

Witt group and nilpotent group of odd order

by

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**Abstract.** The equivariant Witt group  $W_0(D, G)$  of a finite nilpotent group  $G$  over a Dedekind domain  $D$  is studied. We introduce a Morita correspondence on the set of orthogonal representations. We determined the structure of  $W_0(D, G)$  of a finite nilpotent group  $G$  of odd order.

§0. Introduction. Let  $D$  be a Dedekind domain with the quotient field  $K$  and  $A$  be a  $D$ -order in the f.d. semisimple  $K$ -algebra  $A$ . Throughout this paper, we always assume that  $A$  has an anti-involution  $(-)$  with  $\overline{a+b}=\overline{a}+\overline{b}$ ,  $\overline{ab}=\overline{b}\overline{a}$ ,  $\overline{\alpha}=\alpha$  for all  $a, b \in A$ ,  $\alpha \in D$ . We extend this involution into  $A$ , naturally. An orthogonal representation of  $A$  is a pair  $(V, b)$  of the  $A$ -lattice  $V$  and the  $A$ -invariant,  $D$ -valued, nonsingular symmetric bilinear form  $b$  on  $V$ , where " $A$ -invariant" means  $b(\alpha v, w) = b(v, \overline{\alpha} w)$  for  $\alpha \in A$ ,  $v, w \in V$ . An orthogonal representation  $(V, b)$  is metabolic when there is a  $A$ -invariant sublattice  $N$  in  $V$  such that  $N^+ = \{v \in V : b(v, n) = 0 \text{ for all } n \in N\}$  is equal to  $N$ . The equivariant Witt group  $W_0(D, A)$  is the Grothendieck group on the isometry classes of orthogonal representations of  $A$  modulo

the subgroup generated by metabolic forms. Most interesting case is that where  $\Lambda$  is the integral group ring  $DG$  and the involution is given by the inverse of  $G$ . In this case, this group is the same as  $GW(G,D)$  in [3] and  $GW_0(D,G)$  in [2].

At first, we will investigate the structure of  $W_0(D,G)$  of a finite nilpotent group  $G$ . We consider the Lenstra's formula [5] on Witt groups as we have done on the Grothendieck group  $G_0(DG)$  in [6]. Let  $Y$  be the set of all isomorphism classes of irreducible  $K$ -characters  $\theta$  of  $G$ . For the irreducible  $KG$ -module  $V$  corresponding to  $\theta$ , let  $\bar{\theta}$  be the irreducible  $K$ -character of  $G$  corresponding to  $V^* = \text{Hom}_K(V, K)$ . Let  $e_\theta$  be the central primitive idempotent of  $KG$  (or  $KG/G_p$  if  $p = \text{ch}(K)$  is not prime to the order  $|G|$  of  $G$ , where  $G_p$  denotes a Sylow  $p$ -subgroup of  $G$ ) corresponding to  $\theta$ . Set  $D_\theta = D(\frac{1}{\deg \theta})$ . We have an analogue of Theorem 1 in [6].

**Theorem A.** Let  $G$  be a finite nilpotent group. Then we have the following isomorphism:  $W_0(D,DG) \cong \bigoplus_{\theta=\bar{\theta} \in Y} W_0(D_\theta, D_\theta G e_\theta)$ .

We will next introduce a Morita correspondence and we get the following:

**Corollary.** Let  $G$  be a finite nilpotent group of odd order and assume that  $K$  is an algebraic number field. We then have the isomorphism:  $W_0(D,DG) \cong \bigoplus_{\chi \sim_K \bar{\chi} \in T} W_0(D\langle \chi \rangle)$ , where  $T$  is the set of representatives for the  $K$ -conjugacy classes of irreducible complex-characters of  $G$  and  $D\langle \chi \rangle = D(\frac{1}{\deg \chi}, \chi(g): g \in G)$ .

