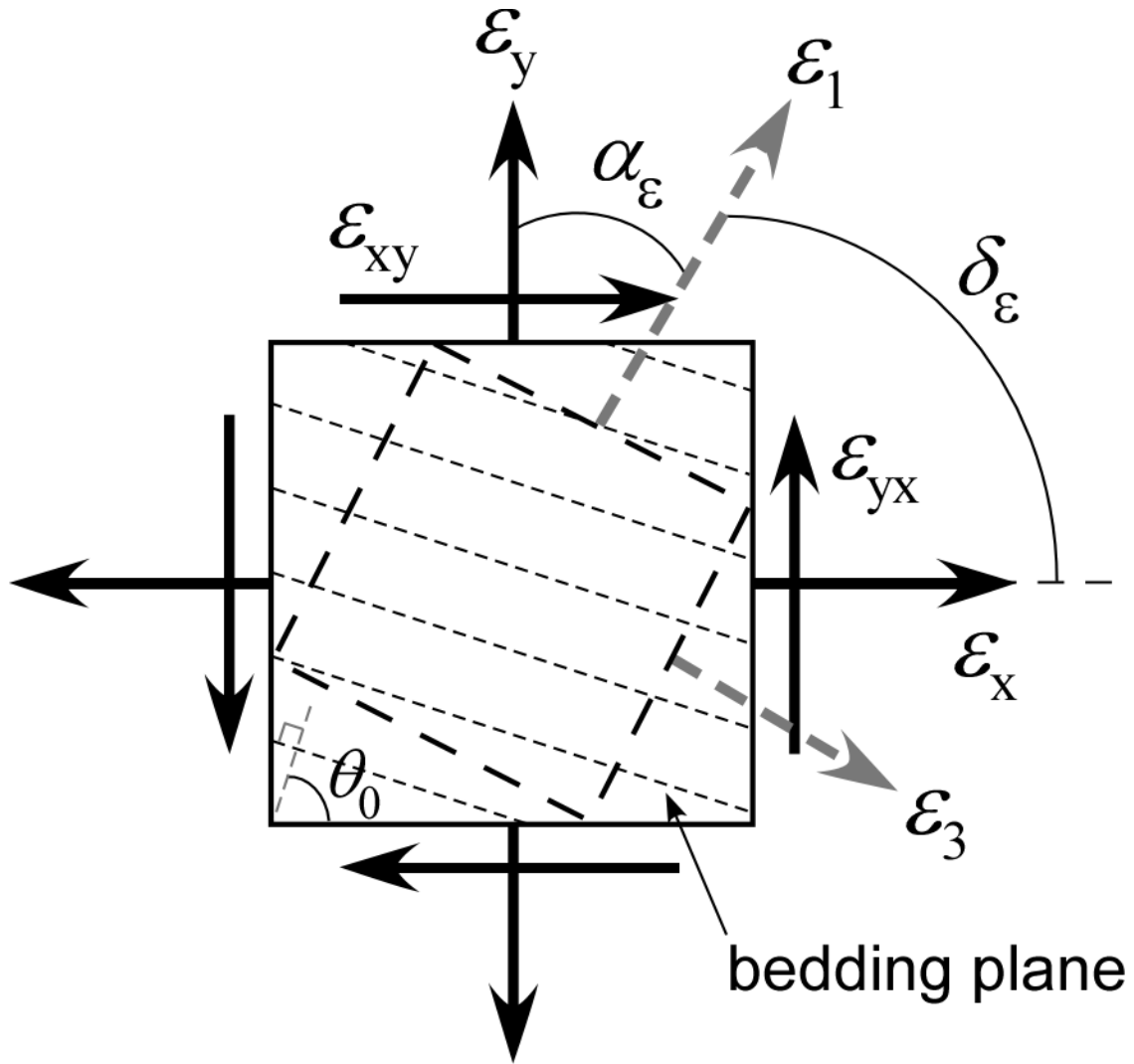


Figure 3. Stress component in the direction of θ to define the two-dimensional mechanism: (a) biaxial shear $\langle \mathbf{n} \otimes \mathbf{n} \rangle$; (b) simple shear $\langle \mathbf{t} \otimes \mathbf{n} \rangle$.

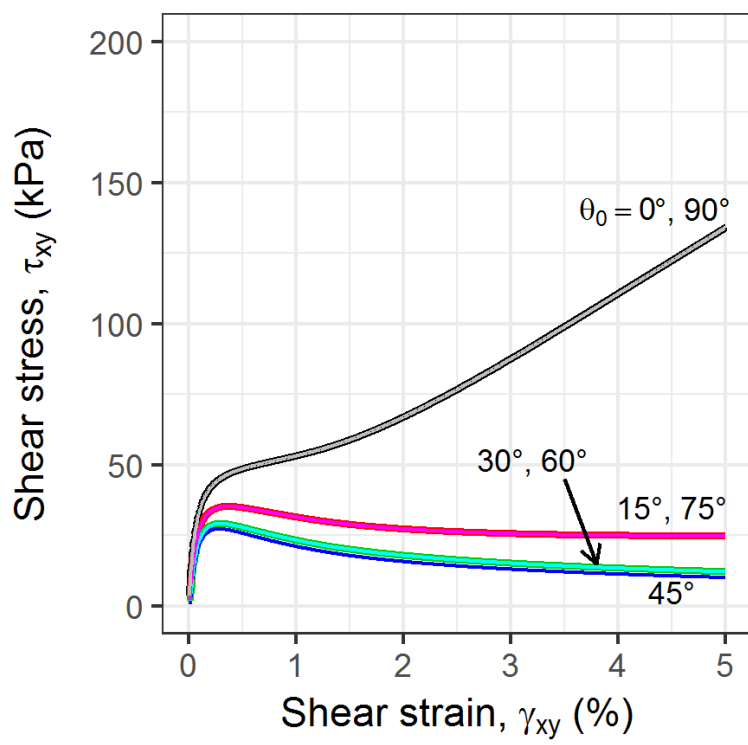




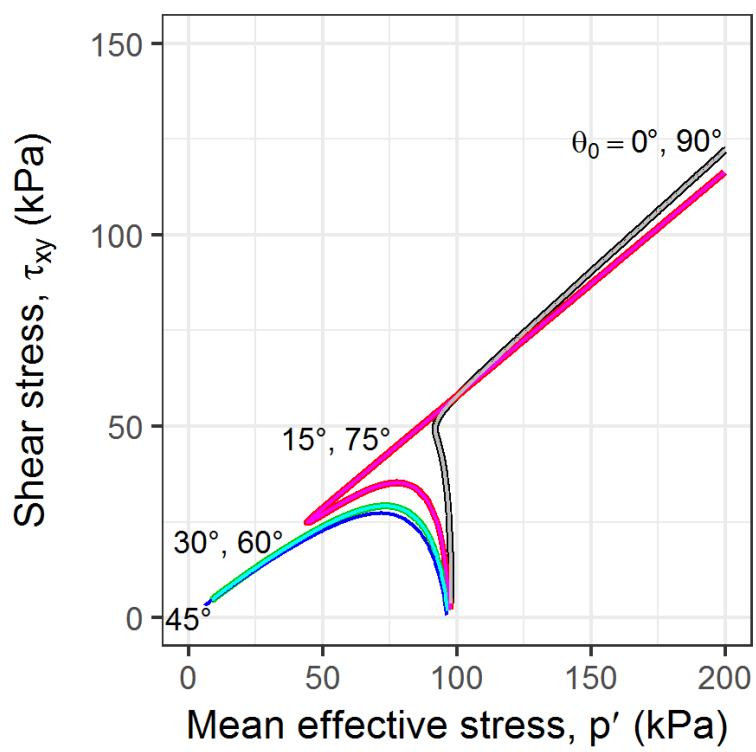
(b)



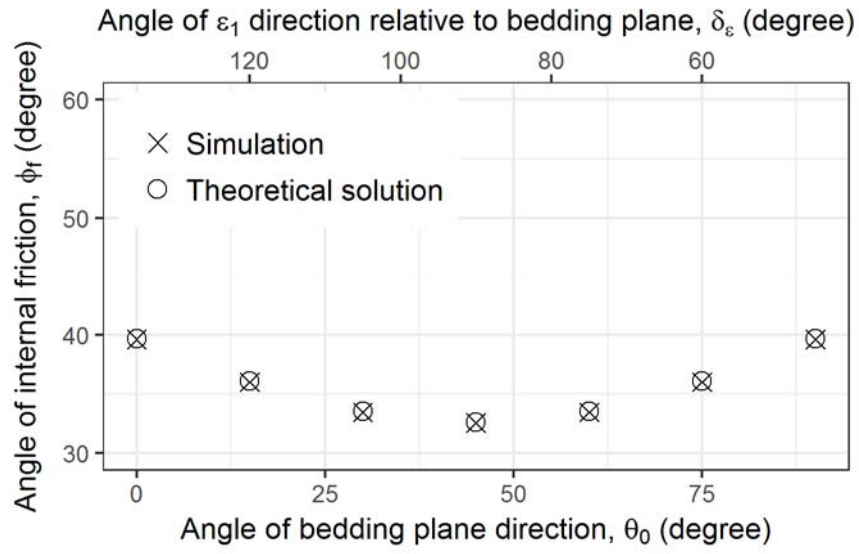
(c)
Figure 4. Schematic image of numerical simulation for strain-controlled laboratory tests under plane-strain condition: (a) simple shear with different bedding-plane angles; (b) biaxial shear with different bedding-plane angles; (c) multi-directional shear (in arbitrary directions) with horizontal bedding planes.



(a)

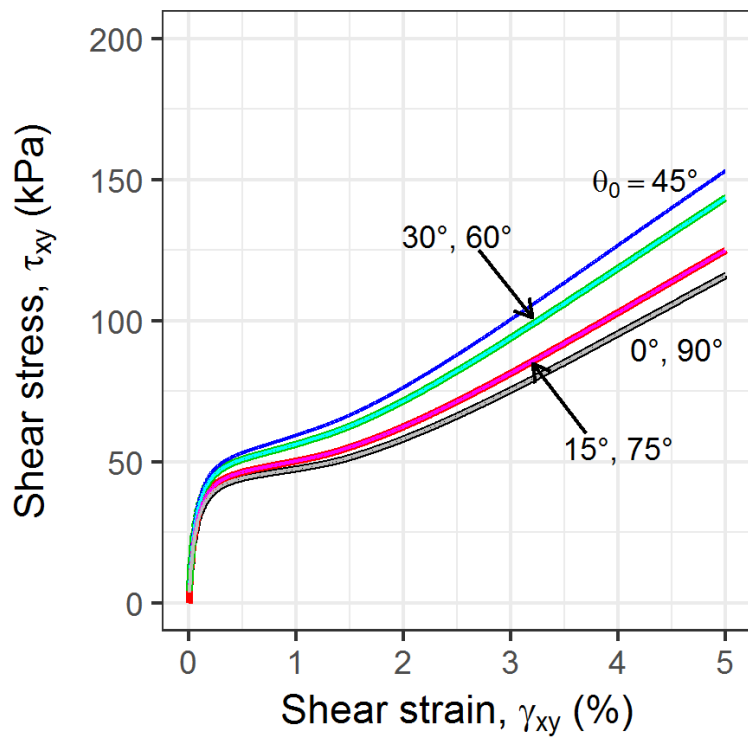


(b)

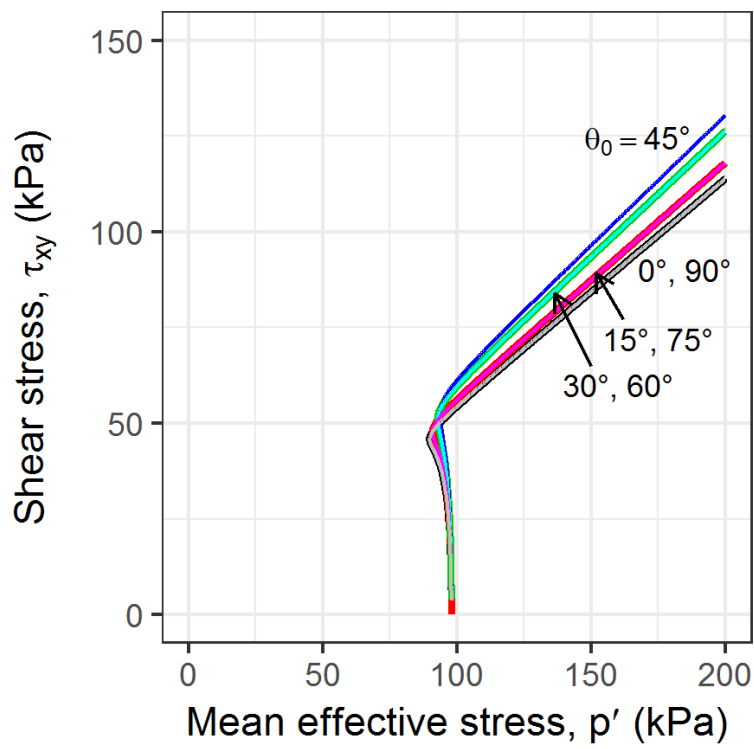


(c)

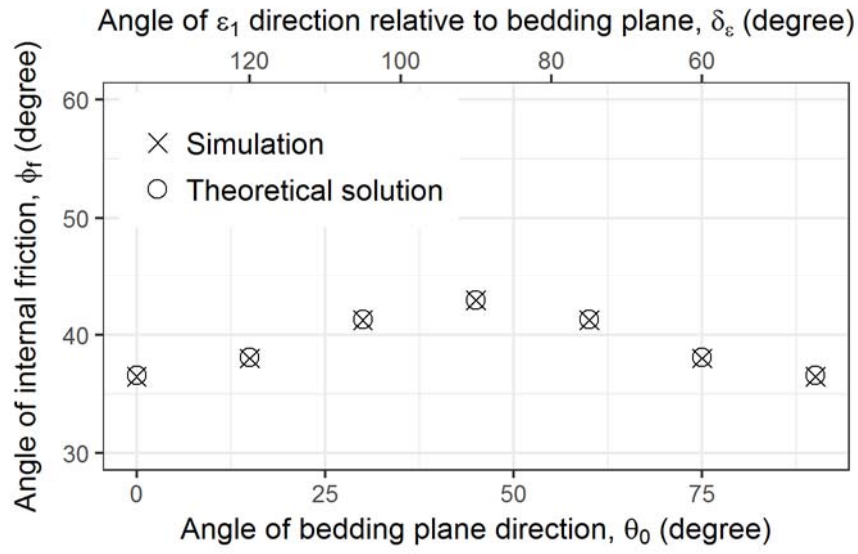
Figure 5. Simulations for undrained behavior under simple shear with $a_1 = 0.2$ and $a_2 = 0.0$: (a) stress–strain relationship; (b) stress path; (c) internal friction angle.



