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On Computation of Calibers of Real Quadratic Fields

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1. The fields of the caliber 2

Let F be a real quadratic field. Each number $x\in F-\mathbb{Q}$ can be expressed in a unique way as continuous fraction:

$$x = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}} \qquad (a_i\in\mathbb{Z},\ a_i \geq 1 \text{ if } i \geq 2).$$

It is well-known that the sequence $\{a_1, a_2, \ldots\}$ becomes periodic. Let m be the period of x. There exists some positive integer i_0 such that $a_{i+m}=a_m$ for all $i\geq i_0$. An element x of F is called *reduced* if $x>1$, $0>x'>-1$. As is well-known x is reduced if and only if the continuous fractional expansion is purely periodic, i.e. $a_{i+m}=a_i$ for all $i\geq 1$. Let A be an ideal class of F (the usual ideal class). One can choose a representative ideal $\mathfrak{a}$ of A such that $\mathfrak{a}=\mathbb{Z} + \mathbb{Z}x$ with a reduced number x. The period m of x depends only on the class A. We write $m(A)$ for m and call $m(A)$ the *caliber* of A. We set

$$\kappa(F) = \sum_A m(A), \tag{1}$$

where A runs over all ideal classes of F. Let D be the discriminant of F. Then we write $\kappa(D)$ instead of $\kappa(F)$ also. The positive integer

$\kappa(F)$ is called the *caliber* of F. Moreover we define a subset DI of F by putting

(2) $DI = \{\theta \in F \mid \theta > 1,\ 0 > \theta' > -1$ and $\mathbb{Z} + \mathbb{Z}\theta$ is an ideal of $F\}$.

It is easy to prove that DI is a finite set and

$$\kappa(D) = \#(DI),$$

where $\#(DI)$ denotes the cardinality of the set DI. It is known by [4] that the number of real quadratic fields with a given caliber is finite. Each element θ of the set DI satisfies the quadratic equation

$$a\theta^2 - b\theta + c = 0 \qquad (a, b, c \in \mathbb{Z},\ a > 0)$$

with the condition $b^2 - 4ac = D$, and we get

(3) $$\theta = \frac{b + \sqrt{D}}{2a},$$

Since $\theta \in DI$, the pair (a, b) satisfies the following condition

(4) $$\begin{cases} b > 2a - \sqrt{D},\ \sqrt{D} > b > -2a + \sqrt{D}, \\ b^2 \equiv D \pmod{4a}. \end{cases}$$

We set

(5) $S(D) = \{ (a, b) \in \mathbb{Z} \mid b > 2a - \sqrt{D},\ \sqrt{D} > b > -2a + \sqrt{D},\ b^2 \equiv D \pmod{4a} \}$.

Thus the caliber $\kappa(D)$ is the number of the pairs (a, b) of integers satisfying the condition (4), i.e. $\kappa(D) = \#S(D)$.

Let $h(D)$ be the *ideal class number* of the quadratic field $F = \mathbb{Q}(\sqrt{D})$ with discriminant D. Then we can prove

Proposition 1.

If $\kappa(D) \leq 3$, *then* $h(D) \leq 1$.

proof) By the definition (1) of $\kappa(D)$ and the fact $m(A)\geq 1$, it is clear that $h(D)\leq 3$. Suppose that $h(D)=2$ (or 3). Let A and B (and C) be the unequivalent ideal classes of F. It is well known that $m(A)\equiv m(B)$ ($\equiv m(C)$) (mod 2). Since $\kappa(D)=m(A)+m(B)+m(C)$, we can show a contradiction.

Proposition 2.

Let $F=\mathbb{Q}(\sqrt{D})$, the quadratic field with discriminant D and $\kappa(D)=2$. If $D\equiv 1$ (mod 4), then D is a product of two primes.

