

# A COMPLETE FILTER ASSOCIATED WITH A THIN LIST

KENTA TSUKUURA

ABSTRACT. In this paper, we construct a  $\kappa$ -complete filter  $F_{\vec{d}}$  over  $\lambda$  using a thin  $\mathcal{P}_\kappa(2^\lambda)$ -list  $\vec{d}$ . We show that if  $\vec{d}$  has no branch, then  $F_{\vec{d}}$  cannot be extended to a  $\kappa$ -complete ultrafilter. This result allows us to demonstrate that if every  $\kappa$ -complete filter over  $\lambda$  can be extended to a  $\kappa$ -complete ultrafilter, then  $\square(2^\lambda)$  fails, without relying on the compactness theorem of  $\mathcal{L}_{\kappa\kappa}$ .

## 1. INTRODUCTION

The notion of strong compactness has been widely studied in the contexts of infinitary logic, large cardinals, and infinitary combinatorics. The original definition is due to Tarski. For a regular cardinal  $\kappa$ , the infinitary logic  $\mathcal{L}_{\kappa\kappa}$  is a logic allows conjunctions of  $<\kappa$ -many formulas and the use of  $<\kappa$ -many quantifiers to define formulas.  $\mathcal{L}_{\omega\omega}$ , as classical first-order logic, satisfies the compactness theorem. For an uncountable cardinal  $\kappa$ ,  $\kappa$  is strongly compact if and only if  $\mathcal{L}_{\kappa\kappa}$  satisfies the compactness theorem. As is widely known, strong compactness has many characterizations. The following are equivalent:

- (1)  $\kappa$  is strongly compact.
- (2)  $\mathcal{P}_\kappa\lambda$  carries a fine ultrafilter for all  $\lambda \geq \kappa$ .
- (3) For every  $\kappa$ -complete filter  $F$ ,  $F$  can be extended to a  $\kappa$ -complete ultrafilter, that is, there exists a  $\kappa$ -complete ultrafilter  $U$  such that  $F \subseteq U$ .
- (4) A two-cardinal tree property  $\text{TP}(\kappa, \lambda)$  holds for all  $\lambda \geq \kappa$ , and  $\kappa$  is inaccessible.

In [1], Hayut analyzed these relations.

**Theorem 1** (Hayut [1]). *For regular cardinals  $\kappa \leq \lambda = \lambda^{<\kappa}$ , the following are equivalent:*

- (1) *Every  $\kappa$ -complete filter over  $\lambda$  can be extended to a  $\kappa$ -complete ultrafilter.*
- (2)  *$\mathcal{L}_{\kappa\kappa}$  satisfies the compactness theorem for a theory of size  $2^\lambda$ . That is, for every  $\mathcal{L}_{\kappa\kappa}$ -theory  $T$  with  $2^\lambda$ -many symbols, if  $T$  is  $<\kappa$ -consistent, then  $T$  is consistent.*

*In particular, if every  $\kappa$ -complete filter over  $\lambda$  can be extended to a  $\kappa$ -complete ultrafilter, then  $\square(\mu, <\kappa)$  fails for all regular  $\lambda \leq \mu \leq 2^\lambda$ .*

Hayut also pointed out the following implications concerning compactness of  $\kappa$ :

$$\begin{array}{ccc} 2^\lambda\text{-strongly compact} & \Rightarrow & \lambda\text{-compact} \\ \Leftrightarrow \mathcal{L}_{\kappa\kappa} \text{ satisfies the comp. thm. for } T \text{ of size } 2^\lambda & \Rightarrow & \lambda\text{-strongly compact} \end{array}$$

---

2020 *Mathematics Subject Classification.* 03E35, 03E40, 03E55.

*Key words and phrases.* Tree property, strongly compact cardinals,  $(\dagger)$ , Namba forcings.

This research was supported by JSPS KAKENHI Grant Number 24K16959.

Here,  $\lambda$ -strongly compactness asserts the existence of a fine ultrafilter over  $\mathcal{P}_\kappa(\lambda)$ .  $\lambda$ -compactness is (1) in Theorem 1.