(a)

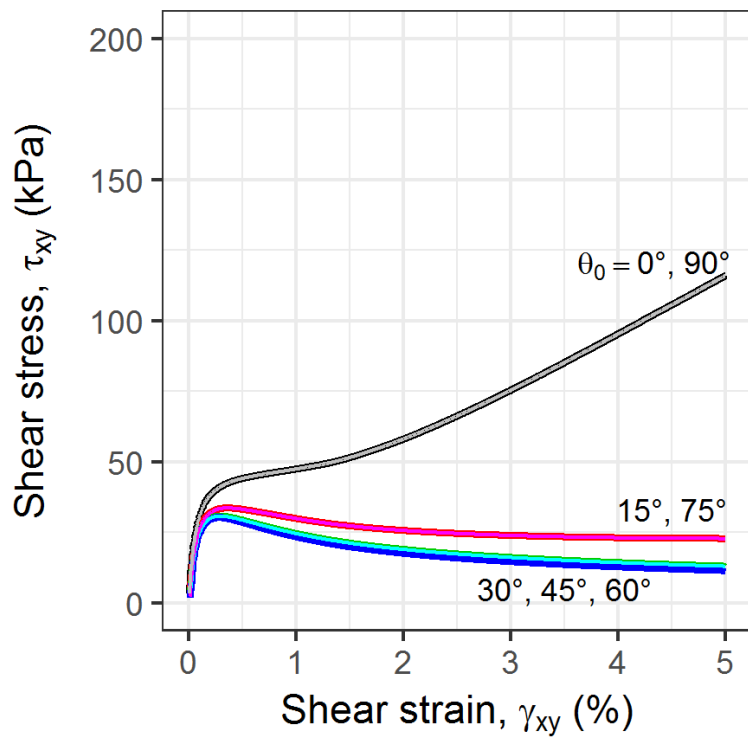


(b)

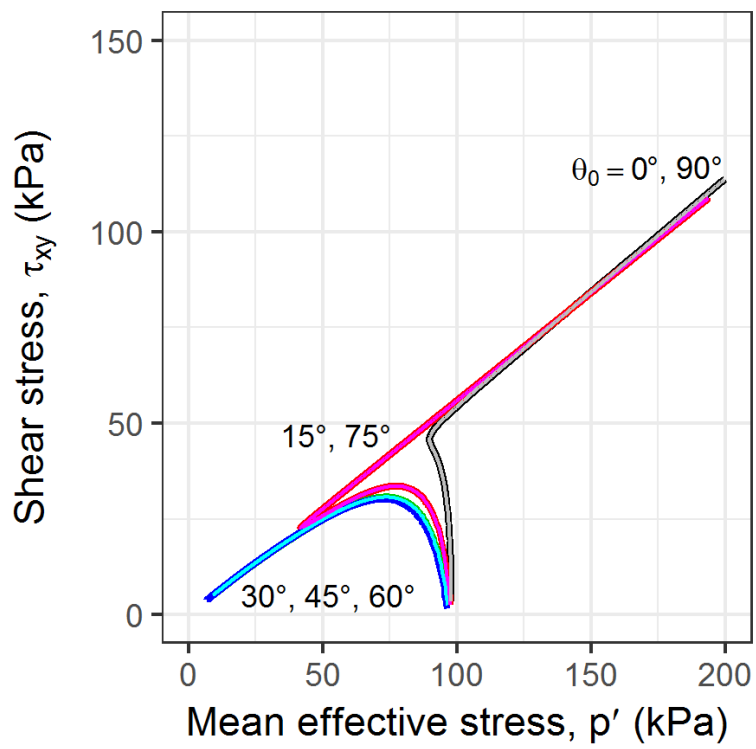


(c)

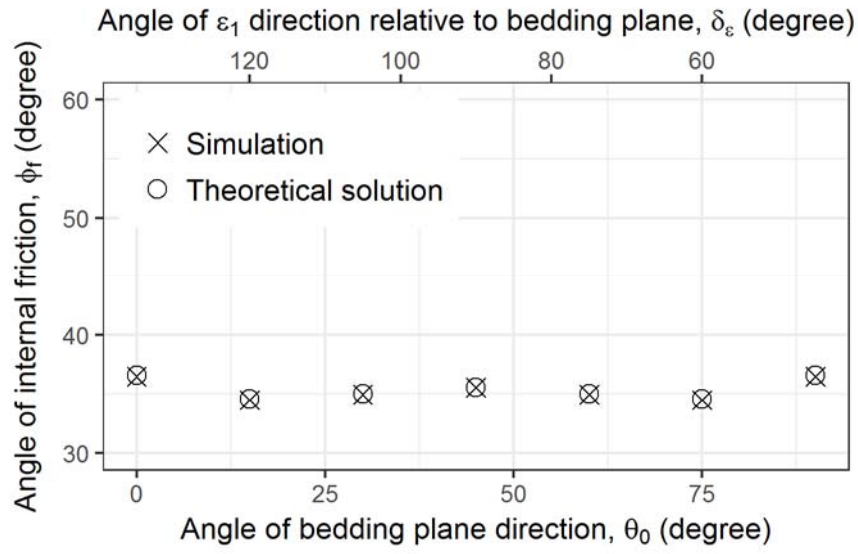
Figure 6. Simulations for undrained behavior under simple shear with $a_1 = 0.0$ and $a_2 = 0.2$: (a) stress–strain relationship; (b) stress path; (c) internal friction angle.



(a)

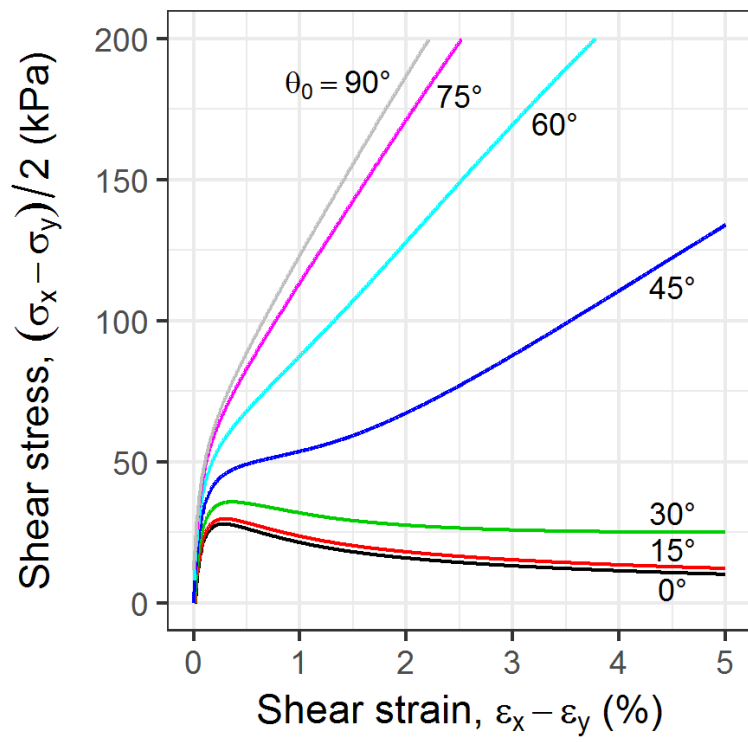


(b)

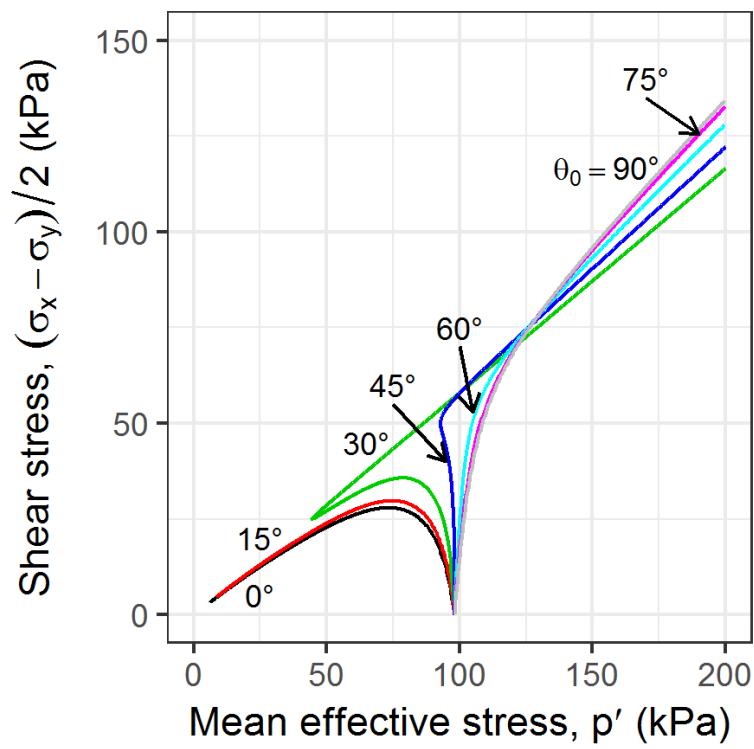


(c)

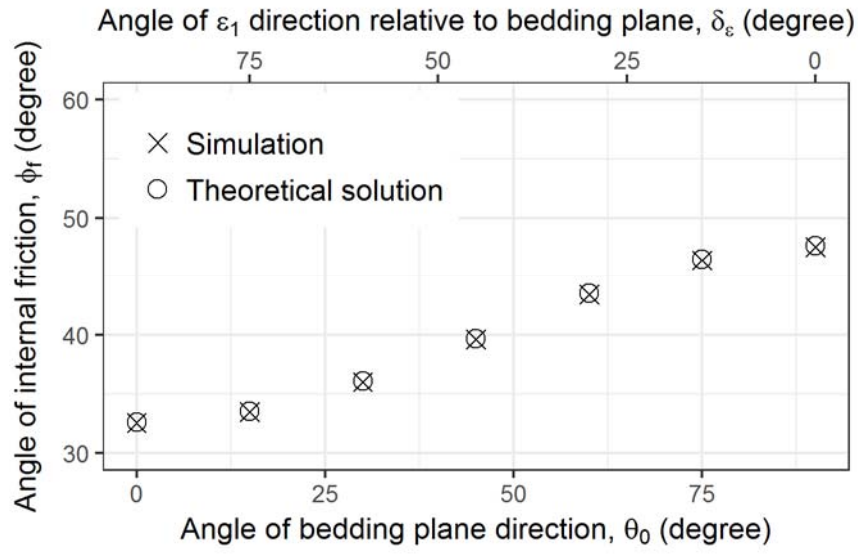
Figure 7. Simulations for undrained behavior under simple shear with $a_1 = 0.2$ and $a_2 = 0.2$: (a) stress–strain relationship; (b) stress path; (c) internal friction angle.



