Infrared secular effects on our local universe and the stochastic approach to inflation

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Abstract

In inflationary paradigm, quantum fluctuations of curvature perturbations generated during inflation give rise to the cosmological density perturbations which are currently measured. Therefore we will evaluate the quantum correlation functions of curvature perturbations during inflation and compare them with the observational results. However, it is known that if we naively evaluate the correlation functions of curvature perturbations in QFT, the result contains large infrared(IR) loop corrections/divergences, leading to the breakdown of perturbative treatment. This is known as the so-called "infrared divergence problems". Specifically, these IR effects can be induced from the loops of curvature perturbations, graviton, or light scalar fields other than inflaton. Recently a consistent quantum field theoretic prescription has been proposed which regularizes the large IR corrections from curvature perturbations and gravitons. However such method does not apply to the case where multiple light scalar fields which are called isocurvature modes exist during inflation. On the other hand, it has been argued that isocurvature's IR effects may not be observable once the stochastic approach to inflation is adopted. In the stochastic approach, IR fields whose wavelength is larger than the Hubble scale obey the Langevin equation: more precisely Markovian process. It is known that the leading order IR divergent pieces can be captured by the Langevin equation. Therefore stochatic treatment might allow us to understand the IR issues. An important assumption adopted in the stochastic approach is that IR fields are "classicalized" in the sense that they take some definite value at each Hubble patch. Once this classical treatment is accepted, the appearance of IR effects is simply interpreted as an artifact of taking an ensemble average of various realizations of IR fields over the *whole* universe, and such effects will be irrelevant for our *local* universe that we actually observe.

However, such classical treatment cannot be justified when the quantum coherence is non-vanishing. It has been also unclear how precisely the stochastic treatment can capture the IR divergent pieces. Therefore, we investigate the theoretical consistency of stochastic approach. We studied a test scalar field theory on de Sitter background with a sufficiently flat potential which will resemble isocurvature modes during inflation. We firstly establish the systematic way of deriving an effective equation of motion (EOM) for IR fields. We then show that an effective EOM for IR fields always takes the form of Langevin equation, and this EOM can correctly reproduce all the IR divergent pieces beyond leading order.

Next, we construct the method which allows us to calculate the degree of quantum decoherence of the time-evolution histories of IR field. We then show that the quantum decoherence occurs very efficiently but not exactly during inflation. We find that the stochastic approach to inflation is theoretically consistent only when considering the coarse-grained histories. This opens the possibility for IR fields to fluctuate in our local universe. We then estimate the amplitude of fluctuations of IR fields in the local universe with taking into account the effects of quantum decoherence, in $\lambda\phi^4$ theory as a toy model. We find that the amplitude of IR fluctuations in our local universe is suppressed

by λ. This indicates that when the field trajectory of a scalar does not shrink after currently observable scales exit the Hubble scale, the amplitude of IR fluctuations in our local universe will be suppressed.

Contents

Chapter 1

Introduction

Inflationary paradigm is one of the leading paradigm in modern cosmology [1, 2, 3, 4]. Although various specific inflationary models have been proposed, the most important outcome of inflation is universal: quantum fluctuations of fields generate the primordial cosmological perturbations. More concretely, long-wavelength modes of quantum fields well beyond the Hubble scale at the end of inflation, generate the observed primordial fluctuations, *e.g.*, in cosmic microwave background (CMB). We refer to the long-wavelength modes well beyond the Hubble scale during inflation as infrared (IR) modes here. In order to extract the information of high-energy physics beyond the standard model from observations of primordial fluctuations, it is very important to check the validity of the theoretical framework to calculate inflationary correlation functions of primordial perturbations. However, this theoretical framework is not fully justified yet because of the issues about "IR divergences" (*e.g.*, see [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25]). It is known that in the theory of a minimally coupled massless scalar field on de Sitter background, which mimics isocurvature perturbations during inflation, correlation functions are IR divergent once IR loop corrections are taken into account.[1] Even if the field has a small positive mass squared, IR loop corrections to correlation functions can be large, leading to the breakdown of perturbation theory. As a result, correlation functions will not be well approximated by tree level amplitudes. Nevertheless, correlation functions without IR loop corrections are usually adopted as observables, which can explain very well the observational results, *e.g.*, slightly red-tilted spectrum at CMB scales. The standard explanation for neglecting IR loop corrections would be that loop contributions from deep IR modes beyond the current observable scale should not affect the correlation functions of currently observed primordial fluctuations because local observer cannot distinguish the deep-IR modes from the homogeneous background. This is true in classical theory, but in quantum theory, one cannot easily justify neglecting the fluctuations of such degrees of freedom, in evaluating observables. In order to clarify if IR loop corrections can largely modify observables, one needs to reconsider which quantities

[1] Correlation functions in position space are IR divergent even at tree level. It should be also noted that loop effects are convergent at each order in perturbation theory for some models, while loop expansion is divergent, precisely speaking.

we actually observe. In the case of single field inflation, it is relatively easy to construct such observables (for example, in [8, 9, 10, 12, 13]): it is indicative that the squeezed bispectrum predicted by the consistency relation vanishes if one construct quantities that can be naturally observed by a local observer [26, 27]. However, the discussions based on the large gauge transformations used in the single field case do not apply to IR loop corrections in the presence of isocurvature perturbations during inflation, as is already pointed out in [7, 27]. Since the existence of such light degrees of freedom other than inflaton is expected in the context of string cosmology [28], it will be necessary to investigate the issues of IR divergences originating from such light fields as well.

It is known that the most dominant part of IR loop corrections can be described by the Brownian motion [29, 30, 31, 32]. At this level of approximation, it is theoretically consistent to treat the inflationary dynamics as if it were a classical stochastic process [33, 34, 35, 36, 37]. This treatment is called stochastic inflation formalism, which explains the appearance of the eternal inflation phase in the very early universe [38]. In this classical stochastic picture, the classicalization of IR fluctuations during inflation is assumed, and the secular growth of IR loop corrections can be regarded as an increase of classical statistical variance. Once this classical stochastic interpretation is accepted, the appearance of IR effects is simply interpreted as an artifact of taking an ensemble average of various realizations of IR fields over the *whole* universe, and such effects will be irrelevant for our *local* universe which we actually observe [17, 18, 19, 20]. This classical interpretation also applies to calculating correlation functions of adiabatic perturbations, and this formalism is called "stochastic-δN formalism" firstly proposed by [39, 40] and developed more in detail in [41, 42]. The important point is that the observables defined based on this formalism are no longer simple QFT expectation values which suffer from large IR loop corrections. Deep-IR modes beyond the observable scales do not contribute to the observables defined based on the classical stochastic picture [17, 18, 19, 20].

However, it has not been clarified yet whether or not all the IR secular growth can really be understood as an increase of classical statistical variance. If statistical interpretation were incorrect, the above stochastic approach to inflation will break down once the sub-leading order IR secular growth terms are taken into account, invalidating the argument of the absence of large IR effects on observables based on the stochastic picture. We will study this aspect in chapter 4.

It is also unclear whether or not such classical treatment is really theoretically consistent even if correlation functions are recovered by classical stochastic process. Regarding the classical aspects of the cosmological perturbations, quantum decoherence of the reduced density matrix for long-wavelength perturbations has also been extensively discussed in the literature [43, 44, 45, 46, 47, 48, 49]. Because the off-diagonal elements of the reduced density matrix for IR fields do not completely vanish during inflation, the classical stochastic picture will not be exactly correct. It is worthwhile investigating this aspect, too. We will study this aspect in chapter 5.

Because the behavior of isocurvature modes will be well captured by a test scalar field on de Sitter background, we investigate a canonically normalized light scalar field with a general sufficiently flat potential including derivative couplings on de Sitter background as a first step. We adopt the units with $c = \hbar = 1$. This thesis is organized as follows:

Chapter 2

We briefly review some basics of inflationary cosmology which will be relevant for the main issues discussed in this thesis.

Chapter 3

We review the IR issues during inflation mainly based on [11, 14]. We firstly introduce how we encounter IR divergences in perturbative evaluation of curvature perturbations in single field inflation. We then explain how the IR effects may be regularized by taking into account the "residual *gauge* d.o.f." which may be respected by local observer. Finally we explain briefly why their method does not apply to multi-field inflationary models where there are light scalars other than inflaton. We also introduce the idea called "stochastic approach to inflation" which may be the key to regularize IR effects in such models.

Chapter 4

Light scalars other than inflaton will be treated as test scalar fields on de Sitter background in a good approximation. We thus construct the method which can capture the non-perturbative aspects of IR effects from a light scalar field on de Sitter background. In sec. 4.1, we develop a systematic derivation of IR dynamics by using the Schwinger-Keldysh formalism. In sec. 4.2, we check how our extend stochastic formalism can recover IR correlation functions beyond leading order. In sec. 4.3, we investigate the general properties of effective EOMs. This chapter is based on the author's previous works:
J. Tokuda and T. Tanaka, "Statistical nature of infrared dynamics on de Sitter background" JCAP **1802** (2018) 014 [50].
J. Tokuda and T. Tanaka, "Can all the infrared secular growth really be understood as increase of classical statistical variance?" JCAP **1811** (2018) 022 [51].

Chapter 5

In this chapter, we explore the role of quantum decoherence for the theoretical consistency of stochastic approach. We firstly analyze the structure of quantum states, by considering the $\lambda\phi^4$ theory in de Sitter space in a flat chart. We then prove that stochastic picture can be theoretically consistent only approximately, and must be violated. We evaluate the degree of this violation by estimating the quantum coherence of IR fields during inflation. We then evaluate the amplitude of the fluctuations of IR fields when adopting the approximate stochastic picture, which will allow us to estimate the possible contributions of IR fluctuations to correlation functions of shorter-wavelength modes. Several implications and future works are listed up at the end of this chapter. This chapter is based on our unpublished work. We also discuss the future directions.

Chapter 6

This chapter is dedicated to conclusion.

Chapter 2

Basics of inflationary cosmology

It is known that our observable universe are well described by the homogeneous and isotropic universe at the background level. On the other hand, we also know that our universe is not exactly homogeneous nor isotropic. In reality, our universe contains rich cosmological structures such as galaxies, stars, and so on. These cosmological structures are formed by nonlinear evolutions at a late time, and are originated in the tiny density perturbations. In the inflationary paradigm, the cosmological density perturbations are generated by the quantum fluctuations of matter fields, whose energy-momentum realizes the inflationary universe. We will firstly review the background dynamics. We will then briefly explain how we can treat the cosmological perturbations and how they are generated quantum mechanically by considering the single-field inflation model which is the simplest model of inflation.

2.1 Background Dynamics

In this section we briefly review some basic equations governing the time evolution of our universe at the background level.

2.1.1 FLRW metric

It is widely known that the our observable universe are homogeneous and isotropic at background level. Such spacetime are described by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric of the form[1]

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j \,. \tag{2.1}$$

$a(t)$ is called a scale factor, and it is a function of cosmic time t. For an expanding universe, we have a positive expansion rate $H \coloneqq \dot{a}/a$ which is called a Hubble parameter. Here, dot denotes the derivative with respect to t: for instance, $\dot{a} \coloneqq \partial_t a$. Because of the

[1] Precisely speaking one can add the spatial curvature, while current cosmological observation suggests that spatial curvature is almost zero. Therefore we neglect the spatial curvature from the beginning here.

symmetry of the spacetime, the energy momentum tensor which is compatible with the Einstein equations takes the following form:

$$T^{\mu}{}_{\nu} = \rho u^{\mu} u_{\nu} + p \left(\delta^{\mu}{}_{\nu} + u^{\mu} u_{\nu}\right) , \tag{2.2}$$

where u^{μ} denotes a time-like unit vector $u^{\mu} = (1/a, 0, 0, 0)$ which is normalized as $u^{\mu} u_{\mu} = -1$. This is the energy momentum tensor of perfect fluid with an energy density ρ and a pressure p. ρ and p are function of time, but they are independent of spatial coordinate $\vec{x}$, as a consequence of the homogeneity of the background FLRW spacetime. Einstein equations are given by

$$H^2 = \frac{1}{3M_{\rm pl}^2}\rho\,, \tag{2.3a}$$

$$\dot{H} = -\frac{1}{2M_{\rm pl}^2}\left(\rho + p\right) . \tag{2.3b}$$

Eq. (2.3a) is called Friedman equation. The above equations are equivalent to

$$\frac{\ddot{a}}{a} = H^2 + \dot{H} = -\frac{1}{6M_{\rm pl}^2}(\rho + 3p)\,. \tag{2.4a}$$

$$\dot{\rho} + 3H(\rho + p) = 0\,. \tag{2.4b}$$

Note that eq. (2.4b) can be derived from the conservation law of energy-momentum tensor:

$$0 = u^{\nu}\nabla_{\mu} T^{\mu}{}_{\nu} = -u^{\mu}\nabla_{\mu}\rho - (\rho + p)\nabla_{\mu} u^{\mu} = -\dot{\rho} - 3H(\rho + p)\,, \tag{2.5}$$

where ∇_{μ} denotes a covariant derivative and we used $\nabla_{\mu} u^{\mu} = \partial_{\mu}\big(\sqrt{-g}\,u^{\mu}\big)/\sqrt{-g}$.

2.1.2 Inflationary universe

In modern cosmology, it is believed that our universe experienced an accelerating expansion era where the scale factor $a(t)$ grows in time almost exponentially as $a(t) \propto \exp[Ht]$. This era is called inflation. Nearly exponential growth of $a(t)$ implies the approximate constancy of the Hubble parameter H. Approximately constant H can be realized if the equation of state becomes $p \approx -\rho$, according to eqs. (2.3). From now on we will briefly review how this equation of state can be realized by introducing a single matter content, base on [52].

Single-field slow-roll inflation

As a matter content which realizes the inflationary universe, we consider the simplest model whose action is given by

$$S_{\rm matter} = \int \mathrm{d}^4 x\,\sqrt{-g}\left[-\frac{1}{2}(\partial\phi)^2 - V(\phi)\right] . \tag{2.6}$$

EOM for ϕ is given by

$$\ddot{\phi} + 3H\dot{\phi} = -V'(\phi)\,. \tag{2.7}$$

Energy density and pressure of ϕ are given by

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi)\,, \quad p = \frac{1}{2}\dot{\phi}^2 - V(\phi)\,. \tag{2.8}$$

In order to realize the inflationary universe, the following conditions must be satisfied:

$$\epsilon_{\rm H} \coloneqq -\frac{\dot{H}}{H^2} = -\frac{3}{2}\frac{\rho + p}{\rho} = \frac{\dot{\phi}^2}{2M_{\rm pl}^2 H^2} \ll 1\,, \tag{2.9}$$

where we used eqs. (2.3b) and (2.8). For this condition being satisfied for sufficiently long period of time, we also require the following condition:

$$\frac{|\dot{\epsilon}_{\rm H}|}{H} = 2\left|\epsilon_{\rm H}\left(\epsilon_{\rm H} - \eta_{\rm H}\right)\right| \ll 1\,. \tag{2.10}$$

This condition is realized when the following condition is also satisfied:

$$\eta_{\rm H} \coloneqq -\frac{\ddot{\phi}}{H\dot{\phi}} \ll 1\,. \tag{2.11}$$

Eqs. (2.9) and (2.11) are called slow-roll conditions. $\epsilon_{\rm H}$ and $\eta_{\rm H}$ are called Hubble slow-roll parameters. Using eq. (2.11), EOM for ϕ which is given by eq. (2.7) reduces to

$$\dot{\phi} \simeq -\frac{1}{3H}V'(\phi)\,. \tag{2.12}$$

This velocity can be understood as the terminal velocity of the system whose EOM is given by eq. (2.7). Therefore, unless $V'(\phi) = H = 0$, the approximation (2.12), or equivalently eq. (2.11) will become valid even if they are not valid at an initial time. We thus assume eq. (2.11) here, although $\eta_{\rm H}$ can be of the order of unity for ensuring eq. (2.10), rigorously speaking.[2] When the slow-roll conditions are satisfied, the temporal variation of slow-roll parameters is also slow-roll suppressed:

$$\frac{{\rm d}^n\epsilon_{\rm H}}{{\rm d}N^n}\,, \frac{{\rm d}^n\eta_{\rm H}}{{\rm d}N^n} \sim \mathcal{O}\left((\epsilon_{\rm H}\,, \eta_{\rm H})^n\right)\,. \tag{2.13}$$

Here, we defined $N \coloneqq \ln a(t)$. Because the condition eq. (2.9) is equivalent to the condition $\dot{\phi}^2 \ll V(\phi)$ which means that the kinetic energy of inflaton is negligibly small compared to the potential energy, slow-roll conditions will be satisfied when the potential $V(\phi)$ is sufficiently flat. Indeed, we have

$$\epsilon_{\rm V} \coloneqq \frac{M_{\rm pl}^2}{2}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 = \epsilon_{\rm H} + \left(\frac{6 - \epsilon_{\rm H} - \eta_{\rm H}}{2(3 - \epsilon_{\rm H})^2}\right)\frac{\dot{\epsilon}_{\rm H}}{H} \simeq \epsilon_{\rm H} \ll 1\,. \tag{2.14}$$

[2] For example in ultra slow-roll inflation, $\eta_{\rm H} \simeq 3/2$. This situation can be realized by considering the potential whose slop changes suddenly enough so that the velocity of scalar field exceeds the end velocity $-V'(\phi)/3H$.

The validity of eq. (2.14) is ensured for a sufficiently long time when

$$\eta_{\rm V} \coloneqq M_{\rm pl}^2 \frac{V''(\phi)}{V(\phi)} \ll 1 \tag{2.15}$$

is satisfied, because $|\dot{\epsilon}_{\rm V}|H^{-1} = 2\sqrt{\epsilon_{\rm H}\epsilon_{\rm V}}(2\epsilon_{\rm V} - \eta_{\rm V})$. This equation can be derived from eq. (2.14) by using the approxiamte constancy of $\epsilon_{\rm H}$ which is expressed by eq. (2.13):

$$\eta_{\rm V} = \eta_{\rm H} + \epsilon_{\rm H} - \frac{1}{2\epsilon_{\rm H}(3-\epsilon_{\rm H})}\left(\frac{{\rm d}^2\epsilon_{\rm H}}{{\rm d}N^2} + (\eta_{\rm H} - 5\epsilon_{\rm H})\frac{{\rm d}\epsilon_{\rm H}}{{\rm d}N}\right) \simeq \eta_{\rm H} + \epsilon_{\rm H} \ll 1\,. \tag{2.16}$$

Smallness of $\epsilon_{\rm V}$ and $\eta_{\rm V}$ implies that it is necessary to make the sufficiently flat potential $V(\phi)$ for realizing the slow-roll inflation. In this thesis we will not discuss how we can obtain such flat potential which is stable under quantum corrections. We will simply assume that such flat potential is realized.

2.2 Perturbations

In this section we describe the very basics of cosmological perturbation theory, especially focusing on how we can define the perturbation and how we can describe their quantum mechanical generation during inflation in the single-field inflation scenario. The content of this section is mainly based on [52, 53].

2.2.1 Gauge invariant variables

Let us consider the perturbations. Perturbed metric is given by

$$ds^2 = -(1+2A)dt^2 + h_{ij}\left(-B^i dt + dx^i\right)\left(-B^j dt + dx^j\right)\,. \tag{2.17}$$

In this expression A, $h_{ij} - a^2\delta_{ij}$, and B_i are perturbed quantities, and let us parameterize h_{ij} as

$$h_{ij} = (1+2D)a^2\delta_{ij} + E_{|ij} + \gamma_{ij}^{\rm TT}\,, \tag{2.18}$$

where $\gamma_{ij}^{\rm TT}$ satisfies traceless and transverse conditions $\gamma_{ii}^{\rm TT} = \partial_i\gamma_{ij}^{\rm TT} = 0$. Here, $E_{|ij} \coloneqq \nabla_i\nabla_j E$ with ∇_i which denotes a spatial covariant derivative with respect to an induced metric h_{ij} on the constant time slicing. Then A, χ, D and $E_{|ij}$ express scalar perturbations, and γ_{ij}^{TT} denotes tensor perturbations. Throughout this thesis we will neglect vector perturbations from the beginning for simplicity, because it decays at large scales in standard inflationary scenario which we will study here. Next we parameterize the perturbations of energy-momentum tensor $T^\mu{}_\nu$ as

$$T^\mu{}_\nu = \rho u^\mu u_\nu + p\left(\delta^\mu{}_\nu + u^\mu u_\nu\right) + \sigma^\mu{}_\nu\,, \tag{2.19}$$

with imposing

$$u^\mu = \frac{1}{a}(1-A, \partial^i v)\,, \quad u^\mu u_\mu = -1\,, \quad \sigma^\mu{}_\nu u^\nu = \sigma^\mu{}_\mu = 0\,, \quad \sigma^{\mu\nu} = \sigma^{\nu\mu}\,. \tag{2.20}$$

Then $u_\mu = a\big(-1-A, \partial_i(v-B)\big)$. Then the components of energy-momentum tensor are given by

$$T^0{}_0 = -\bar{\rho}(1+\delta)\,, \quad T^0{}_i = (\bar{\rho}+\bar{p})\partial_i(v-B)\,, \quad T^i{}_0 = -(\bar{\rho}+\bar{p})\partial^i v\,, \tag{2.21a}$$
$$T^i{}_j = \bar{p}\delta^i_j + \delta p \delta^i_j + \sigma^i{}_j\,, \tag{2.21b}$$

where δ denotes the density contrast $\delta \coloneqq \delta\rho/\bar{\rho}$. $\delta\rho \coloneqq \rho - \bar{\rho}$ and $\delta p \coloneqq p - \bar{p}$ are the perturbations of energy density and pressure, respectively. Quantities with a bar are background quantities which depends only on time.

Using perturbation variables we have introduced above, we can construct the following important gauge invariant quantities $\mathcal{R}$ and ζ:

$$\zeta \coloneqq D + aH(v-B)\,, \quad \mathcal{R} \coloneqq D - \frac{H\bar{\rho}}{\dot{\bar{\rho}}}\delta\,. \tag{2.22}$$

In the comoving gauge which is defined by $v-B=0$ and $E=0$, ζ becomes D, which is related to the perturbations of the three dimensional Ricci scalar $R^{(3)}$: for example in this slicing they are related as[3]

$$R^{(3)} = -\frac{4\nabla^2\zeta}{a^2}\,, \tag{2.23}$$

up to linear order. Therefore ζ is called comoving curvature perturbation. In the uniform density gauge which is defined by $\delta\rho = 0$ and $E=0$, $\mathcal{R}$ coincides with D and hence it is the curvature perturbation. Therefore $\mathcal{R}$ is called the uniform density curvature perturbation.

Gauge fixing

Below we will adopt the unit $M_{\rm pl} = 1$. Let us rewrite the metric perturbation as

$$ds^2 = -N^2 dt^2 + h_{ij}\left(N^i dt + dx^i\right)\left(N^j dt + dx^j\right)\,, \tag{2.24}$$

and consider the simplest single-field inflationary scenario theory described by the following action

$$\begin{aligned} S &= \frac{1}{2}\int \mathrm{d}^4x\,\sqrt{-g}\left[R - (\partial\phi)^2 - 2V(\phi)\right] \\ &= \frac{1}{2}\int \mathrm{d}^4x\,\sqrt{h}\Big[NR^{(3)} - 2NV(\phi) + N\left(K^i{}_jK^j{}_i - (K^i{}_i)^2\right) \\ &\qquad + N^{-1}\left(\dot{\phi} - N^i\partial_i\phi\right)^2 - Nh^{ij}(\partial_i\phi)(\partial_j\phi)\Big]\,, \end{aligned} \tag{2.25}$$

where K_{ij} and $R^{(3)}$ denote an extrinsic curvature and a scalar curvature of constant-t slice, respectively. Explicit form of K_{ij} is given by

$$K_{ij} = \frac{1}{2N}\left(\dot{h}_{ij} - N_{i|j} - N_{j|i}\right)\,. \tag{2.26}$$

[3] Note that $R^{(3)} = 0$ at zeroth order because we consider the flat FLRW spacetime as a background spacetime here.

Because the action does not contain the time derivative of the lapse function N or the shift function N^i, they play the role of Lagrange multipliers rather than dynamical variables. Therefore one obtains the set of constraint equations by taking the variation of the action with respect to N or N_i. This is the generic feature of general relativity, and the corresponding constraints are first class, which reflects the coordinate transformation invariance. Therefore after choosing specific gauge, N and N^i can be determined in terms of dynamical variables.

Here let us work in the comoving gauge. Writing the inflaton fluctuation φ as

$$\phi(x) = \bar{\phi}(t) + \varphi(x)\,, \tag{2.27}$$

the $(0,i)$ components of energy momentum tensor is given by

$$T^0{}_i = -a\dot{\bar{\phi}}\partial_i\varphi\,. \tag{2.28}$$

Therefore the comoving gauge in this setup is specified by $\varphi = 0$ and $E = 0$. The former condition determined the time-slicing and the latter fixes the spatial coordinate. In this gauge we can parametrize the metric perturbations as

$$h_{ij} = a^2 e^{2\zeta} \left(e^{\gamma}\right)_{ij}\,, \tag{2.29}$$

where ζ is comoving curvature perturbation. Now let us derive the expression for the lapse and shift in terms of these dynamical perturbation variables by solving constraint equations. The set of constraint equations are given by

$$R^{(3)} - 2V(\bar{\phi}) - \dot{\bar{\phi}}^2 - K^i{}_j K^j{}_i + (K^i{}_i)^2 = 0\,, \tag{2.30a}$$

$$\nabla_i\left(K^i{}_j - \delta^i{}_j K^k{}_k\right) = 0\,. \tag{2.30b}$$

Eqs. (2.30a) and (2.30b) are called Hamiltonian constraint and momentum constraints, respectively. Let us expand the lapse and shift perturbatively as

$$N = 1 + N_1 + N_2 + \cdots\,, \tag{2.31a}$$

$$N_i = \partial_i\chi\,, \quad \chi = \chi_1 + \chi_2 + \cdots\,, \tag{2.31b}$$

and solve eqs. (2.30) in terms of lapse and shift iteratively. At zeroth order eq. (2.30a) gives rise to the Friedman equation,

$$3H^2 = \frac{1}{2}\dot{\bar{\phi}}^2 + V(\bar{\phi})\,. \tag{2.32}$$

Note that at zeroth order we do not have the momentum constraints. At linear order, eqs. (2.30) give rise to

$$(\partial_i N_1) = \frac{\partial_i\dot{\zeta}}{H}\,, \tag{2.33a}$$

$$H\nabla^2\chi_1 = 3H\dot{\zeta} - \frac{\nabla^2\zeta}{a^2} - \frac{N_1 V(\bar{\phi})}{H}\,. \tag{2.33b}$$

One may notice that the solutions of eqs. (2.33) have ambiguities because these equations are of elliptic type. For example,

$$N_1 = \frac{\dot{\zeta}}{H} + f(t) \tag{2.34}$$

is indeed the solution of eq. (2.33a) with an arbitrary function $f(t)$. This ambiguity is eliminated by imposing the boundary conditions at spatial infinity so that N_1 and χ_1 vanish as other dynamical perturbation variables do: with this boundary condition one has $f(t) = 0$. Substituting the solution of N_1 into eq. (2.33b) one has

$$\chi_1 = -\frac{\zeta}{a^2 H} + \epsilon_{\mathrm{H}} \nabla^{-2}\zeta \,, \tag{2.35}$$

with the same boundary condition at spatial infinity. Here we used

$$3 - \frac{V(\bar{\phi})}{H^2} = \frac{\dot{\bar{\phi}}^2}{2H^2} = \epsilon_{\mathrm{H}} \,.$$

We thus have the solutions for the lapse and shift at linear order as

$$N = 1 + \frac{\dot{\zeta}}{H} \,, \quad N_i = \frac{\partial_i \zeta}{a^2 H} + \epsilon_{\mathrm{H}} \partial_i \nabla^{-2}\zeta \,. \tag{2.36}$$

Substituting eqs. (2.36) into the action (2.25) and expand 3-dimensional geometrical quantities in terms of ζ and γ_{ij}, one can obtain the action for perturbations up to third order. Below we will discuss the quantization of these perturbation variables concentrating on the quadratic action, and we will not discuss the non-linearities any more until chapter 3.

2.2.2 Canonical quantization

From now on we will quantize perturbation variables. Here we concentrate on quantizing the curvature perturbation ζ and forget about the tensor perturbation γ_{ij} for a while. Quadratic action for ζ is given by

$$S_2 = \int \mathrm{d}^4 x \, a^3 \epsilon_{\mathrm{H}} \left[\dot{\zeta}^2 - \frac{(\partial_i \zeta)^2}{a^2} \right] . \tag{2.37}$$

Introducing the gauge invariant variable $v \coloneqq z\zeta$ called Sasaki-Mukhanov variable, where $z \coloneqq \sqrt{2a^2\epsilon_{\mathrm{H}}}$, the above quadratic action becomes the canonically normalized form:

$$S_2 = \frac{1}{2} \int \mathrm{d}\eta \, \mathrm{d}^3 x \, \left[(v')^2 - (\partial_i v)^2 + \frac{z''}{z} v^2 \right] . \tag{2.38}$$

Here, we introduced a conformal time η which is defined by

$$\eta(t_1) \coloneqq \int_{\infty}^{t_1} \frac{\mathrm{d}t}{a(t)} \,, \tag{2.39}$$

and prime denotes the derivative with respect to η: for instance, $v' = \partial_\eta v$. In de Sitter approximation, we have $\eta = -1/(a(t)H)$. One can canonically quantize v by expanding an interaction picture field $\hat{v}_{\rm I}$ in terms of creation and annihilation operators which are referred to as $\hat{a}^\dagger_{\vec{k}}$ and $\hat{a}_{\vec{k}}$, respectively:

$$\hat{v}_{\rm I}(\eta,\vec{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\left[v_k(\eta)\hat{a}_{\vec{k}}e^{i\vec{k}\cdot\vec{x}} + v_k^*(\eta)\hat{a}^\dagger_{\vec{k}}e^{-i\vec{k}\cdot\vec{x}}\right]. \tag{2.40}$$

Here, $v_k(\eta)$ denotes a positive-frequency solution of the EOM of the form

$$v_k'' + \left(k^2 - \frac{z''}{z}\right)v_k = 0\,, \tag{2.41}$$

and we impose the normalization condition:

$$\left(v_k v_k^{*\prime} - v_k' v_k^*\right) = i\,. \tag{2.42}$$

Then the creation and annihilation operators $\hat{a}_{\vec{k}}$ and $\hat{a}^\dagger_{\vec{k}}$ satisfy the commutation relations

$$\left[\,\hat{a}_{\vec{k}}\,,\,\hat{a}^\dagger_{\vec{k}'}\right] = (2\pi)^3\delta^{(3)}(\vec{k}-\vec{k}')\,,\qquad \left[\,\hat{a}_{\vec{k}}\,,\,\hat{a}_{\vec{k}'}\,\right] = 0\,,\qquad \left[\,\hat{a}^\dagger_{\vec{k}}\,,\,\hat{a}^\dagger_{\vec{k}'}\right] = 0\,. \tag{2.43}$$

At leading order in slow-roll parameters, $z''/z \simeq a''/a = 2/\eta^2$. Using this de-Sitter approximation, one has the exact solution for eq. (2.41):

$$v_k(\eta) = \frac{1}{-k\eta}\frac{1}{\sqrt{2k}}\left(1 + ik\eta\right)e^{-ik\eta}\,. \tag{2.44}$$

Vacuum state $|0\rangle$ which is define by

$$\hat{a}_{\vec{k}}\,|0\rangle = 0 \tag{2.45}$$

for $\forall\vec{k}$ is called the adiabatic vacuum state, the WKB vacuum state, or the non-interacting Bunch-Davies vacuum state. This vacuum state is natural because the time-dependence of the final term on the right-hand side of eq. (2.41), which makes the definition of vacuum ambiguous, is negligibly small in the past infinity limit $\eta \to -\infty$. In that limit we have the well-defined notion of positive or negative-frequency modes. The mode function (2.44) asymptotes to the mode function in Minkowski spacetime:

$$v_k(\eta) \to \frac{-i}{\sqrt{2k}}e^{-ik\eta} \quad \text{as} \quad -k\eta \to -\infty\,. \tag{2.46}$$

Usually this vacuum state is adopted as the initial quantum state for cosmological quantum fluctuations. Below let us simply write this adiabatic vacuum state as $|0\rangle$.