The proof of Theorem 1 involves interpreting  $\mathcal{L}_{\kappa 1}$ -formulas and this make the proof a bit complex to understand. To better understand Theorem 1, we focus on the combinatorial aspects of strong compactness. We also provide a combinatorial proof of  $\neg \square(\mu, < \kappa)$ . Specifically:

**Theorem 2.** *For every regular cardinals  $\aleph_2 \leq \kappa \leq \lambda = \lambda^{<\kappa} \leq \mu \leq 2^\lambda$ , if  $\neg \text{TP}(\kappa, \mu)$ , then there exists a  $\kappa$ -complete filter  $F$  over  $\lambda$  (generated by  $\mu^{<\kappa}$ -many sets) that cannot be extended to a  $\kappa$ -complete ultrafilter. In particular, a  $\square(\mu, < \kappa)$ -sequence defines such a  $\kappa$ -complete filter over  $\lambda$ .*

The organization of this paper is as follows: Section 2 provides preliminaries. Section 3 is devoted to the proof of Theorem 2. Section 4 explores Namba forcings.

## 2. PRELIMINARIES

We use [3] as a reference for set theory in general. Our notation is standard. Throughout this paper,  $\kappa$  and  $\lambda$  denote regular cardinals greater than  $\aleph_2$ , unless otherwise stated. Typically, we assume  $\aleph_2 \leq \kappa \leq \lambda = \lambda^{<\kappa}$ . The symbol  $\mu$  is used to denote an infinite cardinal.

An important concept in this paper is the two-cardinal tree property  $\text{TP}(\kappa, \lambda)$ . A  $\mathcal{P}_\kappa \lambda$ -list is a sequence  $\vec{d} = \langle d_a \mid a \in \mathcal{P}_\kappa \lambda \rangle$ , where  $d_a \subseteq a$  for all  $a \in \mathcal{P}_\kappa \lambda$ . We say that  $\vec{d}$  is *thin* if there exists a club  $C \subseteq \mathcal{P}_\kappa \lambda$  such that  $|\{d_a \cap c \mid c \subseteq a\}| < \kappa$  for all  $c \in C$ . We denote by  $\text{Lev}_c(\vec{d})$  the set  $\{d_a \cap c \mid c \subseteq a\}$ . A *branch* of  $\vec{d}$  is a set  $d \subseteq \lambda$  such that, for all  $a \in \mathcal{P}_\kappa \lambda$ ,  $d \cap a \in \text{Lev}_a(\vec{d})$ . The property  $\text{TP}(\kappa, \lambda)$  asserts the *nonexistence* of a thin  $\mathcal{P}_\kappa \lambda$ -list with *no* branches. Note that  $\text{TP}(\kappa, \kappa)$  is equivalent to the *nonexistence* of a  $\kappa$ -Aronszajn tree.

For Theorem 2, the following lemma is central:

**Lemma 3.** *For regular cardinals  $\kappa \leq \lambda = \lambda^{<\kappa}$ , there exists a family  $\langle A_\alpha \mid \alpha < 2^\lambda \rangle$  of subsets of  $\lambda$  such that, for every  $a, b \in \mathcal{P}_\kappa 2^\lambda$ , if  $a \cap b = \emptyset$ , then*

$$|\bigcap_{\alpha \in a} A_\alpha \setminus \bigcup_{\alpha \in b} A_\alpha| = \lambda.$$

*Proof.* See [1]. □

## 3. PROOF OF THEOREM 2

For regular cardinals  $\kappa \leq \lambda = \lambda^{<\kappa} \leq \mu \leq 2^\lambda$  and a thin  $\mathcal{P}_\kappa \mu$ -list  $\vec{d}$ , let us define a  $\kappa$ -complete filter  $F_{\vec{d}}$  over  $\lambda$ . Let  $C$  be a club such that, for all  $a \in \mathcal{P}_\kappa \lambda$ , there exists  $c_a \in C$  with  $a \subseteq c_a$  and  $|\text{Lev}_{c_a}(\vec{d})| < \kappa$ .

For  $a \in \mathcal{P}_\kappa \mu$ , define  $B_a$  by

$$B_a = \bigcup_{d \in \text{Lev}_{c_a}(\vec{d})} \left( \bigcap_{\alpha \in d \cap a} A_\alpha \setminus \bigcup_{\alpha \in a \setminus d} A_\alpha \right).$$

Here,  $\langle A_\alpha \mid \alpha < 2^\lambda \rangle$  comes from Lemma 3. By Lemma 3,  $|B_a| = \lambda$ .

