

*Construction of smooth actions on spheres
for Smith equivalent representations*

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1. PROBLEMS AND RESULTS

Throughout this paper, let G be a finite group. A real G -representation of finite dimension is meant by a *real G -module*, a smooth manifold is meant by a *manifold*, and a smooth G -manifold is meant by a *G -manifold*. For a G -manifold X , let $\mathcal{TR}(X)$ denote the set of all isomorphism classes (as real G -modules) of tangential representations $T_x(X)$, where x runs over the G -fixed point set X^G . We are interested in $\mathcal{TR}(X)$ for manifolds X such that X^G consists of exactly two points. In particular, the case where X are homotopy spheres has been studied as Smith Problem.

Smith Problem. Let Σ be a homotopy sphere with G -action such that the G -fixed point set consists of exactly two points a, b . Are the tangential representations $T_a(\Sigma)$ and $T_b(\Sigma)$ isomorphic to each other (namely $|\mathcal{TR}(\Sigma)| = 1$) ?

We have affirmative answers (e.g. Atiyah-Bott, Milnor, Sanchez) as well as negative answers (e.g. Petrie, Cappell-Shaneson, Petrie-Randall, Petrie-Dovermann, Dovermann-Washington, Dovermann-Suh, Laitinen-Pawałowski, Pawałowski-Solomon), to Smith Problem under various hypotheses. There are surveys relevant to studies on Smith Problem in [24], [18] and [6].

To study the problem, we define the following relations $\sim_{\mathfrak{D}}$, $\sim_{\mathfrak{E}}$ and $\sim_{\mathfrak{DE}}$. In the definition below, V and W are real G -modules.

- (1) $V \sim_{\mathfrak{D}} W$ if there exists a disk D with G -action such that $D^G = \{a, b\}$ and $\{[V], [W]\} = \mathcal{TR}(D)$.
- (2) $V \sim_{\mathfrak{E}} W$ if there exists a homotopy sphere Σ with G -action such that $\Sigma^G = \{a, b\}$ and $\{[V], [W]\} = \mathcal{TR}(\Sigma)$.
- (3) $V \sim_{\mathfrak{DE}} W$ if $V \sim_{\mathfrak{D}} W$ and $V \sim_{\mathfrak{E}} W$.

Here $\sim_{\mathfrak{D}}$ and $\sim_{\mathfrak{DS}}$ may not be equivalence relations, although they stably yield equivalence relations. We have been interested in the relation $\sim_{\mathfrak{S}}$ (namely the Smith equivalence), but in the present paper we will mainly pay our attention to the relation $\sim_{\mathfrak{DS}}$.

Let $\text{RO}(G)$ denote the real representation ring. We define the subsets $\mathfrak{D}(G)$, $\mathfrak{S}(G)$ and $\mathfrak{DS}(G)$ of $\text{RO}(G)$ by

$$\mathfrak{D}(G) = \{V - W \in \text{RO}(G) \mid V \sim_{\mathfrak{D}} W\}$$

$$\mathfrak{S}(G) = \{V - W \in \text{RO}(G) \mid V \sim_{\mathfrak{S}} W\}$$

$$\mathfrak{DS}(G) = \mathfrak{D}(G) \cap \mathfrak{S}(G)$$

The set $\mathfrak{S}(G)$ was usually denoted by $\text{Sm}(G)$. By R. Oliver [16], there exists a disk with G -action with $|D^G| = 2$ if and only if G is an Oliver group (namely, G is not a mod $\mathcal{P}$ hyperelementary group). Thus it is worthwhile to study $\mathfrak{D}(G)$ and $\mathfrak{DS}(G)$ only for Oliver groups G .

If M is a subset of $\text{RO}(G)$ then for families $\mathcal{A}$, $\mathcal{B}$ consisting of subgroups of G we define

$$M_{\mathcal{A}} \stackrel{\text{def}}{=} \{x \in M \mid \text{res}_H^G x = 0 \ \forall H \in \mathcal{A}\}$$

$$M^{\mathcal{B}} \stackrel{\text{def}}{=} \{x = V - W \in M \mid V^K = 0 = W^K \ \forall K \in \mathcal{B}\}$$

$$M_{\mathcal{A}}^{\mathcal{B}} \stackrel{\text{def}}{=} \{x = V - W \in M_{\mathcal{A}} \mid V^K = 0 = W^K \ \forall K \in \mathcal{B}\}.$$

Using the notation with the families

$$\mathcal{P} = \mathcal{P}(G) \stackrel{\text{def}}{=} \{P \leq G \mid |P| = p^a \ (p \text{ a prime})\}$$

$$\mathcal{N}_2 = \mathcal{N}_2(G) \stackrel{\text{def}}{=} \{N \trianglelefteq G \mid |G/N| = 1, 2\}$$

$$\mathcal{N} = \mathcal{N}(G) \stackrel{\text{def}}{=} \{N \trianglelefteq G \mid |G/N| = 1 \text{ or a prime}\}$$

$$\mathcal{L} = \mathcal{L}(G) \stackrel{\text{def}}{=} \{L \leq G \mid L \supseteq G^{\{p\}} \text{ for some prime } p\},$$

we study the subsets $\mathfrak{D}(G)$, $\mathfrak{S}(G)$ and $\mathfrak{DS}(G)$ of $\text{RO}(G)$. Here the group $G^{\{p\}}$ is the smallest normal subgroup of G with prime power index, namely

$$G^{\{p\}} = \bigcap_{H \trianglelefteq G: |G/H| = p^a \text{ for some } a} H.$$

An element in $\mathcal{L}$ defined above is called a *large subgroup* of G .