§1. Notations and Lenstra's formula. We will adopt the notations from [1] and [6]. All modules in this paper are finitely generated left modules, unless otherwise specified. Let  $G$  be a finite nilpotent group and write  $G = \prod_p G_p$  as the direct product of its Sylow  $p$ -subgroups  $G_p$ . For a set  $S$  of primes, set  $G_S = \prod_{p \in S} G_p$  and  $e_S$  denotes an irreducible constituent of  $e_{|G_S}$ . Since  $e_{|G_S}$  is homogeneous,  $e_S$  is defined uniquely. Then, the canonical homomorphisms  $G \rightarrow G_S \rightarrow G$  induce, by restriction, an exact functor  $N_S$ . Namely, for an  $DG$ -module  $M$ ,  $N_S M$  is the  $D$ -module  $M$  on which  $G_p$  acts as given for  $p \in S$  and trivially for  $p \notin S$ . Let  $(M, b)$  be an orthogonal representation of  $DG$ , then  $(N_S M, b)$  is an orthogonal representation of  $DG$ . We set  $N_S(M, b) = (N_S M, b)$ . We have to note that this functor is compatible with the Witt class, that is, for sub  $DG$ -module  $N$  of  $M$ , the orthogonal complement of  $N$  in  $(M, b)$  is equal to that of  $N$  in  $N_S(M, b)$ .

At first, we construct the following diagram:

$$(1.1) \quad \begin{array}{ccccc} W_0(D, DG) & \longrightarrow & W_0(K, KG) & \xrightarrow{\theta} & W_0(K/D, DG) \\ & & \downarrow \phi_K & & \downarrow \oplus \phi_\theta \\ \oplus_{\theta \in \tilde{Y}} W_0(D_\theta, D_\theta Ge_\theta) & \rightarrow & \oplus_{\theta \in \tilde{Y}} W_0(K, KGe_\theta) & \xrightarrow{\oplus \theta} & \oplus_{\theta \in \tilde{Y}} W_0(K/D, D_\theta Ge_\theta) \end{array}$$

where  $\tilde{Y} = \{\theta \in Y : \bar{\theta} = \theta\}$ . Then we will show that this diagram is commutative and that  $\phi_K$  and  $\phi_\theta$  are all isomorphisms. If so, we have the desired isomorphism by the well known result [1],

$$W_0(D, DG) \cong \text{Ker } \theta \cong \text{Ker } \oplus \theta \cong \oplus_{\theta \in \tilde{Y}} W_0(D_\theta, D_\theta Ge_\theta).$$

(1.2). Definition of  $\phi_*$ . Let  $(M, b)$  be an orthogonal representation of  $KG$ . Since every orthogonal representation of

KG decomposes into the orthogonal sum of homogeneous components and metabolic forms, we assume that  $M$  is a  $KG(e_\theta + e_{\bar{\theta}})$ -module and we put  $\phi_K(M, b) = \sum_{S \subset \pi(\theta)} (N_S(M, b)) \in \oplus_{\chi \in Y} W_0(K, KG(e_\chi + e_{\bar{\chi}}))$ . If  $\bar{\theta} \neq \theta$ , then  $M$  and  $\phi_K(M, b)$  are metabolic. Therefore, extending it linearly, we have the homomorphism:

$\phi_K: W_0(K, KG) \rightarrow \oplus_{\theta \in \tilde{Y}} W_0(K, KGe_\theta)$ . On the other hand, let  $\rho$  be a prime ideal of  $D$  and set  $F = D/\rho$  and  $p = \text{ch}(F)$ . Then since every orthogonal representation of  $FG$  is equivalent to some orthogonal representation of  $FG$  on which  $G_p$  acts trivially. We know that this isomorphism:  $W_0(F, FG) \cong W_0(F, FG/G_p)$  is given by  $N_{\pi(G) - \{p\}}$ . Then every orthogonal representation of  $FG/G_p$  decomposes into the orthogonal sum of  $\rho$ -torsion orthogonal representations of  $DGe_\theta$  with  $\theta \in \tilde{Y}$ ,  $p \notin \pi(\theta)$  and hyperbolic forms of  $DG(e_\theta \oplus e_{\bar{\theta}})$  with  $\theta \in \tilde{Y}$ . Let  $(M, b)$  be an orthogonal representation of  $FG$  and assume that  $M$  is a  $DGe_\theta$ -module with  $p \notin \pi(\theta)$ . Now we put  $\phi_\rho((M, b)) = \sum_{S \subset \pi(\theta)} N_S(M, b) \in \oplus_{\chi \in \tilde{Y}} W_0(F, FGe_\chi)$  and extending linealy it, we have the homomorphism:

$$\phi_\rho: W_0(F, FG) \xrightarrow{N_{\pi(G) - \{p\}}} W_0(F, FG/G_p) \longrightarrow \oplus_{\chi \in Y, p \notin \pi(\chi)} W_0(F, FGe_\chi).$$