**Lemma 4.** *The family  $\{B_a \mid a \in \mathcal{P}_\kappa \mu\}$  generates a  $\kappa$ -complete filter.*

*Proof.* For every  $X \in [\mathcal{P}_\kappa \mu]^{<\kappa}$ , fix  $c \in C$  such that  $\bigcup_{a \in X} c_a \subseteq c$ . We can choose  $d \in \text{Lev}_c(\vec{d})$ . Then the following hold:

- $d \cap c_a \in \text{Lev}_{c_a}(\bar{d})$ .
- $|\bigcap_{\alpha \in d} A_\alpha \setminus \bigcup_{\alpha \in c \setminus d} A_\alpha| = \lambda$ .

Let  $B' = \bigcap_{\alpha \in d} A_\alpha \setminus \bigcup_{\alpha \in c \setminus d} A_\alpha$ . It suffices to prove that  $B' \subseteq \bigcap_{a \in X} B_a$ . For each  $a \in X$ ,  $B' \subseteq \bigcap_{\alpha \in (d \cap c_a) \cap a} A_\alpha \setminus \bigcup_{\alpha \in a \setminus (d \cap c_a)} A_\alpha$ . By  $d \cap c_a \in \text{Lev}_{c_a}(\bar{d})$ ,  $B' \subseteq B_a$ . Thus, the proof is complete.  $\square$

Let  $F_{\bar{d}}$  be the filter generated by  $\{B_a \mid a \in \mathcal{P}_\kappa \mu\}$ . The following key lemma is crucial for proving our main theorem:

**Lemma 5.** *If there exists a  $\kappa$ -complete ultrafilter  $U$  over  $\lambda$  such that  $F_{\bar{d}} \subseteq U$ , then  $\bar{d}$  has a branch.*

*Proof.* Define  $d \subseteq \mu$  by  $\alpha \in d$  if and only if  $A_\alpha \in U$ . We claim that  $d$  is a branch of  $\bar{d}$ . For every  $a \in \mathcal{P}_\kappa \mu$ , since  $B_a \in U$  and  $U$  is a  $\kappa$ -complete ultrafilter, there exists  $e \in \text{Lev}_{c_a}(\bar{d})$  such that

$$\bigcap_{\alpha \in e \cap a} A_\alpha \setminus \bigcup_{\alpha \in a \setminus e} A_\alpha \in U.$$

Thus, we have:

- For all  $\alpha \in e \cap a$ ,  $A_\alpha \in U$ .
- For all  $\alpha \in a \setminus e$ ,  $A_\alpha \notin U$ .

By the definition of  $d$ ,  $d \cap a = e \cap a$ . Since  $e \in \text{Lev}_{c_a}(\bar{d})$ , there exists  $c' \in \mathcal{P}_\kappa \mu$  such that  $a \subseteq c_a \subseteq c'$  and  $e = d_{c'} \cap c_a$ . Therefore,  $d \cap a = e \cap a = d_{c'} \cap a \in \text{Lev}_a(\bar{d})$ , as desired.  $\square$

*Proof of Theorem 2.* Suppose there exists a thin  $\mathcal{P}_\kappa \mu$ -list  $\bar{d}$  with no branch. By Lemma 5,  $F_{\bar{d}}$  cannot be extended to a  $\kappa$ -complete ultrafilter.

It is also known that a  $\square(\mu, < \kappa)$ -sequence provides such a thin  $\mathcal{P}_\kappa \mu$ -list. For a proof, see [4]. Therefore, a  $\square(\mu, < \kappa)$ -sequence defines the required  $\kappa$ -complete filter over  $\lambda$ .  $\square$

Our proof can also help in understanding Theorem 1.

**Theorem 6** (Jech [2]). *For regular cardinals  $\kappa \leq \lambda = \lambda^{<\kappa}$ , the following are equivalent:*

- (1)  $\mathcal{L}_{\kappa\kappa}$  satisfies the compactness theorem for a theory of size  $\lambda$ .
- (2)  $\kappa$  is inaccessible and  $\text{TP}(\kappa, \lambda)$ .

**Corollary 7.** *For regular cardinals  $\kappa \leq \lambda$ , the following are equivalent:*

- (1) Every  $\kappa$ -complete filter over  $\lambda$  can be extended to a  $\kappa$ -complete ultrafilter.
- (2)  $\mathcal{L}_{\kappa\kappa}$  satisfies the compactness theorem for a theory of size  $2^\lambda$ .
- (3)  $\kappa$  is inaccessible and  $\text{TP}(\kappa, 2^\lambda)$ .