If $D\equiv 4$ (mod 8), then $D/4$ is a prime. If $D\equiv 0$ (mod 8), then $D/8$ is a prime.

proof)

From $\kappa(D)=2$ and proposition 1, we have $h(D)=1$, so that the caliber of the principal ideal class of F equals to 2. Therefore there exists a number $x\in DI$ whose period is 2, and every number of F has even period. Hence the norm of the fundamental unit of F is +1. Therefore the class number $h^*(D)$ of F in the narrow sense is equal to $2h(D)$, i.e. $h^*(D)=2^{2-1}$. Therefore number of different prime divisors of D is less than 2. From this we can show the assertions of the proposition.

Proposition 3.

Let the notation be as above. Then the following facts hold.

i) *There exists a number $\alpha\in DI$ such that $\alpha=\dfrac{b+\sqrt{D}}{2}$ ($b\in\mathbb{Z}$) and b satisfies $b>2-\sqrt{D}$, $\sqrt{D}>b>-2+\sqrt{D}$, $b^2\equiv D$ (mod 4) (see (2), (3), (4)).*

ii) *If* $D>16$ *and* $D\equiv 0$ *or* 1 *or* 4 (mod 8), *then there exists a number* $\alpha\in DI$ *such that* $\alpha=\dfrac{b+\sqrt{D}}{2\cdot 2}$ $(b\in\mathbb{Z})$ *and* b *satisfies* $b>4-\sqrt{D}$, $\sqrt{D}>b>-4+\sqrt{D}$, $b^2\equiv D$ (mod 8)(*see* (2), (3), (4)).

iii) *If* $D>36$ *and* $D\equiv 0$ *or* 4 (mod 12), *then there exists a number* $\alpha\in DI$ *such that* $\alpha=\dfrac{b+\sqrt{D}}{2\cdot 3}$ $(b\in\mathbb{Z})$ *and* b *satisfies* $b>6-\sqrt{D}$, $\sqrt{D}>b>-6+\sqrt{D}$, $b^2\equiv D$ (mod 12)(*see* (2), (3), (4)).

proof)

We prove only the case iii). Other cases can be proved by the same method. In the case iii), we prove that there exists an integer b satisfying the condition $b>6-\sqrt{D}$, $\sqrt{D}>b>-6+\sqrt{D}$, $b^2\equiv D$ (mod 12), i.e.

$$\sqrt{D}>b>\sqrt{D}-6,\quad b^2\equiv D \pmod{12},$$

because $D>36$. Suppose $D\equiv 0$ (mod 12). Then the condition $b^2\equiv D$ (mod 12) is equivalent to $b\equiv 0$ (mod 6). It is easy to see that there exists an integer b such that $\sqrt{D}>b>\sqrt{D}-6$, and $b\equiv 0$ (mod 6). For the case $D\equiv 4$ (mod 12), the condition $b^2\equiv D$ (mod 12) becomes $b\equiv 2$ or 4 (mod 6). We can find an integer b such that $\sqrt{D}>b>\sqrt{D}-6$, $b\equiv 2$ or 4 (mod 6). Therefore the case iii) is completely proved.

Now we can prove

Theorem 1.

Let F *be* $\mathbb{Q}(\sqrt{D})$, *the quadratic field with the discriminant* D *and let the caliber* $\kappa(D)$ *of* F *be* 2. *If* $D>16$ *and* $D\equiv 1$ (mod 4), *then there exist two positive integer* n,k *satisfying* $D=n^2k^2+4k$ *and* $k\geq 3$. *If*

$D>36$ and $D\equiv 0 \pmod 4$, then $D=36n^2+8$ for some positive integer n.

proof)