Since the comoving curvature perturbation ζ is related to Sasaki-Mukhanov variable v as $\zeta = v/z$, ζ can be also expanded as

$$\hat{\zeta}_{\rm I}(\eta,\vec{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\left[\zeta_k(\eta)\hat{a}_{\vec{k}}\,e^{i\vec{k}\cdot\vec{x}} + \zeta_k^*(\eta)\hat{a}^\dagger_{\vec{k}}\,e^{-i\vec{k}\cdot\vec{x}}\right], \tag{2.47}$$

with

$$\zeta_k(\eta) = \frac{1}{M_{\rm pl}\sqrt{2a^2\epsilon_{\rm H}}} v_k(\eta) \simeq \frac{1}{M_{\rm pl}\sqrt{2\epsilon_{\rm H}}} \frac{1}{-a\eta} \frac{1}{\sqrt{2k^3}} (1+ik\eta)\, e^{-ik\eta}\,, \tag{2.48}$$

where we make the $M_{\rm pl}$ factor explicit for definiteness. Because $\zeta_k(\eta)$ becomes almost constant for $-k\eta \ll 1$, one has

$$\zeta_k(\eta) \simeq \left.\frac{H}{\sqrt{2\epsilon_{\rm H}}}\right|_{\eta_*(k)} \frac{1}{M_{\rm pl}\sqrt{2k^3}} \quad \text{for} \quad -k\eta \ll 1\,, \tag{2.49}$$

where $\eta_*(k)$ denotes the horizon-crossing time for mode k which is defined by $-k\eta_*(k) = 1$. Using this expression, one can calculate the power spectrum of ζ at tree level:

$$\langle 0 |\, \hat{\zeta}_{\vec{k}}(\eta) \hat{\zeta}_{\vec{k}'}(\eta) \,| 0 \rangle_{\rm tree} = (2\pi)^3 \delta\left(\vec{k} + \vec{k}'\right) P_\zeta(k)\,, \tag{2.50}$$

$$P_\zeta(k) = |\zeta_k(\eta)|^2 \simeq \left.\frac{H^2}{2\epsilon_{\rm H} M_{\rm pl}^2}\right|_{\eta_*(k)} \frac{1}{2k^3} \quad \text{for} \quad -k\eta \ll 1\,. \tag{2.51}$$

Here we defined $\hat{\zeta}_{\vec{k}}(\eta)$ by

$$\hat{\zeta}_{\vec{k}}(\eta) \coloneqq \int {\rm d}^3x\, \hat{\zeta}(\eta, \vec{x}) e^{-i\vec{k}\cdot\vec{x}}\,. \tag{2.52}$$

This explains how the simplest single-field inflationary model predicts the nearly scale-invariant spectrum which is confirmed by CMB observation. Deviation from the scale invariant spectrum comes from the k-dependence of the pre-factor $H^2/2\epsilon_{\rm H}$ evaluated at $\eta_*(k)$. Combined with the approximate constancy of slow-roll parameters (2.13), the deviation from the scale invariant spectrum are given by the slow-roll parameters which are much smaller than unity.

Chapter 3

Infrared secular effects in single/multi-field inflation

Here we will review the issues of large IR corrections to the correlation functions of curvature perturbations. Basically we will concentrate on the scalar perturbations. In single inflation model, consistent prescription for resolving IR issues has been already proposed in [5, 6, 7, 10, 11, 13, 14]. We will review this prescription. On the other hand, the consistent QFT prescription for regularizing large IR corrections has not yet been given in case of multi-field inflation where multiple light scalar fields exist. However usually the stochastic approach to inflation are assumed implicitly in the literature as a possible resolution to IR issues in multi-field inflation. We will also briefly explain this. While the IR issue is not yet completely resolved, there seems the universal spirit for resolving IR issues. The point is that the quantities which suffer from large IR loop effects may not be observable quantities: such IR effects may not appear after properly defining observable quantities.

3.1 Infrared issues in inflation

Let us consider the single-field inflationary model. In comoving gauge, the action for the comoving curvature perturbation ζ can be given in the form

$$S = \int \mathrm{d}t \int \mathrm{d}^3x \, \epsilon_{\mathrm{H}} N \sqrt{h} \left[- \left(\partial\zeta\right)^2 + \cdots \right] . \tag{3.1}$$

The non-linear interaction of ζ can be determined by solving the constrain equations iteratively in ζ as is explained in the previous chapter 2. Let us work on the interaction picture and evaluate the loop corrections to the power spectrum. Theory contains the interaction vertex of the form

$$\mathcal{H}_{\mathrm{I}} \ni \zeta_{\mathrm{I}}^2 \left(\partial\zeta_{\mathrm{I}}\right)^2 . \tag{3.2}$$

This vertex stems from $\sqrt{h} = a^3 e^{3\zeta} \simeq a^3(1+3\zeta+9\zeta^2/2+\cdots)$. At leading order, this leads to the IR divergences when evaluating the correlation function of ζ perturbatively. For example, this vertex induces infrared-divergent loop corrections to the $\langle 0|\, \zeta_{\vec{k}}(t)\zeta_{\vec{k}'}(t)\, |0\rangle$ of the form

$$\int \mathrm{d}^3 p\, \Theta\left(a(t')H - p\right)\Theta\left(p - k_0\right)\langle \zeta_{\vec{p}}(t')\zeta_{\vec{p}'}(t')\rangle' \simeq 2\pi H^2 \ln\left(\frac{a(t')H}{k_0}\right) \to \infty \quad \text{as} \quad k_0 \to 0\,. \tag{3.3}$$

Here, $\langle \cdots \rangle'$ denotes the correlation functions with neglecting the delta function which expresses the momentum conservation.[1] As we have explained in the previous chapter, if there exist additional light scalar fields other than inflaton, their loop corrections to the correlation function of ζ also provide large infrared contributions. This infrared issue is sometimes called "infrared divergence problem" in the literature because their loop corrections are divergent in general after eliminating an artificial IR cutoff k_0. This issue can be a serious problem for inflationary paradigm because this issue arises when evaluating correlation functions of curvature perturbations which are observable quantities.

As a one possible resolution to this problem, it has been suggested in the literature that simple correlation functions of ζ may not correspond to observed primordial fluctuations. In the next section, we will explain the possible resolution of IR issues.

3.2 Single field inflation case

We will firstly explain that correlation functions of ζ in fact depend on the information outside the past light cone (more precisely, the boundary conditions at spatial infinity which are necessary to completely fix the gauge condition), based on [6, 11]. Then we will explain how to construct the operators whose correlation functions are independent of the boundary conditions at spatial infinity which is outside the past light cone.

3.2.1 Causality and residual *gauge* degrees of freedom

Complete gauge fixing v.s. non-locality

As we have explained in the previous chapter, it is generically possible to express the lapse and shift in terms of the dynamical variable ζ only, by fixing the gauge completely. However after completely fixing the gauge, the lapse and shift are generally become non-local functionals: we have already encountered this non-locality in the shift N^i in eq. (2.36). Note that beyond linear order this non-locality also appears in the lapse function. This appearance of non-locality will lead to the pathological contributions to the correlation functions of ζ. In order to see this explicitly, let us consider again the loop corrections to the power spectrum of $\hat{\zeta}_{\vec{k}}$ from the interaction vertex of the form

$$\mathcal{H}_{\rm I} \ni H^{-2}\dot{\zeta}_{\rm I}\partial^i\zeta_{\rm I}\nabla^{-2}\left(\dot{\zeta}_{\rm I}\partial_i\zeta_{\rm I}\right)\,. \tag{3.5}$$

[1] For example,

$$\langle 0|\, \zeta_{\vec{k}}(t)\zeta_{\vec{k}'}(t')\, |0\rangle = (2\pi)^3\delta\left(\vec{k}+\vec{k}'\right)\langle \zeta_{\vec{k}}(t)\zeta_{\vec{k}'}(t')\rangle'\,. \tag{3.4}$$

For example two-loop contribution to $\langle 0| \zeta_{\vec{k}}(T)\zeta_{\vec{k}'}(T) |0\rangle$ contains the following piece:

$$\prod_{i=1}^{3}\int \mathrm{d}^3k_i\, \delta\left(\vec{k}_1+\vec{k}_2+\vec{k}_3-\vec{k}\right)\frac{(\vec{k}_1\cdot\vec{k}_2)^2}{\left|\vec{k}-\vec{k}_1\right|^4}\left\langle \zeta_{\vec{k}_1}(t)\zeta_{\vec{k}_1}(t')\right\rangle'\left\langle \zeta_{\vec{k}_2}(t)\zeta_{\vec{k}_2}(t')\right\rangle'\partial_t\partial_{t'}\left\langle \zeta_{\vec{k}_3}(t)\zeta_{\vec{k}_3}(t')\right\rangle'. \tag{3.6}$$

This factor is divergent at $\vec{k}=\vec{k}_1$, unless some non-trivial cancellation mechanism works.

From the causality perspective, it will be unnecessary to impose the regularity of the lapse and shift at spatial infinity which are outside the causal past of us. Observables should be determined only by the information of the causal past. Therefore correlation functions of observable operators should be independent of the boundary conditions at spatial infinity which are outside the causal past. The singular behavior in the infrared can be cured once one considers such correlation functions.

Residual *gauge* degrees of freedom

Because the lapse and shift are the Lagrange multipliers of the generators of coordinate transformations, this ambiguity associated with the boundary conditions lead to the residual *gauge* degrees of freedom. Note that we use an italic font as *gauge* for definiteness, because this coordinate transformation is different from the conventional gauge transformation in discussing cosmological perturbations in the literature: rather residual *gauge* transformations are large gauge transformations which are non-vanishing at spatial boundaries. Since the time slicing has been already fixed in the uniform-field gauge $\varphi=0$, residual *gauge* transformations are spatial coordinate transformations.

Once focusing on the quantities which are invariant under the residual *gauge* transformations, we should consider what happens after fixing this residual *gauge*. Firstly we would like to note that how important the residual *gauge* d.o.f. is for making the causality manifest. Here, causality means that observable correlation functions are independent of the insertion of unitary operators outside the causal past. It will be instructive to note that solutions of the set of first order constraint equations without imposing boundary conditions are given by

$$N_1=\frac{\dot{\zeta}}{H}+f(t)\,,\quad \chi_1=\frac{\zeta}{a^2H}+\nabla^{-2}\left(\epsilon_{\mathrm{H}}\zeta-\frac{f(t)V(\bar{\phi})}{H^2}\right)-G(x)\,, \tag{3.7}$$

where $G(x)$ satisfies the Laplace equation:

$$\nabla^2G(x)=0\,. \tag{3.8}$$

Let us refer to the intersection between the causal past of us and the constant-t slice as $\Sigma_{\mathcal{O}}(t)$. Then we may define the window function $W_t(\vec{x};\mathcal{O})$ which is unity when $\vec{x}\in\Sigma_{\mathcal{O}(t)}$ otherwise vanishes. Then by choosing $G(x)$ as

$$G(x)=\frac{-1}{4\pi}\int \mathrm{d}^3y\,\frac{1-W_t(\vec{y};\mathcal{O})}{|\vec{x}-\vec{y}|}\,\left(\epsilon_{\mathrm{H}}\zeta(t,\vec{y})-\frac{f(t)V(\bar{\phi})}{H^2}\right) \tag{3.9}$$

and substituting this into eq. (3.7), one obtains

$$\chi_1(x) = \frac{\zeta(x)}{a^2 H} - \frac{1}{4\pi}\int \mathrm{d}^3y\, \frac{W_t(\vec{y};\mathcal{O})}{|\vec{x}-\vec{y}|}\left(\epsilon_{\rm H}\zeta(t,\vec{y}) - \frac{f(t)V(\bar{\phi})}{H^2}\right). \tag{3.10}$$

Note that $G(x)$ given in eq. (3.9) indeed satisfies eq. (3.8). This implies that residual-*gauge* invariance will ensure the causality in evaluating observables, in the above-mentioned sense. Next we fix $f(t)$. In order to do so, it is convenient to compare the expressions of N_1 in two different coordinates (t, x^i) and $(t, \tilde{x}^i)$, where $f(t) = 0$ only in the former coordinate choice:

$$N_1(t,x^i) = \frac{\dot{\zeta}(t,x^i)}{H}, \quad \tilde{N}_1(t,x^i) = \frac{\dot{\tilde{\zeta}}(t,x^i)}{H} + f(t). \tag{3.11}$$

The point is that at leading order the lapse function is invariant under the spatial coordinate transformation $(t, x^i) \to (t, \tilde{x}^i)$, implying that $N_1(t, x^i) = \tilde{N}(t, x^i) + \mathcal{O}((\delta x)^2)$ where $\delta x^i := \tilde{x}^i - x^i$. On the other hand, ζ changes under this coordinate transformation as

$$\tilde{\zeta}(t,x^i) = \zeta(t,x^i) - \frac{\partial_i \delta x^i}{3}. \tag{3.12}$$

Therefore we have

$$\tilde{\zeta}(t,x^i) = \zeta(t,x^i) - \int^t \mathrm{d}t'\, Hf(t'). \tag{3.13}$$

Defining the local average of an arbitrary quantities $A(t, x^i)$ as

$$\bar{A}(t) := \frac{\int \mathrm{d}^3y\, W_t(\vec{y};\mathcal{O})A(t,\vec{y})}{\int \mathrm{d}^3y\, W_t(\vec{y};\mathcal{O})}, \tag{3.14}$$

and taking the local average of eq. (3.13), we find that the remaining *gauge* d.o.f. associated with $f(t)$ can be fixed by imposing the vanishing local averaged value of ζ in the coordinate system $(t, \tilde{x}^i)$ by choosing $f(t)$ as

$$\int^t \mathrm{d}t'\, Hf(t') = \bar{\zeta}(t). \tag{3.15}$$

In other words, the condition

$$\bar{\tilde{\zeta}}(t) = 0 \tag{3.16}$$

can be understood as the local gauge condition which fixes the residual *gauge* d.o.f associated with $f(t)$ in eq. (3.7).[2] This fixing can be done only in terms of the perturbation variables which live in the causal past of us. So far we have investigated some important

[2]It is also possible to derive the form of the coordinate transformation associated with $f(t)$ roughly in the following way. Eq. (3.12) and $N_1 = \tilde{N}_1$ give rise to

$$3Hf(t) = \partial_t \partial_i \delta x^i. \tag{3.17}$$

relations between residual *gauge* d.o.f., causality, and its potential role for regularizing IR effects, by considering the lapse and shift at lowest order. However it has been shown that almost the same discussion applies even when considering the higher-order terms. This is because the essential point of the discussion here is the fact that the constraint equations are given by elliptic-type equations which are valid to all orders in perturbation theory. More detailed discussions can be found in [6, 7, 8, 9, 11].

As an explicit example of genuinely *gauge* invariant quantities, [8, 9] proposed to consider the n-point function of the scalar curvature $R^{(3)}$ for the induced metric on the uniform field slice $\varphi = 0$ with specifying n arguments of them in terms of geometrical quantities which are independent of the coordinate choice. They used the three-dimensional geodesic distances between each arguments and some reference point.

3.2.2 Locality condition

So far we discussed importance of the residual *gauge* d.o.f. and it is likely that such d.o.f. are intimately related to the IR effects. This is because, it seems that local average of ζ can be set to zero by using the residual *gauge* transformation. This may indeed indicate some intimate relations between such d.o.f. and IR effects, it is too early to argue that the invariance of observables under this *gauge* transformations ensure the absence of large IR effects. This is because, such transformations are *not* the symmetry of the theory and hence modify the time evolution. One has to deny the possibility that the modification of the time evolution leads to additional large IR effects, while it seems non-trivial in general. Below we will derive the condition for quantum state which ensures the absence of large IR effects in evaluating correlation functions resembling observables which are invariant under the *gauge* transformations. In this subsection, we will work on Schrödinger picture. The content in this section is based on [14].

Dilatation invariance

Let us consider the dilatation transformation $x^i \to \tilde{x}^i = e^s x^i$. Under this transformation, ζ transforms as

$$\zeta_s(t, \vec{x}) = \zeta(t, e^{-s}\vec{x}) - s \simeq \zeta(t, \vec{x}) - s\vec{x} \cdot \partial_{\vec{x}}\zeta(t, \vec{x}) - s + \mathcal{O}(s^2) \,. \tag{3.20}$$

Eq. (3.17) implies that

$$\delta x^i = x^i \int^t \mathrm{d}t' \, Hf(t') \,. \tag{3.18}$$

This is nothing but the time-dependent dilatation transformation of the form

$$x^i \to \tilde{x}^i = e^{s(t)} x^i \,, \quad s(t) = \ln \left| 1 + \int^t \mathrm{d}t' Hf(t') \right| \,. \tag{3.19}$$

Strictly speaking, there are some room for adding other terms on the right-hand side of eq. (3.18) when solving eq. (3.17). However, our purpose here is to derive the form of coordinate transformation corresponding to the ambiguity associated with $f(t)$ only, and hence it will be unnecessary to add such terms. More detailed discussion on the correspondence between the residual *gauge* transformations and the spatial coordinate transformations can be found in [8, 9].

Therefore, under the infinitesimal dilatation transformation, ζ transforms as

$$\Delta\zeta(t,\vec{x}) \coloneqq \frac{\partial}{\partial s}\left[\zeta_s(t,\vec{x}) - \zeta(t,\vec{x})\right] = -1 - \vec{x}\cdot\partial_{\vec{x}}\zeta(t,\vec{x})\,. \tag{3.21}$$

Using this, one can construct the charge $\hat{Q}_\zeta$ which generates dilatation transformation as

$$\hat{Q}_\zeta \coloneqq \frac{1}{2}\int \mathrm{d}^3x\,\left[\Delta\hat{\zeta}(\vec{x})\hat{\pi}(\vec{x}) + \hat{\pi}(\vec{x})\Delta\hat{\zeta}(\vec{x})\right]\,. \tag{3.22}$$

Because the dilatation is just a coordinate transformation, the general covariance says that $\hat{Q}_\zeta$ commutes with the Hamiltonian:

$$\left[\hat{Q}_\zeta, \hat{H}\right] = 0\,. \tag{3.23}$$

Therefore $\hat{Q}_\zeta$ is a conserved Noether charge. By performing the Fourier transformation of eq. (3.22), one obtains the momentum-space representation as

$$\hat{Q}_\zeta = \hat{Q}_{\zeta_{\vec{k}=0}} \oplus \hat{Q}_{\zeta_{\vec{k}\neq 0}}\,, \tag{3.24a}$$

$$\hat{Q}_{\zeta_{\vec{k}=0}} = -\hat{\pi}_{\vec{k}=0}\,,\quad \hat{Q}_{\zeta_{\vec{k}\neq 0}} = -\int\frac{\mathrm{d}^3k}{(2\pi)^3}\left\{\hat{\zeta}_{\vec{k}},\,\vec{k}\cdot\partial_{\vec{k}}\hat{\pi}_{\vec{k}}\right\}\,. \tag{3.24b}$$

Here $\{A,B\} \coloneqq \left(\hat{A}\hat{B}+\hat{B}\hat{A}\right)/2$. Now we impose the dilatation invariant quantum state $|\Psi(t)\rangle$:

$$\hat{Q}_\zeta\,|\Psi(t)\rangle = 0\,. \tag{3.25}$$

It has been argued that this dilatation invariance is essentially important for proving the so-called consistency relation. Because we are interested in the correlation functions of non-zero modes, we investigate how the quantum states for non-zero modes respond to the dilatation transformation from now on. Firstly let us introduce an operator $\hat{\bar{\zeta}}$ which is defined by

$$\hat{\bar{\zeta}} \coloneqq \frac{\int \mathrm{d}^3x\hat{\zeta}(t,\vec{x})}{\int \mathrm{d}^3x} = \frac{\hat{\zeta}_{\vec{k}=0}}{\int \mathrm{d}^3x}\,, \tag{3.26}$$

and consider an eigenstate $|\bar{\zeta}\rangle \in \mathcal{H}_{\vec{k}=0}$ of this operator:

$$\hat{\bar{\zeta}}\left(|\bar{\zeta}\rangle\,|\chi\rangle\right) = \bar{\zeta}\,|\bar{\zeta}\rangle\,|\chi\rangle \quad \text{for} \quad \forall\,|\chi\rangle \in \mathcal{H}_{\vec{k}\neq 0}\,, \tag{3.27}$$

where we decompose a total Hilbert space as $\mathcal{H}_{\mathrm{tot}} = \mathcal{H}_{\vec{k}=0}\otimes\mathcal{H}_{\vec{k}\neq 0}$. Using this eigenstate of zero mode, we decompose the quantum state as

$$|\Psi(t)\rangle = \int \mathrm{d}\bar{\zeta}\,\left|\Psi(\bar{\zeta})\right|\,|\bar{\zeta}\rangle\,\left|\Psi(t);\bar{\zeta}\right\rangle\,, \tag{3.28}$$

with imposing the normalization condition

$$\left\|\left|\Psi(t);\bar{\zeta}\right\rangle\right\|^2 = 1\,. \tag{3.29}$$

Because $\hat{Q}_{\zeta_{\vec{k}=0}}$ is a generator of the dilatation for zero mode which satisfies

$$\left[i\hat{Q}_{\zeta_{\vec{k}=0}}, \hat{\bar{\zeta}}\right] = -1\,, \tag{3.30}$$

one has

$$\exp\left[is\hat{Q}_{\zeta_{\vec{k}=0}}\right] |\bar{\zeta}\rangle\,|\chi\rangle = |\bar{\zeta}+s\rangle\,|\chi\rangle \tag{3.31}$$

for an arbitrary state $|\chi\rangle$. Taking the derivative of above expression with respect to s, one obtains

$$i\hat{Q}_{\zeta_{\vec{k}=0}}\left(|\bar{\zeta}\rangle\,|\chi\rangle\right) = \left(\frac{\partial}{\partial\bar{\zeta}}\,|\bar{\zeta}\rangle\right)|\chi\rangle\ . \tag{3.32}$$

Therefore using eqs. (3.24a) and (3.32), eq. (3.25) gives rise to

$$\begin{aligned} 0 = \hat{Q}_{\zeta}\,|\Psi(t)\rangle = &-\int \mathrm{d}\bar{\zeta}\,|\bar{\zeta}\rangle\left[\left(\frac{\partial}{\partial\bar{\zeta}}\left|\Psi(\bar{\zeta})\right|\right)\left|\Psi(t);\bar{\zeta}\right\rangle\right] \\ &+\int \mathrm{d}\bar{\zeta}\left|\Psi(\bar{\zeta})\right|\,|\bar{\zeta}\rangle\left(i\hat{Q}_{\zeta_{\vec{k}\neq0}}-\frac{\partial}{\partial\bar{\zeta}}\right)\left|\Psi(t);\bar{\zeta}\right\rangle\ . \end{aligned} \tag{3.33}$$

Here, we write $\hat{Q}_{\zeta_{\vec{k}\neq0}}$ as an operator which acts only on $\mathcal{H}_{\vec{k}\neq0}$ because the action onto the zero-mode Hilbert space $\mathcal{H}_{\vec{k}=0}$ is trivial. Strictly speaking, $\hat{Q}_{\zeta_{\vec{k}\neq0}}$ was defined by an operator which acts on total Hilbert space in eqs. (3.24). Below, we will adopt this notation unless it causes any confusions. Operating $\langle\bar{\zeta}|\,\langle\Psi(t);\bar{\zeta}|$ on eq. (3.33) and using the normalization condition (3.29), one has

$$0 = \frac{\partial}{\partial\bar{\zeta}}\left|\Psi(\bar{\zeta})\right| + \left|\Psi(\bar{\zeta})\right|\left[\left\langle\Psi(t);\bar{\zeta}\right|i\hat{Q}_{\zeta_{\vec{k}\neq0}}\left|\Psi(t);\bar{\zeta}\right\rangle - \left\langle\Psi(t);\bar{\zeta}\right|\left(\frac{\partial}{\partial\bar{\zeta}}\left|\Psi(t);\bar{\zeta}\right\rangle\right)\right]\ . \tag{3.34}$$

First term of the right-hand side is a real number, while both of the second term and third term are pure imaginary. This can be verified by $\hat{Q}_{\zeta_{\vec{k}\neq0}} = \hat{Q}^{\dagger}_{\zeta_{\vec{k}\neq0}}$ and the normalization condition (3.29) which leads to

$$\frac{\partial}{\partial\bar{\zeta}}\left\|\left|\Psi(t);\bar{\zeta}\right\rangle\right\|^2 = 0\,. \tag{3.35}$$

Therefore, eq. (3.34) gives rise to the following two conditions:

$$\frac{\partial}{\partial\bar{\zeta}}\left|\Psi(\bar{\zeta})\right| = 0\,, \tag{3.36a}$$

$$i\hat{Q}_{\zeta_{\vec{k}\neq0}}\left|\Psi(t);\bar{\zeta}\right\rangle = \frac{\partial}{\partial\bar{\zeta}}\left|\Psi(t);\bar{\zeta}\right\rangle\ . \tag{3.36b}$$

Eq. (3.36a) says that the wave function is flat with respect to $\bar{\zeta}$. This is an expected result when considering the dilatation invariant quantum state. Eq. (3.36b) says that the effects of dilatation transformation onto the non-zero mode Hilbert space $\mathcal{H}_{\vec{k}\neq0}$ can be expressed by simply changing the value of $\bar{\zeta}$. This is again an expected result.

Inhomogenous dilatation and locality condition

From now on let us construct an operator $\hat{Q}_\zeta^W$ which generates an "inhomogeneous" dilatation:

$$\hat{Q}_\zeta^W(\vec{x}) \coloneqq \frac{1}{2}\int \mathrm{d}^3x'\, W_L(\vec{x}'-\vec{x})\left[\Delta\hat{\zeta}(\vec{x})\hat{\pi}(\vec{x}) + \hat{\pi}(\vec{x})\Delta\hat{\zeta}(\vec{x})\right]. \tag{3.37}$$

Here $W_L(\vec{x})$ is approximately positive constant when $|\vec{x}| \lesssim L$ while it almost vanishes outside this region, and we impose the normalization condition

$$\int \mathrm{d}^3x\, W_L(\vec{x}) = 1\,. \tag{3.38}$$

For later convenience we impose the Fourier component of $W_L(\vec{x})$ as

$$W(\vec{k}_L) \coloneqq \int \mathrm{d}^3x\, W_L(\vec{x})e^{i\vec{k}_L\cdot\vec{x}} = 0 \quad \text{for} \quad k_L \geq L^{-1}\,. \tag{3.39}$$

This operator $\hat{Q}_\zeta^W$ indeed induces the dilatation transformation because it only changes the averaged value of ζ which lives around $\vec{x}$:

$$\left[\hat{Q}_\zeta^W(\vec{x}), \hat{\zeta}(\vec{y})\right] = -\Delta\hat{\zeta}(\vec{y})W_L(\vec{x}-\vec{y})\,. \tag{3.40}$$

By choosing the size L as the causal past, it turns out that $\hat{Q}_\zeta^W(\vec{x})$ generates the residual *gauge* transformations. This is not the symmetry of the theory. Under this transformation the classical action does not remain invariant. Therefore $\hat{Q}_\zeta^W$ does not commute with the Hamiltonian:

$$\left[\hat{Q}_\zeta^W, H\right] \neq 0\,. \tag{3.41}$$

This means that the time evolution is modified under the inhomogeneous dilatation, and hence quantum state is not invariant:

$$\hat{Q}_\zeta^W\left|\Psi(t)\right\rangle \neq \left|\Psi(t)\right\rangle\,. \tag{3.42}$$

In momentum space,

$$\hat{Q}_\zeta^W(\vec{k}_L) \simeq \hat{Q}_{\zeta,L}^W(\vec{k}_L) \oplus \hat{Q}_{\zeta,S}^W(\vec{k}_L)\,, \tag{3.43a}$$

$$\begin{aligned}
\hat{Q}_{\zeta,L}^W(\vec{k}_L) &= -W(\vec{k}_L)\left[\hat{\pi}_{\vec{k}_L} + \int\frac{\mathrm{d}^3k}{(2\pi)^3}\left\{\hat{\zeta}_{\vec{k}},\, \vec{k}\cdot\partial_{\vec{k}}\hat{\pi}_{\vec{k}_L-\vec{k}}\right\}\Theta\left(L^{-1}-k\right)\Theta\left(L^{-1}-\left|\vec{k}-\vec{k}_L\right|\right)\right],\\
\hat{Q}_{\zeta,S}^W(\vec{k}_L) &= -W(\vec{k}_L)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\left\{\hat{\zeta}_{\vec{k}},\, \vec{k}\cdot\partial_{\vec{k}}\hat{\pi}_{\vec{k}_L-\vec{k}}\right\}\Theta\left(k-L^{-1}\right)\Theta\left(\left|\vec{k}-\vec{k}_L\right|-L^{-1}\right)\,.
\end{aligned} \tag{3.43b}$$

Let us decompose the Hilbert space into long-wavelength part and short-wavelength part as

$$\mathcal{H}_{\rm tot} = \mathcal{H}_L\otimes\mathcal{H}_S\,, \quad \mathcal{H}_L \coloneqq \bigotimes_{k_L<L^{-1}}\mathcal{H}_{\vec{k}_L}\,, \quad \mathcal{H}_S \coloneqq \bigotimes_{k_s\geq L^{-1}}\mathcal{H}_{\vec{k}_s}\,. \tag{3.44}$$

Next we decompose $|\Psi(t)\rangle$ as

$$|\Psi(t)\rangle = \int \mathcal{D}\zeta_L \, |\Psi(\zeta_L)| \, |\zeta_L\rangle \, |\Psi(t);\zeta_L\rangle \ , \quad \mathcal{D}\zeta_L \coloneqq \prod_{k_L<L^{-1}} \mathrm{d}\zeta_{\vec{k}_L} \tag{3.45}$$

where $|\zeta_L\rangle$ is an eigenstate of $\hat{\zeta}_{\vec{k}_L}$ for $\forall k_L < L^{-1}$:

$$\hat{\zeta}_{\vec{k}_L} \, |\zeta_L\rangle = \zeta_{\vec{k}_L} \, |\zeta_L\rangle \ . \tag{3.46}$$

Note that $\hat{\zeta}_{\vec{k}_L}$ acts on $\mathcal{H}_{\rm tot}$ strictly speaking, while it trivially acts on $\mathcal{H}_S$ and so we simply write $\hat{\zeta}_{\vec{k}_L}$ as an operator which acts only on $\mathcal{H}_L$ for notational simplicity. Then, as an analogue of (3.36b), we consider the condition

$$i\hat{Q}^W_{\zeta,S}(\vec{k}_L) \, |\Psi(t);\zeta_L\rangle = W(\vec{k}_L)\frac{\delta}{\delta\zeta_{\vec{k}_L}} \, |\Psi(t);\zeta_L\rangle \ . \tag{3.47}$$

Performing the Fourier transformation, eq. (3.47) becomes

$$i\hat{Q}^W_{\zeta,S}(\vec{x}) \, |\Psi(t);\zeta_L\rangle = \int \frac{\mathrm{d}^3k_L}{(2\pi)^3} e^{-i\vec{k}_L\cdot\vec{x}} W(\vec{k}_L)\frac{\delta}{\delta\zeta_{\vec{k}_L}} \, |\Psi(t);\zeta_L\rangle = \frac{\partial}{\partial\zeta_L(\vec{x})} \, |\Psi(t);\zeta_L\rangle \ . \tag{3.48}$$

Here we defined $\zeta_L(\vec{x}) \coloneqq (2\pi)^{-3}\int \mathrm{d}^3k \, W(\vec{k})\zeta_{\vec{k}} \exp[-i\vec{k}\cdot\vec{x}]$. Condition eq. (3.47) or (3.48) is called the "locality condition". Since the inhomogeneous dilatation is not a symmetry of the theory on the contrary to the homogeneous dilatation which is generated by the Noether charge $\hat{Q}_\zeta$, this locality condition are not necessarily respected. Locality condition implies that the effects of time-independent inhomogenous dilatation onto the short-wavelength quantum state can be expressed as the change of the local averaged value of $\zeta_L(\vec{x})$ at a certain time t. Therefore this condition can be satisfied when the constant solution is dominant in the EOM of ζ. Therefore an adiabatic vacuum state satisfies the locality condition, for example.

Below we will see how this locality condition plays an important role for ensuring the IR regularity of correlation functions which are invariant under the residual *gauge* transformations.

Absence of IR divergences

We will show that large IR secular effects are gauge artifact and hence are absent when evaluating correlation functions of genuinely *gauge* invariant variables, assuming that locality condition is satisfied. Following [8, 9], we refer to a genuinely *gauge* invariant operator as ${}^gR$ and evaluate their n-point function $\langle\Psi(t)|\, {}^gR_1\cdots{}^gR_n \, |\Psi(t)\rangle$. Here, the subscript of ${}^gR$ labels the spatial point. *Gauge* invariance of correlation functions implies

$$0 = \frac{\partial}{\partial s} \langle\Psi_s(t)| \, {}^gR_1\cdots{}^gR_n \, |\Psi_s(t)\rangle \ , \tag{3.49}$$

where $|\Psi_s(t)\rangle \coloneqq e^{-is\hat{Q}^W_\zeta}|\Psi(t)\rangle$.[3] Then, using the locality condition, eq. (3.49) gives rise to

$$\int \mathcal{D}\zeta_L\,|\Psi(\zeta_L)|^2\,\frac{\partial}{\partial s}\left(\langle\Psi(t);\zeta_L+s|\,\langle\zeta_L+s|\,{}^g\!R_1\cdots{}^g\!R_n\,|\zeta_L+s\rangle\,|\Psi(t);\zeta_L+s\rangle\right)=0\,, \quad (3.51)$$

for an arbitrary s. Here, we used that

$$\langle\zeta_L|\,{}^g\!R_1\cdots{}^g\!R_n\,|\zeta_L'\rangle \propto \delta\left(\zeta_L-\zeta_L'\right)\,, \quad (3.52)$$

which obeys from the commutativity of ${}^g\!R$ and soft modes $\hat{\zeta}_L$.[4] Because of the validity of eq. (3.51) for an arbitrary choice of s, we have

$$\frac{\partial}{\partial\zeta_L}\Big\{\langle\Psi(t);\zeta_L|\,\langle\zeta_L|\,{}^g\!R_1\cdots{}^g\!R_n\,|\zeta_L\rangle\,|\Psi(t);\zeta_L\rangle\Big\}=0\,. \quad (3.53)$$

This equation says that the terms inside the bracket $\{\cdots\}$ is independent of ζ_L, which means that the correlation function of *gauge* invariant operators are insensitive to the presence of ζ_L. Therefore, the integration over ζ_L will be canceled out by normalization of $|\Psi(t)\rangle$ when evaluating such correlation functions resembling cosmological observables:

$$\begin{aligned}\langle\Psi(t)|\,{}^g\!R_1\cdots{}^g\!R_n\,|\Psi(t)\rangle &= \int\mathcal{D}\zeta_L\,|\Psi(\zeta_L)|^2\left(\langle\Psi(t);\zeta_L|\,\langle\zeta_L|\,{}^g\!R_1\cdots{}^g\!R_n\,|\zeta_L\rangle\,|\Psi(t);\zeta_L\rangle\right)\\ &= \langle\Psi(t);\zeta_L|\,\langle\zeta_L|\,{}^g\!R_1\cdots{}^g\!R_n\,|\zeta_L\rangle\,|\Psi(t);\zeta_L\rangle\,,\end{aligned} \quad (3.54)$$

where we used the normalization condition in the final line

$$\int\mathcal{D}\zeta_L\,|\Psi(\zeta_L)|^2=1\,. \quad (3.55)$$

Eq. (3.54) states that the influence of fluctuations of IR fields on observables are *gauge* artifact in the sense that it vanishes when evaluating *gauge* invariant correlation functions resembling cosmological observables.