(a)

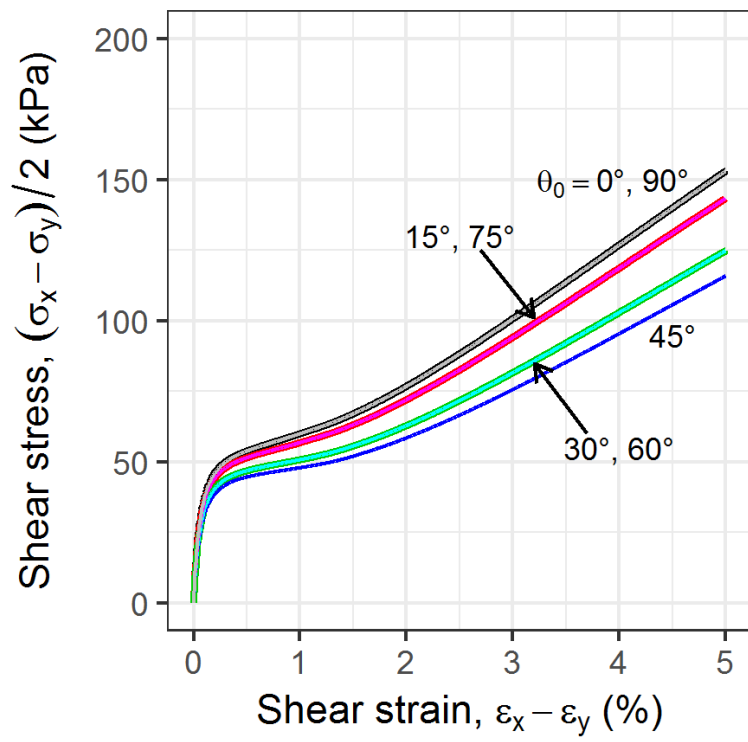


(b)

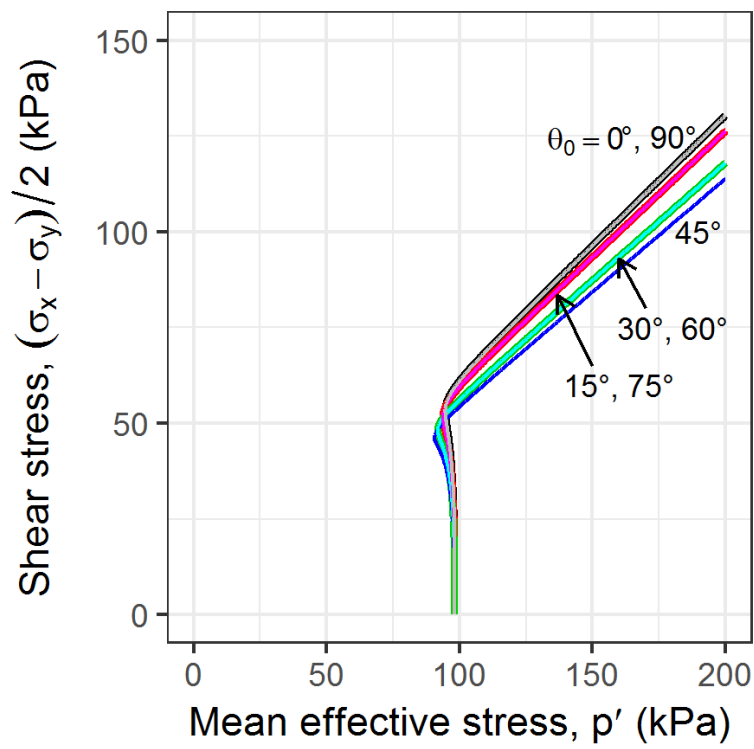


(c)

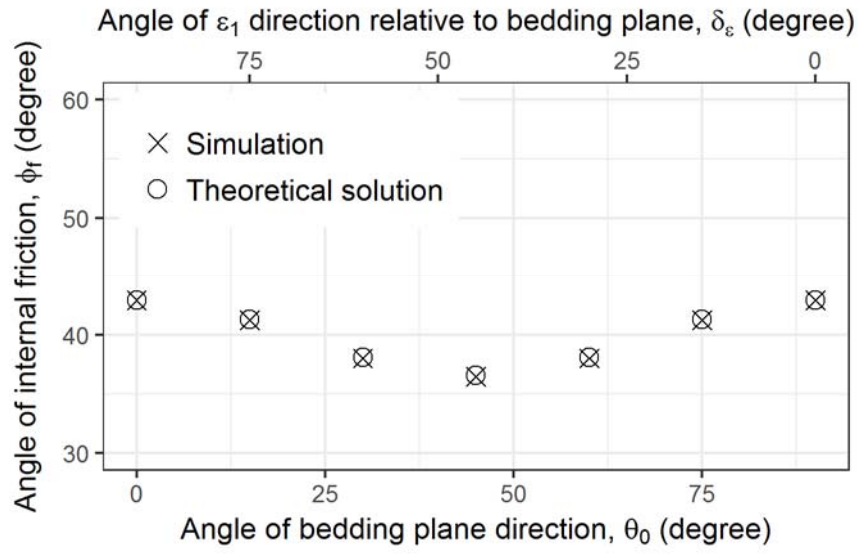
Figure 8. Simulations for undrained behavior under biaxial shear with $a_1 = 0.2$ and $a_2 = 0.0$: (a) stress–strain relationship; (b) stress path; (c) internal friction angle.



(a)

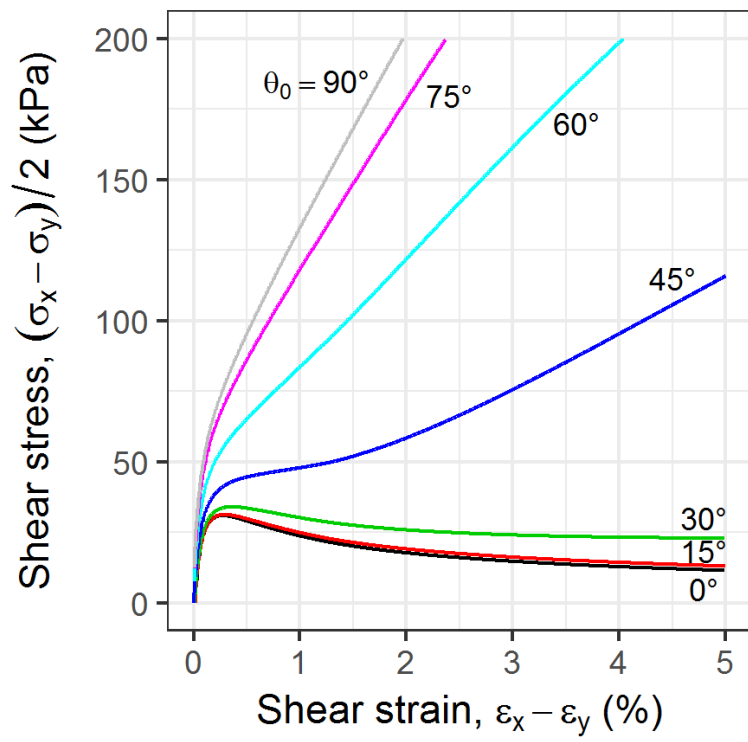


(b)

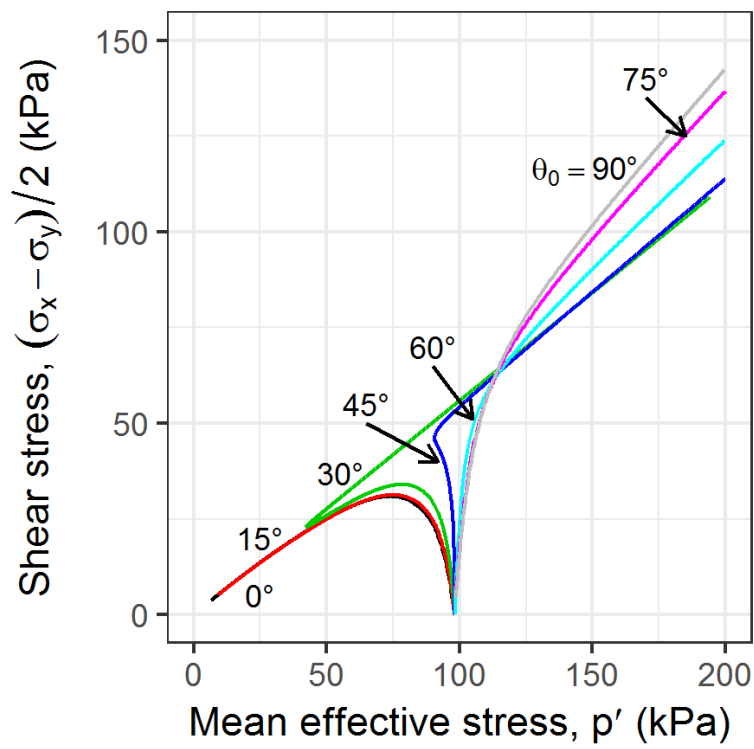


(c)

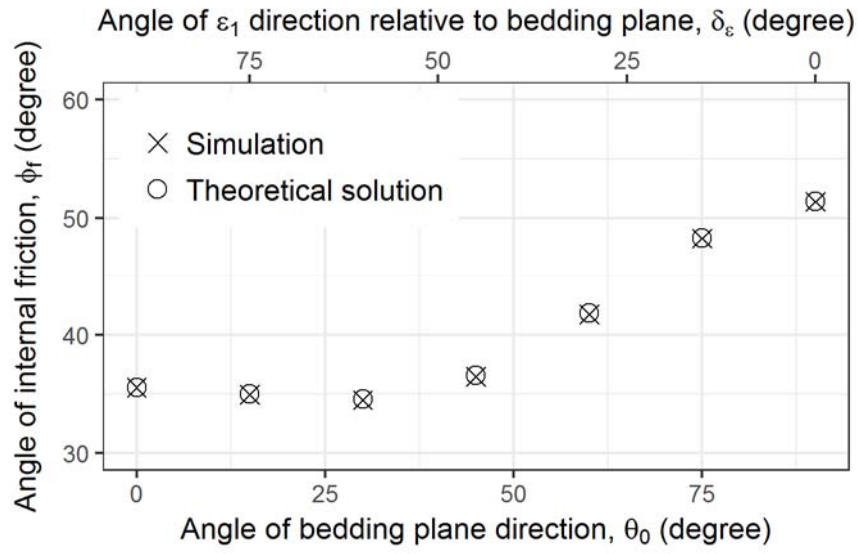
Figure 9. Simulations for undrained behavior under biaxial shear with $a_1 = 0.0$ and $a_2 = 0.2$: (a) stress–strain relationship; (b) stress path; (c) internal friction angle.



(a)

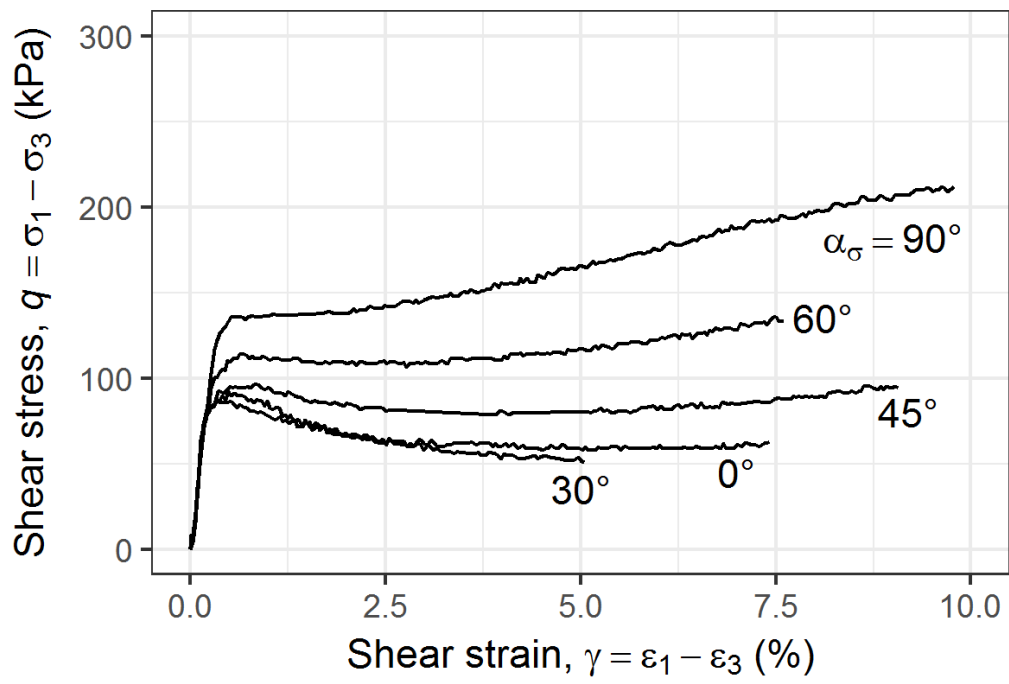


(b)