From proposition 3-i), we can find a number $\alpha=\frac{b+\sqrt{D}}{2}\in DI$ $(b\in\mathbb{Z})$. Since $\kappa(D)=2$, we get $h(D)=1$ by proposition 1, so that the period of α is 2. Put $\alpha=\{\ell,n\}$, where $\ell,\ n\in\mathbb{Z}$, $\ell,\ n>0$ *and* $\ell\neq n$. Then α satisfies the quadratic equation $n^2\alpha^2-\ell\alpha-\ell=0$. Hence $\alpha=\frac{\ell+\sqrt{\ell^2+4(\ell/n)}}{2}$. Compare $\alpha=\frac{b+\sqrt{D}}{2}$ and the above expression of Then we get $b=\ell$ and $D=\ell^2+4(\ell/n)$. It follows from $\alpha\in DI$ that $(b^2-D)/4\in\mathbb{Z}$, i.e $\ell/n\in\mathbb{Z}$. Put $\ell=nk$ with a positive integer k. Since n, $\ell>0$ and $n\neq\ell$, we get $k\geq 2$ and $D=n^2k^2+4k$. If $D\equiv 1 \pmod 4$ then k is odd, so that $k\geq 3$. In the case $D\equiv 0 \pmod 4$, suppose that $k\geq 3$ and put $\gamma=(nk+\sqrt{D})/(2k)$, a root of the equation $k\gamma^2-nk\gamma-1=0$. Then γ is an element of DI and $\gamma\neq\alpha$. From proposition 3-ii), there exists a number $\beta=(c+\sqrt{D})/4\in DI$ $(c\in\mathbb{Z})$. It follows from $\alpha\neq\beta$, $\beta\neq\gamma$ and $\gamma\neq\alpha$ that $\kappa(D)=\#(DI)\geq 3$. This is a contradiction. Therefore we get $k=2$ and $D=4n^2+8$ for an integer n. Furthermore, from proposition 3-iii) and $\#(DI)=\kappa(D)=2$, it is necessary that $D\equiv 8 \pmod{12}$. So that $D=36n^2+8$ for an integer n.

From proposition 1, 2 and theorem 1, we get immediately

Proposition 4.

If $\kappa(D)=2$, then $h(D)=1$. Moreover we have the following facts.

i) *If $D\equiv 1 \pmod 4$ and $D>16$, then there exist two integer n,k such*

that $D=n^2k^2+4k$ *and that* k, n^2k+4 *are primes.*

ii) *If* $D\equiv 0 \pmod 4$ *and* $D>36$, *then there exits an integer* n *suc.* *that* $D=36n^2+8$. *If* n *is odd, then* $9n^2+2$ *is a prime. If* n *is even,* *then* $(9n^2/2)+1$ *is a prime.*

Proposition 5.

i) *If* $D=n^2k^2+4k\equiv 1 \pmod 4$, *then* $\varepsilon=(nk+\sqrt{D})^2/(4k)$ *is a unit o.* $\mathbb{Q}(\sqrt{D})$

ii) *If* $D=36n^2+8$ *then* $\varepsilon=(6n+\sqrt{D})^2/8$ *is a unit of* $\mathbb{Q}(\sqrt{D})$.

The following theorem is proved in [7].

Theorem 2 (Tatsuzawa).

Let $\frac{1}{2}>t>0$ *and* $N>\max(e^{(1/t)}, e^{11.2})$. *Then*

$$L(1,\chi) > 0.655\frac{t}{N^t} \tag{6}$$

with one possible exception χ, *where* χ *is a non-principal primitive* *real character* of *modulus* N *and* $L(s,\chi)$ *is* the *L-function with* *character* χ.

Choose χ_D, the Kronecker character belonging to the quadratic field $\mathbb{Q}(\sqrt{D})$ as χ, $\frac{1}{15}$ as t. Then $N=D$ and we have the inequality

$$h(D) > \frac{0.655D^{(13/30)}}{30\log(\varepsilon_0)} \quad (\text{for } D>e^{15}), \tag{7}$$

where ε_0 is the fundamental unit of $\mathbb{Q}(\sqrt{D})$ (see [3]). If $\kappa(D)=2$, then it follows from theorem 1 and proposition 5 that

$$\log(\varepsilon_0) \leq \log(\varepsilon) < \begin{cases} \log(D) - \log(3) & \text{if } D\equiv 1 \pmod 4 \text{ and } D>16 \\ \log(D) - \log(2) & \text{if } D\equiv 0 \pmod 4 \text{ and } D>36. \end{cases}$$