3.3 Multi-field inflation case

Based on [27], we briefly explain why the above explained method will not apply to IR loop effects from isocurvature modes during inflation. We then introduce the stochastic approach to inflation which may potentially give us the one way of regularizing IR effects from isocurvature modes.

[3] Eq. (3.49) can be expressed as

$$\langle\Psi_s(t)|\left[\hat{Q}^W_\zeta,{}^g\!R_1\cdots{}^g\!R_n\right]|\Psi_s(t)\rangle=0\,. \quad (3.50)$$

[4] Because ${}^g\!R$ is constructed by using purely geometrical quantities where the conjugate momentum of ζ_L is absent, ${}^g\!R$ commutes with ζ_L.

3.3.1 Residual *gauge*-independent IR effects

In sec. 3.2, we discussed a prescription which resolves IR issues. There the residual *gauge* d.o.f.s have played an important role. While it is also quite important to restrict quantum states so as to satisfy the locality condition, the allowed setting the local averaged value of ζ to be zero using the residual *gauge* transformation gives us a good indication for the absence of large IR effects on the *gauge* invariant correlation functions. Therefore, it is worthwhile investigating whether or not the local averaged value of inflaton can be eliminated by using residual *gauge* d.o.f. to get some insight on the physical significance of IR effects on multi-field inflation scenario. Here we consider the multi-field inflation model with the canonical kinetic term whose action is given by

$$S = \frac{1}{2}\int \mathrm{d}^4x\,\sqrt{-g}\left[R - (\partial\phi^I)(\partial\phi_I) = 2V(\phi)\right]\,. \tag{3.56}$$

Here $I = 1, \cdots$ labels inflatons.

For current purpose it is convenient to work on flat gauge in which $D = 0$ is imposed in stead of imposing $\varphi = 0$. In the single field inflation case, the residual *gauge* can be fixed by imposing

$$\bar{\varphi}(t) = \frac{\int \mathrm{d}^3y\, W_t(\vec{y};\mathcal{O})\varphi(t,\vec{y})}{\int \mathrm{d}^3y\, W_t(\vec{y};\mathcal{O})} = 0\,, \tag{3.57}$$

and eliminating the non-local contribution to the lapse and shift by using the remaining residual *gauge* transformations [6]. Note that eq. (3.57) is an analogue of eq. (3.16). In multi-field inflation model which is described by the action (3.56), the residual *gauge* d.o.f. can be fixed by imposing

$$e_I\bar{\varphi}^I(t) = \frac{\int \mathrm{d}^3y\, W_t(\vec{y};\mathcal{O})e_I\varphi^I(t,\vec{y})}{\int \mathrm{d}^3y\, W_t(\vec{y};\mathcal{O})} = 0\,, \tag{3.58}$$

where $e_I := \dot{\bar{\phi}}_I/(\dot{\bar{\phi}}_J\dot{\bar{\phi}}^J)^{1/2}$ [7]. This means that the local averaged value of adiabatic perturbations can be set to zero by using the residual *gauge*. However, the entropy perturbations $\mathcal{S}^I := \varphi^I - e^I e_J\varphi^J$ (or so-called isocurevatur peturbations) which express fluctuations perpendicular to the background trajectory cannot be eliminated. This suggests that the fluctuations of isocurvature modes are not *gauge* artifact, but rather can induce some physical effects on the local dynamics.

3.3.2 Stochastic approach to inflation and IR effects

IR secular growth

Then, how can we resolve IR issues associated with isocurvature modes? Because the field value of the adiabatic direction can be used as a clock field, the dynamics of isocurvature direction can be well captured by considering the test scalar field in the inflationary background. Since the inflationary space time is well approximated by de Sitter space in a flat chart, let us consider a weakly self-interacting test scalar field theory on de Sitter background whose action takes the form of

$$S = \int \mathrm{d}^4x\, a^3 \left(-\frac{1}{2}(\partial\phi)^2 - \frac{\lambda}{4!}\phi^4 \right) , \quad 0 < \lambda \ll 1 , \tag{3.59a}$$

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j = a^2(\eta)\left(-d\eta^2 + \delta_{ij}dx^i dx^j\right) , \tag{3.59b}$$

$$a(t) = e^{Ht} = -\frac{1}{H\eta} , \qquad H := \frac{\dot{a}}{a} = \text{constant} . \tag{3.59c}$$

In order to make the IR issue associated with this theory clearer, we introduce IR field $\phi^<$ defined by

$$\phi^<(x) := \int \frac{\mathrm{d}^3k}{(2\pi)^3} \Theta\left(\epsilon a H - k\right) \phi_{\vec{k}}(t) e^{i\vec{k}\cdot\vec{x}} , \qquad k := \left| \vec{k} \right| , \tag{3.60}$$

where $\Theta(z)$ is the Heaviside function and $\epsilon \ll 1$ is a non-dimensional small but positive parameter. Roughly speaking, $\phi^<(\vec{x}, t)$ is a coarse-grained field averaged over the comoving length scale $1/\epsilon a H$ on a constant time slice. We introduce an IR comoving momentum cutoff

$$k_0 := \epsilon a_0 H , \tag{3.61}$$

where a_0 is the value of a at the initial time t_0 and neglect all modes with a momentum below this cutoff, to avoid the IR divergences due to the momentum integral.[5] Under this setup correlation functions of $\phi^<(\vec{x}, t)$ have secularly growing terms, *e.g.*

$$\begin{aligned}\langle 0 | \phi^{<2n}(x) | 0 \rangle &= \sum_{l=0}^{\infty} \sum_{m=0}^{2l+n-1} c_{lm} \lambda^l \left(\ln \frac{a}{a_0} \right)^{2l+n-m} + (\text{IR regular part}) \\ &\sim H^2 \left(\ln \frac{a}{a_0} \right)^n + \lambda H^2 \left(\left(\ln \frac{a}{a_0} \right)^{n+2} + \left(\ln \frac{a}{a_0} \right)^{n+1} + \cdots \right) + \mathcal{O}\left(\lambda^2\right) ,\end{aligned} \tag{3.62}$$

where $|0\rangle$ is the state vector corresponding to the non-interacting Bunch-Davies vacuum at the initial time. For a later convenience, we refer to $m = 0$ terms in eq. (3.62), which represent the LO IR secular growth, as LO IR terms. Similarly, we refer to $m = 1$ terms as NLO IR terms. In general, $m = m_1$ terms are denoted by N^{m_1}LO IR terms. Since the terms higher order in λ get larger after passing the critical time t_{np} determined by the condition $\lambda(\ln a(t_{\text{np}})/a_0)^2 \approx 1$, which implies

$$t_{\text{np}} - t_0 \approx \frac{1}{\sqrt{\lambda} H} ,$$

perturbative expansion with respect to λ breaks down after t_{np}.

[5] Our expectation is that the initial time can be smoothly sent to the past infinity, once an appropriate effective EoM for IR modes is derived.

Stochastic approach

While the perturbative evaluation of correlation functions of IR field $\phi^<$ suffers from large IR secular effects, it has been argued that an IR field $\phi^<$ will obey an approximate effective EOM for IR field in models with a canonical scalar field with a general potential $V(\phi)$, which takes the form of the stochastic equation [29, 34, 37]:

$$\begin{aligned} \dot{\phi}^<(x) &= -\frac{1}{3H}V'\left(\phi^<(x)\right) + \xi_\phi(x)\,, \\ \langle \xi_\phi(x_1)\xi_\phi(x_2)\rangle &= \frac{H^3}{4\pi^2}\delta(t_1-t_2)\,j_0\left(\epsilon a(t_1)H\left|\vec{x}_1-\vec{x}_2\right|\right)\,, \end{aligned} \tag{3.63}$$

where $j_0(z) = \sin z/z$, and $\langle\cdots\rangle$ means the statistical ensemble average. This equation is obviously equivalent to an EoM of Brownian motion under an external force $V'(\phi)$, and means that each horizon patch evolves independently. Therefore, this equation can be solved non-perturbatively with respect to λ, which makes the obtained solution valid even at a late epoch where perturbative QFT calculations fail to converge. It has been checked explicitly that the stochastic EoM (3.63) correctly recovers LO IR terms (See *e.g.* [29]). Since eq. (3.63) is identical to an equation that describes a stochastic process in classical mechanics, it might be allowed to interpret that the stochastic EoM is the one that attributes probabilities to various possible realizations of the classical time evolution. In this stochastic interpretation, the LO IR secular growth is simply interpreted as an increase of the statistical variance. This stochastic picture of inflation has been utilized in the literatures as a possible resolution to IR problems in multi-field inflationary models [17, 18, 19, 20]. In stochastic approach to inflation, IR field $\phi^<(x)$ which lives in our causal past obeys a stochastic equation eq. (3.63) in a good approximation, and it takes some definite value at around 50 e-folds where the causal past of us shrinks into one-Hubble patch in a good approximation. Referring to this time as t_*, a fixed value of $\phi^<(t_*,\vec{x})$ is assumed in the stochastic approach to inflation. Therefore, once the stochastic picture of inflation is accepted, one will not have the fluctuations of deep-IR modes and hence they cannot contribute to observed fluctuations from the beginning. On the other hand, one can also calculate the probability distribution of $\phi^<(t_*,\vec{x})$ using eq. (3.63) for a given initial value $\phi^<(t_0,\vec{x})$ at an initial time t_0. This is because eq. (3.63) will capture the unitary evolution of IR fields in a good approximation: the recovery of LO IR terms which appears in the evaluation of IR correlation functions as we saw in eq. (3.62) may support this. In the stochastic approach to inflation, however, the probability distribution $P(\phi^<(\vec{x})|t_*)$ is unnecessary for predicting what we measure through the cosmological observations. $P(\phi^<(\vec{x})|t_*)$ will give us just a probability for measuring one particular local universe where $\phi(t_*,\vec{x}) = \phi_1$ is realized, and this is theoretically interesting but $P(\phi^<(\vec{x})|t_*)$ is totally irrelevant for predicting observables [20].

The absence of IR effects on observables in the stochastic approach to inflation is however simply because the quantum coherence of deep-IR modes $k < \epsilon a(t_*)H$ are neglected by hand from the beginning, as has been pointed out in the several previous works [7, 27]. Therefore, from next chapter, we will study the theoretical consistency of the stochastic approach to inflation with taking into account the quantum coherence. We then discuss whether or not large IR effects can be relevant for cosmological observations.

Chapter 4

Extended stochastic formalism

As we explained at the end of the previous chapter, stochastic approach to inflation has been proposed as a possible resolution to IR issues associated with the isocurvature modes. In the stochastic interpretation, the LO IR secular growth is simply interpreted as an increase of the statistical variance, while it will not be observed by a local observer. However, one should recall that the stochastic EOM (3.63) neglects sub-leading order (sub-LO) IR terms, which also increase indefinitely in time. Furthermore, what we are interested in would be the finite part of the correlation functions that after the secular growth effects are removed. Therefore, there seems to be no reason why one can be satisfied with the treatment neglecting the sub-LO IR terms. It seems non-trivial whether or not an effective EOM which can describe all IR correlation functions with sufficient precision can be obtained in the form similar to eq. (3.63).

In this chapter, we will show that in a quite generic setup all the IR secular effects can be recovered by the stochastic equations, based on our previous works [50, 51]. More detailed discussion on the theoretical consistency of stochastic approach will be given in the next chapter 5.

4.1 Systematic derivation of stochastic formalism

In this section, we develop how to derive an effective IR EOM by using the Schwinger-Keldysh formalism. In sec. 4.1.2, we provide a refined way to divide the path integral into the UV part and the IR part. This division becomes non-trivial compared to the case of the usual Wilsonian EFT in flat spacetime, because the current problem is on the EFT for an open system in which the dynamical degrees of freedom are continuously transfered from the environment to the system.

4.1.1 Setup

In this study, we consider a canonically normalized scalar field theory with a general potential $V(\phi, v)$ on de Sitter background, whose Hamiltonian density $\mathcal{H}$ is given by

$$\mathcal{H}(\phi, v) = \mathcal{H}_0(\phi, v) + V(\phi, v) \ , \quad \mathcal{H}_0 = \frac{1}{2}v^2 + \frac{(\nabla\phi)^2}{2a^2} \, , \tag{4.1}$$

where v denotes the conjugate momentum of ϕ divided by a^3. Here, we write $a(t)$ as a for brevity unless it causes any confusion. The mass term $m^2\phi^2$ is also included in the interaction potential $V(\phi, v)$, and we will identify the $\mathcal{H}_0$ and $V(\phi, v)$ as a free part and an interacting part, respectively, when we perform the perturbation theory.[1] We do not specify the concrete form of the interaction potential, while we always consider $\lambda\phi^4$ interaction when performing concrete calculations. We refer to the coupling constant as λ below, assuming that V is proportional to λ. We refer to modes $\vec{k}$ satisfying $k \coloneqq |\vec{k}| \geq \epsilon a H$ and $k < \epsilon a H$ as UV modes and IR modes, respectively, with a small parameter ϵ, where $H \coloneqq \dot{a}/a$ denotes the Hubble parameter.

We make two assumptions on the initial state. Firstly, we assume that the potential $V(\phi, v)$ is turned on at $t = t_0$, and the initial state is set to the non-interacting Bunch-Davies vacuum state $|0\rangle$. Secondly, we neglect modes that are already belonging to IR modes at the time $t = t_0$. This is equivalent to introducing an IR cutoff $k_0 \coloneqq \epsilon a_0 H$ to the comoving momentum k.[2] Under these assumptions, the interaction picture fields $\hat{\phi}_{\rm I}$ and $\hat{v}_{\rm I}$ are expanded as

$$\hat{\phi}_{\rm I}(x) = \int \frac{{\rm d}^3k}{(2\pi)^3} \left[\Phi_k(t) e^{i\vec{k}\cdot\vec{x}} \hat{a}_{\vec{k}} + ({\rm h.c.})\right] , \qquad \hat{v}_{\rm I}(x) = \int \frac{{\rm d}^3k}{(2\pi)^3} \left[\dot{\Phi}_k(t) e^{i\vec{k}\cdot\vec{x}} \hat{a}_{\vec{k}} + ({\rm h.c.})\right] , \tag{4.2}$$

where (h.c.) stands for the hermitian conjugate and the mode function $\Phi_k(t)$ is given by

$$\Phi_k(t) = \frac{H}{\sqrt{2k^3}} (1 + ik\eta)\, e^{-ik\eta} , \tag{4.3}$$

which satisfies the normalization condition

$$\dot{\Phi}_k(t)\Phi_k^*(t) - \Phi_k(t)\dot{\Phi}_k^*(t) = \frac{-i}{a^3} . \tag{4.4}$$

Here, η is the conformal time defined by ${\rm d}\eta = {\rm d}t/a$. Then, the WKB vacuum state $|0\rangle$ is specified by $\hat{a}_{\vec{k}}\,|0\rangle = 0$. The creation and annihilation operators $\hat{a}_{\vec{k}}$ and $\hat{a}^\dagger_{\vec{k}}$ satisfy the commutation relations

$$\left[\,\hat{a}_{\vec{k}}\,,\,\hat{a}^\dagger_{\vec{k}'}\right] = (2\pi)^3\delta^{(3)}(\vec{k} - \vec{k}') , \qquad \left[\,\hat{a}_{\vec{k}}\,,\,\hat{a}_{\vec{k}'}\,\right] = 0 , \qquad \left[\,\hat{a}^\dagger_{\vec{k}}\,,\,\hat{a}^\dagger_{\vec{k}'}\right] = 0 . \tag{4.5}$$

Since all modes are belonging to UV modes at the initial time, each IR mode has the crossing time t_k transferred from a UV mode, which is given by

$$t_k \coloneqq \frac{1}{H} \ln \frac{k}{\epsilon H} .$$

Next, we decompose the Heisenberg picture fields $\phi_{\rm H}$ into the UV part and the IR part, which are denoted by $\phi^>_{\rm H}$ and $\phi^<_{\rm H}$, respectively, as $\phi_{\rm H} = \phi^>_{\rm H} + \phi^<_{\rm H}$, with

$$\phi^<_{\rm H}(x) \coloneqq \int \frac{{\rm d}^3k}{(2\pi)^3}\, \Theta\left(\epsilon a H - k\right) \phi_{\vec{k}}(t) e^{i\vec{k}\cdot\vec{x}} , \tag{4.6}$$

[1] When we consider nonlinear interactions without derivative couplings, the momentum-dependence of V in eq. (4.1) only comes from the field renormalization factor.

[2] After deriving an effective EOM for IR modes, we expect that the initial time t_0 can be smoothly sent to the past infinity.

where $\Theta(z)$ is the Heaviside step function. We also decompose $v_{\rm H}$ as $v_{\rm H} = v_{\rm H}^{>} + v_{\rm H}^{<}$ in the same manner. It is known that the correlation functions of the IR fields $\phi_{\rm H}^{<}$ and $v_{\rm H}^{<}$ contain the IR secular growth terms which are divergent after sending the comoving IR cutoff k_0 to 0. In the next subsection 4.1.2, we demonstrate how to derive the effective EOM for the IR fields which can correctly recover the IR secular growth in all IR correlation functions consisting of $\phi_{\rm H}^{<}$ and $v_{\rm H}^{<}$.

4.1.2 Formulation

In this subsection, we derive the IR dynamics just by integrating out UV modes, by using the path integral. Generating functional for the system defined by eqs. (4.1) is given by

$$
\begin{aligned}
Z[J] = \int & \mathcal{D}\phi_+ \mathcal{D}\phi_- \mathcal{D}v_+ \mathcal{D}v_- \, \Psi_0[\phi_+]\Psi_0^*[\phi_-] \\
& \times \exp\left[i \int \mathrm{d}^4x \, a^3 \left(v_+\dot{\phi}_+ - \frac{1}{2}v_+^2 - \frac{1}{2}\frac{(\nabla\phi_+)^2}{a^2} - V(\phi_+, v_+) + J_+^{(\phi)}\phi_+ + J_+^{(v)}v_+ \right) \right. \\
& \qquad \left. - (+ \to -) \right] ,
\end{aligned} \tag{4.7}
$$

with boundary conditions,

$$
\phi_+(x) = \phi_-(x) \, , \quad \text{for} \quad t = t_f \, , \tag{4.8}
$$

where t_f is an appropriately chosen maximum time. Here, $\Psi_0[\phi_+]\Psi_0^*[\phi_-]$ denotes an initial wave functional, and we will write this initial wave functional dependence simply as $\Psi_0\Psi_0^*$ below, unless it causes any confusions. For brevity, we set $J = 0$ for the moment. Equivalently, the path integral can be written in the Keldysh basis $(\phi_c, v_c, \phi_\Delta, v_\Delta)$ as

$$
Z[0] = \int \mathcal{D}\phi_c \mathcal{D}\phi_\Delta \mathcal{D}v_c \mathcal{D}v_\Delta \, e^{iS_{\rm H,0}[\phi_c,\phi_\Delta,v_c,v_\Delta]} e^{iS_{\rm H,int}[\phi_c,\phi_\Delta,v_c,v_\Delta]} \Psi_0\Psi_0^* \, , \tag{4.9}
$$

$$
S_{\rm H,0}[\phi_c, \phi_\Delta, v_c, v_\Delta] \coloneqq \int \mathrm{d}^4x \, a^3 \left(v_c\dot{\phi}_\Delta + v_\Delta\dot{\phi}_c - v_c v_\Delta - \frac{(\vec{\nabla}\phi_c)\,(\vec{\nabla}\phi_\Delta)}{a^2} \right) , \tag{4.10}
$$

$$
\begin{aligned}
& S_{\rm H,int}[\phi_c, \phi_\Delta, v_c, v_\Delta] \\
& \coloneqq - \int \mathrm{d}^4x \, a^3 \left(V\left(\phi_c + \frac{1}{2}\phi_\Delta, v_c + \frac{1}{2}v_\Delta\right) - V\left(\phi_c - \frac{1}{2}\phi_\Delta, v_c - \frac{1}{2}v_\Delta\right) \right) ,
\end{aligned} \tag{4.11}
$$

with boundary conditions

$$
\phi_\Delta(x) = 0 \quad \text{for} \quad t = t_f \, . \tag{4.12}
$$

Here, the Keldysh basis is defined by

$$
\phi_c \coloneqq \frac{\phi_+ + \phi_-}{2} \, , \quad v_c \coloneqq \frac{v_+ + v_-}{2} \, , \tag{4.13}
$$

$$
\phi_\Delta \coloneqq \phi_+ - \phi_- \, , \quad v_\Delta \coloneqq v_+ - v_- \, . \tag{4.14}
$$

Note that cc propagator and $c\Delta$ propagator correspond to the symmetric propagator and the retarded Green's function, respectively.

We split the above path integral (4.9) into the UV part and the IR part. We focus on the non-interacting part of the path integral $Z_0[0]$, neglecting the interacting part $S_{\mathrm{H,int}}$ for a while. If one simply splits the functional measure into the UV part and the IR part as

$$
\begin{aligned}
Z_0[0] = \mathcal{N}_1 \mathcal{N}_2 \prod_{k \geq k_0} \int \prod_{t > t_k} \left[\mathrm{d}\phi_c\left(t, \vec{k}\right) \mathrm{d}\phi_\Delta\left(t, \vec{k}\right) \mathrm{d}v_c\left(t, \vec{k}\right) \mathrm{d}v_\Delta\left(t, \vec{k}\right) \right] \\
\times \prod_{t \leq t_k} \left[\mathrm{d}\phi_c\left(t, \vec{k}\right) \mathrm{d}\phi_\Delta\left(t, \vec{k}\right) \right] \prod_{t < t_k} \left[\mathrm{d}v_c\left(t, \vec{k}\right) \mathrm{d}v_\Delta\left(t, \vec{k}\right) \right] e^{iS_{\mathrm{H},0}[\phi_c, \phi_\Delta, v_c, v_\Delta]} \Psi_0 \Psi_0^* ,
\end{aligned}
\tag{4.15}
$$

one cannot integrate out UV modes as independent degrees of freedom, because UV modes are to be identified with IR modes at the crossing time, $t = t_k$. Here, we choose the time argument of momentum variables, v_c and v_Δ, not to contain $t = t_k$ in deriving the original path integral $Z_0[0]$ for later convenience. $\mathcal{N}_1$ and $\mathcal{N}_2$ are numerical constants. We show that UV modes can be integrated out as independent degrees of freedom, if the transition from UV modes to IR modes at $t = t_k$ is expressed by interaction vertexes. In order to identifying the required interaction vertexes, firstly, we rewrite the above path integral (4.15) into the following form:

$$
\begin{aligned}
Z_0[0] = \int \mathcal{D}\phi_c^< \mathcal{D}\phi_\Delta^< \mathcal{D}v_c^< \mathcal{D}v_\Delta^< \int \mathcal{D}\phi_c^> \mathcal{D}\phi_\Delta^> \mathcal{D}v_c^> \mathcal{D}v_\Delta^> \, \Psi_0 \left[\phi_c^> + \frac{1}{2}\phi_\Delta^> \right] \Psi_0^* \left[\phi_c^> - \frac{1}{2}\phi_\Delta^> \right] \\
\times e^{iS_{\mathrm{H},0}\left[\phi_c = \phi_c^< + \phi_c^>,\, \phi_\Delta = \phi_\Delta^< + \phi_\Delta^>,\, v_c = v_c^< + v_c^>,\, v_\Delta = v_\Delta^< + v_\Delta^>\right]} ,
\end{aligned}
\tag{4.16}
$$

$$
\mathcal{D}\phi_c^> \mathcal{D}\phi_\Delta^> \mathcal{D}v_c^> \mathcal{D}v_\Delta^> := \mathcal{N}_1 \prod_{k \geq k_0} \prod_{t \leq t_k} \left[\mathrm{d}\phi_c^>\left(t, \vec{k}\right) \mathrm{d}\phi_\Delta^>\left(t, \vec{k}\right) \right] \prod_{t < t_k} \left[\mathrm{d}v_c^>\left(t, \vec{k}\right) \mathrm{d}v_\Delta^>\left(t, \vec{k}\right) \right] ,
$$

$$
\mathcal{D}\phi_c^< \mathcal{D}\phi_\Delta^< \mathcal{D}v_c^< \mathcal{D}v_\Delta^< := \mathcal{N}_2 \prod_{k \geq k_0} \prod_{t \geq t_k} \left[\mathrm{d}\phi_c^<\left(t, \vec{k}\right) \mathrm{d}\phi_\Delta^<\left(t, \vec{k}\right) \right] \prod_{t > t_k} \left[\mathrm{d}v_c^<\left(t, \vec{k}\right) \mathrm{d}v_\Delta^<\left(t, \vec{k}\right) \right] ,
$$

with boundary conditions

$$
\phi_\Delta^>\left(t = t_k, \vec{k}\right) = 0 , \tag{4.17a}
$$

$$
\phi_c^<\left(t = t_k, \vec{k}\right) = 0 , \tag{4.17b}
$$

for all modes $\vec{k}$ with $k \geq k_0$. The boundary condition at final time eq. (4.12) is expressed by $\phi_\Delta^<\left(t_f, \vec{k}\right) = 0$ for $t_k < t_f$. Note that the initial wave functionals are now the functional of UV fields, because all the modes are UV modes at the initial time t_0 in our setup. Again we will below write this initial wave-functional dependence as $\Psi_0^> \Psi_0^{>*}$ for notational simplicity. In order to see the equivalence between the path integral expressions (4.15) and (4.16) with the boundary conditions (4.17), let us remind the derivation of the path integral (4.15). We concentrate on a single mode $\vec{k}$ for a while, because without nonlinear interaction $V(\phi, v)$, each mode $\vec{k}$ evolves independently. The time evolution of a mode $\vec{k}$ from $t = t_2$ to $t = t_1$ is described by the following unitary operator

$$\hat{U}_0\left(t_1, t_2; \vec{k}\right) = \mathrm{T}\exp\left[-i\int_{t_2}^{t_1} \mathrm{d}t\, a^3 \hat{\mathcal{H}}_0\left(t, \vec{k}\right)\right], \tag{4.18}$$

$$\hat{\mathcal{H}}_0\left(t, \vec{k}\right) = \frac{1}{2}\hat{v}_{\mathrm{I}}\left(t, \vec{k}\right)\hat{v}_{\mathrm{I}}\left(t, -\vec{k}\right) + \frac{k^2}{2a^2}\hat{\phi}_{\mathrm{I}}\left(t, \vec{k}\right)\hat{\phi}_{\mathrm{I}}\left(t, -\vec{k}\right). \tag{4.19}$$

Here, T denotes the time-ordered product. We decompose $\hat{U}_0\left(t_1, t_2; \vec{k}\right)$ into a product of the unitary operators with a small time step Δt as

$$\hat{U}_0\left(t_1, t_2; \vec{k}\right) = \lim_{\Delta t\to 0}\hat{U}_0\left(t_1, t_1 - \Delta t; \vec{k}\right)\hat{U}_0\left(t_1 - \Delta t, t_1 - 2\Delta t; \vec{k}\right)\cdots\hat{U}_0\left(t_2 + \Delta t, t_2; \vec{k}\right). \tag{4.20}$$

Inserting identity operators, $\hat{U}_0\left(t, t - \Delta t; \vec{k}\right)$ can be expressed as

$$\begin{aligned}\hat{U}_0\left(t, t - \Delta t; \vec{k}\right) = \int \mathrm{d}\phi\left(t - \Delta t, \vec{k}\right)\mathrm{d}v\left(t', \vec{k}\right)\left|v\left(t', \vec{k}\right)\right\rangle\left\langle v\left(t', \vec{k}\right)\right| \\ \times\hat{U}_0\left(t, t - \Delta t; \vec{k}\right)\left|\phi\left(t - \Delta t, \vec{k}\right)\right\rangle\left\langle\phi\left(t - \Delta t, \vec{k}\right)\right|,\end{aligned} \tag{4.21}$$

where we choose t' as $t - \Delta t < t' < t$. $\left|\phi\left(t, \vec{k}\right)\right\rangle$ and $\left|v\left(t, \vec{k}\right)\right\rangle$ denote the eigenstates of $\hat{\phi}_{\mathrm{I}}\left(t, \vec{k}\right)$ and $\hat{v}_{\mathrm{I}}\left(t, \vec{k}\right)$ with eigenvalues $\phi\left(t, \vec{k}\right)$ and $v\left(t, \vec{k}\right)$, respectively. The path integral (4.15) can be derived from $\hat{U}_0\left(t_f, t_0; \vec{k}\right) = \hat{U}_0\left(t_f, t_k; \vec{k}\right)\hat{U}_0\left(t_k, t_0; \vec{k}\right)$. In the expression with a finite Δt, one can easily understand that the path integral (4.16) with boundary conditions (4.17) is obtained by just applying the following replacement of integration variables in eq. (4.15):

$$\phi_c\left(t, \vec{k}\right) \to \begin{cases}\phi_c^{>}\left(t, \vec{k}\right) & \text{for}\quad t \le t_k \\ \phi_c^{<}\left(t, \vec{k}\right) & \text{for}\quad t > t_k,\end{cases} \qquad \phi_\Delta\left(t, \vec{k}\right) \to \begin{cases}\phi_\Delta^{>}\left(t, \vec{k}\right) & \text{for}\quad t < t_k \\ \phi_\Delta^{<}\left(t, \vec{k}\right) & \text{for}\quad t \ge t_k,\end{cases} \tag{4.22}$$

$$v_c\left(t, \vec{k}\right) \to \begin{cases}v_c^{>}\left(t, \vec{k}\right) & \text{for}\quad t < t_k \\ v_c^{<}\left(t, \vec{k}\right) & \text{for}\quad t > t_k,\end{cases} \qquad v_\Delta\left(t, \vec{k}\right) \to \begin{cases}v_\Delta^{>}\left(t, \vec{k}\right) & \text{for}\quad t < t_k \\ v_\Delta^{<}\left(t, \vec{k}\right) & \text{for}\quad t > t_k.\end{cases} \tag{4.23}$$

In our choice, $\phi_\Delta\left(t_k, \vec{k}\right)$ and $\phi_c\left(t_k, \vec{k}\right)$ belong to IR modes and UV modes, respectively. Physical meaning of this choice can be understood as follows:

1. The condition $\phi_\Delta^{>}\left(t_k, \vec{k}\right) = 0$ ensures that the time path of UV modes $\vec{k}$ is closed at $t = t_k$, which allows one to integrate out UV modes without specifying the trajectories of the path integral of IR modes. Intuitively, it is natural to impose the condition $\phi_\Delta^{>}\left(t_k, \vec{k}\right) = 0$ because the mode $\vec{k}$ is transfered from UV mode to IR mode at $t = t_k$, which implies that the maximum time for UV mode $\vec{k}$ is given by $t = t_k$.

2. One cannot impose any constraint on $\phi_c^{>}\left(t_k,\vec{k}\right)$ in integrating out UV modes. With this requirement, to make (4.16) equivalent to (4.15), we need to fix the value of $\phi_c^{<}\left(t_k,\vec{k}\right)$. Although any choice of the value of $\phi_c^{<}\left(t_k,\vec{k}\right)$ can be absorbed by the redefinition of $\phi_c^{>}\left(t_k,\vec{k}\right)$, the simplest choice is to set $\phi_c^{<}\left(t_k,\vec{k}\right) = 0$ in (4.16), which can be also understood as the initial condition for IR modes.