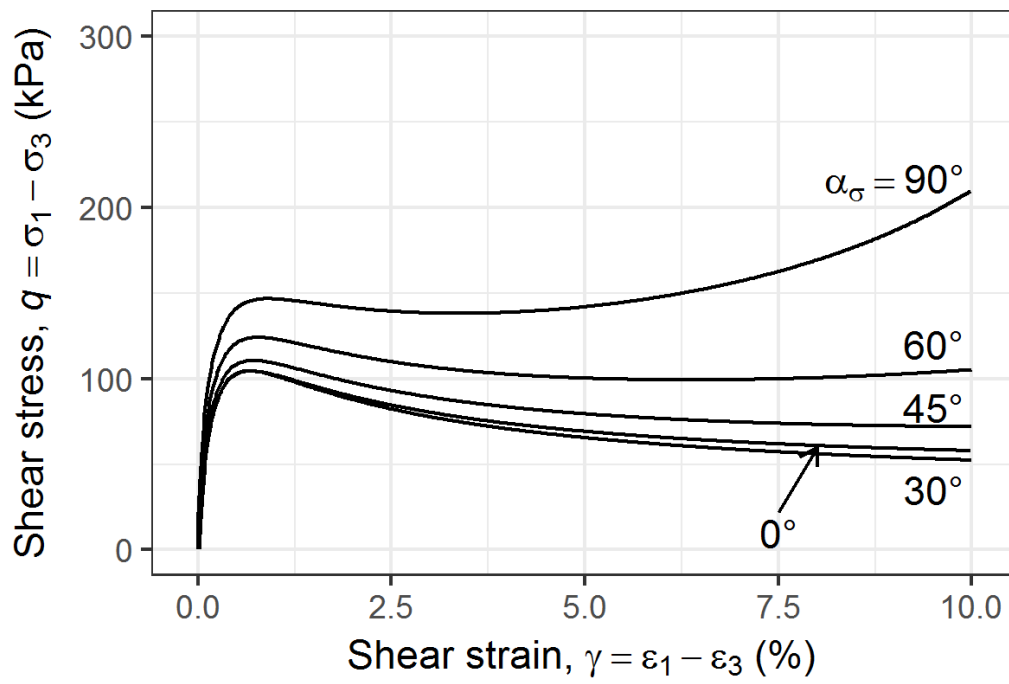


(c)

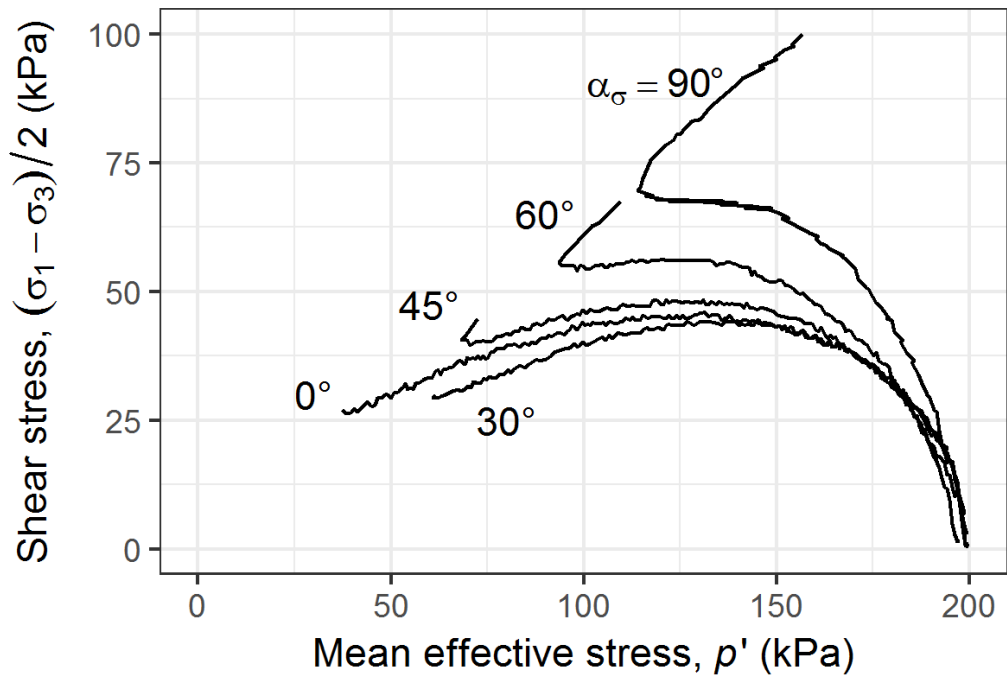
Figure 10. Simulations for undrained behavior under biaxial shear with $a_1 = 0.2$ and $a_2 = 0.2$: (a) stress–strain relationship; (b) stress path; (c) internal friction angle.



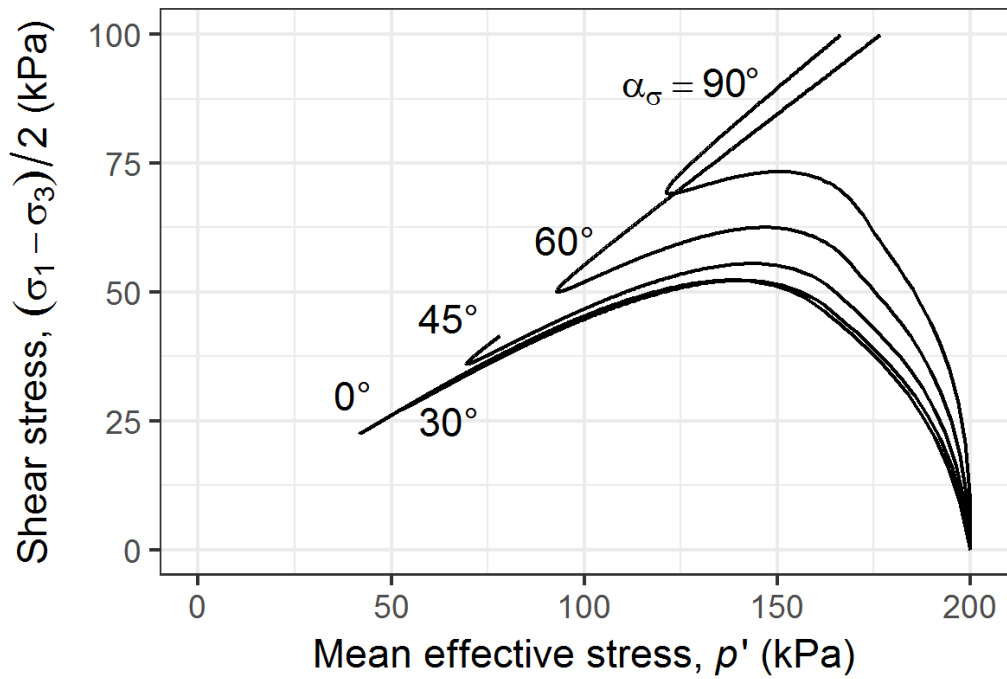
(a)



(b)



(c)



(d)

Figure 11. Experimental results and model simulations for the undrained torsional-shear tests of Fraser River sand with a relative density of 30% (data from Uthayakumar and Vaid [9]): (a) measured stress–strain relationship; (b) computed stress–strain relationship; (c) measured stress path; (d) computed stress path.

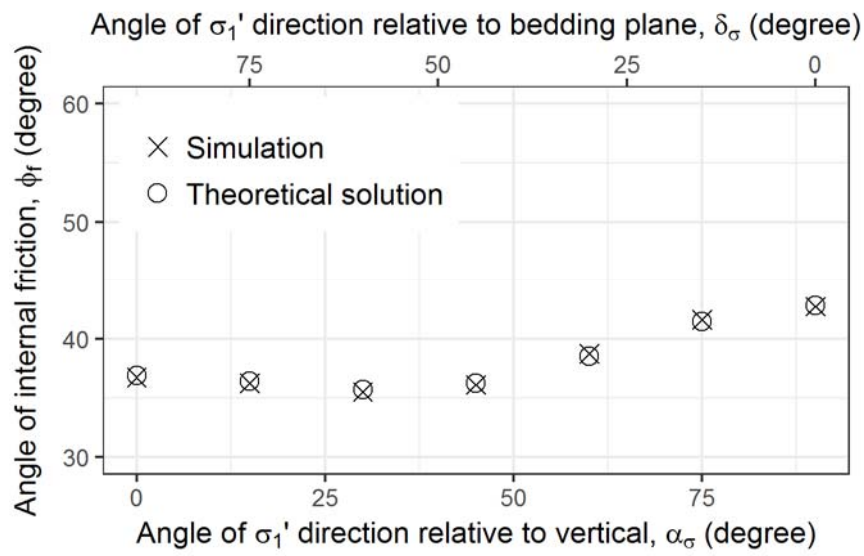
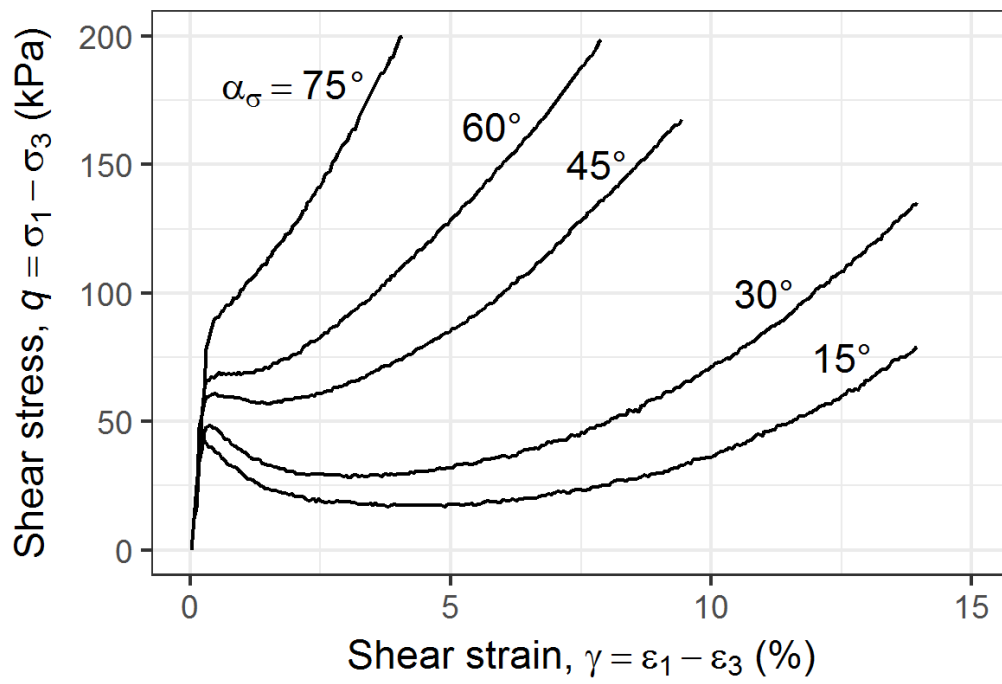
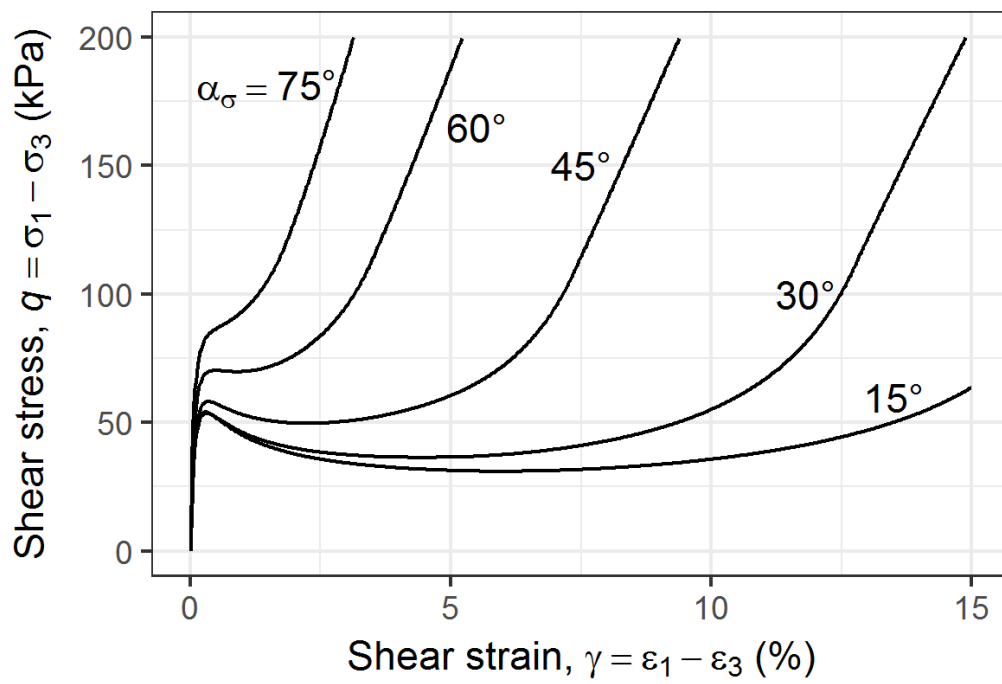


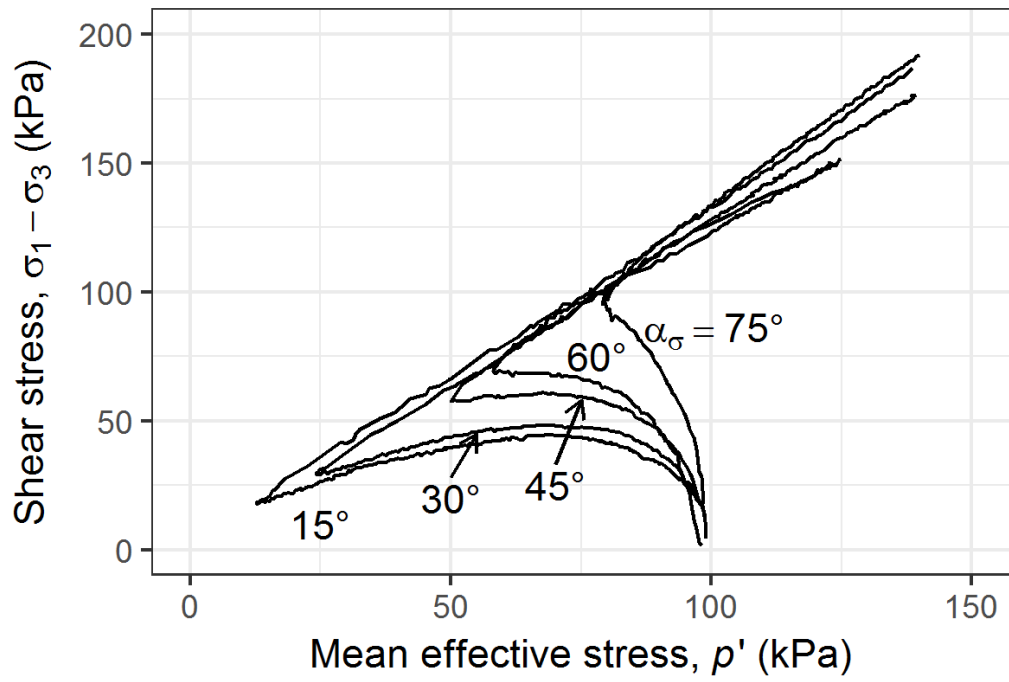
Figure 12. Comparison of simulated internal friction angles with theoretical solutions for Fraser River sand with a relative density of 30%.