Therefore for $F=\mathbb{Q}(\sqrt{D})$ with $\kappa(D)=2$, we have

$$h(D) > \frac{0.655D^{(13/30)}}{30(\log(D)-\log(2))} \quad (\text{for } D>e^{15})$$

with one possible exception D. By proposition 1, we can prove

Proposition 6.

Let F be the real quadratic field $\mathbb{Q}(\sqrt{D})$ with discriminant D. If $\kappa(D)=2$ and $D>e^{14}$, then D satisfies the inequality

$$1 > \frac{0.655D^{(13/30)}}{30(\log(D)-\log(2))}$$

with one possible exception D.

Note that the above inequality does not hold for $D>e^{15}$ (=3269017.23...). So if $\kappa(D)=2$, then $D<3269018$ with one possible exception. If $\kappa(D)=2$, then D satisfies the conditions represented in proposition 4. For these D, by aid of a computer, we can easily caluculate $h(D)$ and $\kappa(D)$. Hence we get

Theorem 3.

Let F a the real quadratic field with the discriminant D. If the caliber $\kappa(D)$ of F = 2, then D=4·3, 4·6, 4·11, 21, 4·38, 77, 4·83, 93, 4·227, 237, 437, 453, 1133, 1253 *or one more possible* D_0(>1253).

2. The fields with caliber 3

Now we consider quadratic fields with caliber 3. Let F be a

quadratic field $\mathbb{Q}(\sqrt{D})$ of the discriminant D.

Proposition 7.

If the caliber $\kappa(D)$ *of* $F=\mathbb{Q}(\sqrt{D})$ *is equal to* 3, *then the discriminant* D *is a prime and* $D\equiv 1 \pmod 4$.

proof)

By the condition $\kappa(D)=3$, we get $h(D)=1$ (see Proposition 1). So that there exists some number α in DI whose period is 3 by using the definition (1) of $\kappa(D)$. Hence the norm of the fundamental unit of F is -1. By the same way as the proof of proposition 2, the number of different primes of D is 1, i.e. D is a prime. Since the discriminant D is a prime, we get $D\equiv 1 \pmod 4$.

For a prime D which satisfies $D\equiv 1 \pmod 4$ and a prime p, we define a symbol $(\frac{D}{p})^*$ as follows:

if $p>2$, then $(\frac{D}{p})^*$ is usual Legendre sumbol $(\frac{D}{p})$,

if $p=2$, then $(\frac{D}{p})^*=\begin{cases} 1 & D\equiv 1 \pmod 8, \\ -1 & D\equiv 5 \pmod 8. \end{cases}$

We denote

$$P(D)=\{p:\text{primes} \mid p<\frac{\sqrt{D}}{2},\ (\frac{D}{p})^*=1\},$$

we put

$$\pi(D)=\#(P(D)).$$

Proposition 8.

If the discriminant D *of* $F=\mathbb{Q}(\sqrt{D})$ *is a prime satisfying* $D\equiv 1 \pmod 4$, *then* $\kappa(D)\geq 1+2\pi(D)$.

proof)

By proposition 3-i) there exists a pair $(1,b)$ which satisfies the conditin (4). If a prime p is an element of $P(D)$, then there exist two b's such that each pair (p, b) is elements in $S(D)$ (see (5)). Hence we obtain $\kappa(D)\geq 1+2\pi(D)$.

Theorem 4.