Next, let us derive the terms which express the transition from UV modes to IR modes, from (4.16). Hamiltonian action $S_{\mathrm{H},0}$ with integration range $t_k - \Delta t \le t \le t_k + \Delta t$ before taking the $\Delta t \to 0$ limit, which is denoted by $S_{\mathrm{H},0}\left(t_k,\vec{k}\right)$, is expressed as

$$S_{\mathrm{H},0}\left(t_k,\vec{k}\right) = \left[a^3\left(t_k+\Delta t'\right)v_\Delta\left(t_k+\Delta t',\vec{k}\right)\left(\phi_c\left(t_k+\Delta t,-\vec{k}\right)-\phi_c\left(t_k,-\vec{k}\right)\right)\right.$$
$$\left. + \left(t_k\to t_k-\Delta t\right)\right] + \left(c\leftrightarrow\Delta\right) + \mathcal{O}(\Delta t)\,, \tag{4.24}$$

with $0 \le \Delta t' < \Delta t$. This choice of $\Delta t'$ corresponds to the choice of t' in eq. (4.21). By applying eqs. (5.29) and (4.23), one obtains

$$\begin{aligned}
&S_{\mathrm{H},0}\left(t_k,\vec{k}\right)\\
&=\left\{\left[a^3\left(t_k+\Delta t'\right)v_\Delta^{<}\left(t_k+\Delta t',\vec{k}\right)\left(\phi_c^{<}\left(t_k+\Delta t,-\vec{k}\right)-\phi_c^{<}\left(t_k,-\vec{k}\right)\right)\right.\right.\\
&\left.\left.+a^3\left(t_k-\Delta\tilde{t}'\right)v_\Delta^{>}\left(t_k-\Delta\tilde{t}',\vec{k}\right)\left(\phi_c^{>}\left(t_k,-\vec{k}\right)-\phi_c^{>}\left(t_k-\Delta t,-\vec{k}\right)\right)\right]+\left(c\leftrightarrow\Delta\right)\right\}\\
&-a^3\left(t_k+\Delta t'\right)v_\Delta^{<}\left(t_k+\Delta t',\vec{k}\right)\phi_c^{>}\left(t_k,\vec{k}\right)+a^3\left(t_k-\Delta\tilde{t}'\right)\phi_\Delta^{<}\left(t_k,\vec{k}\right)v_c^{>}\left(t_k-\Delta\tilde{t}',\vec{k}\right)\\
&+\mathcal{O}(\Delta t)\,,
\end{aligned}\tag{4.25}$$

where $\Delta\tilde{t}' \coloneqq \Delta t - \Delta t' > 0$. The cross terms between UV fields and IR fields which do not vanish after taking the $\Delta t \to 0$ limit only exist at the transition time $t = t_k$, and express the transition from UV modes to IR modes. Therefore, from eqs. (4.16), (4.17), and (4.25), one obtains

$$S_{\mathrm{H},0}\left[\phi_c=\phi_c^{<}+\phi_c^{>},\,\phi_\Delta=\phi_\Delta^{<}+\phi_\Delta^{>},\,v_c=v_c^{<}+v_c^{>},\,v_\Delta=v_\Delta^{<}+v_\Delta^{>}\right]=S_{\mathrm{H},0}^{>}+S_{\mathrm{H},0}^{<}+\tilde{S}_{\mathrm{bilinear}}\,, \tag{4.26}$$

with

$$
\begin{aligned}
S^{>}_{\mathrm{H},0} \coloneqq \int \frac{\mathrm{d}^3k}{(2\pi)^3} \int^{t_k} \mathrm{d}t\, a^3 \Bigg[& v^{>}_{\Delta}\left(t,\vec{k}\right) \dot{\phi}^{>}_{c}\left(t,-\vec{k}\right) + v^{>}_{c}\left(t,-\vec{k}\right) \dot{\phi}^{>}_{\Delta}\left(t,\vec{k}\right) \\
& - v^{>}_{c}\left(t,-\vec{k}\right) v^{>}_{\Delta}\left(t,\vec{k}\right) + \frac{k^2}{a^2}\phi^{>}_{c}\left(t,-\vec{k}\right)\phi^{>}_{\Delta}\left(t,\vec{k}\right) \Bigg],
\end{aligned}
\tag{4.27a}
$$

$$
\begin{aligned}
S^{<}_{\mathrm{H},0} \coloneqq \int \frac{\mathrm{d}^3k}{(2\pi)^3} \int_{t_k} \mathrm{d}t\, a^3 \Bigg[& v^{<}_{\Delta}\left(t,\vec{k}\right) \dot{\phi}^{<}_{c}\left(t,-\vec{k}\right) + v^{<}_{c}\left(t,-\vec{k}\right) \dot{\phi}^{<}_{\Delta}\left(t,\vec{k}\right) \\
& - v^{<}_{c}\left(t,-\vec{k}\right) v^{<}_{\Delta}\left(t,\vec{k}\right) + \frac{k^2}{a^2}\phi^{<}_{c}\left(t,-\vec{k}\right)\phi^{<}_{\Delta}\left(t,\vec{k}\right) \Bigg] \\
= \int \frac{\mathrm{d}^3k}{(2\pi)^3} \int_{t_k} \mathrm{d}t\, a^3 \Bigg[& v^{<}_{\Delta}\left(t,\vec{k}\right) \dot{\phi}^{<}_{c}\left(t,-\vec{k}\right) - \phi^{<}_{\Delta}\left(t,\vec{k}\right)\left(\dot{v}^{<}_{c}\left(t,-\vec{k}\right) + 3Hv^{<}_{c}\left(t,-\vec{k}\right)\right) \\
& - v^{<}_{c}\left(t,-\vec{k}\right) v^{<}_{\Delta}\left(t,\vec{k}\right) + \frac{k^2}{a^2}\phi^{<}_{c}\left(t,-\vec{k}\right)\phi^{<}_{\Delta}\left(t,\vec{k}\right) \Bigg],
\end{aligned}
\tag{4.27b}
$$

$$
\tilde{S}_{\mathrm{bilinear}} \coloneqq -\int \mathrm{d}t\, a^3 \int \frac{\mathrm{d}^3k}{(2\pi)^3}\delta(t-t_k)\left[v^{<}_{\Delta}\left(t,\vec{k}\right)\phi^{>}_{c}\left(t,-\vec{k}\right) - \phi^{<}_{\Delta}\left(t,\vec{k}\right)v^{>}_{c}\left(t,-\vec{k}\right)\right].
\tag{4.27c}
$$

Here, we performed the integration by parts in the second line of eq. (4.27b). The cross terms in the final line in eq. (4.25) lead to the terms $\tilde{S}_{\mathrm{bilinear}}$ defined by (4.27c), after collecting the contributions from all modes with $k \geq k_0$. Substituting eqs. (4.26) and (4.27) into eq. (4.16), one gets the following path integral:

$$
Z_0[0] = \int \mathcal{D}\phi^{<}_{c}\mathcal{D}\phi^{<}_{\Delta}\mathcal{D}v^{<}_{c}\mathcal{D}v^{<}_{\Delta}\, e^{iS^{<}_{\mathrm{H},0}} \int \mathcal{D}\phi^{>}_{c}\mathcal{D}\phi^{>}_{\Delta}\mathcal{D}v^{>}_{c}\mathcal{D}v^{>}_{\Delta}\, e^{iS^{>}_{\mathrm{H},0}} e^{i\tilde{S}_{\mathrm{bilinear}}}\Psi^{>}_{0}\Psi^{>*}_{0},
\tag{4.28}
$$

with boundary conditions (4.17). We emphasize that the boundary condition (4.17a) allows one to regard UV modes and IR modes as independent degrees of freedom from each other and hence $\tilde{S}_{\mathrm{bilinear}}$ terms can be treated as interaction vertexes. $S^{<}_{\mathrm{H},0}$ and $S^{>}_{\mathrm{H},0}$, which are defined by eqs. (4.27a) and (4.27b), denote the non-interacting Hamiltonian action for IR modes and UV modes, respectively. Indeed, we will show that all the propagators can be correctly recovered from the path integral defined by eq. (4.28) in sec. 4.1.3.

The non-triviality of integrating out UV modes even without nonlinear interaction potential stems from the essential difference between the usual Wilsonian EFT in flat spacetime and the current problem. The current problem is on the EFT for an open system in which the dynamical degrees of freedom are continuously transfered from the environment to the system [47, 54, 55, 56]. As a result, in order to integrate out UV modes, one needs to treat the effect of this transition as interaction vertexes.

Since the cross terms in the nonlinear interaction potential $V(\phi, v)$ can contribute to only $\mathcal{O}(\Delta t)$ terms in the exponent of the integrand in the path integral, the path integral for IR fields including nonlinear interactions, which is denoted by $Z^{<}[0]$, can be obtained from eq. (4.28) as

$$Z^{<}[0] = \int \mathcal{D}\phi_c^{<}\mathcal{D}\phi_\Delta^{<}\mathcal{D}v_c^{<}\mathcal{D}v_\Delta^{<} e^{iS_{\mathrm{H},0}^{<}} e^{iS_{\mathrm{H,int}}^{<}\left[\phi_c^{<},\phi_\Delta^{<},v_c^{<},v_\Delta^{<}\right]} e^{i\,\Gamma\left[\phi_c^{<},\phi_\Delta^{<},v_c^{<},v_\Delta^{<}\right]}\,, \tag{4.29}$$

$$\begin{aligned} e^{i\Gamma} &:= \int \mathcal{D}\phi_c^{>}\mathcal{D}\phi_\Delta^{>}\mathcal{D}v_c^{>}\mathcal{D}v_\Delta^{>} e^{iS_{\mathrm{H},0}^{>}} \Psi_0^{>}\Psi_0^{>*} \\ &\quad \times e^{i\left(S_{\mathrm{H,int}}\left[\phi_c=\phi_c^{<}+\phi_c^{>},\phi_\Delta=\phi_\Delta^{<}+\phi_\Delta^{>},v_c=v_c^{<}+v_c^{>},v_\Delta=v_\Delta^{<}+v_\Delta^{>}\right]-S_{\mathrm{H,int}}^{<}\right)} e^{i\tilde{S}_{\mathrm{bilinear}}\left[v_\Delta^{<},\phi_\Delta^{<},v_c^{>},\phi_c^{>}\right]}\,, \end{aligned} \tag{4.30}$$

with boundary conditions (4.17). $\exp[i\Gamma]$ is the Feynman and Vernon's influence functional [57]. Here, $S_{\mathrm{H,int}}^{<}[\phi_c^{<},\phi_\Delta^{<},v_c^{<},v_\Delta^{<}] := S_{\mathrm{H,int}}[\phi_c^{<},\phi_\Delta^{<},v_c^{<},v_\Delta^{<}]$ is the part of the interaction potential purely composed of the IR modes. $S_{\mathrm{H,int}} - S_{\mathrm{H,int}}^{<}$ denotes the non-linear interactions between UV modes and IR modes, and self interactions of UV modes. In these expressions (4.29) and (4.30), the non-interacting parts are $S_{\mathrm{H},0}^{<}$ and $S_{\mathrm{H},0}^{>}$, and the interacting parts to be treated perturbatively are $\tilde{S}_{\mathrm{bilinear}}$, $S_{\mathrm{H,int}}^{<}$, and $S_{\mathrm{H,int}} - S_{\mathrm{H,int}}^{<}$. We will explain how we can perform the perturbation theory in this prescription more in detail in sec. 4.1.3.

One remark is in order in writing the time derivatives of IR fields in real space. In order to make writing

$$\begin{aligned} S_{\mathrm{H},0}^{<} &= \int \mathrm{d}^4x\, a^3 \left(v_c^{<}\dot{\phi}_\Delta^{<} + v_\Delta^{<}\dot{\phi}_c^{<} - v_c^{<}v_\Delta^{<} - \frac{(\vec{\nabla}\phi_c^{<})\,(\vec{\nabla}\phi_\Delta^{<})}{a^2} \right) \\ &= \int \mathrm{d}^4x\, a^3 \left(-\phi_\Delta^{<}\left(\dot{v}_c^{<} + 3Hv_c^{<}\right) + v_\Delta^{<}\dot{\phi}_c^{<} - v_c^{<}v_\Delta^{<} - \frac{(\vec{\nabla}\phi_c^{<})\,(\vec{\nabla}\phi_\Delta^{<})}{a^2} \right)\,, \end{aligned} \tag{4.31}$$

to be consistent, we need to adopt the convention that the time derivatives which are acting on IR fields in real space $\phi^{<}(x)$ or $v^{<}(x)$ do not act on the step function $\Theta(\epsilon aH - k)$ included in the above expressions written in terms of variables in momentum space, *i.e.*,

$$\dot{\phi}^{<}(x) := \int \frac{\mathrm{d}^3k}{(2\pi)^3}\Theta\left(\epsilon aH - k\right)\dot{\phi}^{<}\left(t,\vec{k}\right)e^{i\vec{k}\cdot\vec{x}}\,, \tag{4.32a}$$

$$\dot{v}^{<}(x) := \int \frac{\mathrm{d}^3k}{(2\pi)^3}\Theta\left(\epsilon aH - k\right)\dot{v}^{<}\left(t,\vec{k}\right)e^{i\vec{k}\cdot\vec{x}}\,, \tag{4.32b}$$

where the label c or Δ is abbreviated in eqs. (4.32). This rule is always adopted in our formalism. Regarding the time derivatives of UV fields in real space, the same discussion applies.

Now, we move on to deriving the effective EOM for IR modes for a given $i\Gamma$. We decompose $S_{\mathrm{H}}^{<}$ and Γ as $S_{\mathrm{H,int}}^{<} = S_{\mathrm{H,int\,(d)}}^{<} + S_{\mathrm{H,int\,(s)}}^{<}$ and $\Gamma = \Gamma_{(\mathrm{d})} + \Gamma_{(\mathrm{s})}$, where the terms with the subscript (d) denote the terms linear in $v_\Delta^{<}$ or $\phi_\Delta^{<}$, while the terms with subscript (s) denote the other higher order pieces. The exponent of $\exp[iS_{\mathrm{H,int\,(s)}}^{<} + i\Gamma_{(\mathrm{s})}]$ can be rewritten in the form linear in $\phi_\Delta^{<}$ or $v_\Delta^{<}$ by using the functional Fourier transformation:

$$\begin{aligned} & e^{i\Gamma_{(\mathrm{s})}\left[\phi_c^{<},\phi_\Delta^{<},v_c^{<},v_\Delta^{<}\right]+iS_{\mathrm{H,int\,(s)}}^{<}\left[\phi_c^{<},\phi_\Delta^{<},v_c^{<},v_\Delta^{<}\right]} \\ &= \int \mathcal{D}\xi_\phi\mathcal{D}\xi_v P\left[\xi_\phi,\xi_v;\phi_c^{<},v_c^{<}\right] e^{i\int \mathrm{d}^4x\, a^3\xi_v(x)\phi_\Delta^{<}(x) - i\int \mathrm{d}^4x\, a^3\xi_\phi(x)v_\Delta^{<}(x)}\,, \end{aligned} \tag{4.33}$$

where auxiliary fields ξ_ϕ and ξ_v are introduced, and $\mathcal{D}\xi_\phi := \prod_x \mathrm{d}\xi_\phi(x)$, and $\mathcal{D}\xi_v := \prod_x \mathrm{d}\xi_v(x)$. By performing the path integral over $\phi_\Delta^<$ and $v_\Delta^<$, the path integral for IR modes is rewritten as

$$Z^<[0] \simeq \int \mathcal{D}\phi_c^< \mathcal{D}v_c^< \int \mathcal{D}\xi_\phi \mathcal{D}\xi_v P\left[\xi_\phi, \xi_v; \phi_c^<, v_c^<\right] \\ \times \delta\left(\dot{\phi}_c^< - v_c^< - \mu_1 - \xi_\phi\right) \delta\left(\dot{v}_c^< + 3Hv_c^< + \mu_2 - \xi_v\right) , \tag{4.34}$$

where

$$\mu_1 := \frac{-1}{a^3} \frac{\delta}{\delta v_\Delta^<} \left(S^<_{\mathrm{H,int\,(d)}} + \Gamma_{(\mathrm{d})}\right) , \qquad \mu_2 := \frac{-1}{a^3} \frac{\delta}{\delta \phi_\Delta^<} \left(S^<_{\mathrm{H,int\,(d)}} + \Gamma_{(\mathrm{d})}\right) . \tag{4.35}$$

Here, we neglect the $a^{-2}\nabla^2\phi_c^<$ term because it is suppressed by some power of ϵ.

So far, we derived eq. (4.34) starting from the original expression (4.9). Repeating the same procedures, one can also derive the expression for the generating functional for IR fields $Z^<[J^<]$:

$$Z^<[J^<] \simeq \int \mathcal{D}\phi_c^< \mathcal{D}v_c^< \, e^{i\int \mathrm{d}^4x\, a^3\left(J_\Delta^{(\phi)<}\phi_c^< + J_\Delta^{(v)<} v_c^<\right)} \int \mathcal{D}\xi_\phi \mathcal{D}\xi_v P\left[\xi_\phi, \xi_v; \phi_c^<, v_c^<\right] \\ \times \delta\left(\dot{\phi}_c^< - v_c^< - \mu_1 - \xi_\phi + J_c^{(v)<}\right) \delta\left(\dot{v}_c^< + 3Hv_c^< + \mu_2 - \xi_v - J_c^{(\phi)<}\right) , \tag{4.36}$$

where the superscript $<$ assigned for external fields J means that they only contains IR modes: for example, $J_\Delta^{(\phi)<}(x) := (2\pi)^{-3}\int \mathrm{d}^3k\, \Theta(\epsilon aH - k) J_\Delta^{(\phi<}\big(t, \vec{k}\big) \exp\big[i\vec{k}\cdot\vec{x}\big]$. Therefore, correlation functions consisting of IR-c fields can be calculated by using $P[\xi_\phi, \xi_v; \phi_c^<, v_c^<]$ as

$$\left[\prod_{j=1}^{n}\left(\frac{\delta}{ia^3(x_j)\delta J_\Delta^{(\phi)<}(x_j)}\right)\prod_{l=1}^{m}\left(\frac{-\delta}{ia^3(y_l)\delta J_\Delta^{(v)<}(y_l)}\right)\right] Z^<[J^<] \\ \simeq \int \mathcal{D}\xi_\phi \mathcal{D}\xi_v \; P\left[\xi_\phi, \xi_v; \phi_c^<, v_c^<\right] \prod_{j=1}^{n} \phi_c^<(x_j) \prod_{l=1}^{m} v_c^<(y_l)\Bigg|_{\dot{\phi}_c^< - v_c^< - \mu_1 - \xi_\phi = \dot{\phi}_c^< - v_c^< - \mu_1 - \xi_\phi = 0} . \tag{4.37}$$

Therefore, an effective EOM for IR modes which correctly recovers all IR correlation functions consisting of IR c-fields is

$$\dot{\phi}_c^< = v_c^< + \mu_1 + \xi_\phi \,, \tag{4.38a}$$
$$\dot{v}_c^< = -3Hv_c^< - \mu_2 + \xi_v \,, \tag{4.38b}$$

or equivalently,

$$\ddot{\phi}_c^< + 3H\dot{\phi}_c^< = -\mu_2 + 3H\mu_1 + \dot{\mu}_1 + 3H\xi_\phi + \dot{\xi}_\phi + \xi_v \,, \tag{4.39a}$$
$$\dot{v}_c^< = -3Hv_c^< - \mu_2 + \xi_v \,. \tag{4.39b}$$

If we regard ξ_ϕ and ξ_v as stochastic noises, the above set of equations can be identified as a set of Langevin equations. The noise correlations are given by

$$\langle \xi_v(x_1)\cdots\xi_v(x_n)\xi_\phi(y_1)\cdots\xi_\phi(y_m)\rangle \\ := \int \mathcal{D}\xi_\phi \mathcal{D}\xi_v P\left[\xi_\phi, \xi_v; \phi_c^<, v_c^<\right] \xi_v(x_1)\cdots\xi_v(x_n)\xi_\phi(y_1)\cdots\xi_\phi(y_m) \,, \tag{4.40}$$

and $P\left[\xi_\phi, \xi_v; \phi_c^<, v_c^<\right]$ corresponds to the weight function of the stochastic noises. Using eq. (4.33), we found that the noise correlation which is defined by eq. (4.40) can be calculated once we obtain the influence functional $\Gamma_{(\mathrm{s})}$ as

$$\begin{aligned}
&\langle \xi_v(x_1)\cdots\xi_v(x_n)\xi_\phi(y_1)\cdots\xi_\phi(y_m)\rangle \\
&= \left[\prod_{j=1}^{n}\left(\frac{\delta}{ia^3(x_j)\delta\phi_\Delta^<(x_j)}\right)\prod_{l=1}^{m}\left(\frac{-\delta}{ia^3(y_l)\delta v_\Delta^<(y_l)}\right)\right] e^{i\Gamma_{(\mathrm{s})}+iS^<_{\mathrm{H,int(s)}}}\Big|_{\phi_\Delta^<=v_\Delta^<=0}\,. \qquad (4.41)
\end{aligned}$$

Reality of the noise probability distributions

Here, we will briefly show that the probability distribution P of the noise defined above is real. Because flipping the sign of $\phi_\Delta^<$ and $v_\Delta^<$ is equivalent to take the complex conjugate of the path integral $Z[0]$, and the functional measure $\mathcal{D}\phi_\Delta^<\mathcal{D}v_\Delta^<$ is invariant under these transformations, $i\Gamma_{(\mathrm{s})}[\phi_c^<, \phi_\Delta^<, v_\Delta^<]$ and $iS^<_{\mathrm{H,int(s)}}[\phi_c^<, \phi_\Delta^<]$ must obey

$$i\Gamma_{(\mathrm{s})}\left[\phi_c^<, \phi_\Delta^<, v_\Delta^<\right] + iS^<_{\mathrm{H,int(s)}}\left[\phi_c^<, \phi_\Delta^<\right] = \Big(i\Gamma_{(\mathrm{s})}\left[\phi_c^<, -\phi_\Delta^<, -v_\Delta^<\right] + iS^<_{\mathrm{H,int(s)}}\left[\phi_c^<, -\phi_\Delta^<\right]\Big)^*\,. \qquad (4.42)$$

On the other hand, from eq. (4.33), P and its complex conjugate P^* can be written as

$$P\left[\xi_\phi, \xi_v, \phi_c^<\right] = \int \mathrm{d}\phi_\Delta^< \int \mathrm{d}v_\Delta^< e^{i\Gamma_{(\mathrm{s})}[\phi_c^<,\phi_\Delta^<,v_\Delta^<]+iS^<_{\mathrm{H,int(s)}}[\phi_c^<,\phi_\Delta^<]} e^{i\int \mathrm{d}^4x a^3(v_\Delta^<(x)\xi_\phi(x)-\phi_\Delta^<(x)\xi_v(x))}\,, \qquad (4.43a)$$

$$\begin{aligned}
&\left(P\left[\xi_\phi, \xi_v, \phi_c^<\right]\right)^* \\
&= \int \mathrm{d}\phi_\Delta^< \int \mathrm{d}v_\Delta^< e^{\left(i\Gamma_{(\mathrm{s})}[\phi_c^<,-\phi_\Delta^<,-v_\Delta^<]+iS^<_{\mathrm{H,int(s)}}[\phi_c^<,-\phi_\Delta^<]\right)^*} e^{i\int \mathrm{d}^4x a^3(v_\Delta^<(x)\xi_\phi(x)-\phi_\Delta^<(x)\xi_v(x))}\,. \qquad (4.43b)
\end{aligned}$$

Combining eqs. (4.42) and (4.43), P turns out to be real,

$$P\left[\xi_\phi, \xi_v, \phi_c^<\right] = \left(P\left[\xi_\phi, \xi_v, \phi_c^<\right]\right)^* \in \mathbb{R}\,. \qquad (4.44)$$

4.1.3 Role of bilinear vertexes: recovery of propagators

From now on, let us show the recovery of correlation functions by considering the recovery of propagators by bilinear interaction vertexes. As is noted in sec. 4.1.2, both UV part and IR part are treated as independent variables and the transition from UV to IR is expressed as the bilinear interaction vertexes. We thus define the free part of the path integral in our formalism excluding the bilinear interaction by

$$Z_0[J=0] = \int \mathcal{D}\phi_{c,\Delta}^<\mathcal{D}v_{c,\Delta}^<\ e^{iS^<_{\mathrm{H},0}} \int \mathcal{D}\phi_{c,\Delta}^>\mathcal{D}v_{c,\Delta}^>\ e^{iS^>_{\mathrm{H},0}}\Psi_0^>\Psi_0^{>*}\,, \qquad (4.45)$$

with boundary conditions (4.17). In this expression without interaction terms, it is obvious that UV modes and IR modes are treated independently.

Free parts

The UV free propagators are given by a completely standard form as

$$G^{>ij}_{cc}(x,x') \coloneqq \int \mathcal{D}v^{>}_{c,\Delta}\mathcal{D}\phi^{>}_{c,\Delta}\ \phi^{>i}_{c}(x)\phi^{>j}_{c}(x')e^{iS^{>}_{\mathrm{H},0}}\Psi^{>}_{0}\Psi^{>*}_{0} = \frac{1}{2}\left\langle 0\left|\left\{\phi^{>i}_{\mathrm{I}}(x),\phi^{>j}_{\mathrm{I}}(x')\right\}\right|0\right\rangle\,, \tag{4.46a}$$

$$G^{>ij}_{c\Delta}(x,x') \coloneqq \int \mathcal{D}v^{>}_{c,\Delta}\mathcal{D}\phi^{>}_{c,\Delta}\ \phi^{>i}_{c}(x)\phi^{>j}_{\Delta}(x')e^{iS^{>}_{\mathrm{H},0}}\Psi^{>}_{0}\Psi^{>*}_{0} = \Theta(t-t')\left[\phi^{>i}_{\mathrm{I}}(x),\phi^{>j}_{\mathrm{I}}(x')\right]\,, \tag{4.46b}$$

$$\begin{aligned}G^{>ij}_{\Delta c}(x,x') &\coloneqq \int \mathcal{D}v^{>}_{c,\Delta}\mathcal{D}\phi^{>}_{c,\Delta}\ \phi^{>i}_{\Delta}(x)\phi^{>j}_{c}(x')e^{iS^{>}_{\mathrm{H},0}}\Psi^{>}_{0}\Psi^{>*}_{0}\\ &= -\Theta(t'-t)\left[\phi^{>i}_{\mathrm{I}}(x),\phi^{>j}_{\mathrm{I}}(x')\right] = G^{>ij}_{c\Delta}(x',x)\,,\end{aligned} \tag{4.46c}$$

$$G^{>ij}_{\Delta\Delta}(x,x') \coloneqq \int \mathcal{D}v^{>}_{c,\Delta}\mathcal{D}\phi^{>}_{c,\Delta}\ \phi^{>i}_{\Delta}(x)\phi^{>j}_{\Delta}(x')e^{iS^{>}_{\mathrm{H},0}}\Psi^{>}_{0}\Psi^{>*}_{0} = 0\,, \tag{4.46d}$$

with boundary conditions (4.17a). Here, $\phi^1 \coloneqq \phi, \quad \phi^2 \coloneqq v$. From eq. (4.46), one can see that the cc propagator is the symmetric propagator, the $c\Delta$ propagator is the retarded Green's function, and the Δc propagator is the advanced Green's function. These propagators can be written more explicitly in terms of the mode functions as

$$\begin{aligned}G^{>ij}_{cc}(x,x') &= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\Theta(k-\epsilon aH)\,\Theta\left(k-\epsilon a(t')H\right)\,\mathrm{Re}\left[\Phi^i_k(t)\Phi^{*j}_k(t')\right]e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\,,\\ G^{>ij}_{c\Delta}(x,x') &= 2i\,\Theta(t-t')\int \frac{\mathrm{d}^3k}{(2\pi)^3}\Theta(k-\epsilon aH)\,\mathrm{Im}\left[\Phi^i_k(t)\Phi^{*j}_k(t')\right]e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\,,\end{aligned}$$

where

$$\Phi^1_k(t) \coloneqq \Phi_k(t),\ \ \Phi^2_k(t) \coloneqq \dot{\Phi}_k(t)\,. \tag{4.48}$$

Next, we consider IR free propagators. Since the propagators including $v^<$ can be obtained by taking the derivative of $G^{<11}$ with respect to time, we concentrate on $G^{<11}$.[3] As in the ordinary case, the retarded propagator obeys the equation

$$\Box_x\,G^{<11}_{c\Delta}(x,x') = \frac{i}{a^3}\delta(t-t')\int \frac{\mathrm{d}^3k}{(2\pi)^3}\Theta\left(\epsilon a(t')H-k\right)\,e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\,,$$

and the boundary conditions are specified by $G^{<11}_{c\Delta}(x,x') = 0$ for $t < t'$. Therefore, $G^{<ij}_{c\Delta}(x,x')$ can be written by means of the mode functions as

$$G^{<ij}_{c\Delta}(x,x') = 2i\,\Theta(t-t')\int \frac{\mathrm{d}^3k}{(2\pi)^3}\Theta\left(\epsilon a(t')H-k\right)\,\mathrm{Im}\left[\Phi^i_k(t)\Phi^{*j}_k(t')\right]e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\,.$$

On the other hand, the cc propagator obeys

$$\Box_x\,G^{<11}_{cc}(x,x') = 0\,,$$

[3] We should not take the time derivative of a step function $\Theta\left(\epsilon a(t')H-k\right)$ in a momentum integral: see eqs. (4.49).

with the boundary conditions that $G_{cc}^{<11}(x,x')=0$ at $t=t_0$, because we do not have the initial wave functional for IR modes. This reflects our assumtion that there is no IR mode at $t=t_0$. Therefore, $G_{cc}^{<11}(x,x')$ vanishes identically. Since the $\Delta\Delta$ propagator $G_{\Delta\Delta}^{<11}(x,x')$ also obeys $\Box_x\, G_{\Delta\Delta}^{<11}(x,x')=0$, $G_{\Delta\Delta}^{<ij}(x,x')$ is identically zero for the same reason.

To summarize, the free propagators of IR modes are

$$G_{cc}^{<ij}(x,x') = 0\,, \tag{4.49a}$$

$$G_{c\Delta}^{<ij}(x,x') = 2i\Theta(t-t')\int\frac{\mathrm{d}^3k}{(2\pi)^3}\Theta\left(\epsilon a(t')H-k\right)\,\mathrm{Im}\left[\Phi_k^i(t)\Phi_k^{*j}(t')\right]e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\,, \tag{4.49b}$$

$$G_{\Delta\Delta}^{<ij}(x,x') = 0\,. \tag{4.49c}$$

The IR parts of $G_{cc}^{ij}(x,x')$, which exist in the ordinary QFT, are recovered by taking into account the bilinear interaction, as we shall prove soon.

We explain how we describe these free propagators diagrammatically. We write the IR and UV free propagators as solid and dashed, respectively. Lines associated with an arrow correspond to $c\Delta$ propagators, while lines without an arrow correspond to cc propagators, as shown in fig. 4.1. The arrow indicates the time direction. We explicitly associate ϕ or v to the end of propagators when the distinction is necessary.

Interaction vertexes

Next we discuss the vertexes. In our formalism, there are two types of interaction terms. One is the usual self-interaction, and the other is the bilinear interaction, which is peculiar to the present formulation. Self-interaction is described by $S_{\mathrm{H,int}}$. This vertex is treated just in the same way as in the usual perturbation theory. Note that we have not only nonlinear self interactions of IR fields or UV fields, but also have nonlinear interactions between UV fields and IR fields.

We describe the bilinear interaction as a two-point vertex, which we call bilinear vertex, in the Feynman diagrams, as shown in fig. 4.2. Owing to the factor $\delta(t-t_k)$ in (4.27c), the integrand is non-vanishing only for k with $k=\epsilon aH$. As a Feynman rule, we assign $-i\delta(t-t_k)$ to the vertex $v_\Delta^<\left(\vec{k},t\right)\phi_c^>\left(-\vec{k},t\right)$, and $i\delta(t-t_k)$ to the vertex $\phi_\Delta^<\left(\vec{k},t\right)v_c^>\left(-\vec{k},t\right)$, and the vertex integral is taken over $\int\mathrm{d}t\,a^3$.

Now, we prove that the bilinear interaction vertexes correctly reproduce the transition from UV to IR. For this purpose, it is sufficient to show that the ordinary two point functions for a free field, $G_{IJ}(x,x')$, with I,J being c or Δ, can be reproduced by taking into account the bilinear interaction. It is more convenient to focus on the Fourier component , $G_{IJ}\left(\vec{k},t,\vec{k}',t'\right)$. We show the equivalence between $G_{IJ}\left(\vec{k},t,\vec{k}',t'\right)$ and

$$\begin{aligned}&\left\langle\phi_I^i\left(\vec{k},t\right)\phi_J^j\left(\vec{k}',t'\right)\right\rangle^{(0)}\\ &:=\int\mathcal{D}\phi_{c,\Delta}^<\,\mathcal{D}v_{c,\Delta}^<\,e^{iS_{\mathrm{H},0}^<}\int\mathcal{D}\phi_{c,\Delta}^>\,\mathcal{D}v_{c,\Delta}^>\,e^{iS_{\mathrm{H},0}^>}\,e^{i\tilde{S}_{\mathrm{bilinear}}}\,\phi_I^i\left(\vec{k},t\right)\phi_J^j\left(\vec{k}',t'\right)\Psi_0^>\Psi_0^{>*}\,.\end{aligned} \tag{4.50}$$

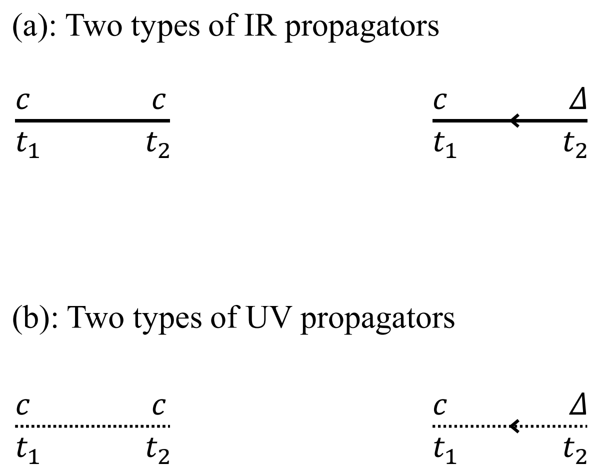

Figure 4.1: Solid and dashed lines denote IR and UV free propagators, respectively. Therefore, figures (a) express two types of IR propagators, and figures (b) express two types of UV propagators. Lines associated with an arrow correspond to $c\Delta$ propagators, while lines without an arrow correspond to cc propagators: the left-hand side of this figure shows cc propagators, and propagators on the right hand side are $c\Delta$ propagators with $t_1 \geq t_2$.