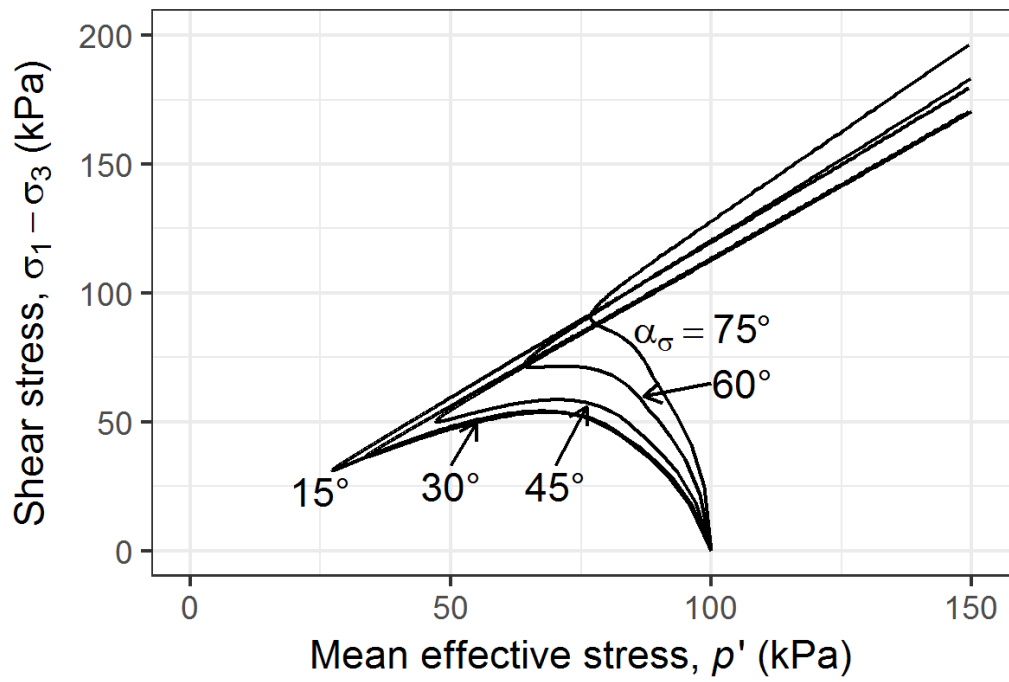
(a)



(b)



(c)



(d)

Figure 13. Experimental results and model simulations for undrained torsional-shear tests of Toyoura sand with a relative density of 40% (data from Yoshimine et al. [10]): (a) measured stress–strain relationship; (b) computed stress–strain relationship; (c) measured stress path; (d) computed stress path.

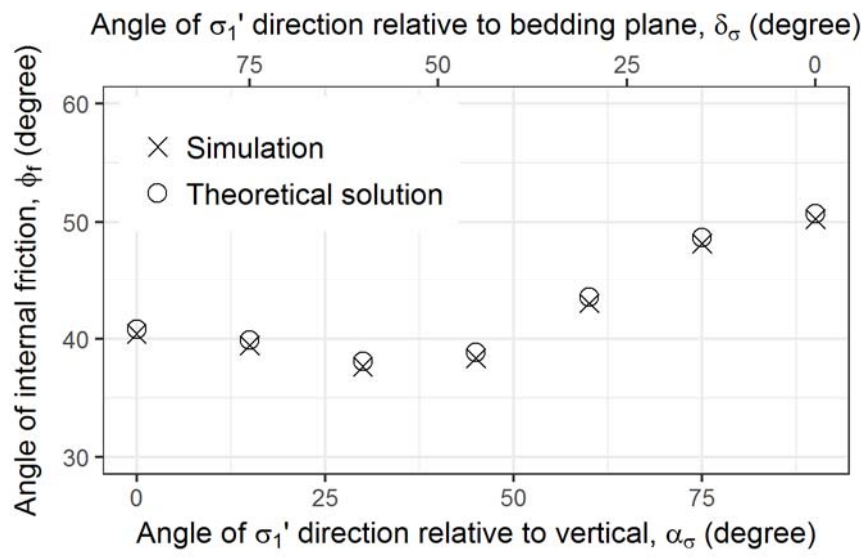
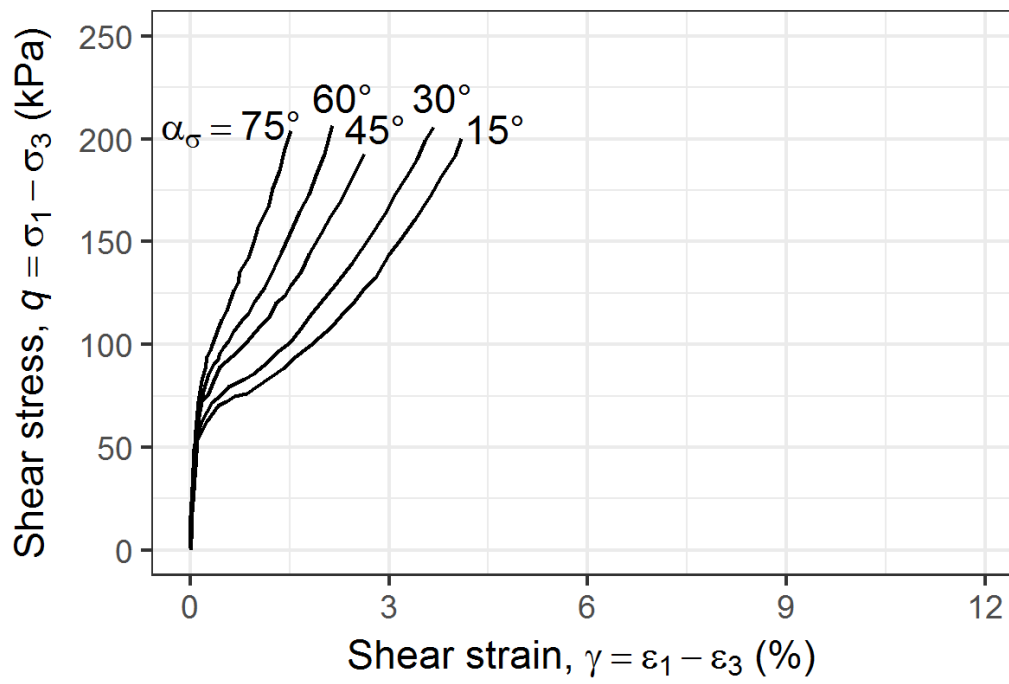
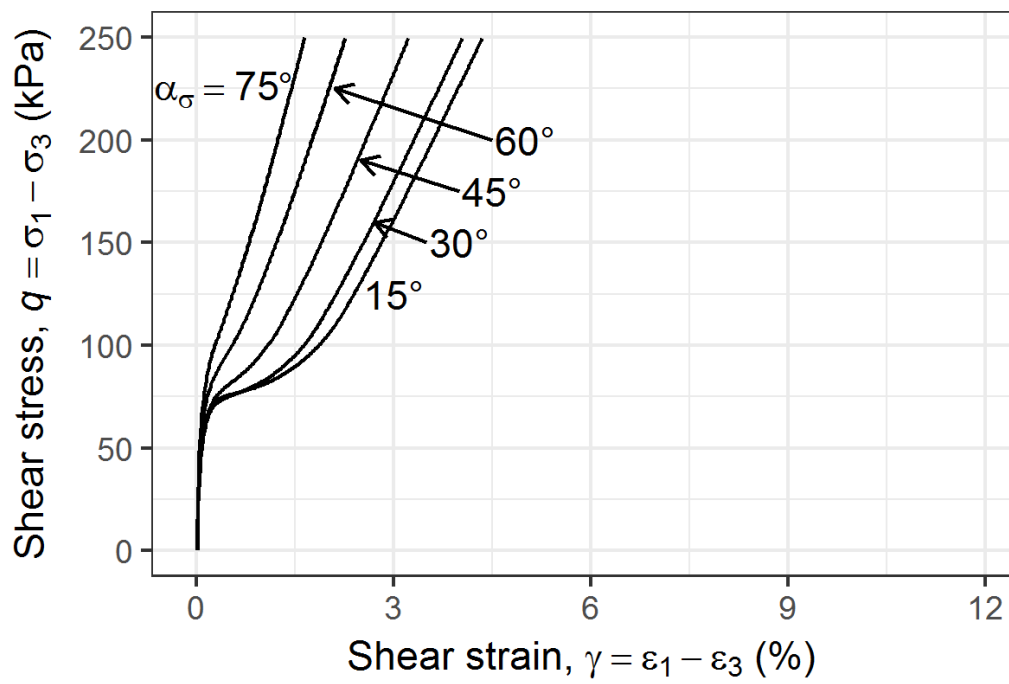


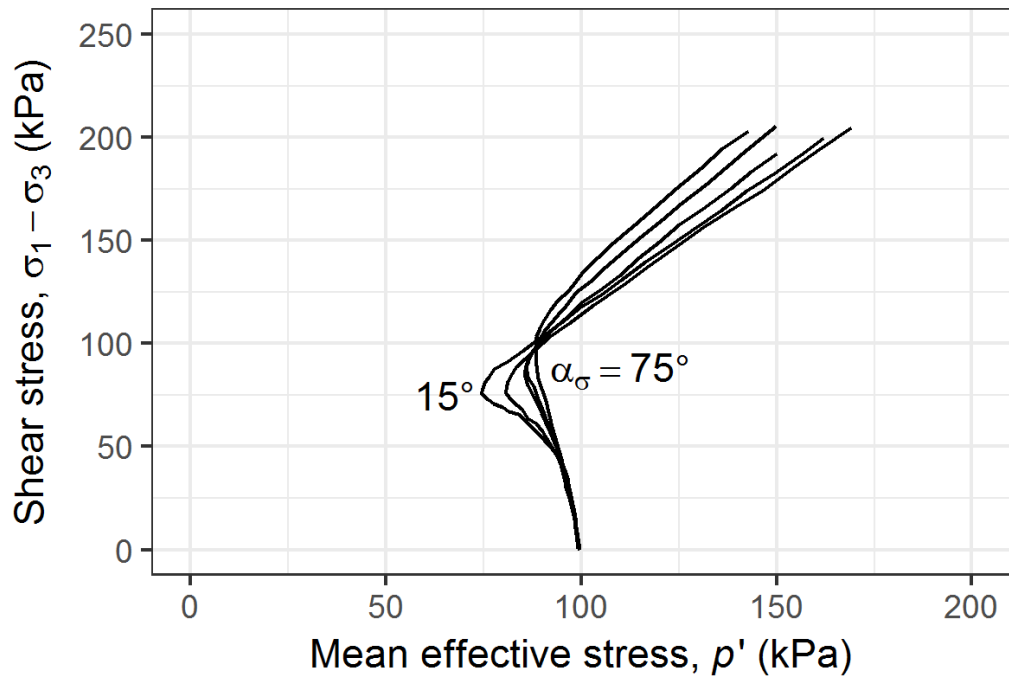
Figure 14. Comparison of simulated internal friction angles with theoretical solutions for Toyoura sand with a relative density of 40%.