Lemma 1.3.  $\phi_*$  are the isomorphisms.

Proof. We will give the inverse maps. Let  $(H, h)$  be an orthogonal representation of  $KGe_\theta$  (or  $FGe_\theta$  with  $p \notin \pi(\theta)$ ) and we put  $\phi_K(H, h)_\theta$  (or  $\phi_\rho(H, h)_\theta$ ) =  $\sum_{S \subset \pi(\theta)} (-1)^{|\pi(\theta) - S|} N_S(H, h)$ , where  $|\pi(\theta) - S|$  denotes the number of elements in  $\pi(\theta) - S$ . We can easily check  $\phi\phi = 1$  and  $\phi\phi = 1$ .

We next show that the diagram (1.1) is commutative. Let

$(V, b)$  be an orthogonal representation of  $KG$  and assume that  $V$  is a  $KG e_\theta$ -module. Let  $I$  be a full DG-lattice in  $V$  with  $b(I, I) \subset D$  and set  $J = \{v \in V : b(v, I) \subset D\}$ . Then  $\Theta(V, b)$  is given by  $(J/I, \tilde{b})$ , where  $\tilde{b}$  is  $b + D/D$ . Decomposing into the  $\beta$ -torsion parts, we have  $\Theta(V, b) = \oplus_\beta (J_\beta / I_\beta, \tilde{b}_\beta)$ , where  $( )_\beta$  denotes the  $\beta$ -localization, that is,  $J_\beta = D_\beta \otimes J$ . Therefore, let  $p_\beta = \text{ch}(D/\beta)$  and we have  $(\oplus_\gamma) \Theta(V, b) = \sum_\beta \phi_\beta (J_\beta / I_\beta, \tilde{b}_\beta) = \sum_\beta \sum_{S \subset \pi(\theta) - \{p_\beta\}} N_S(J_\beta / I_\beta, \tilde{b}_\beta)$ . On the other hand,  $\phi_K(V, b) = \sum_{S \subset \pi(\theta)} N_S(V, b)$  and  $(\oplus_\chi) \phi_K(V, b) = \sum_{S \subset \pi(\theta)} \Theta_{\theta_S}(N_S(V, b)) = \sum_{S \subset \pi(\theta)} ((D_{\theta_S} \otimes J) / (D_{\theta_S} \otimes I), \tilde{b}_S) = \sum_{S \subset \pi(\theta)} \sum_{p_\beta \notin \pi(\theta_S)} (J_\beta / I_\beta, \tilde{b}_\beta) = (\sum_\beta \phi_\beta) \Theta(V, b)$ . This completes the proof of Theorem A.

§2. Morita correspondence. Let  $\Delta$  be a maximal D-order in the semisimple  $K$ -algebra  $A$ . Let  $M_\Delta$  be a progenerator of  $\Delta$  and set  $\Lambda = \text{End}_\Delta(M)$ . Then  $M$  can be considered as a  $(\Lambda, \Delta)$ -bimodule and  $M^* = \text{Hom}(M, \Delta)$  is a  $(\Delta, \Lambda)$ -bimodule. As well known, the Morita correspondence states that the tensoring  $M \otimes_\Delta$  gives an equivalence functor of the category of  $\Delta$ -left modules to the category of  $\Lambda$ -left modules and the tensoring  $M^* \otimes_\Lambda$  gives the inverse functor. The purpose in this section is to construct a Morita correspondence from the set of orthogonal representations of  $\Delta$  onto the set of orthogonal representations of  $\Lambda$ . Since  $M_\Delta$  is a progenerator, there are isomorphisms:

$$\mu: M \otimes_\Delta M^* \rightarrow \Delta \quad \text{given by} \quad \mu(m \otimes f)m_1 = m(fm_1), \quad \text{and}$$

$$\tau: M^* \otimes_\Lambda M \rightarrow \Delta \quad \text{given by} \quad \tau(f \otimes m) = fm, \quad \text{for } m, m_1 \in M, f \in M^*.$$

We assume that  $\Delta$  and  $\Lambda$  have anti-involutions, respectively.