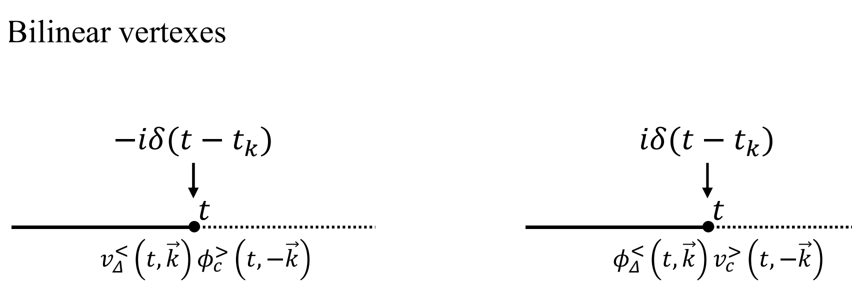

Figure 4.2: This diagram is the bilinear vertex. The corresponding vertex factor is also noted in this figure. Vertex factors assigned to $v_\Delta^< \phi_c^>$ bilinear vertex and $\phi_\Delta^< v_c^>$ bilinear vertex are $-i\delta(t-t_k)$ and $i\delta(t-t_k)$, respectively. Vertex integral is $\int \mathrm{d}t\, a^3$.

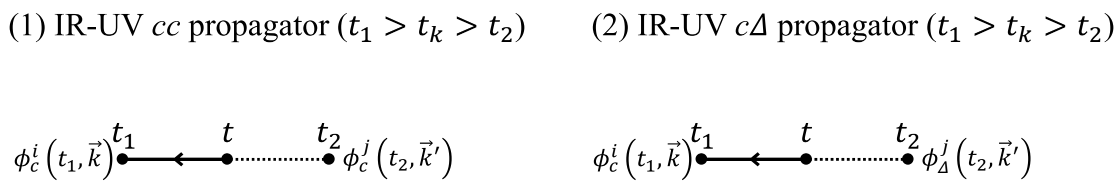

Figure 4.3: These figures show the contributions of bilinear vertexes to the two point functions with mode k which satisfies $\epsilon a(t_1)H \leq k \leq \epsilon a(t_2)H$. The left figure shows the generation of IR-UV cc two-point function and the right figure shows the generation of IR-UV $c\Delta$ two-point function.

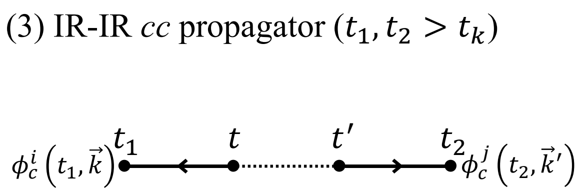

Figure 4.4: This figure shows the contributions of bilinear vertexes to the two point functions with mode k which satisfies $k = \epsilon a(t_1)H = \epsilon a(t_2)H$.

Here, we stress that the bilinear interaction that we have introduced connects $\phi_\Delta^{<i}$ and $\phi_c^{>j}$. As the IR propagator vanishes for $G_{\Delta\Delta}^{<ij}$, the insertion of bilinear vertexes is relevant only when we consider the two-point functions that include $\phi_c^i\left(\vec{k},t\right)$ with $t > t_k$. In fact, the free propagators that we derived in the preceding subsection agree with the ordinary ones, except for $\left\langle \phi_c^i\left(\vec{k},t\right)\phi_c^j\left(\vec{k}',t'\right)\right\rangle^{(0)}$ and $\left\langle \phi_c^i\left(\vec{k},t\right)\phi_\Delta^j\left(\vec{k}',t'\right)\right\rangle^{(0)}$ with $t > t_k$ and $t' < t_{k'}$, and $\left\langle \phi_c^i\left(\vec{k},t\right)\phi_c^j\left(\vec{k}',t'\right)\right\rangle^{(0)}$ with $t > t_k$ and $t' > t_{k'}$, which we call 0-th order IR-UV cc and $c\Delta$, and IR-IR cc two-point functions, respectively. Here, 0-th order means the order with respect to λ, and 0-th order two-point functions are distinguished from the free propagators. The Feynman diagrams that contribute to these 0-th order two-point functions from the bilinear interaction are presented in figs. 4.3 and 4.4. We discuss these diagrams in turn.

1. IR-UV cc two-point function
   Assuming that $t_1 > t_k$ and $t_2 < t_{k'}$, the contribution of the diagram (1) in fig. 4.3 corresponding to $\left\langle \phi_c^i\left(\vec{k},t_1\right)\phi_c^j\left(\vec{k}',t_2\right)\right\rangle^{(0)}$, excluding the trivial factor $\delta^3\left(\vec{k}-\vec{k}'\right)$,

is evaluated as

$$\int \mathrm{d}t\, a^3 \delta\left(t-t_k\right) \frac{i}{2}\Big[\left(\Phi_1^i \Phi^* - {\Phi_1^i}^* \Phi\right)\left(\dot{\Phi}{\Phi_2^j}^* + \dot{\Phi}^* \Phi_2^j\right)$$
$$- \left(\Phi_1^i \dot{\Phi}^* - {\Phi_1^i}^* \dot{\Phi}\right)\left(\Phi{\Phi_2^j}^* + \Phi^* \Phi_2^j\right)\Big]$$
$$= \int \mathrm{d}t\, a^3 \delta\left(t-t_k\right) \frac{i}{2}\left({\Phi_1^i}^* \Phi_2^j + \Phi_1^i {\Phi_2^j}^*\right)\left(\dot{\Phi}\Phi^* - \Phi\dot{\Phi}^*\right) = \frac{1}{2}\left({\Phi_1^i}^* \Phi_2^j + \Phi_1^i {\Phi_2^j}^*\right), \tag{4.51}$$

where $\Phi^1 = \Phi$, $\Phi^2 = \dot{\Phi}$ as defined in eq. (4.48). The subscripts 1, 2 associated with Φ^i label the time coordinate in its argument, and the subscript on the mode function Φ to specify the momentum k is abbreviated, for notational simplicity. In the last equality, we used the normalization condition of mode functions (4.4). The above result reproduces the ordinary expression for IR-UV cc two-point function.

2. IR-UV $c\Delta$ two-point function
In a completely analogous manner, the contribution of the diagram (2) in fig. 4.3 correponding to $\left\langle \phi_c^i\left(\vec{k}, t_1\right)\phi_\Delta^j\left(\vec{k}', t_2\right)\right\rangle^{(0)}$ with $t_1 > t_k$ and $t_2 < t_{k'}$ is evaluated as

$$\int \mathrm{d}t\, a^3 \delta\left(t-t_k\right) i\Big[\left(\Phi_1^i \Phi^* - {\Phi_1^i}^* \Phi\right)\left(\dot{\Phi}{\Phi_2^j}^* - \dot{\Phi}^* \Phi_2^j\right)$$
$$- \left(\Phi_1^i \dot{\Phi}^* - {\Phi_1^i}^* \dot{\Phi}\right)\left(\Phi{\Phi_2^j}^* + \Phi^* \Phi_2^j\right)\Big]$$
$$= \int \mathrm{d}t\, a^3 \delta\left(t-t_k\right) i\left(\Phi_1^i {\Phi_2^j}^* - {\Phi_1^i}^* \Phi_2^j\right)\left(\dot{\Phi}\Phi^* - \Phi\dot{\Phi}^*\right) = \left(\Phi_1^i {\Phi_2^j}^* - {\Phi_1^i}^* \Phi_2^j\right), \tag{4.52}$$

which reproduces the ordinary expression for IR-UV $c\Delta$ two-point function.

3. IR-IR cc two-point function
We evaluate the contribution of the diagram (3) in fig. 4.4. This contributions corresponds to $\left\langle \phi_c^i\left(\vec{k}, t_1\right)\phi_c^j\left(\vec{k}', t_2\right)\right\rangle^{(0)}$ with $t_1 > t_k$ and $t_2 > t_{k'}$. We first perform the integration over t', using the result of eq. (4.51). Then, the remaining integration over t is evaluated as

$$\int \mathrm{d}t\, \delta\left(t-t_k\right) \frac{-ia^3}{2}\Big[\left(\Phi_1^i \Phi^* + {\Phi_1^i}^* \Phi\right)\left(\dot{\Phi}^* \Phi_2^j - \dot{\Phi}{\Phi_2^j}^*\right)$$
$$- d\left(\Phi_1^i \dot{\Phi}^* + {\Phi_1^i}^* \dot{\Phi}\right)\left(\Phi^* \Phi_2^j - \Phi{\Phi_2^j}^*\right)\Big]$$
$$= \int \mathrm{d}t\, \delta\left(t-t_k\right) \frac{-ia^3}{2}\Big[\left({\Phi_1^i}^* \Phi_2^j + \Phi_1^i {\Phi_2^j}^*\right)\left(\Phi\dot{\Phi}^* - \dot{\Phi}\Phi^*\right)\Big] = \frac{1}{2}({\Phi_1^i}^* \Phi_2^j + \Phi_1^i {\Phi_2^j}^*), \tag{4.53}$$

which reproduces the ordinary expression for IR-IR cc two-point function.

The above discussions show that all 0-th order two point functions are properly recovered by introducing the bilinear interaction $\tilde{S}_{\text{bilinear}}$, and hence eq. (4.50) is proved.

This recovery of propagator implies that our method including bilinear vertexes can correctly reproduce the Wigner function of IR fields. This implies that the reduced density matrix for IR fields can be also derived using our method. In the next chapter 5, we will derive the explicit relation between the quantum states and the influence functional. Before investigating these relations, in the next sec. 4.2, we will see the recovery of infrared correlation functions using our extended stochastic method more concretely in $\lambda\phi^4$ theory.

4.2 Recovery of infrared correlation functions

In this section we consider the $\lambda\phi^4$ theory as a concrete example. In order to see how our formalism developed in sec 4.1 works well, we firstly derive the leading order corrections to the noise correlations. Next we calculate correlation functions of IR fields $\phi_c^<$ perturbatively in λ by using two methods independently: one is our stochastic equations, and another is conventional perturbative QFT method. Then, we obtained the same answer up to NLO. This means that our stochastic equations correctly reproduce NLO secularly-growing terms, perturbatively in λ.

4.2.1 Derivation of NLO stochastic equations

In order to derive an effective EOM for IR modes, one needs to calculate the influence functional $\exp[i\Gamma]$. Here, we propose a systematic way of estimating the order of the IR secular growth for each diagram which constitutes the influence functional and $S^<_{\text{H,int}}$. Then, using this proposed method, we explicitly derive the effective EOM for IR modes accurate enough to recover IR secular growth up to NLO in $\lambda\phi^4$ theory.

Systematic way of estimating the order of IR secular growth for each diagram

All the diagrams which constitute the influence functional are classified into two: diagrams which contribute to μ_2 and diagrams which contribute to the noise correlation. We especially call the latter diagrams as noise diagrams, for a later convenience.

All the connected diagrams are represented as a whole by the left diagram in fig. 4.5. Δ-fields of IR modes are written on the left and c-fields of IR modes on the right. The gray blob consists of diagrams connected only by UV propagators. A concrete example is given by the right diagram in fig. 4.5. Since all the vertexes are connected by UV propagators, it is expected that the influence functional $i\Gamma$ is approximately local. That is, if one consider a certain diagram with n external vertexes $x_1 \cdots x_n$, the support of the diagram will be approximately given by $\vec{x}_i \sim \vec{x}_j$ and $t_i \sim t_j$ for arbitrary pairs of (i,j), $i,j = 1, \cdots, n$, and outside this region the value of diagram will decay exponentially. Here, $\vec{x}_i \sim \vec{x}_j$ and $t_i \sim t_j$ mean that $|\vec{x}_i - \vec{x}_j| \lesssim 1/\epsilon a(t_i)H \sim 1/\epsilon a(t_j)H$ and $|t_i - t_j| \lesssim \ln(1/\epsilon)/H$, respectively. In this section, we assume this approximate locality is satisfied by all diagrams.[4] Under

[4] Validity of this assumption is discussed in the appendix D of [50].

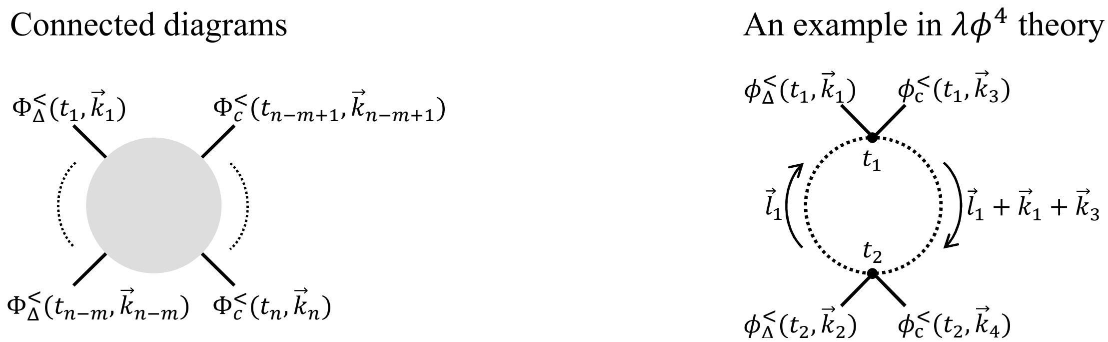

Figure 4.5: The diagram on the left-hand side represents the whole set of connected diagrams which constitute $i\Gamma$. Here, we adopt a rule that IR Δ-fields are written on the left and IR c-fields on the right. In this figure, IR Δ-fields and IR c-fields are represented by $\Phi_\Delta^<$ and $\Phi_c^<$, respectively. The gray blob consists of diagrams connected only by UV propagators. A concrete example of the connected diagram in $\lambda\phi^4$ theory is shown on the right-hand side.

this assumption, we develop a systematic way to estimate the order of IR secular growth for each diagram, or equivalently counting the number of the factor of $\ln(a/a_0)$.

Since the time evolution of correlation functions is determined by eq. (4.39), one can evaluate them in terms of the time integration of the noise correlations for given initial conditions. Then, the factor of $\ln(a/a_0) \propto t - t_0$ comes only from the integration over time $\int \mathrm{d}t'$ from t_0 to t, assuming the approximate temporal-locality of noise correlations. Therefore, in order to estimate the order of IR secular growth for each diagram, we need to count the number of the time integrals included in the diagram whose effective integration range is comparable to the whole range of time that we are concerned with. Firstly, we express $\phi_c^<$ in terms of noise fields. Both ξ_ϕ and ξ_v can be treated simultaneously by the unified noise $\tilde{\xi}$, which is defined by

$$\tilde{\xi} \coloneqq 3H\xi_\phi + \dot{\xi}_\phi + \xi_v \,. \tag{4.54}$$

By using the unified noise $\tilde{\xi}$, eq. (4.39) is rewritten as

$$\ddot{\phi}_c^< + 3H\dot{\phi}_c^< = -\mu_2 + \dot{\mu}_1 + 3H\mu_1 + \tilde{\xi} \,. \tag{4.55}$$

From this equation, $\phi_c^<(T)$ can be expressed by using $\tilde{\xi}(t)$ formally as

$$\phi_c^<(T) = i\int_{t_0}^{T} \mathrm{d}t\, a^3 G(T,t)\left[-\mu_2 + \dot{\mu}_1 + 3H\mu_1 + \tilde{\xi}(t)\right] , \tag{4.56}$$

$$G(T,t) \coloneqq \frac{i}{3H}\left(\frac{1}{a^3(T)} - \frac{1}{a^3}\right)\Theta(T-t)\,, \quad -\left(\partial_T^2 + 3H\partial_T\right)G(T,t) = \frac{i}{a^3(T)}\delta(T-t)\,. \tag{4.57}$$

Here, we have abbreviated the spatial argument for notational simplicity. Eq. (4.56) allows us to express $\phi_c^<(T)$ in terms of $\tilde{\xi}$ iteratively expanding with respect to the coupling constant. We will take into account the self-interaction terms later and concentrate on

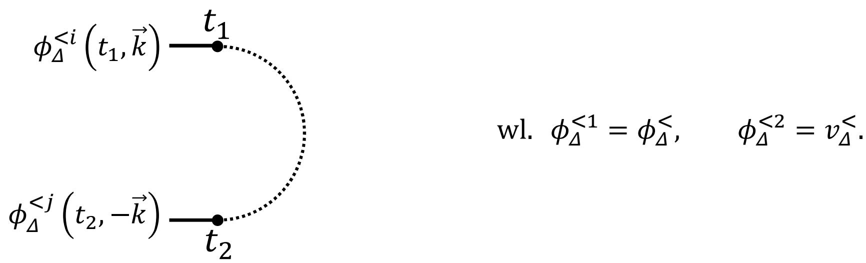

Figure 4.6: This figure shows the noise diagrams with $(i,j)=(1,1),(1,2),(2,1),(2,2)$ which contribute to the LO noise correlation which is described by eq. (3.63). More precisely, the diagram with $(i,j)=(2,2)$ which does not contain $\phi_\Delta^<$ fields contribute to the LO noise correlation, and the remaining diagrams are $\mathcal{O}(\epsilon)$ suppressed.

0-th order with respect to the coupling constant now. Then, the number of time integrals to be assigned to $\phi_c^<$ is counted as $1-(1/2)=1/2$ because the noise correlation $\left\langle\tilde{\xi}\tilde{\xi}\right\rangle$ is given by eq. (3.63) at 0-th order. Next, we establish how to assign the number of time integrals to each diagram with Δ-fields of IR modes. When a Δ-field of IR modes is contracted with $\phi_c^<$, one gets retarded Greens functions $G_{c\Delta}^{<1i}$. Although the number of time integrals attributed to $\phi_c^<$ is counted as $1/2$, retarded Green's functions $G_{c\Delta}^{<1i}$ themselves do not grow at a late time, which implies that the number of time integrals attributed to $\phi_c^<$ is canceled by $\phi_\Delta^<$, and hence the number of time integral to be assigned to a Δ-field of IR modes is counted as $-1/2$ at 0-th order. Finally, from vertex integrals included in the gray blob in fig. 4.5, only one time integral remains unconstrained because of the approximate locality of $i\Gamma$. From now on, we refer to this time integral as the time integral of the vertex.

For example, let us consider $\lambda\phi^4$ theory. From the above discussion, the number of time integrals attributed to the diagram with $n-m$ Δ-fields and m c-fields of IR modes, which is shown in fig. 4.5, is $1+(m/2)-((n-m)/2)$. Suppose that this diagram is $\mathcal{O}\left(\lambda^l\right)$. Then, this diagram contributes to IR secular growth terms in correlation functions to

$$\mathrm{N}^{2l-[1+(m/2)-((n-m)/2)]}\mathrm{LO}=\mathrm{N}^{(n/2)+2l-(m+1)}\mathrm{LO}\,, \tag{4.58}$$

because the l-th order vertex in λ which consists only of LO vertex $\lambda\phi_c^{<3}\phi_\Delta^<$ has $(3l/2)-(l/2)+l=2l$ time integrals, and the difference between $2l$ and $1+(m/2)-((n-m)/2)$ is $(n/2)+2l-(m+1)$. We can gather the diagrams necessary to derive the stochastic formalism described by eq. (3.63), which is valid up to LO, by using eq. (4.58): see appendix A.2. Here, we just mention that the diagram which is relevant to LO IR effects is shown in fig. 4.6: this diagram exists even without nonlinear interaction vertexes.

We have established the counting rule, but the vertexes with one UV leg suffer from further suppression. We assign momenta as shown in fig. 4.7. Although three IR propagators are attached to this vertex, the range of the time integral is suppressed unless two of the momenta of IR propagators are around ϵaH due to the momentum conservation. If we assume that an inequality $k_1\sim k_2\lesssim\delta\epsilon aH\ll k_3\sim\epsilon aH$ with $\delta\ll 1$ and $\ln(1/\delta)\ll\ln(a/a_0)$ holds, the integration range of the integral over $\vec{k}_3$ is restricted to very

Vertex with restricted IR fields

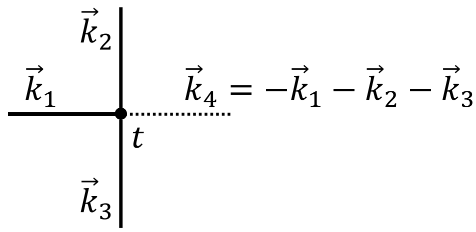

Figure 4.7: Three IR propagators and one UV propagator are combined via this vertex. Because of the momentum conservation law, one cannot take the zero-momentum limit about two or three IR momenta, with escaping from the phase-space volume suppression. This implies at least two IR momenta assigned to this vertex are restricted to the range $k \sim \epsilon a H$.

narrow region:

$$\epsilon a H \left(1 - \frac{\left|\vec{k}_1 + \vec{k}_2\right|}{\epsilon a H}\right) \lesssim k_3 \leq \epsilon a H \,,$$

where higher order terms with respect to $\left|\vec{k}_1 + \vec{k}_2\right|/k_3$ are neglected. When one evaluates the contribution from this vertex to correlation functions, this restriction of the integration range implies that the remaining integrand after performing the integration over $\vec{k}_3$ is suppressed by factor $\left|\vec{k}_1 + \vec{k}_2\right|/\epsilon a H$, compared to the case without this restriction. Therefore, contribution from this vertex is suppressed by factor $\mathcal{O}(\delta)$ when two of three momenta are deep IR modes, which means that one has to put two momenta around $\epsilon a H$ in evaluating the contribution from this vertex to correlation functions. For this reason, one needs to change the counting rule mentioned above in evaluating the diagram containing the vertexes with one UV leg. From now on, we refer to a field whose assigned momentum is restricted to $\delta\epsilon a H \lesssim k < \epsilon a H$ as a restricted field. In estimating the order of IR secular growth, IR propagators including restricted fields can be replaced by UV propagators with mode $\vec{k}$ satisfying $k \sim \epsilon a H$. This is because the contribution from IR propagators including restricted $\phi_c^<$ are simply obtained by changing the lower bound of momentum integration of UV propagators from $\epsilon a H$ to $\delta \epsilon a H$ with $\delta \ll 1$ and $\ln(1/\delta) \ll \ln(a/a_0)$, and this replacement does not change the order of magnitude. Therefore, in evaluating the order of IR secular growth, one can perform the following replacements in effect:

$$\lambda \phi_\Delta^< \phi_c^{<2} \phi_c^> \to \lambda \phi_c^< \phi_\Delta^> \phi_c^{>2} \text{ or } \lambda \phi_\Delta^< \phi_c^{>3} \,, \quad \lambda \phi_c^{<3} \phi_\Delta^> \to \lambda \phi_c^< \phi_\Delta^> \phi_c^{>2} \,.$$

There are also vertexes which can only contribute to $\mathcal{O}(\epsilon)$ terms in correlation functions and hence are irrelevant in our analysis. The Fourier components of IR propagators $\tilde{G}_{cc}^{<ij}(t,t';k)$ and $\tilde{G}_{c\Delta}^{<ij}(t,t';k)$ satisfy the following relations:

$$\frac{\tilde{G}_{c\Delta}^{<i1}(t,t';k)}{\tilde{G}_{cc}^{<i1}(t,t';k)} \sim \begin{cases} \mathcal{O}\left(\epsilon^3\right) & i = 1 \,, \\ \mathcal{O}(\epsilon) & i = 2 \,. \end{cases}$$

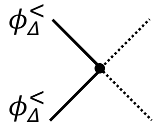

Figure 4.8: Examples of vertexes which have several IR Δ-fields.

Therefore, in evaluating correlation functions which consist of $\phi_c^<$ and $v_c^<$, we can neglect diagrams which include vertexes with more than one $\phi_\Delta^<$. For example, we can neglect $S_{\mathrm{H,int(s)}}^<$ and the vertexes shown in fig. 4.8. Note that we give an alternative proof of this in sec. 4.3.

NLO stochastic dynamics

Now, let's move on to derive an effective EOM which can describe IR secular growth up to NLO. From now on, we refer to this effective EOM as NLO stochastic EOM. In order to derive NLO stochastic EOM, let us consider the number of time integrals to be assigned to the vertexes with UV legs which constitutes the diagrams shown in fig. 4.5. From the discussion in sec. 4.2.1, the number of time integrals attributed to a vertex with p UV legs and q $\phi_\Delta^<$ fields is counted as $-(p/2) - q$ for $p \geq 2$ compared to the leading order vertex $\lambda\phi_\Delta^<\phi_c^{<3}$. We need to take care when $p = 1$, as we discussed above: $p = 1$ vertex is equivalent to $p = 3$ vertex regarding this counting because of the presence of restricted fields. Therefore, maximum value attributed to a vertex with UV legs is -1, which is realized for $(p, q) = (2, 0)$, and hence the order of IR secular growth decreases as the number of these vertexes increases. Then, one may think that the most dominant l-th order diagram at a late time would be given by the diagram which only consists of $\lambda\phi_c^{<2}\phi_c^>\phi_\Delta^>$ vertex, which would contribute to N^{l-1}LO IR secular growth where -1 coming from the time integral of the vertex. However, all the diagrams must have at least one Δ-field of IR modes from the requirement of causality, and this cancel the time integral of the vertex. Thus, l-th order diagrams can contribute to N^{l}LO at most, and we need to consider only first order diagrams with respect to λ to derive NLO stochastic EOM. By writing down all the $\mathcal{O}(\lambda)$ diagrams and using the method developed above, we can show that the diagrams shown in fig. 4.9 can contribute to NLO IR secular growth.[5] However, since the value of UV loop integral in diagram (e) is time independent, the contribution from diagram (e) can be exactly canceled by an appropriate choice of the finite term in the mass renormalization δm^2, and we choose so in our study. Thus, what we need to calculate are only diagrams (a), (b), (c), and (d).

[5]Diagrams which are not shown in fig. 4.9 do not contribute to NLO IR secular growth, except for the LO diagrams shown in fig. 4.6.

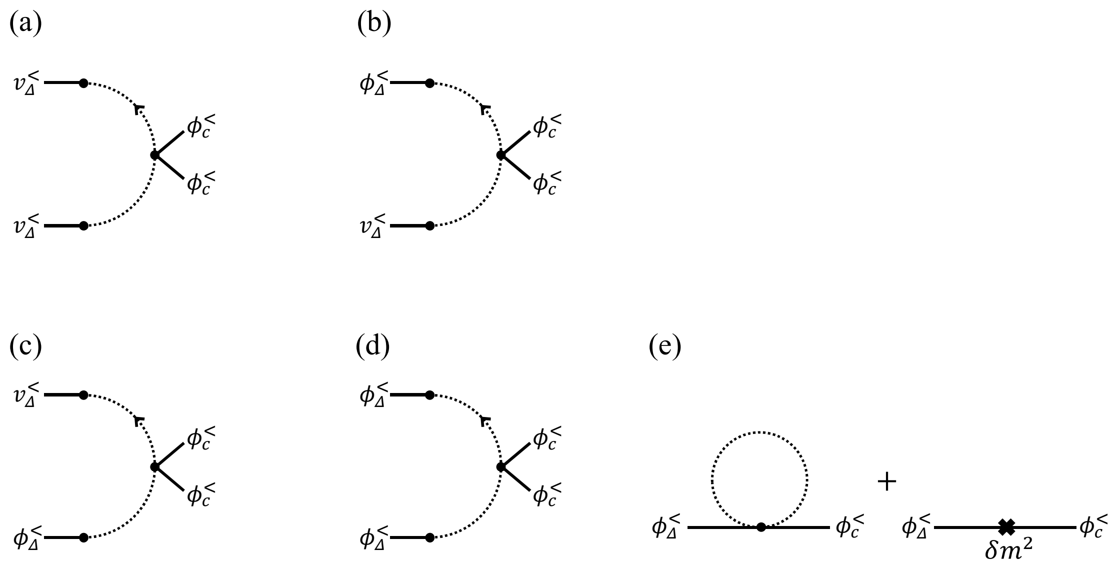

Figure 4.9: These diagrams can contribute to NLO IR secular growth according to the estimation by using eq. (4.58).

The contribution from diagram (a) to $i\Gamma$, which we denote as $i\Gamma_{(\mathrm{a})}$, is evaluated as

$$
\begin{aligned}
i\Gamma_{(\mathrm{a})} &= \int \mathrm{d}^4x_1\, a^3(t_1) v_\Delta^<(x_1) \int \mathrm{d}^4x_2\, a^3(t_2) v_\Delta^<(x_2) N_{(\mathrm{a})}(x_1, x_2)\,, \\
N_{(\mathrm{a})}(x_1, x_2) &\sim \int \mathrm{d}^3x \int_{t_0}^{t_m} \mathrm{d}t\, a^3 \phi_c^{<2}(\vec{x}, t) \int \frac{\mathrm{d}^3k_1}{(2\pi)^3} \int \frac{\mathrm{d}^3k_2}{(2\pi)^3} \\
&\quad \times \tilde{G}_{c\Delta}^{>11}(t_1, t; k_1) \tilde{G}_{cc}^{>11}(t_1, t; k_2) e^{i\vec{k}_1\cdot(\vec{x}_1 - \vec{x})} e^{i\vec{k}_2\cdot(\vec{x}_2 - \vec{x})} \delta(t_1 - t_c(k_1)) \delta(t_2 - t_c(k_2))\,,
\end{aligned}
\tag{4.59}
$$

$$
G^{>ij}(x, x') = \int \frac{\mathrm{d}^3k}{(2\pi)^3} \tilde{G}^{>ij}(t, t'; k) e^{-i\vec{k}\cdot(\vec{x} - \vec{x}')}\,.
$$

Here, $t_m := \min(t_1, t_2)$ and we omit a proportionality coefficient in the 2nd line. Up to NLO, we only need to extract the most dominant IR secular growth from $\phi_c^<(\vec{x}, t)$ in the integrand because these noise diagrams contribute to NLO at most. In this case, when the range of the vertex integral $\int \mathrm{d}^4x$ is restricted to the coarse-graining scale around a certain space-time point $X = (\vec{X}, T)$, we can replace $\phi_c^<(\vec{x}, t)$ by $\phi_c^<(\vec{X}, T)$. This is because the most dominant part of $\phi_c^<(\vec{x}, t)$ at a late time is identical to that of $\phi_c^<(\vec{X}, T)$ if $\vec{x} \sim \vec{X}$ and $t \sim T$.[6] Now, we prove that the range of vertex integral $\int \mathrm{d}^4x$ in eq. (4.59)

[6] This statement can be shown as follows. First, let us consider the time dependence of $\phi_c^<$. From eqs. (4.56) and (4.57), the relationship between $\phi_c^<(\vec{x}, t)$ and $\phi_c^<(\vec{x}, T)$ is given by

$$
\phi_c^<(\vec{x}, t) = \phi_c^<(\vec{x}, T) - i \int_t^T \mathrm{d}t'\, a^3(t') G(T, t') \left[-\mu_2 + \dot{\mu}_1 + 3H\mu_1 + \tilde{\xi}(t') \right].
$$

If $T - t \ll T - t_0$ is satisfied, it is obvious that only $\phi_c^<(\vec{x}, T)$ is the most dominant term on the right-hand is restricted to the coarse-graining scale around a space-time point $x_1 \sim x_2 \sim X$. First, we consider the spatial integral. From the angular integrals of $\vec{k}_1$ and $\vec{k}_2$ in eq. (4.59), $j_0(k_1|\vec{x}_1-\vec{x}|)$ and $j_0(k_2|\vec{x}_2-\vec{x}|)$ appear in the integrand, which lead to exponential decay for $|\vec{x}_1-\vec{x}|,\ |\vec{x}_2-\vec{x}| \gg 1/(\epsilon a H)$ after time coarse-graining. Next, we show that the integrand in eq. (4.59) is exponentially suppressed for $t_m - t \gg 1/H$. For $t_m - t \gg 1/H$, the momenta k_3 and k_4 which are assigned to $\phi_c^{<2}(\vec{x},t)$ are much less than k_1 and k_2 because the upper bound for k_3 and k_4 is given by $\epsilon a H$. In this case, we can set $\vec{k}_1+\vec{k}_2=\vec{0}$, which results in

$$N_{(\mathrm{a})}(x_1,x_2) \sim \int_{t_0}^{t_m} \mathrm{d}t\, \tilde{\phi_c^<}(\vec{k}_3,t)\tilde{\phi_c^<}(-\vec{k}_3,t)\cdot H^2\delta(t_1-t_2)j_0(\epsilon a(t_1)H|\vec{x}_1-\vec{x}_2|)$$
$$\times\left(\frac{a}{a(t_1)}\right)^3\frac{1}{\epsilon^3}\left(f_I(\Delta t_1)+\frac{\epsilon^3}{3}f_R(\Delta t_1)\right)f_R(\Delta t_1)\,,$$

where f_R and f_I are defined in eq. (A.7), $\Delta t_1 \coloneqq t_1 - t$, and $j_0(z) \coloneqq \sin z/z$. From the above equation and eq. (A.7), one can see that the integrand in eq. (4.59) decays exponentially for $t_m - t \gg 1/H$. It is straightforward to apply the above discussion to other diagrams in fig. 4.9.