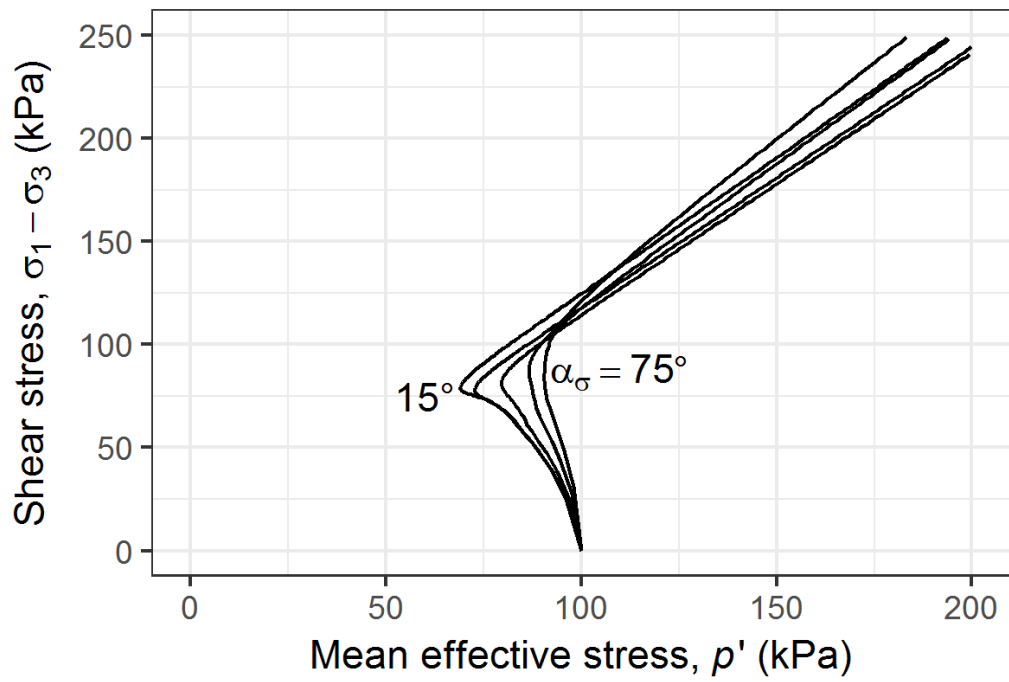
(a)



(b)

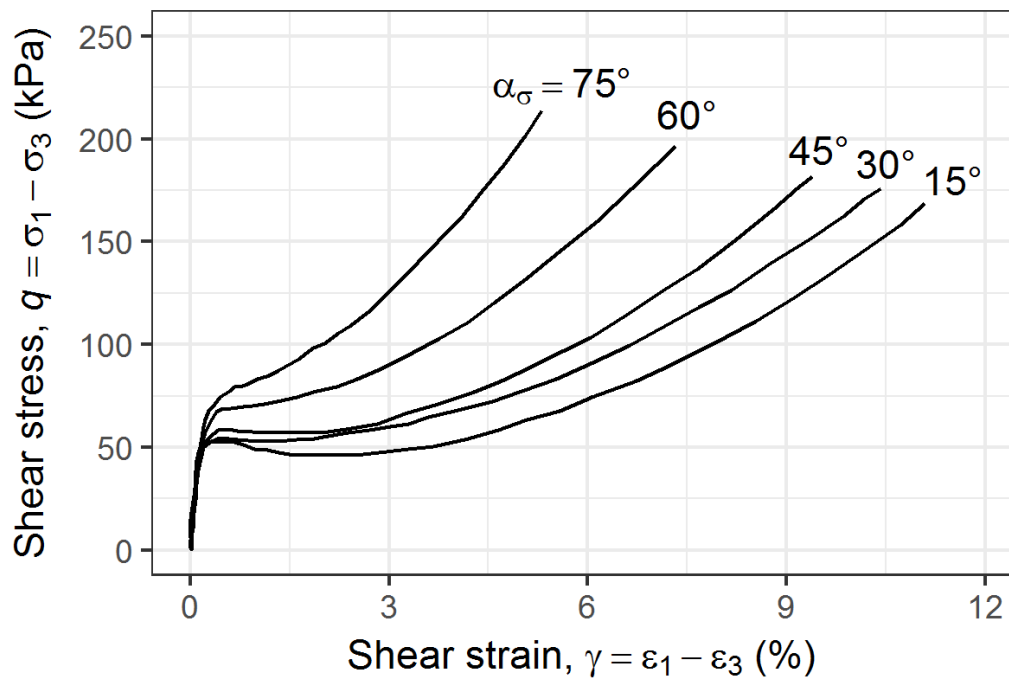


(c)

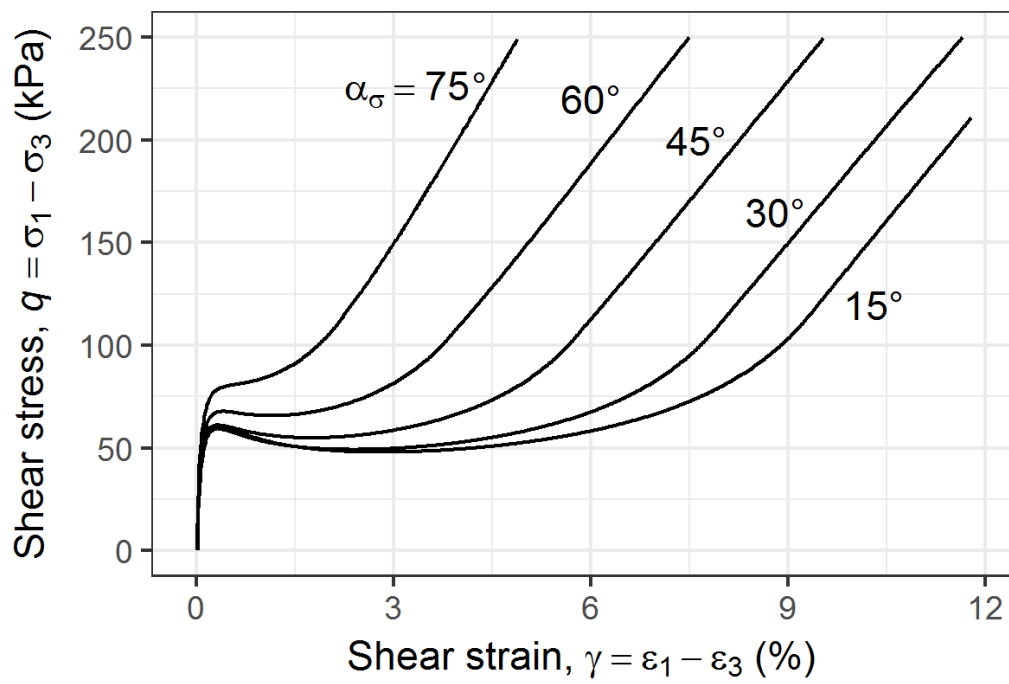


(d)

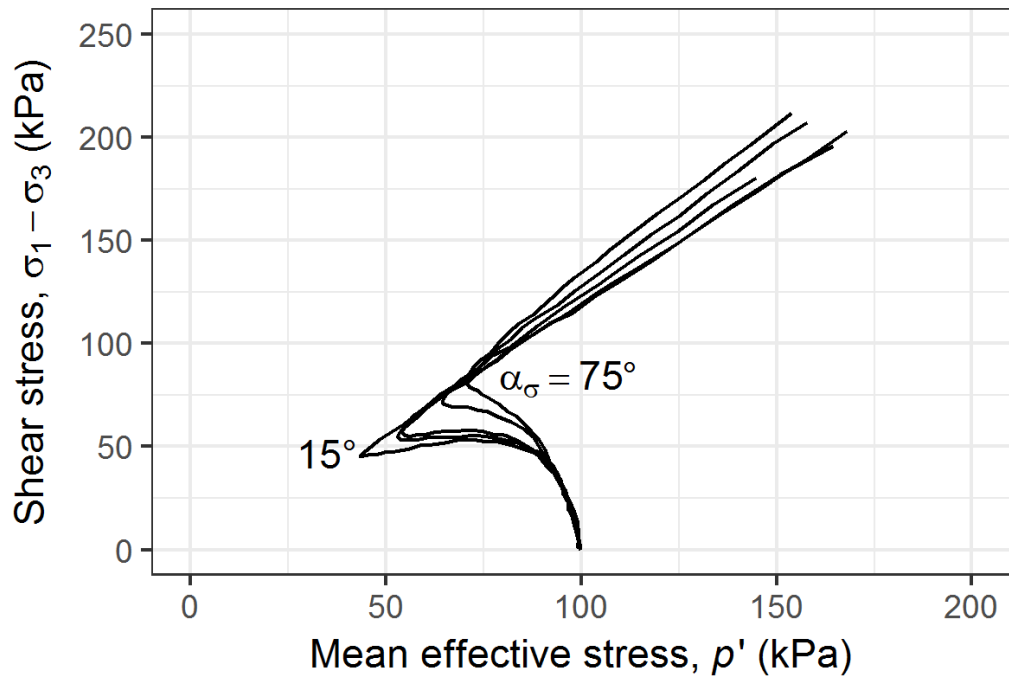
Figure 15. Experimental results and model simulations for undrained torsional-shear tests of Toyoura sand with a relative density of 90% (data from Nakata et al. [8]): (a) measured stress–strain relationship; (b) computed stress–strain relationship; (c) measured stress path; (d) computed stress path.



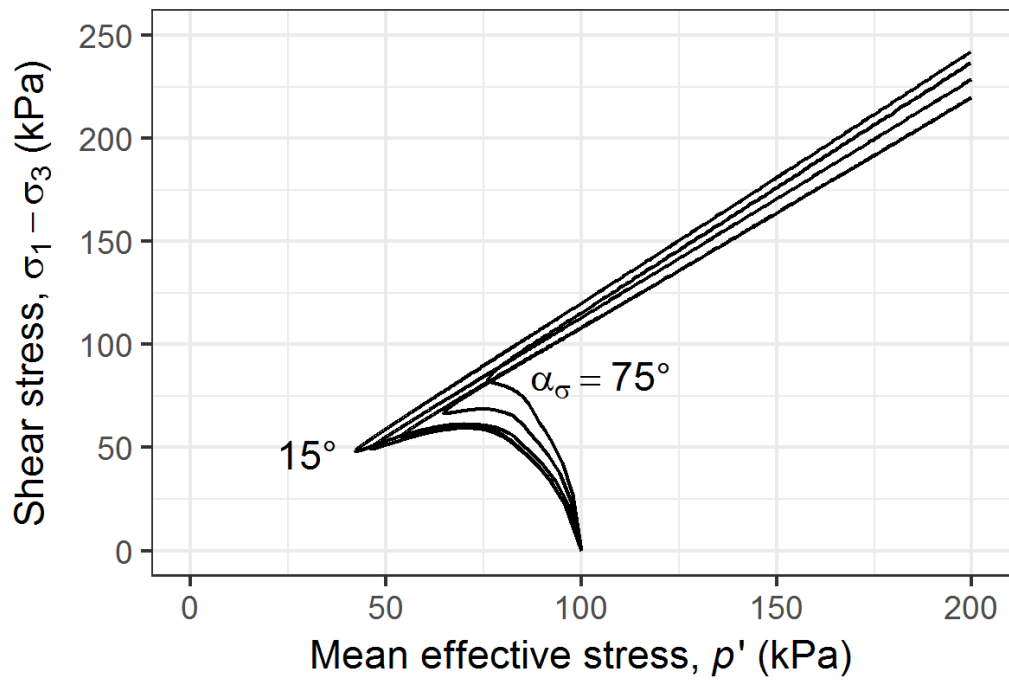
(a)



(b)

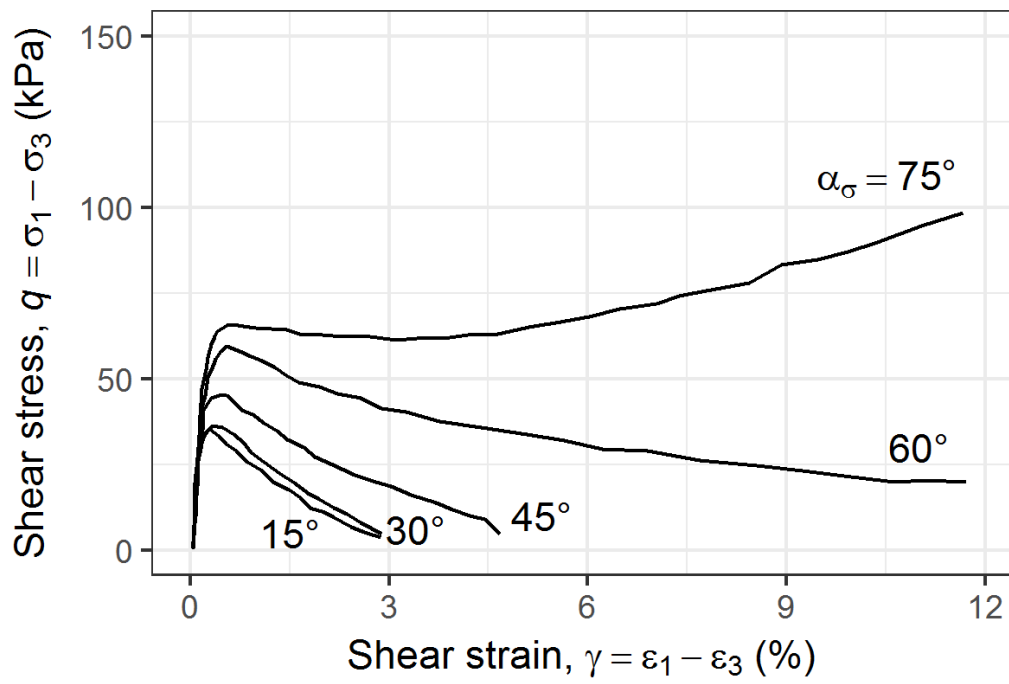


(c)