Let D be the discriminant of $F=\mathbb{Q}(\sqrt{D})$ and the caliber $\kappa(D)$ be equal to 3. Put $D=4d+1$ (see Proposition 7). Then d is written by p, p^2, p^3, pq or p^2q for odd primes p and q $(p\neq q)$.

proof)

Assume $d=p_1p_2p_3\ell$, where p_1, p_2 and p_3 $(p_1<p_2<p_3)$ are primes and ℓ is a positive integer. Then it is easy to see that p_1 and p_2 are elements of $P(D)$. So that $\kappa(D)\geq 1+2\pi(D)\geq 5$. Assume $d=p_1^{e_1}p_2^{e_2}\ell$, where p_1 and p_2 are primes, e_1, e_2 and ℓ are positive integer such that e_1, $e_2\geq 2$. Then p_1 and p_2 are elements of $P(D)$, i.e. $\kappa(D)\geq 5$. Assume $D=p^e\ell$, where p is a prime, e and ℓ are positive integers such that $e\geq 4$. Then the fact $p\in P(D)$ holds. Furthermore there exists two elements (p^2, b) in $S(D)$. Hence we get the fact $\kappa(D)\geq 5$ for $D=4p^e\ell+1$ $(e\geq 1)$. Assume that d is even. If we put $d=2f$ $(0<f\in\mathbb{Z})$, then $d=8f+1$. So that we get $D\equiv 1 \pmod 8$ and $D\equiv 1$ or $9 \pmod{16}$. For $D\equiv 1 \pmod 8$, we can see that there exist two pairs $(2,b)$ which satisfy (4) by proposition 3-ii). Further it is easy to show that for $D\equiv 1$ or $9 \pmod{16}$, there exist two pairs $(4,b)$ which are elements in $S(D)$.

Thus we have obtained $K(D)\geq 5$ for even d and proved the assertion of the proposition.

Proposition 9.

If $k(D)\geq 3$, *then there exists a number* $\alpha=\frac{b+\sqrt{D}}{2}\in DI$ $(b\in Z)$ *(Propositipon* 3-i)). *Moreover* $\alpha=\{\ell,n,n\}$, *where* ℓ,n *are positive integers and* n^2+1 *divides* $\ell n+1$.

proof) It follows from $K(D)=3$ and $h(D)=1$ (see Proposition 1) that $\alpha=\{\ell,m,n\}$, where ℓ,m and n are positive integers excluding the case $\ell=m=n$. Therefore

$$\alpha = \ell + \cfrac{1}{m + \cfrac{1}{n + \cfrac{1}{\alpha}}}, \text{ i.e.}$$

α satisfies the equation

$$(mn+1)\alpha^2 + (m-\ell-n-\ell mn)\alpha - (\ell m+1) = 0.$$

We obtain

$$\alpha = \frac{\ell + \frac{n-m}{mn+1} \pm \sqrt{\left(1 + \frac{n-m}{mn+1}\right)^2 + 4\left(\frac{\ell m+1}{n^2+1}\right)}}{2}$$

Compare above representation of α and $\alpha=\frac{b+\sqrt{D}}{2}$. Then we get $m=n$, $\ell\neq m$, $n^2+1 \mid \ell m+1$. Therfore we obtain

$$\alpha = \frac{\ell + \sqrt{\ell^2 + \frac{\ell n+1}{n^2+1}}}{2} \text{ and } \alpha = \{\ell,n,n\} \ (\ell\neq n).$$

By the equality

$$\ell + \cfrac{1}{n + \cfrac{1}{n + \cfrac{1}{\alpha}}} = \frac{\{\ell(n^2+1)+n\}\alpha + (\ell n+1)}{(n^2+1)\alpha + n}.$$

and Theorem 3.9 in [6, p.215], we can prove immediately

Proposition 10.

Let the notation be as above. Then $(n^2+1)\alpha+n$ *is a unit of* $\mathbb{Q}(\sqrt{D})$.