Now it is justified to replace $\phi_c^<(\vec{x},t)$ by $\phi_c^<(\vec{x}_1,t_1)$ in eq. (4.59), and we get

$$N_{(\mathrm{a})}(x_1,x_2) \simeq -\frac{H\lambda}{24\pi^2}\phi_c^{<2}(x_1)\left(-\ln\frac{1}{\epsilon}+\ln 2+\gamma-2+\mathcal{O}\left(\epsilon^2\right)\right)$$
$$\times\,\delta(t_1-t_2)j_0(\epsilon a(t_1)H|\vec{x}_1-\vec{x}_2|)\,. \tag{4.60}$$

Note that the approximation adopted in the first line holds up to NLO.
Similarly, diagram (b) is calculated as follows:

$$i\Gamma_{(\mathrm{b})} = \int \mathrm{d}^4x_1\, a^3(t_1)\phi_\Delta^<(x_1)\int \mathrm{d}^4x_2\, a^3(t_2)v_\Delta^<(x_2)N_{(\mathrm{b})}(x_1,x_2)\,,$$
$$N_{(\mathrm{b})}(x_1,x_2) \simeq \frac{-\lambda H^2\phi_c^{<2}(x_1)}{24\pi^2}\left(1+\mathcal{O}\left(\epsilon^2\right)\right)\delta(t_1-t_2)j_0(\epsilon a(t_1)H|\vec{x}_1-\vec{x}_2|)\,. \tag{4.61}$$

Similarly we can evaluate diagrams (c) and (d), and it turns out that they are $\mathcal{O}(\epsilon)$ suppressed.

Therefore, combining eqs. (4.38), (4.41), (4.60), and (4.61), we obtain a set of effective

side. Next, let us consider the spatial dependence of $\phi_c^<(\vec{x},t)$. The most dominant part of $\phi_c^<(\vec{x},t)$ at a late time appears only when the momentum k which is assigned to $\phi_c^<$ is much less than $\epsilon a H$. And if both $k \ll \epsilon a H$ and $|\vec{x}_1-\vec{x}|,\ |\vec{x}_2-\vec{x}| \lesssim 1/(\epsilon a H)$ are satisfied, we can replace $e^{i\vec{k}\cdot\vec{x}}$ by $e^{i\vec{k}\cdot\vec{x}_1}$ or $e^{i\vec{k}\cdot\vec{x}_2}$ in a good approximation. Therefore if $|\vec{x}_1-\vec{x}|,\ |\vec{x}_2-\vec{x}| \lesssim 1/(\epsilon a H)$ are satisfied, we can use the equality $\phi_c^<(\vec{x},t) \simeq \phi_c^<(\vec{x}_1,t) \simeq \phi_c^<(\vec{x}_2,t)$ regarding the most dominant part of $\phi_c^<(\vec{x},t)$ at a late time.

EOM for IR modes which can describe IR secular growth up to NLO:

$$\dot{\phi}_c^< = v_c^< + \xi_\phi\,, \quad \dot{v}_c^< = -3Hv_c^< - \frac{\lambda\phi_c^{<3}}{6} + \xi_v\,, \tag{4.62a}$$

$$\langle\xi_\phi(x_1)\xi_\phi(x_2)\rangle = \frac{H}{4\pi^2}\left[H^2 + \lambda l(\epsilon)\phi_c^{<2}(x_1)\right]\delta(t_1-t_2)j_0(\epsilon a(t_1)H|\vec{x}_1-\vec{x}_2|)\,, \tag{4.62b}$$

$$\langle\xi_\phi(x_1)\xi_v(x_2)\rangle = -\frac{H^2\lambda}{24\pi^2}\phi_c^{<2}(x_1)\delta(t_1-t_2)j_0(\epsilon a(t_1)H|\vec{x}_1-\vec{x}_2|)\,, \tag{4.62c}$$

$$\langle\xi_v(x_1)\xi_v(x_2)\rangle = \mathcal{O}\left(\epsilon^2\right) \simeq 0\,, \quad l(\epsilon) \coloneqq -\frac{1}{3}\left[\ln\frac{1}{\epsilon} - \ln 2 - \gamma + 2\right]. \tag{4.62d}$$

Since the time derivative of ξ_v cannot contribute to NLO, we can further simplify the above stochastic EOM by redefining the $v_c^<$, up to NLO, as

$$\dot{\phi}_c^< = v_c^< + \xi'_\phi\,, \quad \dot{v}_c^< = -3Hv_c^< - \frac{\lambda\phi_c^{<3}}{6}\,, \tag{4.63a}$$

$$\begin{aligned}\left\langle\xi'_\phi(x_1)\xi'_\phi(x_2)\right\rangle &= \frac{H}{4\pi^2}\left[H^2 + \lambda\phi_c^{<2}(x_1)\left(l(\epsilon)-\frac{1}{9}\right)\right]\delta(t_1-t_2)j_0(\epsilon a(t_1)H|\vec{x}_1-\vec{x}_2|)\\ &\coloneqq N\delta(t_1-t_2)j_0(\epsilon a(t_1)H|\vec{x}_1-\vec{x}_2|)\,.\end{aligned} \tag{4.63b}$$

Here, ξ'_ϕ is related to ξ_ϕ and ξ_v as

$$\xi'_\phi = \xi_\phi + \frac{1}{3H}\xi_v\,.$$

Stochastic calculation of IR correlation functions up to NLO

Next, we calculate $\langle 0|\,\phi^{<2}(x)\,|0\rangle$ perturbatively with respect to λ up to NLO, by using our NLO stochastic eq. (4.63). It is convenient to rewrite eq. (4.63) in the form of Fokker-Planck equation in evaluating correlation functions. When we derive the corresponding Fokker-Planck equation, we need to define the integral of the stochastic noise because the amplitude of the stochastic noise depends on field $\phi_c^<$. Since the physical origin of this field dependence of the stochastic noise is that UV modes feel the background value of IR modes $\phi_c^<$, we should adopt the Itô integral from causality. Therefore, we treat eq. (4.63) as the Itô type Langevin equation, and the corresponding Fokker-Planck equation can be obtained as [58]

$$\dot{\rho}(\phi_c^<, v_c^<, t) = \frac{1}{2}\frac{\partial^2}{\partial\phi_c^{<2}}(N\rho) - \frac{\partial}{\partial\phi_c^<}(v_c^<\rho) + \frac{\partial}{\partial v_c^<}\left(3Hv_c^< + \frac{\lambda}{6}\phi_c^{<3}\rho\right). \tag{4.64}$$

By multiplying both sides of eq. (4.64) by $\phi_c^{<n}v_c^{<m}$ and performing partial integral with respect to $\phi_c^<$ and $v_c^<$, one can get the following recurrence relation:

$$\begin{aligned}\frac{\partial}{\partial t}\left\langle\phi_c^{<n}v_c^{<m}\right\rangle &= \left[\frac{\lambda Hn(n-1)}{8\pi^2}\left(l(\epsilon)-\frac{1}{9}\right) - 3mH\right]\left\langle\phi_c^{<n}v_c^{<m}\right\rangle\\ &\quad + \frac{n(n-1)H^3}{8\pi^2}\left\langle\phi_c^{<n-2}v_c^{<m}\right\rangle + n\left\langle\phi_c^{<n-1}v_c^{<m+1}\right\rangle - \frac{m\lambda}{6}\left\langle\phi_c^{<n+3}v_c^{<m-1}\right\rangle.\end{aligned} \tag{4.65}$$

Here, initial conditions are given by

$$\left\langle \phi_c^{<n} v_c^{<m} \right\rangle = 0 \,, \qquad \text{for} \quad t = t_0 \,, \; (n, m) \neq (0, 0) \,, \tag{4.66}$$

and n and m are non-negative integers. Next, we calculate $\left\langle \phi_c^{<2}(x) \right\rangle$ up to first order with respect to λ by using eqs. (4.65) and (4.66). Substituting $(n, m) = (2, 0), (1, 1), (0, 2)$ into eq. (4.65), one obtains

$$\frac{\partial}{\partial t} \left\langle \phi_c^{<2} \right\rangle = \frac{H^3}{4\pi^2} + \frac{\lambda H}{4\pi^2} \left(l(\epsilon) - \frac{1}{9} \right) \left\langle \phi_c^{<2} \right\rangle + 2 \left\langle \phi_c^< v_c^< \right\rangle \,, \tag{4.67}$$

$$\frac{\partial}{\partial t} \left\langle \phi_c^< v_c^< \right\rangle = -3H \left\langle \phi_c^< v_c^< \right\rangle + \left\langle v_c^{<2} \right\rangle - \frac{\lambda}{6} \left\langle \phi_c^{<4} \right\rangle \,, \tag{4.68}$$

$$\frac{\partial}{\partial t} \left\langle v_c^{<2} \right\rangle = -6H \left\langle v_c^{<2} \right\rangle - \frac{\lambda}{3} \left\langle \phi_c^{<3} v_c^< \right\rangle \,. \tag{4.69}$$

From eq. (4.69), the 0-th order term of $\left\langle v_c^{<2} \right\rangle$ with respect to λ satisfies

$$\frac{\partial}{\partial t} \left\langle v_c^{<2} \right\rangle = -6H \left\langle v_c^{<2} \right\rangle \,.$$

Since the solution of this differential equation with an intial condition (4.66) is only a trivial solution $\left\langle v_c^{<2} \right\rangle = 0$, $\left\langle v_c^{<2} \right\rangle$ is, at least, first order in λ. By using this fact and eq. (4.68), it is also obvious that $\left\langle \phi_c^< v_c^< \right\rangle$ is, at least, first order in λ. Using the fact $\left\langle \phi_c^< v_c^< \right\rangle \sim \mathcal{O}(\lambda)$ and eq. (4.69), it turns out that $\left\langle v_c^{<2} \right\rangle$ is, at least, second order in λ. Using these facts and $\left\langle \phi_c^{<4} \right\rangle = 3 \left\langle \phi_c^{<2} \right\rangle^2 + \mathcal{O}(\lambda)$, eqs. (4.67) and (4.68) reduce to

$$\frac{\partial}{\partial t} \left\langle \phi_c^{<2} \right\rangle = \frac{H^3}{4\pi^2} + \frac{\lambda H}{4\pi^2} \left(l(\epsilon) - \frac{1}{9} \right) \left\langle \phi_c^{<2} \right\rangle + 2 \left\langle \phi_c^< v_c^< \right\rangle \,, \tag{4.70}$$

$$\frac{\partial}{\partial t} \left\langle \phi_c^< v_c^< \right\rangle = -3H \left\langle \phi_c^< v_c^< \right\rangle - \frac{\lambda}{2} \left\langle \phi_c^{<2} \right\rangle^2 + \mathcal{O}\left(\lambda^2\right) \,. \tag{4.71}$$

Let us put the ansatz as follows:

$$\left\langle \phi_c^{<2} \right\rangle = \frac{H^2}{4\pi^2} \ln \frac{a}{a_0} + \lambda H^2 \left(c_1 \left(\ln \frac{a}{a_0} \right)^3 + c_2 \left(\ln \frac{a}{a_0} \right)^2 + \mathcal{O}\left(\ln \frac{a}{a_0} \right) \right) + \mathcal{O}\left(\lambda^2\right) \,, \tag{4.72}$$

$$\left\langle \phi_c^< v_c^< \right\rangle = \lambda H^3 \left(d_1 \left(\ln \frac{a}{a_0} \right)^2 + d_2 \ln \frac{a}{a_0} + d_3 + \mathcal{O}\left(\frac{a_0}{a} \right) \right) + \mathcal{O}\left(\lambda^2\right) \,, \tag{4.73}$$

where $c_1, \cdots, d_3$ are constants. Substituting eqs. (4.72) and (4.73) into eqs. (4.70) and (4.71), and focusing on the coefficients of the first order terms with respect to λ, one obtains

$$
\begin{aligned}
&3c_1\left(\ln\frac{a}{a_0}\right)^2+2c_2\ln\frac{a}{a_0}+\mathcal{O}\left(\left(\ln\frac{a}{a_0}\right)^0\right)\\
&=2d_1\left(\ln\frac{a}{a_0}\right)^2+\left[\frac{l(\epsilon)-1/9}{16\pi^4}+2d_2\right]\ln\frac{a}{a_0}+\mathcal{O}\left(\left(\ln\frac{a}{a_0}\right)^0\right)\,,\\
&2d_1\ln\frac{a}{a_0}+d_2=-\left(3d_1+\frac{1}{32\pi^4}\right)\left(\ln\frac{a}{a_0}\right)^2-3d_2\ln\frac{a}{a_0}+\mathcal{O}\left(\left(\ln\frac{a}{a_0}\right)^0\right)\,.
\end{aligned}
$$

From the above two identities, the following equations follow:

$$
3c_1=2d_1\,,\quad 2c_2=2d_2+\frac{l(\epsilon)-1/9}{16\pi^4}\,,\quad d_1=-\frac{1}{96\pi^4}\,,\quad 3d_2=-2d_1\,.
$$

Therefore, one gets

$$
\begin{aligned}
&d_1=-\frac{1}{96\pi^4}\,,\quad d_2=\frac{1}{144\pi^4}\,,\quad c_1=-\frac{1}{144\pi^4}\,,\\
&c_2=\frac{1}{32\pi^4}\left(l(\epsilon)+\frac{1}{9}\right)=\frac{1}{96\pi^4}\left(-\ln\frac{1}{\epsilon}+\ln 2+\gamma-\frac{5}{3}\right)\,,
\end{aligned}
$$

which results in

$$
\begin{aligned}
\left\langle\phi_c^{<2}\right\rangle=\frac{H^2}{4\pi^2}\ln\frac{a}{a_0}+\lambda H^2\Bigg[&-\frac{1}{144\pi^4}\left(\ln\frac{a}{a_0}\right)^3\\
&+\frac{1}{96\pi^4}\left(-\ln\frac{1}{\epsilon}+\ln 2+\gamma-\frac{5}{3}\right)\left(\ln\frac{a}{a_0}\right)^2+\mathcal{O}\left(\ln\frac{a}{a_0}\right)\Bigg]\,.
\end{aligned}
\tag{4.74}
$$

As we will see in the next subsection, this result coincides with the result eq. (4.83) which is calculated by perturbative QFT technique.

4.2.2 Perturbative QFT calculations

From now on we perform a standard perturbative QFT calculation up to NLO. The first order corrections to $\langle 0|\,\phi^{<2}(x)\,|0\rangle$ with respect to λ are shown in fig. 4.10 when the renormalization scheme is chosen so that the contribution from δm^2 term and that from UV loop integral are canceled out. Then, $\langle 0|\,\phi^{<2}(x)\,|0\rangle$ is evaluated up to first order with respect to λ as

$$
\begin{aligned}
\langle 0|\,\phi^{<2}(x)\,|0\rangle\simeq&\int_{\epsilon a_0 H}^{\epsilon a H}\frac{\mathrm{d}^3k}{(2\pi)^3}\mathrm{Re}[\Phi_k(t)\Phi_k^*(t)]\\
&-i\lambda\int_{t_0}^{t}\mathrm{d}t'\,a^3(t')\frac{H^2}{4\pi^2}\ln\frac{a(t')}{a_0}\int_{\epsilon a_0 H}^{\epsilon a H}\frac{\mathrm{d}^3k}{(2\pi)^3}2i\,\mathrm{Im}[\Phi_k(t)\Phi_k^*(t')]\mathrm{Re}[\Phi_k(t)\Phi_k^*(t')]\,,
\end{aligned}
\tag{4.75}
$$

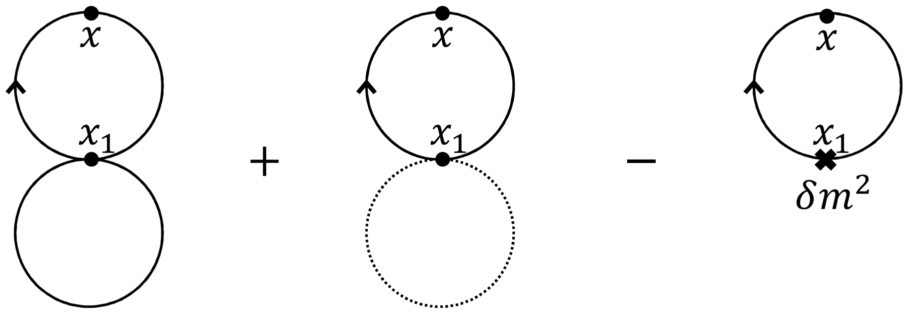

Figure 4.10: These diagrams are first order corrections with respect to λ to $\langle 0|\, \phi^{<2}(x)\, |0\rangle$. An effective EOM for IR modes in our formalism can recover IR correlation functions.

where we neglect $\mathcal{O}(\epsilon)$ terms. Next, by using the following equations

$$\mathrm{Re}[\Phi_k(t)\Phi_k^*(t')]_{k\leq\epsilon aH} = \frac{H^2}{2k^3}\left(\frac{k}{a(t')H}\sin\!\left(\frac{k}{a(t')H}\right)+\cos\!\left(\frac{k}{a(t')H}\right)\right)\left(1+\mathcal{O}\!\left(\left(\frac{k}{aH}\right)^2\right)\right), \tag{4.76a}$$

$$\begin{aligned}&\mathrm{Im}[\Phi_k(t)\Phi_k^*(t')]_{k\leq\epsilon aH}\\ &=\frac{H^2}{2k^3}\left[\left(\frac{k}{a(t')H}\cos\!\left(\frac{k}{a(t')H}\right)-\sin\!\left(\frac{k}{a(t')H}\right)\right)\right.\\ &\quad\left.+\frac{1}{3}\left(\frac{k}{aH}\right)^3\left(\frac{k}{a(t')H}\sin\!\left(\frac{k}{a(t')H}\right)+\cos\!\left(\frac{k}{a(t')H}\right)\right)\right]\left(1+\mathcal{O}\!\left(\left(\frac{k}{aH}\right)^2\right)\right),\end{aligned} \tag{4.76b}$$

and neglecting higher order terms with respect to (k/aH), the first order terms of $\langle 0|\, \phi^{<2}(x)\, |0\rangle$ with respect to λ are given by

$$\begin{aligned}\frac{\lambda H^3}{16\pi^4}\Bigg\{&\int_{t_0}^{t}\mathrm{d}t'\ln\frac{a(t')}{a_0}\left[F\left(\frac{\epsilon a_0}{a(t')}\right)-F\left(\frac{\epsilon a}{a(t')}\right)\right]\\ &+\frac{1}{3}\int_{t_0}^{t}\mathrm{d}t'\left(\frac{a(t')}{a}\right)^3\ln\frac{a(t')}{a_0}\int_{\frac{\epsilon a_0}{a(t')}}^{\frac{\epsilon a}{a(t')}}\frac{\mathrm{d}K}{K}(K\sin K+\cos K)^2\Bigg\}.\end{aligned} \tag{4.77}$$

Here, we define new dimensionless variable K by $K := k/a(t')H$ and function F by

$$F(x) := \int_x^\infty \frac{\mathrm{d}K}{K^4}(K\sin K+\cos K)(K\cos K-\sin K)\,. \tag{4.78}$$

From the definition of K, $K\geq 1$ corresponds to UV modes at time t', and $K\leq 1$ corresponds to IR modes at time t'. Let us calculate the first term in eq. (4.77). $F(\epsilon a/a(t'))$ decays with oscillating for $t_0+\ln(1/\epsilon)/H \lesssim t' \leq t$, which is the dominant part of the time integral region, and hence does not contribute to $\lambda(\ln a/a_0)^2$ terms in $\langle 0|\, \phi^{<2}(x)\, |0\rangle$. On the other hand, $F(\epsilon a_0/a(t'))$ can contribute to $\lambda(\ln a/a_0)^2$ terms in $\langle 0|\, \phi^{<2}(x)\, |0\rangle$.

$F(\epsilon a_0/a(t'))$ can be expanded analytically as

$$F\left(\frac{\epsilon a_0}{a(t')}\right) = \frac{1}{3}\left[\ln\left(\frac{\epsilon a_0}{a(t')}\right) + \ln 2 + \gamma - \frac{7}{3} + \mathcal{O}\left(\left(\frac{\epsilon a_0}{a(t')}\right)^2\right)\right] . \tag{4.79}$$

$\mathcal{O}\left(\epsilon a_0/a(t')\right)$ terms can be neglected because an inequality $\frac{\epsilon a_0}{a(t')} < \epsilon \ll 1$ is always satisfied in the integral region $t_0 \leq t' \leq t$. Here, γ is an Euler's constant. Therefore,

$$\begin{aligned}
&\int_{t_0}^{t} \mathrm{d}t' \ln\frac{a(t')}{a_0}\left[F\left(\frac{\epsilon a_0}{a(t')}\right) - F\left(\frac{\epsilon a}{a(t')}\right)\right] \\
&= -\frac{1}{9H}\left(\ln\frac{a}{a_0}\right)^3 + \frac{1}{6H}\left(-\ln\frac{1}{\epsilon} + \ln 2 + \gamma - \frac{7}{3}\right)(\ln\frac{a}{a_0})^2 + \mathcal{O}\left(\ln\frac{a}{a_0}\right) .
\end{aligned} \tag{4.80}$$

Next, let us calculate the second term in eq. (4.77). This term is calculated as

$$\begin{aligned}
&\int_{t_0}^{t} \mathrm{d}t' \left(\frac{a(t')}{a}\right)^3 \ln\frac{a(t')}{a_0} \int_{\frac{\epsilon a_0}{a(t')}}^{\frac{\epsilon a}{a(t')}} \frac{\mathrm{d}K}{K}(K\sin K + \cos K)^2 \\
&= \int_{t_0}^{t} \mathrm{d}t' \left(\frac{a(t')}{a}\right)^3 \ln\frac{a(t')}{a_0} \cdot \left\{\frac{1}{2}\ln\frac{a}{a_0} + \frac{1}{2}\left[\mathrm{Ci}\left(\frac{2\epsilon a}{a(t')}\right) - \mathrm{Ci}\left(\frac{2\epsilon a_0}{a(t')}\right)\right]\right. \\
&\qquad + \frac{5}{8}\left[\cos\left(\frac{2\epsilon a}{a(t')}\right) - \cos\left(\frac{2\epsilon a_0}{a(t')}\right)\right] + \frac{1}{4}\left[\left(\frac{2\epsilon a}{a(t')}\right)^2 - \left(\frac{2\epsilon a_0}{a(t')}\right)^2\right] \\
&\qquad \left. - \frac{1}{4}\left[\frac{2\epsilon a}{a(t')}\sin\left(\frac{2\epsilon a}{a(t')}\right) - \frac{2\epsilon a_0}{a(t')}\sin\left(\frac{2\epsilon a_0}{a(t')}\right)\right]\right\} .
\end{aligned} \tag{4.81}$$

Here, $\mathrm{Ci}(x)$ is defined by

$$\mathrm{Ci}(x) \coloneqq \int_{\infty}^{x} \mathrm{d}x \frac{\cos(x)}{x} .$$

Because the powers of $a(t')$ in the first factor of the integrand in eq. (4.81) are positive, $(\ln a/a_0)^2$ terms appear from this integral only when this positive powers of $a(t')$ are canceled or the powers of $\ln(a/a_0)$ in the integrand becomes equal to or more than two. For this reason, the terms in the bracket $\{\cdots\}$ in eq. (4.81) which contribute to $(\ln a/a_0)^2$ terms are $\ln(a/a_0)$ term and $\mathrm{Ci}(\frac{2\epsilon a_0}{a(t')})$ term. Indeed, $\mathrm{Ci}(x)$ can be expanded for small argument x as

$$\mathrm{Ci}(x) = \ln(x) + \gamma + \mathcal{O}\left(x^2\right) ,$$

which shows logarithmic divergence for small x and this divergence becomes the origin of $(\ln a/a_0)^2$ terms in eq. (4.81).[7] Using this equation, eq. (4.81) is calculated as

$$\int_{t_0}^{t} \mathrm{d}t' \left(\frac{a(t')}{a}\right)^3 \ln\frac{a(t')}{a_0} \int_{\frac{\epsilon a_0}{a(t')}}^{\frac{\epsilon a}{a(t')}} \frac{\mathrm{d}K}{K}(K\sin K + \cos K)^2 = \frac{1}{3H}\left(\ln\frac{a}{a_0}\right)^2 + \mathcal{O}\left(\ln\frac{a}{a_0}\right) . \tag{4.82}$$

[7] Although $\mathrm{Ci}(x)$ is logarithmically divergent as $\ln x$, it decays rapidly for $x \geq 1$. This is why $\mathrm{Ci}(\frac{2\epsilon a}{a(t')})$ does't contribute to $(\ln a/a_0)^2$ terms in eq. (4.81).

Therefore, by substituting eqs.(4.80) and (4.82) into eq. (4.77), and using eq. (4.75), one obtains

$$\langle 0|\, \phi^{<2}(x)\, |0\rangle = \frac{H^2}{4\pi^2}\ln\frac{a}{a_0} + \lambda H^2\Bigg[-\frac{1}{144\pi^4}\left(\ln\frac{a}{a_0}\right)^3 + \frac{1}{96\pi^4}\left(-\ln\frac{1}{\epsilon}+\ln 2+\gamma-\frac{5}{3}\right)\left(\ln\frac{a}{a_0}\right)^2 + \mathcal{O}\left(\ln\frac{a}{a_0}\right)\Bigg] + \mathcal{O}\left(\lambda^2\right)\,. \tag{4.83}$$

4.2.3 Operator ordering

So far we have derived the effective EOM of $\phi_c^<$ and $v_c^<$ fields and checked its validity. However, the operator orderings of correlation functions which can be calculated by using our extended stochastic formalism are uniquely fixed: we can calculate only correlation functions consisting of $\phi_c^<$ and $v_c^<$ fields in our formalism, as we mentioned in sec. 4.1.2. In order to obtain correlation functions with arbitrary operator orderings, we also need to know correlation functions which include Δ-fields. However, such correlation functions are $\mathcal{O}(\epsilon)$ suppressed compared to that consisting only of c-fields, owing to the squeezing. This can be seen explicitly as

$$\frac{\tilde{G}^{<ij}_{c\Delta}(t,t';k)}{\tilde{G}^{<ij}_{cc}(t,t';k)} \sim \begin{cases} \mathcal{O}\left(\epsilon^3\right) & (i,j)=(1,1) \\ \mathcal{O}(\epsilon) & (i,j)=(1,2),(2,1),(2,2)\,. \end{cases}$$

Therefore, our formalism can evaluate IR secular growth which appears in *all* the IR correlation functions in a good approximation even though the operator orderings of calculable correlation functions are uniquely fixed.

4.3 General properties of effective IR dynamics

In sec. 4.2, we showed that the IR dynamics of $\phi_c^<$ can be written in the form of a set of Langevin equations. Here, we describe the generic properties of them by analyzing an influence functional. Under several assumptions, we will show the absence of secular enhancement of the correlations in the real-space stochastic noise.

In this section, we only consider the region where

$$\left|\frac{\partial^n}{\partial\phi_c^{<n}}\frac{\partial^m}{\partial v_c^{<m}}V(\phi_c^<, v_c^<)/H^{4-n-2m}\right| \ll 1$$

is satisfied for $n,m \geq 0\,,\,(n,m)\neq(0,0)$. This is because UV modes and IR modes are strongly interacting to each other outside this region, and we cannot handle the theory perturbatively there. While we considered $\lambda\phi^4$ interaction as a concrete example in sec. 4.2.1, here we do not specify the concrete form of the interaction potential. We simply refer to the coupling constant as λ below, assuming that V is proportional to λ. In this section, we set $H=1$.

From the definition (4.33), the weight function $P\left[\xi_\phi, \xi_v; \phi_c^<, v_c^<\right]$ is obtained by the Fourier transformation of $\exp\left[i\Gamma_{(\mathrm{s})} + iS^<_{\mathrm{H,int(s)}}\right]$ with respect to $v_\Delta^<$ and $\phi_\Delta^<$. When we perform this functional Fourier transformation, we can treat IR c-fields as given functions, and hence we neglect the dependence of $i\Gamma_{(\mathrm{s})} + iS^<_{\mathrm{H,int(s)}}$ on IR c-fields below in order to make the expression clearer.

We refer to diagrams constituting $i\Gamma_{(\mathrm{s})}$ as noise diagrams. $i\Gamma_{(\mathrm{s})}$ can be expanded in terms of $v_\Delta^<$ and $\phi_\Delta^<$ as

$$\begin{aligned} i\Gamma_{(\mathrm{s})} = \sum_{V=1} \prod_{i=1}^{V} &\left[\int \mathrm{d}t_i \int_{k_0}^{\epsilon a(t_i)} \frac{\mathrm{d}^3 k_i}{(2\pi)^3}\, \mathcal{O}_i\left(t_i, \vec{k}_i\right)\right] \\ &\times (2\pi)^3 \delta^{(3)}\left(\vec{k}_1 + \cdots + \vec{k}_V\right) i\tilde{V}_V\left(t_1, \cdots, t_V, \vec{k}_1, \cdots, \vec{k}_V\right), \end{aligned} \tag{4.84}$$

where V denotes the number of external vertexes, and $\mathcal{O}_i\left(t_i, \vec{k}_i\right)$ is the Fourier component of the product of IR-Δ fields $\Phi_\Delta^<$ contained in the i-th external vertex. We assume that $i\Gamma_{(\mathrm{s})}$ is approximately local, *i.e.*, $\tilde{V}_V\left(t_1, \cdots, t_V, \vec{k}_1, \cdots, \vec{k}_V\right)$ decays exponentially for $|t_i - t_j| \gtrsim \ln(1/\epsilon)$ for $i, j = 1, \cdots, V$. This approximate locality is expected because all the external IR-Δ legs are connected only by UV propagators, and these UV propagators are rapidly oscillating before the horizon crossing. Nontrivial phase cancellation of this rapid oscillation is not expected well before the horizon crossing. Under this assumption, we estimate the order of magnitude of $\tilde{V}_V$ below.