Many authors (e.g. Petrie-Randall, Petrie-Dovermann, Dovermann-Washington, Dovermann-Suh, Laitinen-Pawałowski, Pawałowski-Solomon) found various pairs (V, W) of nonisomorphic $\mathfrak{DS}$ -related real G -modules V, W . But their (V, W) with $V \sim_{\mathfrak{DS}} W$ satisfy $V^N = 0 = W^N$ for all $N \triangleleft G$ with prime index. In other words, they showed

$$\mathfrak{DS}(G)^{\mathcal{N}} \neq 0$$

for various G . Now we recall the next proposition.

Proposition 1 ([12], [13]). *The implications $\mathfrak{S}(G) \subseteq \mathrm{RO}(G)_{\mathbb{Q}}^{\mathcal{N}_2}$ and $\mathfrak{DS}(G) \subseteq \mathrm{RO}(G)_{\mathbb{P}}^{\mathcal{N}_2}$ hold.*

These facts motivate us to study the following problem.

Problem A. Does there exist a finite group G satisfying $\mathfrak{DS}(G) \neq \mathfrak{DS}(G)^{\mathcal{N}}$?

The notion *gap module* is convenient to study this problem as well as Smith Problem. A real G -module V is called a *gap module* if it satisfies the following conditions.

- (1) $V^L = 0$ for all $L \in \mathcal{L}(G)$.
- (2) $\dim V^P > \dim V^H$ for all pairs (P, H) of subgroups of G such that $P \in \mathcal{P}(G)$ and $H > P$.

A finite group G is called a *gap group* if G admits a gap real G -module. Pawałowski-Solomon showed in [18] that for an arbitrary nonsolvable gap group G with $a_G \geq 2$ and $G \not\cong P\Sigma L(2, 27)$,

$$\mathfrak{DS}(G) \supseteq \mathrm{RO}(G)_{\mathbb{P}}^{\mathcal{L}} \neq 0.$$

Since the appearance of this result, the next problem has been asked.

Problem B. Are the sets $\mathfrak{S}(G)$ and $\mathfrak{DS}(G)$ nontrivial in the case $G = P\Sigma L(2, 27)$?

The purpose of the present paper is to answer to Problems A and B, and we obtained the following results.

Theorem 2. *For each odd prime p , there exist a gap Oliver group G and real G -modules V and W such that $V \sim_{\mathfrak{DS}} W$, $\dim V^N > 0$ and $\dim W^N = 0$ for some $N \triangleleft G$ with $|G/N| = p$, hence $\mathfrak{DS}(G) \neq \mathfrak{DS}(G)^{\mathcal{N}}$.*

Let $SG(m, n)$ denote the small group of order m and type n appearing in the computer software GAP [5].

Theorem 3. *Let $G = P\Sigma L(2, 27)$, $SG(864, 2666)$, or $SG(864, 4666)$. Then $\mathrm{RO}(G)_{\mathbb{P}}^{\mathcal{L}} = 0$ but*

$$\mathfrak{S}(G) = \mathfrak{D}(G) = \mathfrak{DS}(G) = \mathrm{RO}(G)_{\mathbb{P}}^{\{G\}} \cong \mathbb{Z}.$$

2. ADDITIONAL INFORMATION

For $g \in G$, let (g) denote the conjugacy class of g in G . The *real conjugacy class* $(g)^{\pm}$ of g is the union of (g) and (g^{-1}) . Let a_G denote the number of all real conjugacy classes

of elements g of G such that g does not have prime power order. By the representation theory, we have

$$a_G = \text{rank RO}(G)_{\mathcal{P}}.$$

Let δ denote the homomorphism from $\text{RO}(G)_{\mathcal{P}}$ to $\mathbb{Z}$ given by

$$\delta([V] - [W]) = \dim V^G - \dim W^G.$$

Then by definition,

$$\text{RO}(G)_{\mathcal{P}}^{\{G\}} = \text{Ker}[\delta : \text{RO}(G)_{\mathcal{P}} \rightarrow \mathbb{Z}].$$

B. Oliver [17] showed that if $a_G \geq 1$ then

$$\text{Image}[\delta : \text{RO}(G)_{\mathcal{P}} \rightarrow \mathbb{Z}] \supseteq 2\mathbb{Z}.$$

Thus the next proposition immediately follows.