Put $\varepsilon=(n^2+)\alpha+n$. Then it folllows from $D=\ell^2+4(\ell n+1)/(n^2+1)$ and $\alpha=(1+\sqrt{D})/2$ that $\log \varepsilon < \frac{3}{2} \log D$. In the unequation (6), put $\chi=\chi_D$ and $t=\frac{1}{16}$. Then we get

$$h(D) > \frac{0.655 D^{7/16}}{32 \log \varepsilon_0}.$$

So that we can prove, by proposition 1,

Proposition 11.

Let F be the real quadartic field $\mathbb{Q}(\sqrt{D})$ with the discriminant D. If $\kappa(D)=3$ and $D>e^{16}$, then D satisfies the inequality

$$1 > \frac{0.655 D^{\frac{3}{7}}}{48 \log D}$$

with one possible exception D.

It is easy to show that the above inequality does not hold for $D>e^{16.18}$ (=10638605.805..). So that if $\kappa(D)=3$, then $D<10638617$ and D satisfies the condition repersented in Theorem 4. Furthermore by proposition 8, and $\kappa(D)=3$, we get $\#P(D)=0$ or 1. By aid of a computer, we get

Theorem 5.

Let F be the real quadratic field with discriminant D. If the

caliber $\kappa(D)$ *of* F *is equals to 3, then* D=17, 37, 61, 101, 197, 317, 461, 557, 677, 773, 1877 *or one more possible* D_0(>1877). Cororally (of Theorem 3 and Theorem 5)

Let the notation be as above. Then at least two of the following assertions hold.

i) $\kappa(D)=1$ *if and only if* D=4·2, 5, 13, 29, 53, 173, 293.

ii) $\kappa(D)=2$ *if and only if* D=4·3, 4·6, 4·11, 21, 4·38, 77, 4·83, 93, 4·227, 237, 437, 453, 1133, 1253.

iii) $\kappa(D)=3$ if and only if D=17, 37, 61, 101, 197, 317, 461, 57, 677, 773, 1877

proof)

If $\kappa(D)=1$, then it follows from § 3 in [4] and theorem 2 in [3] that D=4·2, 5, 13, 29, 53, 173, 293 or one more possible D_0(>293). D_0 is same as in Theorem 3 and Theorem 5 if it exists.

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LISP FLUID('(DD));
LISP FLUID('(FF));
LISP FLUID('(EE));
LISP FLUID('(LSTAB));
LISP FLUID('(EE2));


SYMBOLIC PROCEDURE ISQRT(I);
BEGIN
      IF I<0 THEN RETURN NIL;
      IF I=0 THEN RETURN 0;
      IF I<4 THEN RETURN 1;
      RETURN ISQRT1(QUOTIENT(I,2),I);
END;

SYMBOLIC PROCEDURE ISQRT1(J,I);
 BEGIN
      INTEGER K;
     K:=QUOTIENT(I,J);
      IF K<J THEN RETURN ISQRT1(QUOTIENT(J+K,2),I);
      RETURN J;
END;


SYMBOLIC PROCEDURE ABLST0(DD,EE,EE2,BB);
BEGIN
       INTEGER Y,A,I,B;
       SCALAR LL,LSTAB;
       I:=1;
       B:=BB;
       LSTAB:=NIL;

F1:
          Y:=(DD-B**2)/4;

        IF NOT (Y>0) THEN <<FF:=I;RETURN LSTAB>>;
           A:=QUOTIENT(EE-B,2);
F2:
          A:=A+1;
           IF A>(EE+B)/2 THEN <<B:=B+2;GOTO F1>>;
          IF REMAINDER(Y,A) NEQ 0 THEN GOTO F2;
           LL:=CONS(A, B);
          LSTAB:= CONS(LL,LSTAB);
          I:=I+1;
         GOTO F2;
END;



SYMBOLIC PROCEDURE EQUIV0(LL,EE,EE2);
BEGIN
    SCALAR N,R,H,AB,A,B,LL1,AB0;
    H:=0;

F1:
     IF NULL(LL) THEN RETURN (-1)**N*H;
      AB:=CAR(LL);
      LL:=CDR(LL);