Since $\tilde{V}_V$ can be expressed as

$$\begin{aligned} &\tilde{V}_V\left(t_1, \cdots, t_V, \vec{k}_1, \cdots, \vec{k}_V\right)(2\pi)^3\delta^{(3)}\left(\vec{k}_1 + \cdots + \vec{k}_V\right) \\ &= A\int \mathcal{D}\phi_+^> \mathcal{D}\phi_-^> \mathcal{D}v_+^> \mathcal{D}v_-^> \prod_{i=1}^{V}\left[\frac{\delta\tilde{S}}{\delta\mathcal{O}_i\left(t_i, \vec{k}_i\right)}\right] e^{iS^>_{\mathrm{H},0}} e^{i\tilde{S}}\Bigg|_{v_\Delta^< = \phi_\Delta^< = 0} \Psi_0^> \Psi_0^{>*}, \end{aligned} \tag{4.85}$$

$$\begin{aligned} \tilde{S} \coloneqq &\left(S_{\mathrm{H,int}}\left[\phi_c = \phi_c^< + \phi_c^>, \phi_\Delta = \phi_\Delta^< + \phi_\Delta^>, v_c = v_c^< + v_c^>, v_\Delta = v_\Delta^< + v_\Delta^>\right] - S^<_{\mathrm{H,int}}\right) \\ &+ i\tilde{S}_{\mathrm{bilinear}}, \end{aligned}$$

where A denotes the normalization constant, one can evaluate an arbitrarily given noise diagram by solving the Yang-Feldman equation iteratively. This means that one can construct an arbitrarily given noise diagram by connecting tree-shaped components composed of UV retarded Green's functions, using the UV Wightman functions $\tilde{G}^{>ij}_{+-}$ defined by

$$\tilde{G}^{>ij}_{+-}(t, t'; k)(2\pi)^3\delta^{(3)}\left(\vec{k} - \vec{k}'\right) \coloneqq \int \mathcal{D}\phi_+^> \mathcal{D}\phi_-^> \mathcal{D}v_+^> \mathcal{D}v_-^>\, \phi_+^{>i}\left(t, \vec{k}\right) \phi_-^{>j}\left(t', \vec{k}'\right) e^{iS^>_{\mathrm{H},0}} \Psi_0^> \Psi_0^{>*},$$

with $\phi_\pm^{>1} \coloneqq \phi_\pm^>$ and $\phi_\pm^{>2} \coloneqq v_\pm^>$. The i-th component is shown in fig. 4.11. By construction, the i-th component is connected to $\mathcal{O}_i$. The number of UV-$\pm$ legs can be zero, *i.e.*, $Y_i = 0$ is allowed. Assuming the approximate locality, the order of magnitude of the i-th component shown in fig. 4.11 with Y_i UV-$\pm$ legs including g_i nonlinear interaction

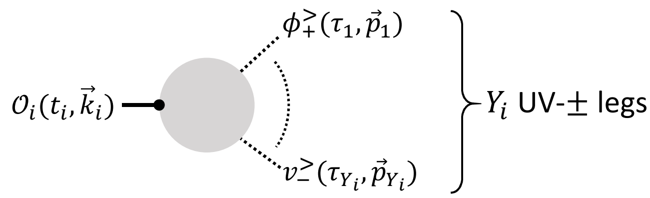

Figure 4.11: We can construct all the noise diagrams from the components shown in this figure, by connecting the UV-$\pm$ legs included in those components to other UV-$\pm$ legs which stems from themselves or from other components. The gray blob is the tree diagram connected only by the UV retarded Green's functions. $Y_i \geq 0$ denotes the number of UV legs.

vertexes can be estimated as

$$\mathcal{O}_i\left(t_i,\vec{k}_i\right)(2\pi)^3\delta^{(3)}\left(\vec{k}_i+\vec{p}_1+\cdots+\vec{p}_{Y_i}\right) \\ \times W_{Y_i}^{(g_i)}\left(t_i,\tau_1,\cdots,\tau_{Y_i},\vec{k}_i,\vec{p}_1,\cdots,\vec{p}_{Y_i}\right)\,, \tag{4.86a}$$

$$W_{Y_i}^{(g_i)} \sim \lambda^{g_i}\prod_{b=1}^{I_{\rm r}^{(i)}}\left[\int_{t_i-\ln(1/\epsilon)}^{t_i}\mathrm{d}T_b'\,a^3(T_b')\tilde{G}_{c\Delta}^{>}(T_b,T_b';K_b)\right]\prod_{j=1}^{Y_i}\left[\Phi^{>}\left(\tau_j,\vec{p}_j\right)\right]\,. \tag{4.86b}$$

Here, $W_{Y_i}^{(g_i)}$ expresses the amplitude of the gray blob with UV legs, and $\sim$ here represents the estimation of the leading order amplitude neglecting the factor independent of ϵ, λ, and the scale factor a. We just refer to the UV-$\pm$ field as $\Phi^>$. $I_{\rm r}^{(i)}$ is the number of the UV retarded Green's functions which is identical to the number of the internal vertexes contained in the gray blob, and then $I_{\rm r}^{(i)}=g_i$ when the external vertex to which $\mathcal{O}_i$ is attached is the bilinear vertex, and otherwise $I_{\rm r}^{(i)}=g_i-1$. T_b and T_b' are, respectively, the times assigned to the vertexes connected by each UV retarded Green's function. $K_b \coloneqq \left|\vec{K}_b\right|$ and $\vec{K}_b$ is given by some linear combination of $\vec{p}_1,\cdots,\vec{p}_{Y_i}$. We refer to the UV retarded Green's function as $\tilde{G}_{c\Delta}^{>}(t,t';k)$ symbolically. Since the typical behavior of $\tilde{G}_{c\Delta}^{>}(t,t';k)$ for $k\lesssim a(t')$ can be written as

$$\left.\tilde{G}_{c\Delta}^{>}(t,t';k)\right|_{k\lesssim a(t')} \sim \frac{i}{a^3(t')}\Theta(t-t')\,, \tag{4.87}$$

$W_{Y_i}^{(g_i)}$ can be estimated as

$$W_{Y_i}^{(g_i)} \sim \lambda^{g_i}\prod_{b=1}^{I_{\rm r}^{(i)}}\left[\int_{t_i-\ln(1/\epsilon)}^{t_i}\mathrm{d}T_b'\right]\prod_{j=1}^{Y_i}\left[\Phi^{>}\left(\tau_j,\vec{p}_j\right)\right]\,. \tag{4.88}$$

Thus, the amputated diagram $\tilde{V}_V$ defined by eq. (4.84) composed of these effective vertexes $W^{(g_i)}_{Y_i}$ can be estimated as

$$
\begin{aligned}
&\prod_{i=1}^{V}\left[\frac{1}{a^3(t_i)}\int_{k_0}^{\epsilon a(t_i)}\frac{\mathrm{d}^3k_i}{(2\pi)^3}\right]\tilde{V}_V\left(t_1,\cdots,t_V,\vec{k}_1,\cdots,\vec{k}_V\right)(2\pi)^3\delta^{(3)}\left(\vec{k}_1+\cdots+\vec{k}_V\right)\\
&\sim\sum_{\{g_i,Y_i\}}\prod_{i=1}^{V}\lambda^{g_i}\prod_{b_i=1}^{I_{\mathrm{r}}^{(i)}}\left[\int_{t_i-\ln(1/\epsilon)}^{t_i}\mathrm{d}T'^{(i)}_{b_i}\right]\prod_{d=1}^{I_{\mathrm{w}}}\left[\int_{\epsilon a(\tau_{d1})}^{a(\tau_{d2})}\frac{\mathrm{d}^3p_d}{(2\pi)^3}\tilde{G}^{>}_{\mathrm{w}}\left(\tau_{d1},\tau_{d2};p_d\right)\right],
\end{aligned}
\tag{4.89}
$$

where we refer to the UV Wightman function just symbolically as $\tilde{G}^{>}_{\mathrm{w}}$. $I_{\mathrm{w}}=\frac{1}{2}\sum_i Y_i$ is the number of the UV Wightman functions. τ_{d1} and τ_{d2} denote the times at endpoints of the UV Wightman function with $\tau_{d1}\geq\tau_{d2}$. Here, we assumed that the dominant contribution comes from $\epsilon a(\tau_{d2})\lesssim p_d\lesssim a(\tau_{d1})$ after subtracting UV divergences by renormalization. Since the decaying mode is always smaller than the growing mode after the horizon crossing, $\left|\tilde{G}^{>11}_{cc}(t,t';p)\right|\gtrsim\left|\tilde{G}^{>ij}_{c\Delta}(t,t';p)\right|$ always holds for $\epsilon a\lesssim p\lesssim a(t')$. This allows one to estimate the maximum of the order of magnitude of (4.89) by replacing $\tilde{G}^{>}_{\mathrm{w}}$ by $\tilde{G}^{>11}_{cc}$, where the symmetric propagator $\tilde{G}^{>ij}_{cc}$ is defined by

$$
\tilde{G}^{>ij}_{cc}(t,t';k)(2\pi)^3\delta^{(3)}\left(\vec{k}-\vec{k}'\right):=\int\mathcal{D}\phi^{>}_{c}\mathcal{D}\phi^{>}_{\Delta}\mathcal{D}v^{>}_{c}\mathcal{D}v^{>}_{\Delta}\,\phi^{>i}_{c}\left(t,\vec{k}\right)\phi^{>j}_{c}\left(t',\vec{k}'\right)e^{iS^{>}_{\mathrm{H},0}},
$$

with $\phi^{>1}_{c}:=\phi^{>}_{c}$ and $\phi^{>2}_{c}:=v^{>}_{c}$. Then, from the behavior of $\tilde{G}^{>11}_{cc}(t,t';p)$ after the horizon crossing, which is given by

$$
\tilde{G}^{>11}_{cc}(t,t';p)\Big|_{\epsilon a\lesssim p\lesssim a(t')}\sim\frac{1}{p^3},
\tag{4.90}
$$

eq. (4.89) can be estimated as

$$
\begin{aligned}
&\prod_{i=1}^{V}\left[\frac{1}{a^3(t_i)}\int_{k_0}^{\epsilon a(t_i)}\frac{\mathrm{d}^3k_i}{(2\pi)^3}\right]\tilde{V}_V\left(t_1,\cdots,t_V,\vec{k}_1,\cdots,\vec{k}_V\right)(2\pi)^3\delta^{(3)}\left(\vec{k}_1+\cdots+\vec{k}_V\right)\\
&\sim\sum_{\{g_i,Y_i\}}\lambda^{g_i}\left(\ln(1/\epsilon)\right)^{I_{\mathrm{r}}+I_{\mathrm{w}}},
\end{aligned}
\tag{4.91}
$$

neglecting the higher order terms in eq. (4.89). Here, $I_{\mathrm{r}}:=\sum_i I^{(i)}_{\mathrm{r}}$ is the total number of the UV retarded Green's functions included in $\tilde{V}_V$. Strictly speaking, in eq. (4.91), the power of $\ln(1/\epsilon)$ can be even smaller. When the largest number of legs attached to a vertex is N, $I_{\mathrm{r}}+I_{\mathrm{w}}\leq\frac{N}{2}\sum_i g_i$ because $I_{\mathrm{r}}+I_{\mathrm{w}}$ equals to the total number of propagators included in $\tilde{V}_V$, and at most N propagators can be attached to each nonlinear vertex. Therefore, in such a model, we choose an ϵ parameter so as to satisfy $\lambda\big(\ln(1/\epsilon)\big)^{N/2}\ll 1$:[8]

$$
\exp\left[-\frac{1}{\lambda^{2/N}}\right]\ll\epsilon\,.
\tag{4.92}
$$

[8]Thanks to the condition (4.92), one can estimate the order of magnitude of $i\Gamma_{(s)}$ by counting the powers of ϵ and λ.

Then, the terms with minimum power of λ become the leading order terms in eq. (4.91).

When the number of external IR-Δ fields $v_\Delta^<$ and $\phi_\Delta^<$ are n and m, respectively, $\tilde{V}_V$ with the largest V, *i.e.*, $V = n + m$ becomes the leading term in eq. (4.84) because of the factor

$$\prod_{i=1}^{V}\left[\frac{1}{a^3(t_i)}\int_{k_0}^{\epsilon a(t_i)}\frac{\mathrm{d}^3 k_i}{(2\pi)^3}\right] \sim \epsilon^{3V}\,.$$

Noise diagrams satisfying $V = n + m$ can be constructed from the components shown in fig. 4.11 with $\mathcal{O}_i\left(t_i, \vec{k}_i\right) = \Phi_\Delta^<\left(t_i, \vec{k}_i\right)$. The same argument gives an alternative proof of the fact that the contributions of $S^<_{\mathrm{H,int\,(s)}}$ to the influence functional are suppressed compared to the leading order contributions. Furthermore, since $\tilde{G}_{cc}^{>2i}(t,t';k)$ is suppressed by ϵ^2 compared to $\tilde{G}_{cc}^{>1i}(t,t';k)$ for $t = t_k$ as

$$\frac{\tilde{G}_{cc}^{>2i}\left(t_k,t';k\right)}{\tilde{G}_{cc}^{>1i}\left(t_k,t';k\right)} \sim \mathcal{O}(\epsilon^2) \quad \text{for} \quad i = 1, 2\,, \tag{4.93}$$

the contributions of the bilinear vertex with $\phi_\Delta^<$ to $i\Gamma_{(\mathrm{s})}$ are suppressed by $\epsilon^2\lambda^{-1}$ when the associated $v_c^>$ field is contracted with other UV-$\pm$ legs, compared to the contributions of the other components with $\phi_\Delta^<$ shown in fig. 4.11. Here, the suppression factor is associated with λ^{-1} because the bilinear vertex with $\phi_\Delta^<$ has no λ factor in itself while the other components with $\phi_\Delta^<$ are associated with λ. Thus, when ϵ satisfies

$$\epsilon \ll \lambda^{1/2}\,, \tag{4.94}$$

which is compatible with eq. (4.92), the contributions of the bilinear vertex with $\phi_\Delta^<$ to $i\Gamma_{(\mathrm{s})}$ are always suppressed compared to another component with $\phi_\Delta^<$. This means that all the components contributing to $i\Gamma_{(\mathrm{s})}$ at leading order are given by fig. 4.12. In the following discussions, we assume that eqs. (4.92) and (4.94) are satisfied.

Then, referring to the contribution from n and m effective vertexes associated with $v_\Delta^<$ and $\phi_\Delta^<$ fields to $\tilde{V}_{n+m}$ as $\tilde{V}_{nm}$, eq. (4.91) reads

$$\begin{aligned}&\prod_{i=1}^{n+m}\left[\frac{1}{a^3(t_i)}\int_{k_0}^{\epsilon a(t_i)}\frac{\mathrm{d}^3 k_i}{(2\pi)^3}\right]\tilde{V}_{nm}\left(t_1,\cdots,t_{n+m},\vec{k}_1,\cdots,\vec{k}_{n+m}\right)(2\pi)^3\delta^{(3)}\left(\vec{k}_1+\cdots+\vec{k}_{n+m}\right)\\ &\sim \lambda^{g(n,m)}\left(\ln(1/\epsilon)\right)^{I_{\mathrm{r}}+I_{\mathrm{w}}}\,,\end{aligned} \tag{4.95}$$

where $g(n,m)$ is the power of λ included in the leading order terms of $\tilde{V}_{nm}$, which can be estimated as

$$g(n,m) = f(n,m) + m\,, \tag{4.96a}$$

$$f(n,m)\begin{cases} = 0 & \text{for} \quad n = 2\,,\ m = 0\,,\\ \geq 1 & \text{for} \quad n \geq 3\,,\ m = 0\,,\\ \geq 0 & \text{for} \quad m \geq 1\,,\end{cases} \tag{4.96b}$$

because $\phi_\Delta^<$ is always associated with λ in fig. 4.12.

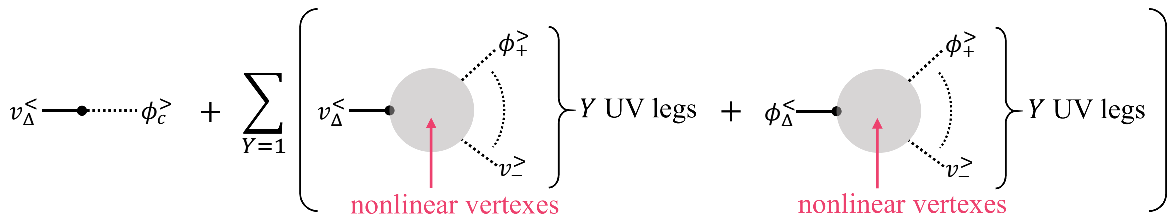

Figure 4.12: We can construct all the leading-order noise diagrams from the components shown in this figure, by connecting the UV legs included in those components to other UV legs which stems from themselves or from other components. The gray blob consists of the tree diagram connected only by the UV retarded Green's functions. The gray blob contains at least one nonlinear vertex. Here, diagrams with $Y = 0$ are not included, because they do not contribute to $\Gamma_{(\mathrm{s})}$ but to $\Gamma_{(\mathrm{d})}$.

Now, we move on to translating the above results into the expression of $i\Gamma_{(\mathrm{s})}$ in real space assuming the approximate locality of $i\Gamma_{(\mathrm{s})}$. Applying

$$\Theta\left(\epsilon a H - k_i\right) \Phi_\Delta^<\left(t, \vec{k}_i\right) = \int \mathrm{d}^3 x_i\, \Phi_\Delta^<(t, \vec{x}_i) e^{-i\vec{k}_i\cdot\vec{x}_i} \tag{4.97}$$

to eq. (4.84) and neglecting the terms with $V < n+m$, one obtains

$$i\Gamma_{(\mathrm{s})} \simeq \sum_{(n,m), n+m\geq 2} \prod_{i=1}^{n} \left[\int \mathrm{d}^4 x_i\, a^3(t_i) v_\Delta^<(x_i)\right] \prod_{j=n+1}^{n+m} \left[\int \mathrm{d}^4 x_j\, a^3(t_j) \phi_\Delta^<(x_j)\right] iV_{nm}(x_1, \cdots, x_{n+m})\,, \tag{4.98a}$$

$$V_{nm}(x_1, \cdots, x_{n+m}) := \frac{1}{a^3(t_{n+m})} \prod_{j=1}^{n+m-1} \left[\frac{1}{a^3(t_j)} \int_{k_0}^{\epsilon a(t_j)} \frac{\mathrm{d}^3 k_j}{(2\pi)^3} e^{-i\vec{k}_j\cdot(\vec{x}_j - \vec{x}_{n+m})}\right]$$
$$\times \tilde{V}_{nm}\left(t_1, \cdots, t_{n+m}, \vec{k}_1, \cdots, \vec{k}_{n+m-1}, \vec{k}_{n+m} = -\vec{k}_1 - \cdots - \vec{k}_{n+m-1}\right). \tag{4.98b}$$

Strictly speaking, the integrand and the integration region of $\vec{p}_d$-integrals in eq. (4.89) depend on $\vec{k}_i$. However, they become almost independent of $\vec{k}_j$ when $k_j \ll \epsilon a(t_j)$ with $j = 1, \cdots, n+m$. Therefore, $\tilde{V}_{nm}(t_1, \cdots, t_{n+m}, \vec{k}_1, \cdots, \vec{k}_{n+m-1}, \vec{k}_{n+m} = -\vec{k}_1 - \cdots - \vec{k}_{n+m-1})$ becomes almost independent of $\vec{k}_j$ for $k_j \ll \epsilon a(t_j)$, which implies that in eq. (4.98b), the dominant contribution of the $\vec{k}_j$-integrals comes from $k_j \sim \epsilon a(t_j)$. Thus, it is expected that $V_{nm}(x_1, \cdots, x_{n+m})$ decays exponentially for $|\vec{x}_i - \vec{x}_j| \gg (\epsilon a)^{-1}$ for $i, j = 1, \cdots, n+m$ when $t_1 \sim \cdots \sim t_{n+m} \sim t$, and hence we assume this spatial approximate locality. For $|\vec{x}_i - \vec{x}_j| \lesssim (\epsilon a)^{-1}$ for $i, j = 1, \cdots, n+m$ when $t_1 \sim \cdots \sim t_{n+m} \sim t$, the order of magnitude of V_{nm} can be estimated from eqs. (4.95) and (4.96) as

$$\begin{aligned}V_{nm}(x_1,\cdots,x_{n+m}) \sim \prod_{i=1}^{n+m}&\left[\frac{1}{a^3(t_i)}\int_{k_0}^{\epsilon a(t_i)}\frac{\mathrm{d}^3k_i}{(2\pi)^3}\right]\\ &\times\tilde{V}_{nm}\left(t_1,\cdots,t_{n+m},\vec{k}_1,\cdots,\vec{k}_{n+m}\right)(2\pi)^3\delta^{(3)}\left(\vec{k}_1+\cdots+\vec{k}_{n+m}\right)\\ \sim\lambda^{m+f(n,m)}\,,\end{aligned} \tag{4.99}$$

neglecting the $\ln(1/\epsilon)$ dependence. $V_{nm}(x_1,\cdots,x_{n+m})$ corresponds to the vertex function with n $v_\Delta^<$ legs and m $\phi_\Delta^<$ legs in real space which contributes to the connected noise correlation as

$$\langle\xi_\phi(x_1)\cdots\xi_\phi(x_n)\xi_v(x_{n+1})\cdots\xi_v(x_{n+m})\rangle_{\mathrm{c}}\sim V_{nm}(x_1,\cdots,x_{n+m})\,, \tag{4.100}$$

and especially for $t_1\sim\cdots\sim t_{n+m}\sim t$, $|\vec{x}_i-\vec{x}_j|\lesssim(\epsilon a)^{-1}$ for $i,j=1,\cdots,n+m$,

$$\langle\xi_\phi(x_1)\cdots\xi_\phi(x_n)\xi_v(x_{n+1})\cdots\xi_v(x_{n+m})\rangle_{\mathrm{c}}\sim\epsilon^0\cdot a^0\cdot\lambda^{m+f(n,m)}\,, \tag{4.101}$$

neglecting the $\ln(1/\epsilon)$ dependence. Eq. (4.101) implies that neither IR diveregence nor late time suppression exists in the noise amplitudes in real space to all order in the coupling constant λ.

4.4 Summary

Let us summarize what we have established in this chapter 4. Here, we have investigated the dynamics which can describe all the IR secular effects beyond LO, in the theory of a canonically normalized light scalar field on de Sitter background. In sec. 4.1, we have constructed the systematic way of deriving an effective EOM for IR fields by integrating out UV fields. The point of our method is the introduction of bilinear interaction vertexes, which allows us to treat UV fields and IR fields independently. Without introduction of the bilinear vertexes, UV fields and IR fields are no longer independent to each other, because UV fields become IR fields at a late time because of an accelerating expansion of de Sitter space in a flat chart. Our method developed in sec. 4.1 showed that an effective EOM for IR fields which can correctly reproduce all the IR secular effects takes the form of a set of Langevin equations.

In sec. 4.2, we played with $\lambda\phi^4$ theory and we derived an effective IR dynamics which is valid up to NLO which are given by eqs. (4.62). We then explicitly check the coincidence between an IR two-point function evaluated using this effective EOM and the one evaluated using perturbative QFT: see eqs. (4.74) and (4.83). This shows how our formalism works well. More detailed analysis of obtained Langevin equations are also performed in sec. 4.3 for definiteness.

In this chapter, we established the method how we can handle IR secular effects. Our discussion in this chapter however will be insufficient for ensuring the theoretical consistency of stochastic approach to inflation. It is unclear how exactly the stochastic description of IR secular effects justify the classical treatment of the system. It seems that the possibility of stochastic description does not necessarily justify the classical treatment.[9] In the discussion of the classical aspects of the quantum fluctuations during inflation, the quantum decoherence of the reduced density matrix for IR fields has been extensively investigated in the literature [43, 44, 45, 46, 47, 48, 49]. In the next chapter 5, we will clarify how the quantum decoherence of IR fields and the stochastic description what we found here is related to each other.

[9] We see that the operator ordering becomes irrelevant owing to the squeezing in sec. 4.2.3, but it does not necessarily mean that the system is almost classical: for example see [59].

Chapter 5

Quantum decoherence and infrared fluctuations in local universe

As a possible resolution to IR problems in multi-field inflationary models, the stochastic picture of inflation has been utilized in the literatures [17, 18, 19, 20]. We have investigated the validity of the stochastic picture even after taking into account the UV-IR non-linearities. Once the stochastic picture of inflation is accepted, one will not have the deep-IR fluctuations and hence they cannot contribute to observed fluctuations from the beginning. This absence of IR effects however is simply because the quantum coherence of deep-IR fields are neglected by hand from the beginning, as has been pointed out in the several previous works [7, 27]. Thus, we will below concentrate on the test scalar ϕ resembling the iso-curvature modes in de Sitter background from a certain initial time t_0 to t_f. The setup is essentially the same as the one in the previous chapter. In this setup we will quantitatively discuss the theoretical validity of stochastic picture of inflation by precisely analyzing the structure of quantum states, by considering the $\lambda\phi^4$ theory in de Sitter space in a flat chart.

We will prove that stochastic picture can be theoretically consistent only approximately, and must be violated. We will evaluate the degree of this violation by estimating the quantum coherence of IR fields during inflation. We will then evaluate the amplitude of the fluctuations of IR fields when adopting the approximate stochastic picture, which will allow us to estimate the possible contributions of IR fluctuations to correlation functions of shorter-wavelength modes. Several implications and future works are listed up at the end of this chapter.

5.1 Structure of quantum states and the origin of large infrared effects

5.1.1 Quantum coherence of the history

In this study we would like to focus on the quantity called *decoherence functional*, because it is useful for clarifying the quantum nature of IR fields. In order to explain the definition and physical meaning of decoherence functional, let us consider a quantum system whose initial state is given by a pure state $\rho(t_0) = |\Psi(t_0)\rangle \langle\Psi(t_0)|$ and its time evolution is governed by the unitary evolution $\hat{U}(t,t_0)$. Note that we work on Shrödinger picture in this chapter. Suppose that we are interested in the variable $\hat{\chi}$ and refer to its eigenstate with eigenvalue χ as $|\chi\rangle$: $\hat{\chi}\,|\chi\rangle = \chi\,|\chi\rangle$. Then, the decoherence functional of the variable $\hat{\chi}$ is defined by

$$D_l\left(\chi,\chi'\right) \coloneqq \mathrm{Tr}\left(\hat{C}^l_{\chi}\rho(t_0)\hat{C}^{l\,\dagger}_{\chi'}\right)\,, \tag{5.1}$$

where the class operator $\hat{C}^l_{\chi}$ is defined by the successive insertion of the projection operator $P_\chi \coloneqq |\chi\rangle\langle\chi|$ at each time step t_i $(i = 1,\cdots,l)$:

$$\hat{C}^l_{\chi} \coloneqq \hat{P}_{\chi_l}\hat{U}_{l,l-1}\hat{P}_{\chi_{l-1}}\cdots\hat{P}_{\chi_1}\hat{U}_{1,0}\,. \tag{5.2}$$

Here, $\hat{U}_{j,j-1}$ expresses the unitary time evolution from t_{j-1} to t_j. Note that we consider the discretized time $t_0, t_1, \cdots, t_l$ to define the class operator $\hat{C}^l_{\chi}$ here, we will take the continuous limit later. $\hat{C}^l_{\chi}\,|\Psi(t_0)\rangle$ corresponds to the quantum state labeled by a specific trajectory $\chi_l\cdots\chi_1$, and this label is so-called "history". We will use this terminology below.[1] "Off-diagonal" element of the decoherence functional, which is denoted by $D_l(\chi,\chi')$ with $\chi\neq\chi'$, expresses the quantum coherence between two different histories $\chi_l\,\chi_{l-1}\cdots\chi_1$ and $\chi_l\,\chi'_{l-1}\cdots\chi'_1$. This can be understood by considering the double-slit experiment. Classically any physical prediction with opening both slits 1 and 2 must coincide with the statistical ensemble average of the prediction with opening either slit 1 or 2. However the latter procedure cannot reproduce the contribution from the $D(1,2)$, $D(2,1)$ factor. Therefore the off-diagonal components of the decoherence functional express the quantum coherence.

On the other hand, the reduced density matrix is often used to discuss the quantum decoherence. It is possible to relate the decoherence functional to the reduced density matrix when one can perform the "system-environment" split and concentrates only on the system variables.[2] For example, when considering the quantum system which is composed of two subsystems S and E, and total Hilbert space $\mathcal{H}_{\rm tot}$ is given by the tensor product

$$\mathcal{H}_{\rm tot} = \mathcal{H}_S\otimes\mathcal{H}_E\,, \tag{5.4}$$

and suppose that we are interested in the system variable $\hat{\chi}$ which can be written as $\hat{\chi} = \hat{\chi}_S\otimes\hat{I}_E$. Here, $\hat{I}_E$ denotes an identity operator which acts on the environment sector

[1] Note that one can consider more generic histories: one can insert projection operator of different variables at different time step. For example when considering the system consists of two-fields $\chi^{(1)}$ and $\chi^{(2)}$ (and their conjugate momenta), one can consider the following class operator

$$\hat{C}^l = \hat{P}^l_{\chi_l}U_{l,l-1}P^{l-1}_{\chi_{l-1}}\cdots\hat{P}^1_{\chi_1}\hat{U}_{1,0}\,, \tag{5.3}$$

where $\hat{P}^j_{\chi_j} = |\chi^{(\epsilon_j)}_j\rangle\langle\chi^{(\epsilon_j)}_j|$ with $\epsilon_j = 1\,\text{or}\,2$. One can also choose ϵ_j in a history-dependent manner.

[2] Originally the decoherence functional is introduced to discuss the emergence of classical world in the *closed* quantum system in the context of decoherent histories approach (for example, see [60, 61, 62]). No system-environment split is assumed there, generically speaking, and thus the variables which are used to discuss the decoherence functional can be operators which act non-trivially on the total Hilbert space.

$\mathcal{H}_E$. In this setup the quantum state corresponding to one particular history $\chi_l \cdots \chi_1$ is given by

$$\hat{C}^l_\chi \left|\Psi(t_0)\right\rangle = \left|\chi_l\right\rangle \left|z\left(\chi_l \cdots \chi_1\right)\right\rangle , \tag{5.5}$$

where $|\chi_l\rangle \in \mathcal{H}_S$ denotes the eigenstate of $\hat{\chi}_S$ with an eigenvalue χ_l, and $|z\left(\chi_l \cdots \chi_1\right)\rangle \in \mathcal{H}_E$ is defined by[3]

$$\left|z\left(\chi_l \cdots \chi_1\right)\right\rangle \coloneqq \int \left\langle\chi'_l\right| \hat{C}^l_\chi \left|\Psi(t_0)\right\rangle \mathrm{d}\chi'_l \,. \tag{5.6}$$

Then, from the definition (5.2), the decoherence functional of $\hat{\chi}$ can be written in terms of $|z(\chi_l \cdots \chi_1)\rangle$ as

$$D_l\left(\chi, \chi'\right) = \left\langle z\left(\chi_l\, \chi'_{l-1} \cdots \chi'_1\right) \middle| \, z\left(\chi_l\, \chi_{l-1} \cdots \chi_1\right)\right\rangle . \tag{5.7}$$

On the other hand, the reduced density matrix for the system which is defined by $\rho_S(t) \coloneqq \mathrm{Tr}_E \rho(t)$, can be also written in terms of $|z(\chi_l \cdots \chi_1)\rangle$ as

$$\left\langle\chi_l\right| \rho_S(t_l) \left|\chi'_l\right\rangle = \prod_{i=1}^{l-1} \int \mathrm{d}\chi_i \mathrm{d}\chi'_i \left\langle z\left(\chi'_l \cdots \chi'_1\right) \middle| \, z\left(\chi_l \cdots \chi_1\right)\right\rangle . \tag{5.8}$$

Here, Tr_E means the partial trace over $\mathcal{H}_E$. Therefore, when it is possible to perform the system-environment split, the inner product of $|z\rangle$ can capture the degree of quantum coherence.

Decoherence and classical statistical interpretation

Let us assume that the quantum coherence between any two different histories is exactly lost, that is, the following orthogonality condition[4]

$$\frac{\left|\left\langle z\left(\chi'_l\, \chi'_{l-1} \cdots \chi'_1\right) \middle| \, z\left(\chi_l\, \chi_{l-1} \cdots \chi_1\right)\right\rangle\right|}{\sqrt{\left\| \left|z\left(\chi_l\, \chi_{l-1} \cdots \chi_1\right)\right\rangle \right\|^2 \left\| \left|z\left(\chi'_l\, \chi_{l-1} \cdots \chi'_1\right)\right\rangle \right\|^2}} \propto \prod_{i=1}^{l-1} \delta\left(\chi_i - \chi'_i\right) \tag{5.9}$$

is satisfied exactly. In this case, we will see that the contributions from various histories to expectation values of an system operator $\hat{O}$ which formally takes the form of $\hat{O} = \hat{O}_S \otimes \hat{I}_E$ can be treated within the validity of classical statistics. Let us define the quantum state corresponding to a history $\chi_{l-1} \cdots \chi_1$ by

$$\left|\Psi\left(\chi_{l-1} \cdots \chi_1\right)\right\rangle \coloneqq \int \mathrm{d}\chi_l \left|\chi_l\right\rangle \left|z\left(\chi_l \cdots \chi_1\right)\right\rangle , \tag{5.10}$$

and an expectation value of $\hat{O}$ for this state by

$$O\left(\chi_{l-1} \cdots \chi_1\right) \coloneqq \frac{\left\langle\Psi\left(\chi_{l-1} \cdots \chi_1\right)\right| \hat{O} \left|\Psi\left(\chi_{l-1} \cdots \chi_1\right)\right\rangle}{p\left(\chi_l \cdots \chi_1\right)} , \tag{5.11}$$

$$p\left(\chi_{l-1} \cdots \chi_1\right) \coloneqq \left\| \left|\Psi\left(\chi_{l-1} \cdots \chi_1\right)\right\rangle \right\|^2 \geq 0 \,. \tag{5.12}$$

[3] When the spectrum of $\hat{\chi}_S$ is discrete, the integration over χ'_l should be understood as a discrete sum.

[4] This condition is called "strong decoherence" conditions [63, 64].

Here, $p\left(\chi_l \cdots \chi_1\right)$ denotes the probability assigned to a history $\chi_{l-1} \cdots \chi_1$. Then, one can understand the contribution of various histories as a classical statistical variances:

$$
\begin{aligned}
&\langle \Psi(t_l)| \hat{O} |\Psi(t_l)\rangle \\
&= \int \mathrm{d}\chi_l^+ \mathrm{d}\chi_l^- \langle \chi_l^- | \hat{O}_S | \chi_l^+ \rangle \prod_{i=1}^{l-1} \int \mathrm{d}\chi_i \left\langle z\left(\chi_l^- \chi_{l-1} \cdots \chi_1\right) \middle| z\left(\chi_l^+ \chi_{l-1} \cdots \chi_1\right) \right\rangle \\
&= \prod_{i=1}^{l-1} \int \mathrm{d}\chi_i \, O\left(\chi_{l-1} \cdots \chi_1\right) p\left(\chi_{l-1} \cdots \chi_1\right) . \qquad (5.13)
\end{aligned}
$$

However, such decoherence condition (5.9) is not exactly satisfied in a generic physical setup. In this case, it is necessary to perform the coarse-graining of the history. We will analyze the quantum nature of IR modes during inflation by playing with $\lambda\phi^4$ theory in de Sitter background and see explicitly that the decoherence does not occur exactly, while it occurs quite efficiently. We will then explain how we should appropriately perform the coarse-graining of the history to explain the theoretical consistency of stochastic approach.