We will use the same symbol  $(-)$  to denote them.

Now we assume that there is an isomorphism  $b: M \rightarrow M^*$  satisfying the following four conditions.

- 1)  $b(rm) = b(m)\bar{r}$  for  $r \in A, m \in M$ ,
- 2)  $b(rs) = \bar{s}b(m)$  for  $s \in A$ ,
- 3)  $\tau(b(n) \otimes_A m) = \overline{\tau(b(m) \otimes_A n)}$  for  $n, m \in M$ , and
- 4)  $\mu(m \otimes_A b(n)) = \overline{\mu(n \otimes_A b(m))}$ .

We will show that under the conditions (C), the tensoring  $M \otimes_A$  gives an equivalence functor of category of orthogonal representations of  $A$  to that of  $A$ . It will be easily proved that this functor sends the set of metabolic forms of  $A$  onto the set of metabolic forms of  $A$ . Therefore, we have the following:

**Theorem B.** Under the above conditions, we have;

$W_0(D, A) \cong W_0(D, A)$  and  $WH_0(D, A) \cong WH_0(D, A)$ , where  $WH_0(D, A)$  is the Grothendieck group on the isometry classes of orthogonal representations of  $A$  modulo the subgroup generated by hyperbolic forms.

We start the proof of Theorem B. Let  $(H, h)$  be an orthogonal representation of  $A$ . Let  $\phi(h)$  be the bilinear form on  $M \otimes_A N$  defined by  $\phi(h)(m_1 \otimes_A n_1, m_2 \otimes_A n_2) = h(n_1, \tau(b(m_1) \otimes_A m_2) n_2)$ .

**Lemma 2.1.** The above definition is well defined and  $\phi(h)$  is a  $A$ -invariant, symmetric bilinear form.

**Proof.** For  $\alpha, \beta \in A$  and  $m, u \in M, n, v \in H$ , we have

$$\phi(h)(m\alpha \otimes n, u\beta \otimes v) = h(n, \tau(b(m\alpha) \otimes u\beta)v) = h(n, \alpha \tau(b(m) \otimes u)\beta v)$$

$$\begin{aligned}
&=h(\alpha n, \tau(b(m) \otimes u) \beta v) = \phi(h)(m \otimes \alpha n, u \otimes \beta v). \text{ And for } r \in A, \text{ we get} \\
&\phi(h)(r(m \otimes n), u \otimes v) = h(n, \tau(b(rm) \otimes u) v) = h(n, \tau(b(m) \otimes \bar{r}u) v) \\
&= \phi(h)(m \otimes n, \bar{r}(u \otimes v)). \text{ Moreover, we obtain} \\
&\phi(h)(m \otimes n, u \otimes v) = h(n, \tau(b(m) \otimes u) v) = h(\tau(b(m) \otimes u) v, n) = h(v, \overline{\tau(b(m) \otimes u) n}) \\
&= h(v, \tau(b(u) \otimes m) n) = \phi(h)(u \otimes v, m \otimes n).
\end{aligned}$$

Therefore, the mapping  $\Phi[(H, h)] = (M \otimes_A N, \phi(h))$  sends the set of isometry classes of orthogonal representations of  $A$  (containing singular forms) into the set of isometry classes of orthogonal representations of  $A$  (containing singular forms). Similarly, we can define the mapping  $\Psi[(S, s)] = (M^* \otimes_A S, \phi(s))$ , where  $\phi(s)(f_1 \otimes t_1, f_2 \otimes t_2) = s(t_1, \mu(b^{-1}(f_1) \otimes f_2) t_2)$  for  $f_1, f_2 \in M^*$ ,  $t_1, t_2 \in S$  and  $(S, s)$  is an orthogonal representation of  $A$ .