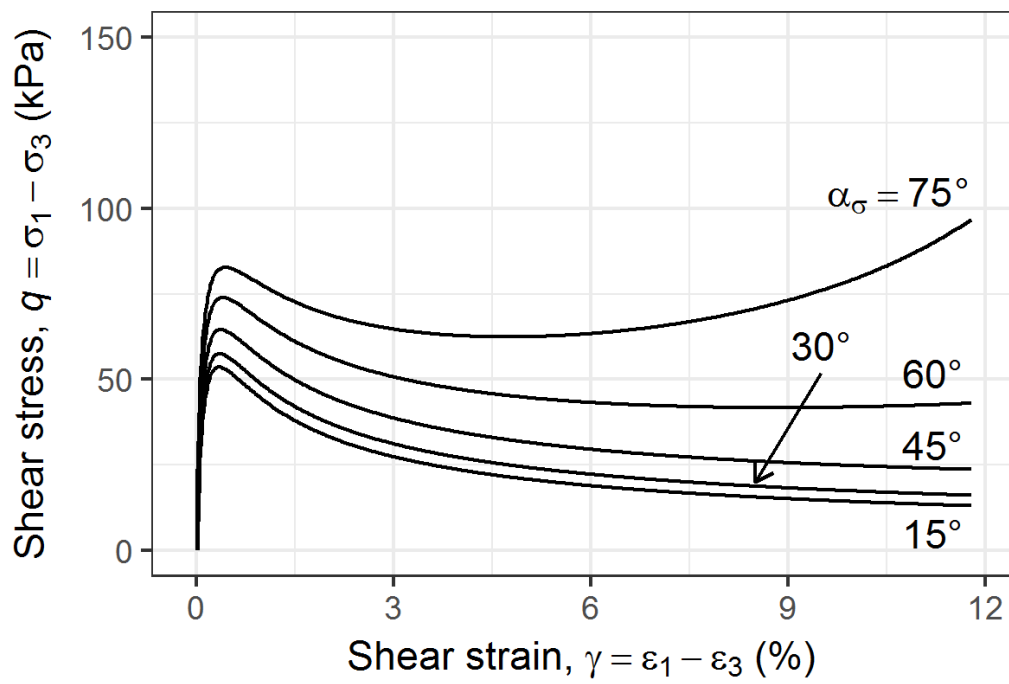


(d)

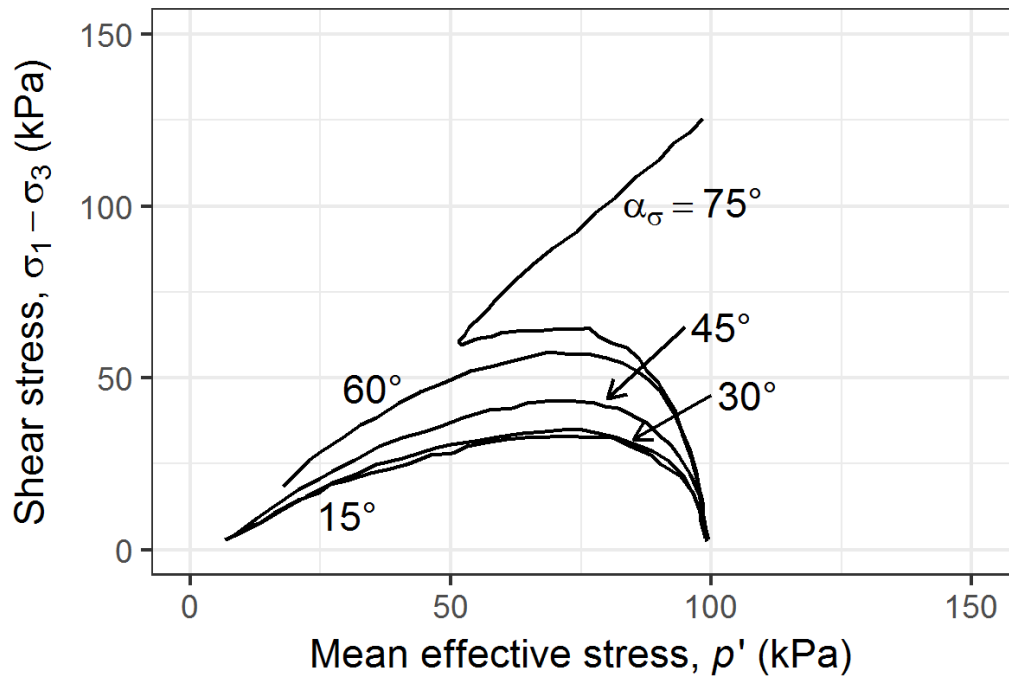
Figure 16. Experimental results and model simulations for undrained torsional-shear tests of Toyoura sand with a relative density of 60% (data from Nakata et al. [8]): (a) measured stress–strain relationship; (b) computed stress–strain relationship; (c) measured stress path; (d) computed stress path.



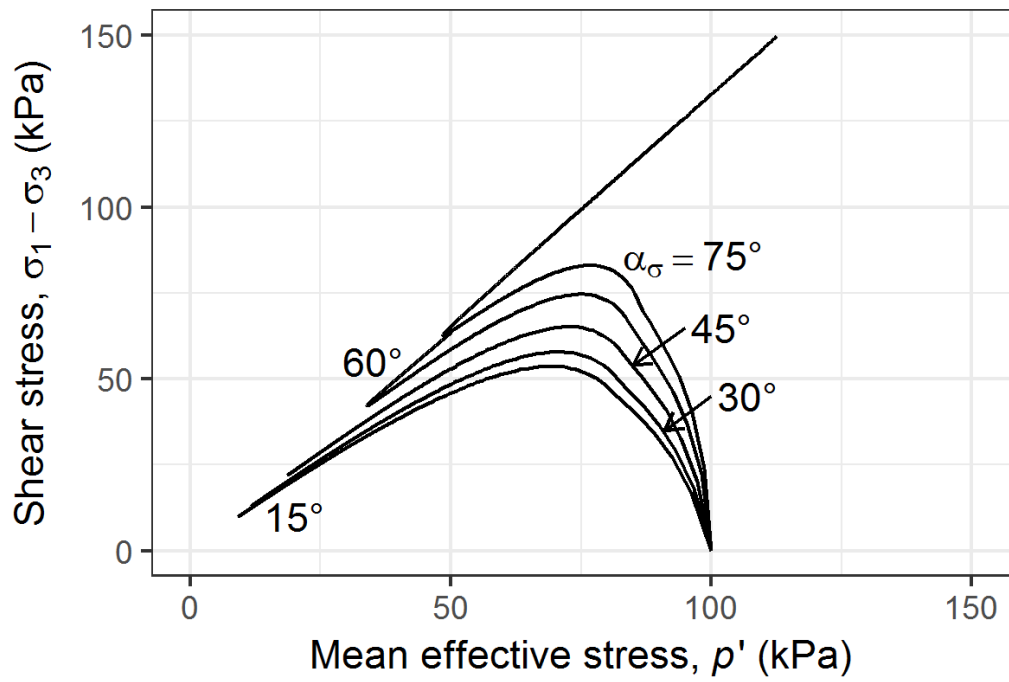
(a)



(b)

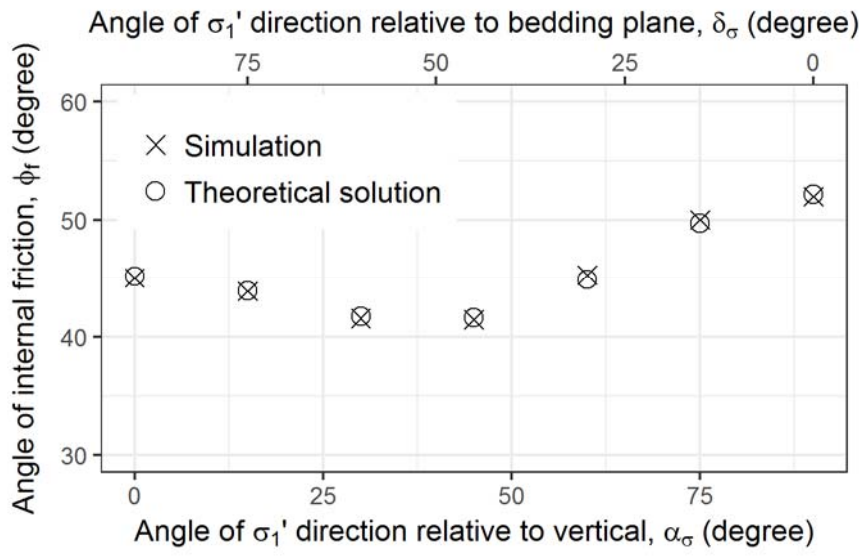


(c)

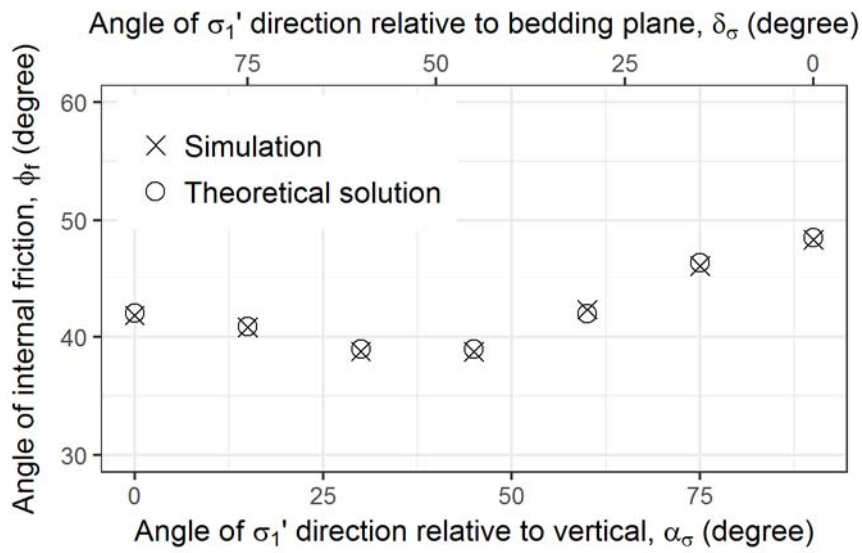


(d)

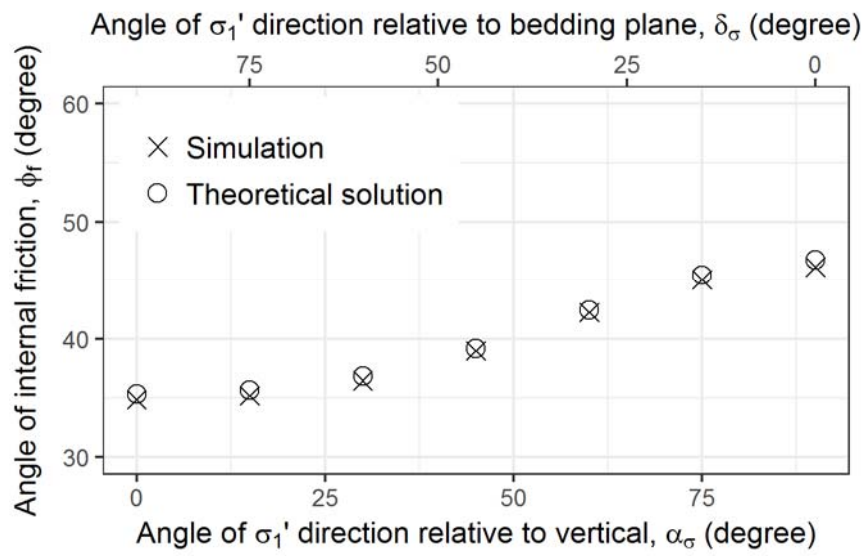
Figure 17. Experimental results and model simulations for the undrained torsional-shear tests of Toyoura sand with a relative density of 30% (data from Nakata et al. [8]): (a) measured stress–strain relationship; (b) computed stress–strain relationship; (c) measured stress path; (d) computed stress path.



(a)



(b)



(c)

Figure 18. Comparison between simulated internal friction angles and theoretical results for Toyoura sand with a relative density of: (a) 90%; (b) 60%; (c) 30%.