#### 4. SEMIPROPERNESS OF NAMBA FORCINGS

The notions of thin  $\mathcal{P}_\kappa \lambda$ -lists  $\bar{d}$  and  $F_{\bar{d}}$  can be defined even if  $\kappa$  is a successor cardinal. To study the compactness properties underlying  $F_{\bar{d}}$ , we introduce Namba forcing.

For a  $\kappa$ -complete fine ideal  $I$  over  $Z \subseteq \mathcal{P}(X)$ , an  $(I)$ -Namba tree  $p$  is a set  $p \subseteq [Z]^{<\omega}$  satisfying the following conditions:

- (1)  $p$  is a tree. That is, each  $s \in p$  is  $\subseteq$ -increasing, and  $p$  is closed under initial segments.

- (2) There exists a maximal  $s \in p$  such that  $\forall t \in p (s \subseteq t \vee t \subseteq s)$ . This  $s$  is called the trunk, denoted  $\text{tr}(p)$ , of  $p$ .
- (3) For each  $s \in p$ , if  $s \sqsupseteq \text{tr}(p)$ , then  $\text{Suc}(s) = \{a \in Z \mid s \smallfrown \langle a \rangle \in p\} \in I^+$ .

Let  $\text{Nm}(Z, I)$  denote the set of all  $I$ -Namba trees. For a filter  $F$ ,  $\text{Nm}(Z, F)$  is defined as  $\text{Nm}(Z, F^*)$ . We write  $\text{Nm}(\kappa)$  and  $\text{Nm}(\kappa, \lambda)$  to refer to  $\text{Nm}(\kappa, \text{bounded ideal})$  and  $\text{Nm}(\mathcal{P}_\kappa \lambda, \text{bounded ideal})$ , respectively.  $\text{Nm}(\aleph_2)$  corresponds to the original Namba forcing. Since every Namba forcing  $\text{Nm}(Z, I)$  is  $\omega_1$ -stationary preserving, every Namba forcing can be semiproper.

The semiproperness of Namba forcings is connected to reflection principles (see Lemma 9). A typical example is the following:

**Theorem 8.** *For a  $\kappa$ -complete ideal  $I$  over  $\lambda$ , if  $\text{Nm}(\lambda, I)$  is semiproper, then  $\square(\lambda)$  fails.*

Note that  $\text{Nm}(\lambda, I)$  forces  $\dot{\text{cf}}(\lambda) = \omega$ . Theorem 8 follows from Lemma 9 and Theorems 10 and 11.

**Lemma 9** (Tsukuura [7]). *For regular cardinals  $\aleph_2 \leq \kappa \leq \lambda$  and a semistationary subset  $S \subseteq [\lambda]^\omega$ , if  $\text{Nm}(\kappa, \lambda)$  forces that  $S$  is semistationary, then there exists  $a \in \mathcal{P}_\kappa \lambda$  such that  $S \cap [a]^\omega$  is semistationary. Conversely, if we assume  $\forall \mu < \kappa (\mu^\omega < \kappa)$  or  $\kappa = \aleph_2$ , the reverse direction also holds.*

**Theorem 10** (Sakai–Veličković [5]). *If every semistationary subset  $S \subseteq [\lambda]^\omega$  reflects to some  $a \in \mathcal{P}_\kappa \lambda$ , then  $\square(\lambda)$  fails.*

**Theorem 11** (Shelah [6] for (1)  $\leftrightarrow$  (3), Tsukuura [7] for (1)  $\leftrightarrow$  (2)). *For a regular cardinal  $\kappa \geq \aleph_2$ , the following are equivalent:*

- (1)  $\text{Nm}(\kappa)$  is semiproper.
- (2)  $\text{Nm}(\kappa, \kappa)$  is semiproper.
- (3) There exists a semiproper forcing that forces  $\dot{\text{cf}}(\kappa) = \omega$ .

Strong compactness directly effects the semiproperness of Namba forcings.

**Proposition 12.** *For a  $\kappa$ -complete filter  $F$  over  $\lambda$ , if  $F$  can be extended to a  $\kappa$ -complete ultrafilter, then  $\text{Nm}(\lambda, F)$  is semiproper.*

The extendability of  $F_{\bar{d}}$  connects both the semiproperness of  $\text{Nm}(\lambda, F_{\bar{d}})$  and the existence of a branch in  $\bar{d}$ . However, these two properties do not imply each other.