Proposition (Laitinen-Pawałowski [8]). *If $a_G \geq 1$ then $\text{rank RO}(G)_{\mathcal{P}}^{\{G\}} = a_G - 1$.*

In addition, B. Oliver [17] implies the next result.

Theorem (Oliver). *If G is an Oliver group then $\mathfrak{D}(G) = \text{RO}(G)_{\mathcal{P}}^{\{G\}}$.*

Viewing these facts, E. Laitinen conjectured the next.

Laitinen's Conjecture. *If G is an Oliver group with $a_G \geq 2$ then $\mathfrak{D}\mathfrak{S}(G) \neq 0$.*

This conjecture had been positively expected until 2006. We, however, have a negative example.

Theorem 4 ([12], [13]). *Let $G = \text{Aut}(A_6)$. Then Laitinen's Conjecture fails, in fact $a_G = 2$ and $\mathfrak{S}(G) = 0 = \mathfrak{D}\mathfrak{S}(G)$.*

Most finite Oliver groups are gap groups, but neither S_5 nor $\text{Aut}(A_6)$ is a gap group, where S_5 is the symmetric group on five letters and A_6 is the alternating group on six letters. Pawałowski-Solomon [18] showed the next theorem using a deleting-inserting theorem of G -fixed point sets to disks ([10], [15, Appendix]).

Theorem (Pawałowski-Solomon [18]). *If G is a gap Oliver group then*

$$\text{RO}(G)_{\mathcal{P}}^{\mathcal{L}} \subseteq \mathfrak{D}\mathfrak{S}(G).$$

On the other hand, they also showed the next result using the finite group theory.

Theorem (Pawałowski-Solomon [18]). *Let G be a nonsolvable gap group with $a_G \geq 2$. If $G \not\cong P\Omega L(2, 27)$ then*

$$\text{RO}(G)_{\mathcal{P}}^{\mathcal{L}} \neq 0.$$

Putting these results together, we obtain a corollary.

Corollary (Pawałowski-Solomon [18]). *Let G be a nonsolvable gap group with $a_G \geq 2$. If $G \not\cong P\Sigma L(2, 27)$ then $\mathfrak{DS}(G) \neq 0$.*

Since $S_5 \times C_2$, where C_2 is the cyclic group of order 2, is not a gap group, the next result is also interesting.

Theorem (X.M. Ju [6]). *In the case $G = S_5 \times C_2$, the equalities*

$$\mathfrak{S}(G) = \mathfrak{DS}(G) = \text{RO}(G)_P^{\mathbb{Z}} \cong \mathbb{Z}$$

hold.

We obtained a deleting-inserting theorem [14] of new kind by employing an equivariant interpretation of Cappell-Shaneson's surgery obstruction theory for getting homology (possibly, not homotopy) equivalences as well as employing the induction theory of Wall's surgery obstruction groups. We state here the theorem in a simplified form.

Theorem 5. *Let G be an Oliver group and Y a disk with G -action. Suppose the following conditions are satisfied.*

- (1) $Y^G = \{y_1, \dots, y_m\}$, where $m \geq 1$.
- (2) $\partial Y^L = \emptyset$ for all $L \in \mathcal{L}(G)$.
- (3) $\dim Y^H \geq 5$ for all mod $\mathcal{P}$ cyclic subgroups H , i.e. $1 \triangleleft_{\mathcal{P}} P \triangleleft_{\text{cyclic}} H$.
- (4) $\dim Y^P > 2(\dim Y^H + 1)$ for all $P \in \mathcal{P}(G)$ and $H > P$.
- (5) $|\pi_1(Y^P)| < \infty$ and $(|\pi_1(Y^P)|, |P|) = 1$ for all $P \in \mathcal{P}(G)$.
- (6) *The inclusion induced maps $\pi_1(\partial Y^P) \rightarrow \pi_1(Y^P)$ are isomorphisms for all $P \in \mathcal{P}(G)$.*

Then there exists a disk X with G -action such that $\partial X = \partial Y$ and $X^G = \emptyset$.

Remark that the union $\Sigma = X \cup_{\partial} Y$ identified along the boundaries of X and Y in the theorem above is a homotopy sphere such that $\mathcal{TR}(\Sigma) = \mathcal{TR}(Y)$. Since various G -actions on disks Y are constructed by Oliver's theory [17], we would obtain G -actions on homotopy spheres Σ from those on disks. In fact, the next result is an outcome of Theorem 5.

Theorem 6. *Let p be an odd prime. Let G be an Oliver group such that $G = G^{\{q\}}$ for all primes $q \neq p$ and $|G/G^{\{p\}}| = p$. If G has a dihedral subquotient D_{2qr} (order $2qr$) with distinct primes q and r and further that G contains distinct real G -conjugacy classes*

$(x)^\pm$, $(y)^\pm$ of elements x, y not of prime power order, then $\mathcal{DS}(G)$ contains a direct summand of $\mathrm{RO}(G)$ of rank 1.

Theorems 2 and 3 follow from Theorem 6. In addition, we conclude the next.

Theorem 7. *Laitinen's Conjecture is affirmative for any finite nonsolvable gap group.*

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