APPENDIX A

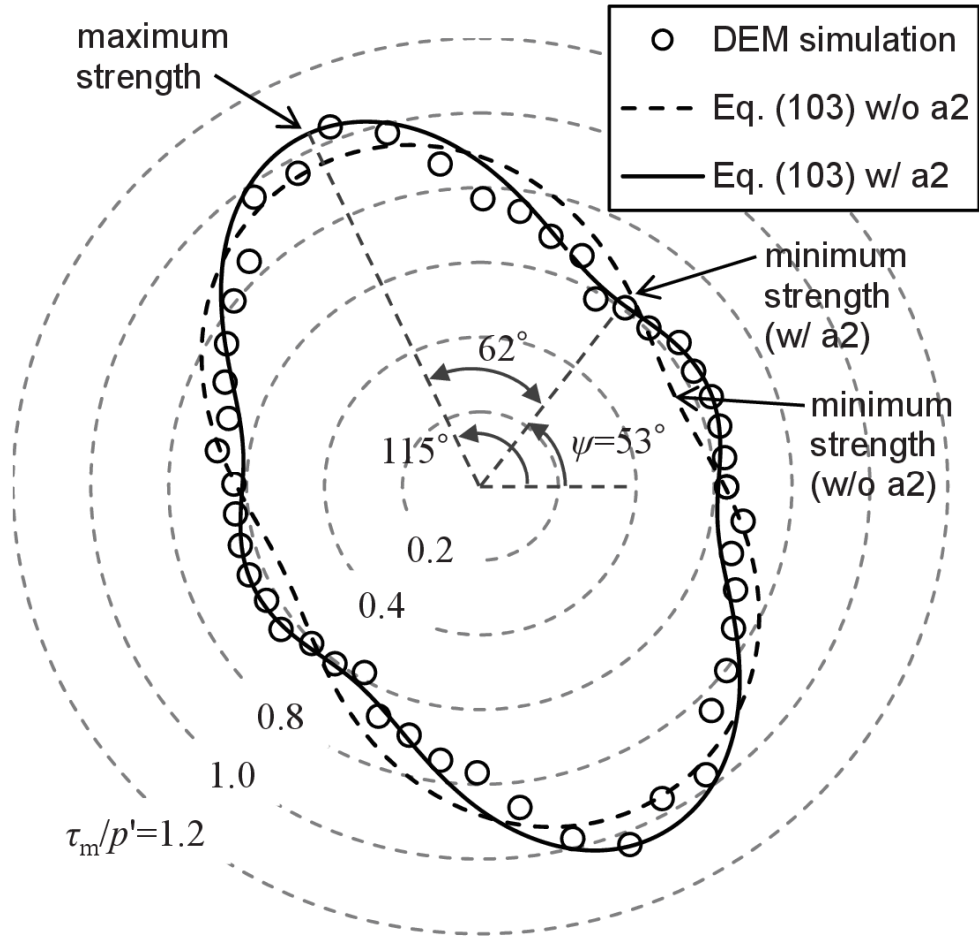


Figure A.1. Variation of peak shear strength with respect to inclination angles (DEM data from Fu and Dafalias [17]).

APPENDIX B

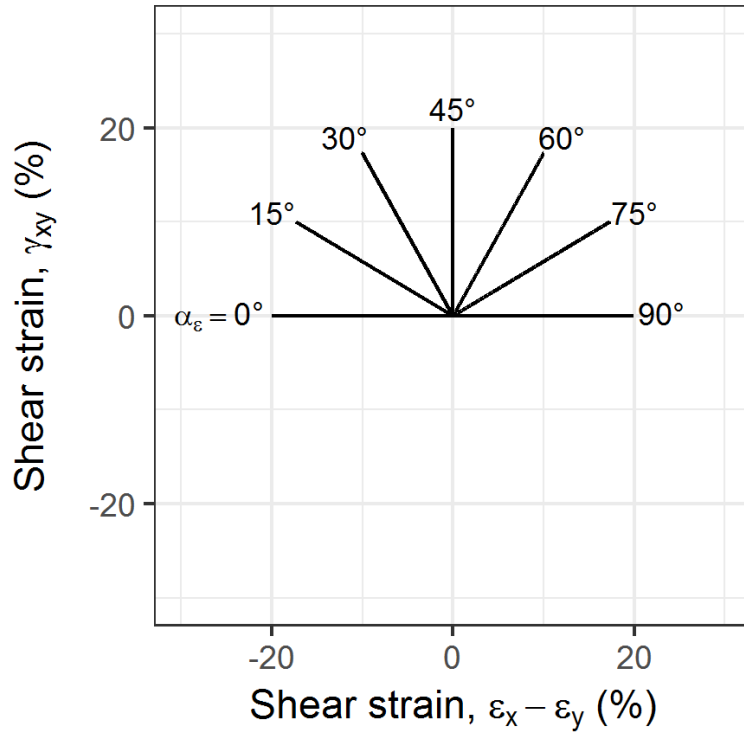


Figure B.1. Strain path applied in simulations of undrained monotonic behavior under general-directional-shear loading.

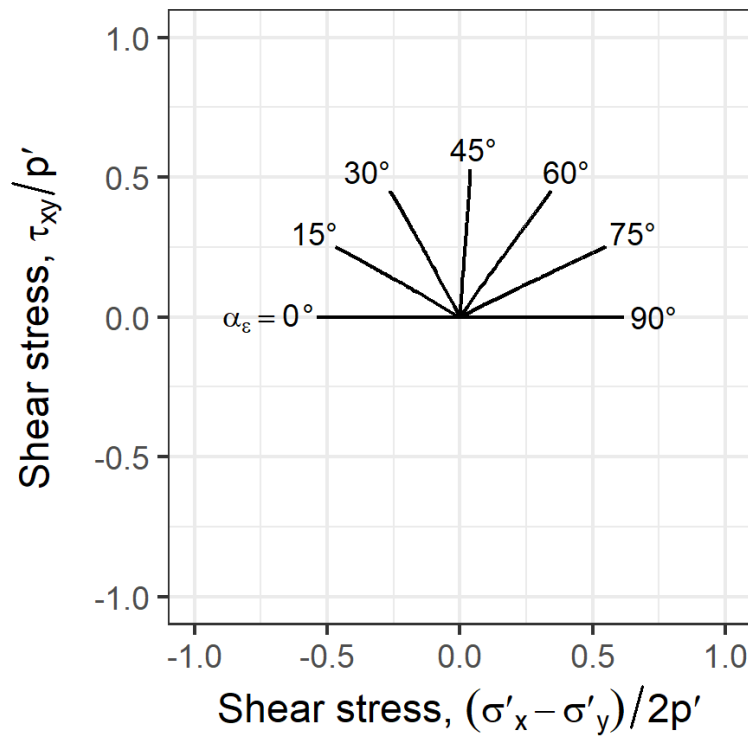
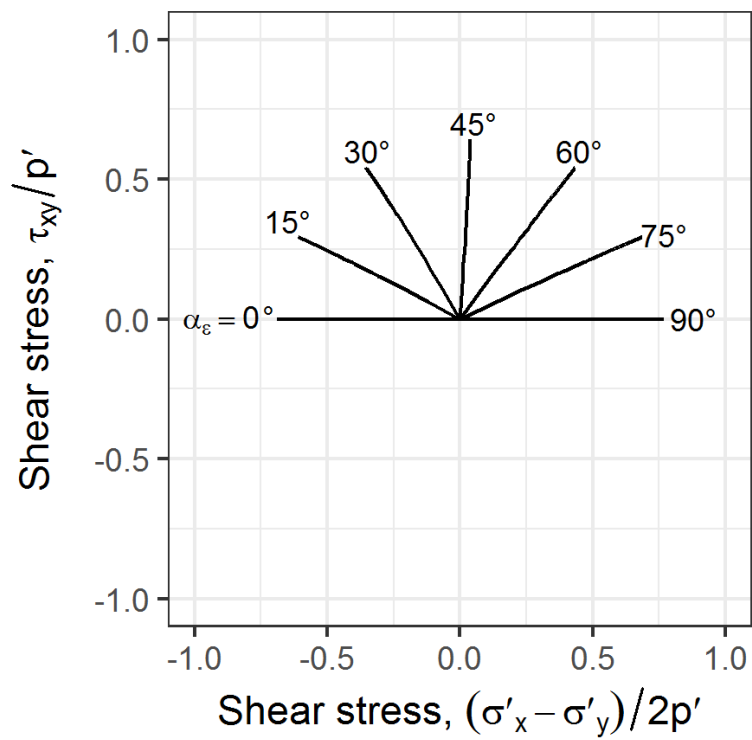
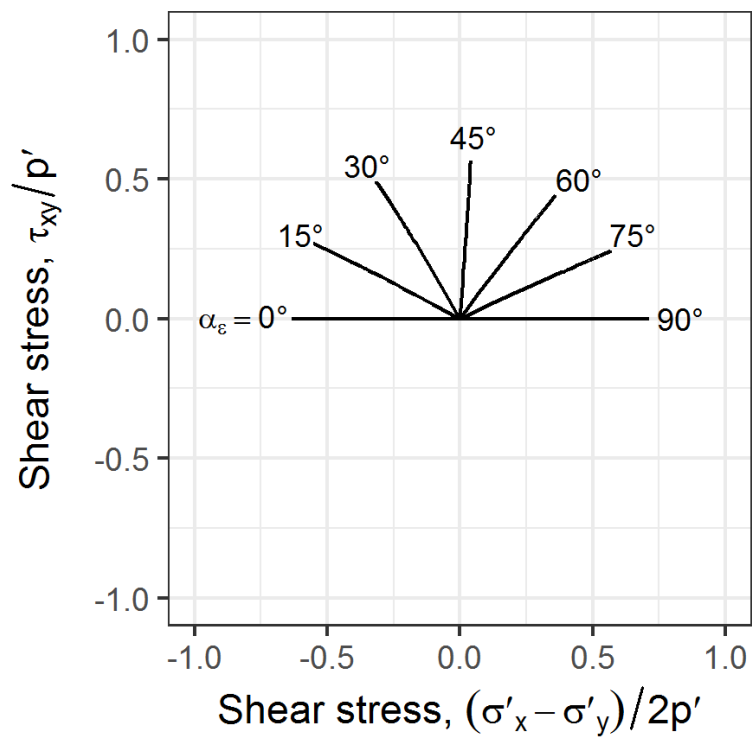


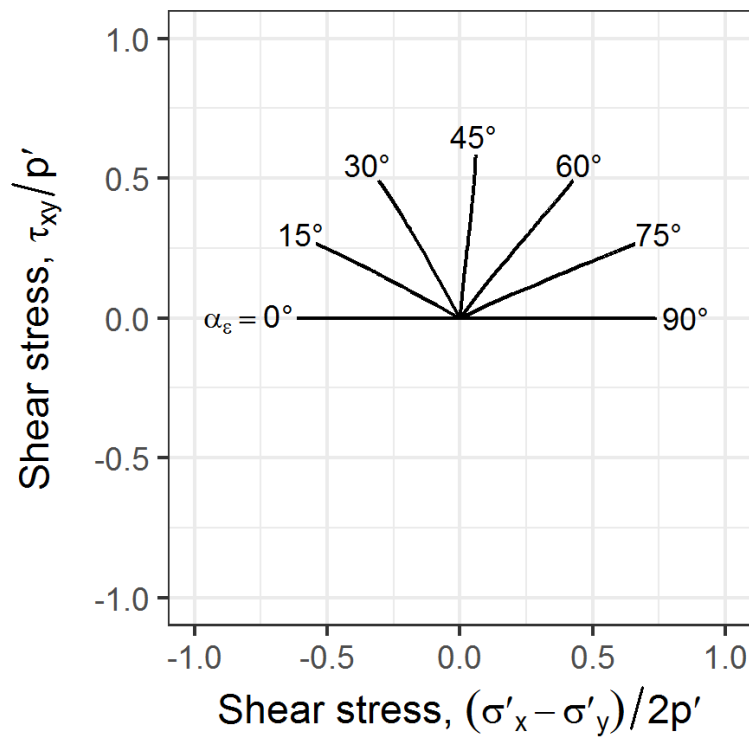
Figure B.2. Stress path obtained from the simulation of Fraser River sand with a relative density of 30%.



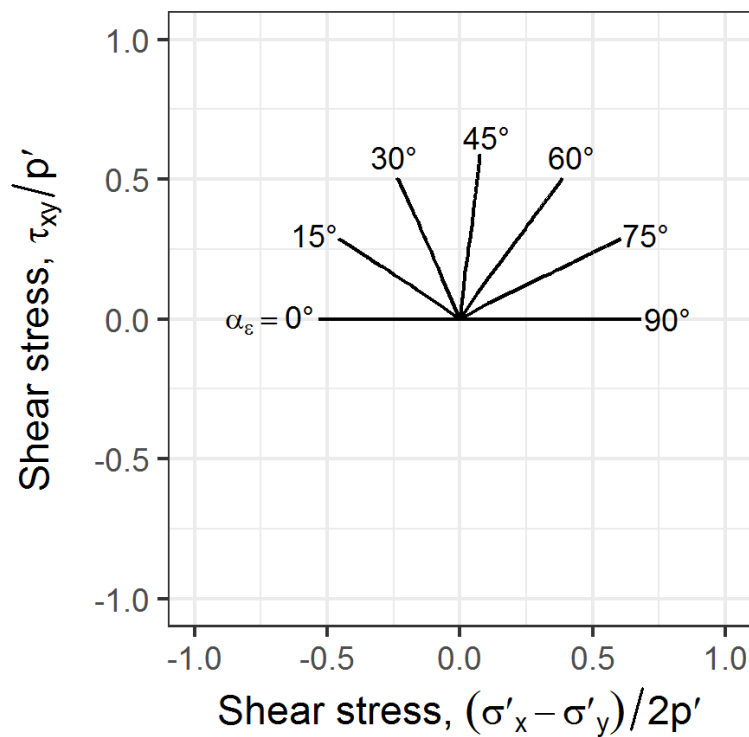
(a)



(b)



(c)



(d)

Figure B.3. Stress path obtained from the simulation of Toyoura sand with a relative density of: (a) 90%; (b) 60%; (c) 40%; (d) 30%.