

Reflection groups of geodesic spaces and Coxeter groups

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The purpose of this note is to introduce a result of my recent paper [8] about *cocompact discrete reflection groups of geodesic spaces*.

A metric space (X, d) is called a *geodesic space* if for each $x, y \in X$, there exists an isometry $\xi : [0, d(x, y)] \rightarrow X$ such that $\xi(0) = x$ and $\xi(d(x, y)) = y$ (such ξ is called a *geodesic*). We say that an isometry r of a geodesic space X is a *reflection* of X , if

- (1) r^2 is the identity of X ,
- (2) $X \setminus F_r$ has strictly two convex components X_r^+ and X_r^- , and
- (3) $\text{Int } F_r = \emptyset$,

where F_r is the fixed-point set of r which is called the *wall* of r . An isometry group Γ of a geodesic space X is called a *reflection group*, if some set of reflections of X generates Γ . Let Γ be a reflection group of a geodesic space X and let R be the set of all reflections of X in Γ . We note that R generates Γ by definition. Now we suppose that the action of Γ on X is proper, that is, $\{\gamma \in \Gamma \mid \gamma x \in B(x, N)\}$ is finite for each $x \in X$ and $N > 0$ (cf. [2, p.131]). Then the set $\{F_r \mid r \in R\}$ is locally finite. Let C be a component of $X \setminus \bigcup_{r \in R} F_r$, which is called a *chamber*. Here we can show that $\Gamma C = X \setminus \bigcup_{r \in R} F_r$. Then $\Gamma \overline{C} = X$ and for each $\gamma \in \Gamma$, either $C \cap \gamma C = \emptyset$ or $C = \gamma C$. We say that Γ is a *cocompact discrete reflection group* of X , if $\overline{C}$ is compact and $\{\gamma \in \Gamma \mid C = \gamma C\} = \{1\}$.

Definition. A group Γ is called a *cocompact discrete reflection group* of a geodesic space X , if

- (1) Γ is a reflection group of X ,
- (2) the action of Γ on X is proper,
- (3) for a chamber C , $\overline{C}$ is compact, and
- (4) $\{\gamma \in \Gamma \mid C = \gamma C\} = \{1\}$.

For example, every Coxeter group is a cocompact discrete reflection group of some geodesic space.

A *Coxeter group* is a group W having a presentation

$$\langle S \mid (st)^{m(s,t)} = 1 \text{ for } s, t \in S \rangle,$$

where S is a finite set and $m : S \times S \rightarrow \mathbb{N} \cup \{\infty\}$ is a function satisfying the following conditions:

- (1) $m(s, t) = m(t, s)$ for each $s, t \in S$,
- (2) $m(s, s) = 1$ for each $s \in S$, and
- (3) $m(s, t) \geq 2$ for each $s, t \in S$ such that $s \neq t$.

The pair (W, S) is called a *Coxeter system*. H.S.M. Coxeter showed that a group Γ is a finite reflection group of some Euclidean space if and only if Γ is a finite Coxeter group. Every Coxeter system (W, S) induces the Davis-Moussong complex $\Sigma(W, S)$ which is a CAT(0) space ([6], [7], [10]). Then the Coxeter group W is a cocompact discrete reflection group of the CAT(0) space $\Sigma(W, S)$.

Here we obtained the following theorem in [8].

Theorem. *A group Γ is a cocompact discrete reflection group of some geodesic space if and only if Γ is a Coxeter group.*

Let Γ be a cocompact discrete reflection group of a geodesic space X , let C be a chamber and let S be a *minimal* subset of R such that $C = \bigcap_{s \in S} X_s^+$ (i.e. $C \neq \bigcap_{s \in S \setminus \{s_0\}} X_s^+$ for each $s_0 \in S$). Then we can show that $\langle S \rangle C = X \setminus \bigcup_{r \in R} F_r = \Gamma C$. Since $\{\gamma \in \Gamma \mid C = \gamma C\} = \{1\}$, S generates Γ . In [8], we have proved that the pair (Γ, S) is a Coxeter system.

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