Lemma 2.2.  $\Phi\Psi = 1$  and  $\Psi\Phi = 1$ .

Proof. Let  $(H, h)$  be an orthogonal representation of  $A$ . Then we have  $(\phi\phi)(h)(f \otimes m \otimes n, g \otimes u \otimes v) = \phi(h)(m \otimes n, \mu(b^{-1}(f) \otimes g)(u \otimes v))$   
 $= h(n, \tau(b(m) \otimes_A (\mu(b^{-1}(f) \otimes_A g) u)) v) = h(n, \tau(b(m) \otimes_A b^{-1}(f)(gu)) v)$   
 $= h(n, \tau(b(m) \otimes b^{-1}(f))(gu) v) = \overline{h(\tau(b(m) \otimes b^{-1}(f)) n, (gu) v)}$   
 $= h(\tau(b(b^{-1}(f)) \otimes m) n, guv) = h(fmn, guv), \text{ for } f, g \in M^*, m, u \in M, \text{ and } n, v \in H.$   
Identifying  $H$  and  $M^* \otimes_A M \otimes_A H$ , we have  $(\phi\phi)(h) = h$  and so  $(\Phi\Psi) = 1$ . Similarly, we get  $(\Psi\Phi) = 1$ .

By Lemma 2.2, we see that if  $(H, h)$  is nonsingular, then  $\Phi(H, h)$  is also nonsingular, which proves Theorem B. This completes the proof of Theorem B.

We now start the proof of Corollary. Let  $\theta$  be a faithful irreducible KG-character with  $\bar{\theta} = \theta$  and  $T$  be the simple component

of  $KG$  corresponding to  $\theta$ . Then it follows from Feit [Theorem 14.4 and 14.5] that  $T$  is the full matrix algebra  $M_n(K(\chi))$  over the field  $k(\chi) = K(\chi(g) : g \in G)$  and there is a representation  $\zeta : G \rightarrow T$  satisfying  $\zeta(g^{-1}) = {}^t \overline{\zeta(g)}$  and  $\zeta(D_\theta G) = M_n(D\langle \chi \rangle)$ , where  $\chi$  is an irreducible complex character of  $G$  whose  $K$ -conjugacy class is  $\theta$  and  $t$  denotes the transpose and  $(-)$  denotes the complex conjugate. Taking  $M$  as a row vector  $nD\langle \chi \rangle = D\langle \chi \rangle \oplus \dots \oplus D\langle \chi \rangle$  and  $M^*$  as a column vector, we can apply the Morita correspondence on  $M_n(D\langle \chi \rangle)$  and  $D\langle \chi \rangle$  and we get the isomorphism:

$W_0(D_\theta, Ge_\theta) \cong W_0(D_\chi, D\langle \chi \rangle)$ . Since the relative different of  $D\langle \chi \rangle / D_\theta$  is a unit, we have that the trace  $\text{tr}_{D\langle \chi \rangle / D_\theta}$  gives the isomorphism:  $\kappa_0(D\langle \chi \rangle) \cong W_0(D_\chi, D\langle \chi \rangle)$ .

This completes the proof of Corollary.

## Reference.

- [1]. J.P.Alexander, P.E.Conner, G.C.Hamrick, "Odd order group actions and Witt classification of innerproducts", Lecture Notes in Mathematics 625, Springer-Verlag, Heidelberg, (1977).
- [2]. A.Bak, "K-theory of forms", Annals of Math. Studies, No. 98, Princeton (1981).
- [3]. A.Dress, Induction and structure theorems for orthogonal representation of finite groups, Annals of Math.,(2),102 (1975), 291-325.
- [4]. W.Feit, "Characters of finite groups", Math. Lecture Notes Series, Benjamin, New York, 1967.
- [5]. H.Lenstra, Grothendieck groups of Abelian group rings, J.Pure Appl. Algebra 20 (1981), 173-193.
- [6]. M.Miyamoto, Grothendieck groups of integral nilpotent group rings, J.Algebra 91 (1984), 32-35.