It is straightforward to construct a model where  $\text{Nm}(\lambda, F_{\bar{d}})$  is *not* semiproper, but  $\bar{d}$  has a branch. Indeed, a thin  $\mathcal{P}_\kappa \lambda$ -list  $\bar{d}$  with a branch always exists. On the other hand, obtaining the semiproperness of  $\text{Nm}(\lambda, F_{\bar{d}})$  requires the failure of  $\square(\lambda)$ .

**Theorem 13.** *It is consistent that there exists a thin  $\mathcal{P}_\kappa \lambda$ -list  $\bar{d}$  such that  $\bar{d}$  has no branch, but  $\text{Nm}(\lambda, F_{\bar{d}})$  is semiproper.*

*Proof.* Every  $\text{Nm}(\lambda, F_{\bar{d}})$  is  $\omega_1$ -stationary preserving. It is known that if a strongly compact cardinal  $\kappa$  is collapsed to  $\aleph_2$  by  $\text{Coll}(\aleph_1, < \kappa)$ , then  $(\dagger)$  and CH hold in the extension. Let us consider the extension. Note that CH implies the existence of an  $\aleph_2$ -Aronszajn tree, and thus  $\text{TP}(\aleph_2, \aleph_2)$  fails. Let  $\bar{d}$  be a thin  $\mathcal{P}_{\aleph_2} \aleph_2$ -list with *no* branches. By  $(\dagger)$ ,  $\text{Nm}(\aleph_2, F_{\bar{d}})$  must be semiproper.  $\square$

The next question concerns whether there exists a thin  $\mathcal{P}_\kappa \lambda$ -list  $\bar{d}$  such that  $\bar{d}$  has *no* branches, and  $\text{Nm}(\lambda, F_{\bar{d}})$  is *not* semiproper. It seems that stronger anti-compactness principles are needed.

**Question 14.** Suppose that  $\square(2^\lambda)$  holds. Is there an  $\aleph_2$ -complete filter  $F$  over  $\lambda$  such that  $\text{Nm}(\lambda, F)$  is not semiproper? What about  $F_{\vec{a}}$  for a thin  $\mathcal{P}_\kappa 2^\lambda$ -list defined by  $\square(2^\lambda)$ ?

We conclude this paper with the following diagram, which motivates and illustrates our question. For details on the implications of Namba forcings, see [7]. Note that  $F$  denotes a  $\kappa$ -complete filter, and we assume  $\aleph_2 \leq \kappa \leq \lambda = \lambda^{<\kappa}$ .

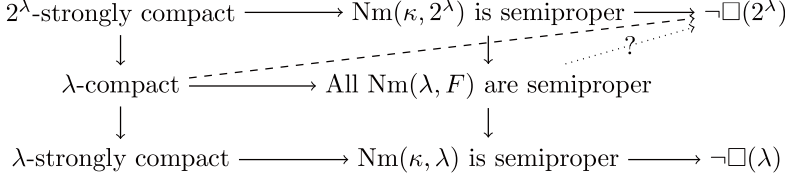


FIGURE 1. Our question concerns the “dotted” arrow. The “dashed” arrow comes from Theorem 1.

#### REFERENCES

- [1] Yair Hayut. Partial strong compactness and squares. *Fund. Math.*, 246(2):193–204, 2019.
- [2] Thomas Jech. Some combinatorial problems concerning uncountable cardinals. *Ann. Math. Logic*, 5(3):165–198, 1973.
- [3] Akihiro Kanamori. *The higher infinite*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, second edition, 2003. Large cardinals in set theory from their beginnings.
- [4] Hiroshi Sakai. TP and weak square (accessed: 2025-01-24). [https://www2.kobe-u.ac.jp/~hsakai/Research/notes/tp\\_weak\\_square.pdf](https://www2.kobe-u.ac.jp/~hsakai/Research/notes/tp_weak_square.pdf).
- [5] Hiroshi Sakai and Boban Velićković. Stationary reflection principles and two cardinal tree properties. *J. Inst. Math. Jussieu*, 14(1):69–85, 2015.
- [6] Saharon Shelah. *Proper and improper forcing*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, second edition, 1998.
- [7] Kenta Tsukuura. On semiproperness of namba forcings and ideals in prikry extensions. submitted.

DEPARTMENT OF FISHERIES DISTRIBUTION AND MANAGEMENT, NATIONAL FISHERIES UNIVERSITY, SHIMONOSEKI, 759-6533, JAPAN

Email address: [tsukuura@fish-u.ac.jp](mailto:tsukuura@fish-u.ac.jp)