F2:
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     H:=H+1;
     N:=0;
     A:=CAR(AB);
     B:=CDR(AB);

F3:
     N:=N+1;
     A:=(DD-B**2)/(4*A);
     R:=REMAINDER(EE+B,2*A);
     B:=EE-R;
     IF A>(EE+B)/2 THEN GOTO F3;

      AB0:= A . B;

     LL1:=TAKEOUT(LL, AB0);

     IF LL1=LL THEN GOTO F1 ELSE <<LL:=LL1; GOTO F3>>;

END;


SYMBOLIC PROCEDURE TAKEOUT(L,A);
BEGIN
     SCALAR LT,B;
LOOP:
     IF  (MEMBER(A,L) = NIL) THEN RETURN REVERSE(APPEND(REVERSE(L),LT));

     B:=CAR(L);
     L:=CDR(L);

     IF A NEQ B THEN LT:=CONS(B,LT);
GO TO LOOP;
END;



SYMBOLIC PROCEDURE SQFREE(MK);
BEGIN
  integer i,i2;
  IF REMAINDER(MK,4) =0 THEN RETURN NIL;
  IF REMAINDER(MK,9) =0 THEN RETURN NIL;
  IF REMAINDER(MK,25) =0 THEN RETURN NIL;
  IF REMAINDER(MK,49) =0 THEN RETURN NIL;
  IF REMAINDER(MK,121) =0 THEN RETURN NIL;
  IF REMAINDER(MK,169) =0 THEN RETURN NIL;
  IF REMAINDER(MK,289) =0 THEN RETURN NIL;
  IF REMAINDER(MK,361) =0 THEN RETURN NIL;
  IF REMAINDER(MK,529) =0 THEN RETURN NIL;
  IF REMAINDER(MK,841) =0 THEN RETURN NIL;
  IF REMAINDER(MK,961) =0 THEN RETURN NIL;
  IF REMAINDER(MK,1369) =0 THEN RETURN NIL;
  IF REMAINDER(MK,1681) =0 THEN RETURN NIL;
  i:=47;
   loop:
    i2:=i*i;
    if i2>mk then return t;
    if gcd(mk,i2)=i2 then return nil;
    i:=i+2;
   goto loop;
 END;
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SYMBOLIC PROCEDURE CALB(DIS,PF,XX);
   BEGIN
      INTEGER B,DD,DD0,EE,EE2;
      SCALAR LL,PF;
      DD0:=REMAINDER(DIS,4);
      IF DD0=0 THEN RETURN NIL;
      IF DD0=1 THEN <<DD:=DIS;B:=1>> ELSE <<DD:=4*DIS;B:=2>>;

      EE:=ISQRT(DD);
      EE2:=EE/2;
      LL:=ABLST0(DD,EE,EE2,B);
      IF XX=T THEN WRITE("h=",EQUIV0(LL,EE,EE2),"       ");
      IF PF=T THEN RETURN <<WRITE(LL);TERPRI();LENGTH(LL)>>;
      RETURN <<write("caliber=");LENGTH(LL)>>;


END;

SYMBOLIC PROCEDURE CCALB(M,N,CLP);
BEGIN
   INTEGER KK;
   KK:=M;
   LOOP:
      IF KK>N THEN RETURN;
      IF NOT SQFREE(KK) THEN <<KK:=KK+1;GOTO LOOP>>;
      IF CLP=NIL THEN <<WRITE("d=",KK,"       ",CALB(KK,NIL,NIL));TERPRI();>>
                 ELSE <<WRITE("d=",KK,"       ",CALB(KK,NIL,T));TERPRI();>>;
      KK:=KK+1;
   GOTO LOOP;
END;




SYMBOLIC OPERATOR CALB;
SYMBOLIC OPERATOR CCALB;

CCALB(10,100,T);
CCALB(10,100,nil);
END;
```