5.1.2 Decoherent histories of IR fields during inflation

From now on, let us apply the above discussion to the inflationary case and calculate the inner product of $|z\rangle$, after performing the system-environment split which is appropriate for understanding the relation between the IR effects and quantum coherence. The setup we consider in this chapter is completely the same as the one in the previous chapter. That is, we will consider a scalar field theory on de Sitter background with an sufficiently flat potential $V(\phi)$, whose hamiltonian density $\mathcal{H}$ is

$$
\mathcal{H} = \frac{1}{2Z_\phi a^6}\pi^2 + \frac{Z_\phi}{2}\left(\frac{\nabla\phi}{a}\right)^2 + V(\phi) . \qquad (5.14)
$$

Note that $V(\phi)$ also contains the mass term, generically, while we will later treat the $V = \lambda\phi^4$ case. Z_ϕ a denotes field-renormalization.[5] We introduce an initial time t_0 and assume that interaction term V is turned on at t_0. We also introduce an IR cutoff $k_0 \coloneqq \epsilon a(t_0)H$ with $\epsilon \ll 1$. We take an initial quantum state $|\Psi_0\rangle$ as a non-interacting Bunch-Davies vacuum state. For later convenience, we normalize $|\Psi_0\rangle$ so that $\| \, |\Psi_0\rangle \, \|^2 = 1$.

In inflationary paradigm, operators corresponding to observables are the field operators whose momenta becomes super-horizon modes during inflation. This motivates us to split the total Hilbert space into IR part and UV part as

$$
\mathcal{H}_{\text{tot}} = \mathcal{H}^<(t_f) \otimes \mathcal{H}^>(t_f) , \qquad (5.15)
$$

$$
\mathcal{H}^<(t) \coloneqq \bigotimes_{k<\epsilon a(t)H} \mathcal{H}_{\vec{k}} , \quad \mathcal{H}^>(t) \coloneqq \bigotimes_{k\geq\epsilon a(t)H} \mathcal{H}_{\vec{k}} , \qquad (5.16)
$$

[5] Note that we did not explicitly write the field renormalization factor and just include it in an interaction potential term in the previous chapter. In this chapter, we explicitly write the field renormalization factor to make it clear that we are presently treating the non-derivatively coupled theory.

and regard the IR part of $\mathcal{H}^<(t_f)$ and UV part $\mathcal{H}^>(t_f)$ as system and environmental sector, respectively. Here, t_f denotes the final time put by hand, which resembles the time when the inflation ends. Then we may consider the class operator $\hat{C}^f_\phi$ which is defined by the successive insertion of the IR projection operators $\hat{P}^<_{\boldsymbol{\phi}}$:

$$\hat{C}^f_{\boldsymbol{\phi}} \coloneqq \hat{P}^<_{\boldsymbol{\phi}_f}(t_f)\,\hat{U}_{f,f-1}\hat{P}^<_{\boldsymbol{\phi}_{f-1}}(t_{f-1})\cdots\hat{P}^<_{\boldsymbol{\phi}_1}(t_1)\,\hat{U}_{1,0}\,, \tag{5.17a}$$

$$\hat{P}^<_{\boldsymbol{\phi}_l}(t_l) \coloneqq \prod_{k\geq\epsilon a(t_l)H}\int \mathrm{d}\phi_l(\vec{k})\,|\boldsymbol{\phi}_l\rangle\langle\boldsymbol{\phi}_l|\,. \tag{5.17b}$$

Here, we have used a bold letter $\boldsymbol{\phi}$ to express the collective modes for notational simplicity. $|\boldsymbol{\phi}_l\rangle$ denotes the eigenstate of $\{\hat{\phi}(\vec{k})\}_{\forall\vec{k}}$ with an eigenvalue $\{\phi_l(\vec{k})\}_{\forall\vec{k}}$ which is properly normalized as

$$\langle\boldsymbol{\phi}_l|\,\boldsymbol{\phi}'_l\rangle = \prod_{\vec{k}}\delta\left(\phi_l(\vec{k})-\phi'_l(\vec{k})\right)\,. \tag{5.18}$$

From the definition of $\hat{C}^f_{\boldsymbol{\phi}}$, one can express the quantum state $\hat{C}^f_\phi\,|\Psi(t_0)\rangle$ corresponding to a certain history of IR fields $\hat{\phi}^<$ as

$$\begin{aligned}
&\hat{C}^f_{\boldsymbol{\phi}}\,|\Psi_0\rangle\\
&=\int\prod_{i=0}^{f}\prod_{k\geq\epsilon a(t_i)H}\mathrm{d}\phi_i(\vec{k})\,\hat{U}_{f,0}\,|\boldsymbol{\phi}_f,t_f\rangle\times\langle\boldsymbol{\phi}_f,t_f|\boldsymbol{\phi}_{f-1},t_{f-1}\rangle\langle\boldsymbol{\phi}_{f-1},t_{f-1}|\cdots|\boldsymbol{\phi}_0,t_0\rangle\big\langle\boldsymbol{\phi}_0,t_0\big|\Psi_0\big\rangle\\
&=\int\prod_{i=0}^{f}\prod_{k\geq\epsilon a(t_i)H}\mathrm{d}\phi_i(\vec{k})\int\prod_{j=0}^{f-1}\prod_{\vec{k}}\frac{\mathrm{d}\pi_j(\vec{k})}{2\pi}\,|\boldsymbol{\phi}_f\rangle\\
&\qquad\times e^{i\sum_{n=0}^{f-1}\left[\int\frac{\mathrm{d}^3k}{(2\pi)^3}\pi_n(\vec{k})\left(\phi_{n+1}(-\vec{k})-\phi_n(-\vec{k})\right)\right]-H\left(\pi_n,\frac{\phi_{n+1}+\phi_n}{2}\right)\Delta t_n}\,\big\langle\boldsymbol{\phi}_0,t_0\big|\Psi_0\big\rangle\,.
\end{aligned}\tag{5.19}$$

Here, $|\boldsymbol{\phi}_j,t_j\rangle$ is defined by

$$|\boldsymbol{\phi}_j,t_j\rangle \coloneqq \hat{U}_{j,0}{}^\dagger\,|\boldsymbol{\phi}_j\rangle\,, \tag{5.20}$$

and we used

$$\begin{aligned}
&\big\langle\boldsymbol{\phi}_{j+1},t_{j+1}|\,\boldsymbol{\phi}_j,t_j\big\rangle\\
&\simeq\int\prod_{\vec{k}}\frac{\mathrm{d}\pi_j(\vec{k})}{2\pi}\,e^{i\int\frac{\mathrm{d}^3k}{(2\pi)^3}\pi_j(\vec{k})\left(\phi_{j+1}(-\vec{k})-\phi_j(-\vec{k})\right)}\left(1-iH\left(\boldsymbol{\pi}_j,\frac{\boldsymbol{\phi}_{j+1}+\boldsymbol{\phi}_j}{2}\right)\Delta t+\mathcal{O}\left((\Delta t)^2\right)\right)\,,
\end{aligned}\tag{5.21}$$

with $\Delta t \coloneqq t_{j+1}-t_j$ for an arbitrary j. Because we will take the limit $\Delta t\to 0$ later with fixing the value of t_f and t_0, we omit below the terms which vanish after taking this limit. In this expression, the functional form of the c-number Hamiltonian $H\left(\pi_j,\frac{\phi_{j+1}+\phi_j}{2}\right)$ included in the expression (5.19) or (5.21) is the same as that of the Weyl-ordered Hamiltonian operator H. Note that eq. (5.21) is compatible with the normalization condition

(5.18). In order to make the notation simpler, let us define an eigenstate of UV fields and IR fields at time t_l by

$$|\boldsymbol{\phi}_l^{>}\rangle \coloneqq \left|\left\{\phi_{l,\vec{k}}\right\}_{k\geq\epsilon a(t_l)H}\right\rangle \in \mathcal{H}^{>}(t_l)\,, \quad |\boldsymbol{\phi}_l^{<}\rangle \coloneqq \left|\left\{\phi_{l,\vec{k}}\right\}_{k<\epsilon a(t_l)H}\right\rangle \in \mathcal{H}^{<}(t_l)\,. \tag{5.22}$$

Then, for given initial state $|\Psi_0\rangle$, the quantum state corresponding to one particular history $\hat{C}_{\boldsymbol{\phi}}^f|\Psi_0\rangle$ can be written as

$$\hat{C}_{\boldsymbol{\phi}}^f|\Psi_0\rangle = \left|\boldsymbol{\phi}_f^{<}\right\rangle\left|z\left(\boldsymbol{\phi}_f^{<}\cdots\boldsymbol{\phi}_1^{<}\right)\right\rangle\,, \tag{5.23}$$

where $\left|z\left(\boldsymbol{\phi}_f^{<}\cdots\boldsymbol{\phi}_1^{<}\right)\right\rangle \in \mathcal{H}^{>}(t_f)$. This state $\left|z\left(\boldsymbol{\phi}_f^{<}\cdots\boldsymbol{\phi}_1^{<}\right)\right\rangle$ is nothing but the analogue of $|z(\chi_f\cdots\chi_1)\rangle$ defined in eq. (5.6). As is discussed in the previous subsection 5.1.1, the inner product of the $|z(\chi_f\cdots\chi_1)\rangle$ expresses for the quantum coherence of the variable $\hat{\chi}$. In our case, by evaluating the inner product of $\left|z\left(\boldsymbol{\phi}_f^{<}\cdots\boldsymbol{\phi}_1^{<}\right)\right\rangle$, one can identify the quantum coherence of IR fields $\hat{\phi}^{<}$, and thus let us explain how to calculate this quantity from now on. Using $|\boldsymbol{\phi}_l\rangle = |\boldsymbol{\phi}_l^{<}\rangle\otimes|\boldsymbol{\phi}_l^{>}\rangle$, one can obtain the expression for $\left|z\left(\boldsymbol{\phi}_f^{<}\cdots\boldsymbol{\phi}_1^{<}\right)\right\rangle$ from eq. (5.19) as

$$\begin{aligned}\left|z\left(\boldsymbol{\phi}_f^{<}\cdots\boldsymbol{\phi}_1^{<}\right)\right\rangle = \int&\left[\prod_{\vec{k}}\prod_{i=0}^{f}\mathrm{d}\phi_i^{+}(\vec{k})\prod_{j=0}^{f-1}\frac{\mathrm{d}\pi_j^{+}(\vec{k})}{2\pi}\right]\\ &\times\left[\prod_{l=0}^{f}\prod_{k<\epsilon a(t_i)H}\delta\left(\phi_l^{+}(\vec{k})-\phi_l(\vec{k})\right)\right]\left|\boldsymbol{\phi}_f^{>}\right\rangle e^{iS_{\mathrm{H}}[\phi^{+},\pi^{+}]}\left\langle\boldsymbol{\phi}_0,t_0\middle|\Psi_0\right\rangle\,,\end{aligned} \tag{5.24}$$

where $S_{\mathrm{H}}[\phi,\pi]$ is defined by

$$S_{\mathrm{H}}[\phi\,,\pi] \coloneqq \sum_{n=0}^{f-1}\left\{\left[\int\frac{\mathrm{d}^3k}{(2\pi)^3}\pi_n(\vec{k})\left(\phi_{n+1}(-\vec{k})-\phi_n(-\vec{k})\right)\right]-H\left(\boldsymbol{\pi}_n,\frac{\boldsymbol{\phi}_{n+1}+\boldsymbol{\phi}_n}{2}\right)\Delta t\right\}\,. \tag{5.25}$$

Let us introduce a transition time step m_k for each mode $\vec{k}$ by $\epsilon\left(t_{m_k}\right)H \leq k < \epsilon a\left(t_{m_k+1}\right)H$. Then, each mode $\vec{k}$ is belonging to a UV mode for $t \leq t_{m_k}$, while it becomes an IR mode for $t \geq t_{m_k+1}$. Then, the transition time $t_{m_k} = t_i$ when the mode $\vec{k}$ satisfies $\epsilon a(t_i)H \leq k < \epsilon a(t_{i+1})H$.[6] With definition of the transition time, eq. (5.24) can be rewritten as

$$\begin{aligned}\left|z\left(\boldsymbol{\phi}_f^{<}\cdots\boldsymbol{\phi}_1^{<}\right)\right\rangle = &\left[\prod_{\vec{k}}\prod_{i=0}^{f}\int\mathrm{d}\phi_i^{+}(\vec{k})\prod_{j=0}^{f-1}\frac{\mathrm{d}\pi_j^{+}(\vec{k})}{2\pi}\right]\\ &\times\left[\prod_{k<\epsilon a\left(t_f\right)H}\prod_{l=m_k+1}^{f}\delta\left(\phi_l^{+}(\vec{k})-\phi_l(\vec{k})\right)\right]\left|\boldsymbol{\phi}_f^{>}\right\rangle e^{iS_{\mathrm{H}}[\phi^{+},\pi^{+}]}\left\langle\boldsymbol{\phi}_0,t_0\middle|\Psi_0\right\rangle\,.\end{aligned} \tag{5.26}$$

[6]For example, $t_{m_k} = t_{i-1}$ when $k = \epsilon a\big(t_i - \Delta t/2\big)H$.

Using this expression (5.26), one can evaluate the quantum interference between the history specified by $\left\{\boldsymbol{\alpha}_i^<\right\}_{i=1\cdots f}$ and the different one specified by $\left\{\boldsymbol{\beta}_i^<\right\}_{i=1\cdots f}$ as

$$
\begin{aligned}
&\left\langle z\left(\boldsymbol{\phi}_f'^< \cdots \boldsymbol{\phi}_1'^<\right)\right| z\left(\boldsymbol{\phi}_f^< \cdots \boldsymbol{\phi}_1^<\right)\Big\rangle \\
&= \left[\prod_{\vec{k}}\prod_{i=0}^{f}\int \mathrm{d}\phi_i^+\left(\vec{k}\right)\mathrm{d}\phi_i^-\left(\vec{k}\right)\prod_{j=0}^{f-1}\int\frac{\mathrm{d}\pi_j^+\left(\vec{k}\right)}{2\pi}\int\frac{\mathrm{d}\pi_j^-\left(\vec{k}\right)}{2\pi}\right]\prod_{k\geq\epsilon a(t_f)H}\delta\left(\phi_f^+\left(\vec{k}\right)-\phi_f^-\left(\vec{k}\right)\right) \\
&\quad\times\left[\prod_{k<\epsilon a\left(t_f\right)H}\prod_{i=m_k+1}^{f}\delta\left(\phi_i^+\left(\vec{k}\right)-\phi_i\left(\vec{k}\right)\right)\delta\left(\phi_i^-\left(\vec{k}\right)-\phi_i'\left(\vec{k}\right)\right)\right]e^{i\left(S_{\mathrm{H}}^+-S_{\mathrm{H}}^-\right)} \\
&\quad\times\left\langle\boldsymbol{\phi}_0^+,t_0\middle|\Psi_0\right\rangle\left\langle\Psi_0\middle|\boldsymbol{\phi}_0^-,t_0\right\rangle\,.
\end{aligned}
\tag{5.27}
$$

By rewriting eq. (5.27) in terms of variables $\left(\phi_i^c\left(\vec{k}\right),\phi_i^\Delta\left(\vec{k}\right),\pi_i^c\left(\vec{k}\right),\pi_i^\Delta\left(\vec{k}\right)\right)$, which are related to original integration variables as

$$
\phi_i^c\left(\vec{k}\right)\coloneqq\frac{\phi_i^+\left(\vec{k}\right)+\phi_i^-\left(\vec{k}\right)}{2}\,,\quad \phi_i^\Delta\left(\vec{k}\right)\coloneqq\phi_i^+\left(\vec{k}\right)-\phi_i^-\left(\vec{k}\right)\,, \tag{5.28a}
$$

$$
\pi_i^c\left(\vec{k}\right)\coloneqq\frac{\pi_i^+\left(\vec{k}\right)+\pi_i^-\left(\vec{k}\right)}{2}\,,\quad \pi_i^\Delta\left(\vec{k}\right)\coloneqq\pi_i^+\left(\vec{k}\right)-\pi_i^-\left(\vec{k}\right)\,, \tag{5.28b}
$$

and further performing the following replacement of integration variables

$$
\phi_i^c\left(\vec{k}\right)\to\begin{cases}\phi_i^{>c}\left(\vec{k}\right) & \text{for}\quad i\leq m_k\\ \phi_i^{<c}\left(\vec{k}\right) & \text{for}\quad i\geq m_k+1\,,\end{cases}\quad \phi_i^\Delta\left(\vec{k}\right)\to\begin{cases}\phi_i^{>\Delta}\left(\vec{k}\right) & \text{for}\quad i\leq m_k-1\\ \phi_i^{<\Delta}\left(\vec{k}\right) & \text{for}\quad i\geq m_k\,,\end{cases} \tag{5.29}
$$

$$
\pi_i^c\left(\vec{k}\right)\to\begin{cases}\pi_i^{>c}\left(\vec{k}\right) & \text{for}\quad i\leq m_k-1\\ \pi_i^{<c}\left(\vec{k}\right) & \text{for}\quad i\geq m_k\,,\end{cases}\quad \pi_i^\Delta\left(\vec{k}\right)\to\begin{cases}\pi_i^{>\Delta}\left(\vec{k}\right) & \text{for}\quad i\leq m_k-1\\ \pi_i^{<\Delta}\left(\vec{k}\right) & \text{for}\quad i\geq m_k\,,\end{cases} \tag{5.30}
$$

one obtains

$$
\begin{aligned}
&\left\langle z\left(\boldsymbol{\phi}_f'^< \cdots \boldsymbol{\phi}_1'^<\right)\right| z\left(\boldsymbol{\phi}_f^< \cdots \boldsymbol{\phi}_1^<\right)\Big\rangle \\
&= \int\left[\prod_{k<\epsilon a\left(t_f\right)H}\mathrm{d}\phi_{m_k,\vec{k}}^{<(\Delta)}\prod_{j=m_k}^{f-1}\frac{\mathrm{d}\pi_j^{<c}\left(\vec{k}\right)}{2\pi}\frac{\mathrm{d}\pi_j^{<\Delta}\left(\vec{k}\right)}{2\pi}\right] \\
&\quad\times e^{i\int_{k<\epsilon a(t_f)H}\frac{\mathrm{d}^3k}{(2\pi)^3}\left[\pi_{m_k}^{<\Delta}(\vec{k})\phi_{m_k+1}^{<c}(-\vec{k})+\pi_{m_k}^{<c}(-\vec{k})\left(\phi_{m_k+1}^{<\Delta}(-\vec{k})-\phi_{m_k}^{<\Delta}(-\vec{k})\right)\right]}e^{i\left(S_{\mathrm{H}}^<+\Gamma\right)}\,,
\end{aligned}
\tag{5.31}
$$

with substituting

$$
\begin{aligned}
&\phi_{m_k}^{<c}\left(\vec{k}\right)=0\,,\quad\left\{\phi_i^{<c}\left(\vec{k}\right)\right\}_{i=m_k+1,\cdots,f}=\left\{\frac{\phi_i^<\left(\vec{k}\right)+\phi_i'^<\left(\vec{k}\right)}{2}\right\}_{i=m_k+1,\cdots,f}\,, \\
&\left\{\phi_i^{<\Delta}\left(\vec{k}\right)\right\}_{i=m_k+1,\cdots,f}=\left\{\phi_i^<\left(\vec{k}\right)-\phi_i'^<\left(\vec{k}\right)\right\}_{i=m_k+1,\cdots,f}
\end{aligned}
\tag{5.32}
$$

for $\forall k < \epsilon a(t_f) H$. Let us note here that this procedure is equivalent to the procedure performed in deriving the extended stochastic formalism in the previous chapter 4. We will later see the consistency between the results obtained in this chapter and those in the previous chapter. In eq. (5.31), $S_{\rm H}^{<}$ denotes an IR part of Hamiltonian action $\left(S_{\rm H}^{(+)} - S_{\rm H}^{(-)}\right)$ consisting purely of IR fields,[7] and $e^{i\Gamma}$ is an influence functional. Their explicit forms are given by

$$\begin{aligned}
S_{\rm H}^{<} :=& \int_{k<\epsilon a\left(t_{f-1}\right)H} \frac{{\rm d}^3 k}{(2\pi)^3} \Bigg\{ \sum_{j=m_k+1}^{f-1} \bigg[\pi_j^{<\Delta}(\vec{k}) \left(\phi_{j+1}^{<c}(-\vec{k}) - \phi_j^{<c}(-\vec{k}) \right) \\
&+ \pi_j^{<c}(\vec{k}) \left(\phi_{j+1}^{<\Delta}(-\vec{k}) \phi_j^{<\Delta}(-\vec{k}) \right) \bigg] \Delta t \Bigg\} \\
&- \int_{k<\epsilon a(t_f)H} \frac{{\rm d}^3 k}{(2\pi)^3} \sum_{i=m_k+1}^{f} H_i^{<}(\vec{k})[\phi^{<c}, \phi^{<\Delta}, \pi^{<c}, \pi^{<\Delta}] \Delta t \,,
\end{aligned} \tag{5.33}$$

$$\begin{aligned}
e^{i\Gamma} :=& \int \left[\prod_{k\geq\epsilon a\left(t_f\right)H} \prod_{i=0}^{f} {\rm d}\phi_i^{>c}(\vec{k}) {\rm d}\phi_i^{>\Delta}(\vec{k}) \prod_{j=0}^{f-1} \frac{{\rm d}\pi_j^{>c}(\vec{k})}{2\pi} \frac{{\rm d}\pi_j^{>\Delta}(\vec{k})}{2\pi} \delta\left(\phi_f^{>\Delta}(\vec{k})\right) \right] \\
&\times \left[\prod_{k<\epsilon a\left(t_f\right)H} \prod_{i=0}^{m_k} {\rm d}\phi_i^{>c}(\vec{k}) {\rm d}\phi_i^{>\Delta}(\vec{k}) \prod_{j=0}^{m_k-1} \frac{{\rm d}\pi_j^{>c}(\vec{k})}{2\pi} \frac{{\rm d}\pi_j^{>\Delta}(\vec{k})}{2\pi} \delta\left(\phi_{m_k}^{>\Delta}(\vec{k})\right) \right] \\
&\times e^{i\left(S_{\rm H}^{>} + S_{\rm H,\,int} + S_{\rm bilinear}\right)} \left\langle \boldsymbol{\phi}_0^{>c} + \frac{\boldsymbol{\phi}_0^{>\Delta}}{2}, t_0 \middle| \Psi_0 \right\rangle \left\langle \Psi_0 \middle| \boldsymbol{\phi}_0^{>c} - \frac{\boldsymbol{\phi}_0^{>\Delta}}{2}, t_0 \right\rangle \,,
\end{aligned} \tag{5.34}$$

In this expression, $H_j^{<}(\vec{k})$ denotes an IR part of the total Hamiltonian density consisting of purely IR fields.[8] $S_{\rm H}^{>}$ denotes a purely UV part of Hamiltonian action, $S_{\rm H,\,int}$ consists of UV-IR non-linear interaction terms, and $S_{\rm bilinear}$ contains UV-IR bilinear in-

[7] Strictly speaking, we have factored out the contributions from IR-π fields at the transition time t_{m_k} to the final line of the right-hand side of eq. (5.31), which corresponds to "$j = m_k$" part of the right-hand side of eq. (5.33). This is because $t_{m_k+1} = t_f$ for such modes. Therefore, terms $\sum_{j=m_k+1}^{f-1}[\cdots]$ in the integrand of eq. (5.33), which become $\pi\dot{\phi}$ terms after taking the limit $\Delta t \to 0$, vanish for modes satisfying $\epsilon a(t_{f-1})H \leq k < \epsilon a(t_f)H$. Corresponding terms are instead written in the final line of the right-hand side of eq. (5.31). This is the reason why the upper bound of momentum integral of the first term of eq. (5.33) is given by $\epsilon a(t_{f-1})H$ rather than $\epsilon a(t_f)H$.

[8] For example in $\lambda\phi^4$ theory, $H_j^{<}(\vec{k})$ is given by

$$H_j^{<}(\vec{k})[\phi^{<c}, \phi^{<\Delta}, \pi^{<c}, \pi^{<\Delta}] = H_j^{<}(\vec{k})[\phi^{+}, \pi^{+}] - H_j^{<}(\vec{k})[\phi^{-}, \pi^{-}] \,, \tag{5.35}$$

$$\begin{aligned}
H_j^{<}(\vec{k})[\phi, \pi] =& \frac{1}{2a^3(t_j)} \pi_j(\vec{k}) \pi_j(-\vec{k}) + \frac{a^3}{2} \left(\frac{k}{a(t_j)}\right)^2 \phi_j(\vec{k}) \phi_j(-\vec{k}) \\
&+ \lambda \phi_j^{<}(\vec{k}) \prod_{i=1}^{3} \int \frac{{\rm d}^3 k_i}{(2\pi)^3} \Theta\left(\epsilon a(t_j) H - k_i\right) \phi_j^{<}(\vec{k}_i) (2\pi)^3 \delta^{(3)}\left(\vec{k} + \sum_{l=1}^{3} \vec{k}_l\right) + ({\rm c.t.}) \,.
\end{aligned} \tag{5.36}$$

Here (c.t.) expresses counterterms which are necessary for renormalization.

teraction terms which correctly reproduce the UV to IR transition due to an accelerating expansion of the universe. Their explicit expressions are

$$S_{\mathrm{H}}^{>} \coloneqq \int \frac{\mathrm{d}^3k}{(2\pi)^3} \sum_{j=0}^{\min(f,m_k)-1} \left\{ \left[\pi_j^{>\Delta}(\vec{k}) \left(\phi_{j+1}^{>c}(-\vec{k}) - \phi_j^{>c}(-\vec{k}) \right) \right.\right.$$
$$\left.\left. + \pi_j^{>c}(\vec{k}) \left(\phi_{j+1}^{>\Delta}(-\vec{k}) - \phi_j^{>\Delta}(-\vec{k}) \right) \right] - H_{j,\vec{k}}^{>} \Delta t \right\} , \tag{5.37a}$$

$$S_{\mathrm{bilinear}} \coloneqq \int_{k<\epsilon a\left(t_f\right)H} \frac{\mathrm{d}^3k}{(2\pi)^3} \left[\pi_{m_k}^{<\Delta}(\vec{k}) \phi_{m_k}^{>c}(-\vec{k}) - \phi_{m_k}^{<\Delta}(\vec{k}) \pi_{m_k-1}^{>c}(-\vec{k}) \right] + \mathcal{O}(\Delta t) \, . \tag{5.37b}$$

Eq. (5.34) shows that an influence functional $e^{i\Gamma}$ is independent of $\left\{\pi_{m_k}^{<c}(\vec{k})\right\}_{k<\epsilon a(t_f)H}$. This is a consequence of the absence of IR-c fields in the bilinear interaction vertexes. Therefore, one can perform the integration over $\left\{\pi_{m_k}^{<c}(\vec{k})\right\}_{k<\epsilon a(t_f)H}$ in eq. (5.31), yielding the product of delta functions $\prod_{k<\epsilon a(t_f)H} \delta\left(\phi_{m_k+1}^{<\Delta}(\vec{k}) - \phi_{m_k}^{<\Delta}(\vec{k})\right)$. Thus, eq. (5.31) reads

$$\left\langle z\left(\boldsymbol{\phi}_f'^{<} \cdots \boldsymbol{\phi}_1'^{<}\right) \middle| \, z\left(\boldsymbol{\phi}_f^{<} \cdots \boldsymbol{\phi}_1^{<}\right) \right\rangle$$
$$= \prod_{k<\epsilon a\left(t_f\right)H} \prod_{j=m_k+1}^{f-1} \int \frac{\mathrm{d}\pi_j^{<c}(\vec{k})}{2\pi} \frac{\mathrm{d}\pi_j^{<\Delta}(\vec{k})}{2\pi} e^{iS_{\mathrm{H}}^{<}} \int \frac{\mathrm{d}\pi_{m_k}^{<\Delta}(\vec{k})}{2\pi} \, e^{i\Gamma} \, e^{i\int_{k<\epsilon a(t_f)H} \frac{\mathrm{d}^3k}{(2\pi)^3} \pi_{m_k}^{<\Delta}(\vec{k}) \phi_{m_k+1}^{<c}(-\vec{k})} , \tag{5.38}$$

with substituting

$$\phi_{m_k}^{<c}(\vec{k}) = 0 \, , \quad \phi_{m_k+1}^{<\Delta}(\vec{k}) = \phi_{m_k}^{<\Delta}(\vec{k}) \, , \tag{5.39a}$$

$$\left\{\phi_i^{<c}(\vec{k})\right\}_{i=m_k+1,\cdots,f} = \left\{ \frac{\phi_i^{<}(\vec{k}) + \phi_i'^{<}(\vec{k})}{2} \right\}_{i=m_k+1,\cdots,f} , \tag{5.39b}$$

$$\left\{\phi_i^{<\Delta}(\vec{k})\right\}_{i=m_k+1,\cdots,f} = \left\{ \phi_i^{<}(\vec{k}) - \phi_i'^{<}(\vec{k}) \right\}_{i=m_k+1,\cdots,f} \tag{5.39c}$$

for $\forall k < \epsilon a(t_f)H$. In order to rewrite eq. (5.38) into simpler form, we would like to perform the integration over $\pi_{m_k}^{<\Delta}(\vec{k})$. In order to do so, we firstly decompose Γ into $\left\{\pi_{m_k}^{<\Delta}(\vec{k})\right\}$-dependent part $\tilde{\Gamma}$ and $\left\{\pi_{m_k}^{<\Delta}(\vec{k})\right\}$-independent part Γ': $\Gamma = \tilde{\Gamma} + \Gamma'$. Let us further decompose $\tilde{\Gamma}$ as $\tilde{\Gamma} = \tilde{\Gamma}_{(\mathrm{d})} + \tilde{\Gamma}_{(\mathrm{s})}$, where the terms with subscript (d) are linear in $\pi_{m_k}^{<\Delta}(\vec{k})$, while the terms with subscript (s) denote other remaining pieces which are higher order in $\pi_{m_k}^{<\Delta}(\vec{k})$. We may explicitly write $\tilde{\Gamma}_{(\mathrm{d})}$ as

$$\tilde{\Gamma}_{(\mathrm{d})} = -\int_{k<\epsilon a(t_f)H} \frac{\mathrm{d}^3k}{(2\pi)^3} \, \pi_{m_k}^{<\Delta}(\vec{k}) \tilde{\mu}\left(t_{m_k}, -\vec{k}\right) \, , \tag{5.40}$$

where $\tilde{\mu}$ generically depends on IR fields except for $\pi_{m_k}^{<\Delta}(\vec{k})$.

From below, we consider the theory which does not contain derivative interactions to make the following discussion simpler. In this case, $\tilde{\mu}$, $\tilde{\Gamma}_{(\mathrm{s})}$, and Γ' are independent of IR π fields. In other words, they are functionals of $\left\{\phi_j^{<c}(\vec{k})\right\}_{j=m_k+1,\cdots,f-1}$ or

$\left\{\phi_j^{<\Delta}(\vec{k})\right\}_{j=m_k,\cdots,f-1}$. By defining P as

$$
\begin{aligned}
&P\left[\boldsymbol{\xi}_\phi\,,\left\{\phi_j^{<c}(\vec{k})\right\}_{j=m_k+1,\cdots,f}\,;\,\left\{\phi_j^{<\Delta}(\vec{k})\right\}_{j=m_k+1,\cdots,f}\right]\\
&\coloneqq \prod_{k<\epsilon a\left(t_f\right)H}\int\frac{\mathrm{d}\pi_{m_k}^{<\Delta}(\vec{k})}{2\pi}\;e^{i\tilde{\Gamma}_{(\mathrm{s})}}e^{i\int_{k<\epsilon a(t_f)H}\frac{\mathrm{d}^3k}{(2\pi)^3}\,\pi_{m_k}^{<\Delta}(\vec{k})\xi_\phi(-\vec{k})}\Big|_{\phi_{m_k}^{<\Delta}(\vec{k})=\phi_{m_k+1}^{<\Delta}(\vec{k})}\,,
\end{aligned}
\tag{5.41}
$$

we obtain the formal expression for $e^{i\tilde{\Gamma}_{(\mathrm{s})}}$ in terms of $P[\xi_\phi]$ by performing the inverse Fourier transformation of eq. (5.41). Substituting that expression into eq. (5.38), one obtains

$$
\begin{aligned}
\left\langle z\left(\boldsymbol{\phi}_f'^{<}\cdots\boldsymbol{\phi}_1'^{<}\right)\right|\left.z\left(\boldsymbol{\phi}_f^{<}\cdots\boldsymbol{\phi}_1^{<}\right)\right\rangle &= e^{i\left(\Gamma'+S_{\mathrm{H,int}}^{<}\right)}\prod_{k<\epsilon a\left(t_f\right)H}\prod_{j=m_k+1}^{f-1}\int\frac{\mathrm{d}\pi_j^{<c}(\vec{k})}{2\pi}\frac{\mathrm{d}\pi_j^{<\Delta}(\vec{k})}{2\pi}e^{iS_{\mathrm{H,kin}}^{<}}\\
&\times\int\mathrm{d}\xi_\phi(\vec{k})\delta\left(\phi_{m_k+1}^{<c}(\vec{k})-\tilde{\mu}\left(t_{m_k},\vec{k}\right)-\xi_\phi(\vec{k})\right)\\
&\times P\left[\boldsymbol{\xi}_\phi\,;\left\{\phi_j^{<c}(\vec{k})\right\}_{j=m_k+1,\cdots,f}\,,\,\left\{\phi_j^{<\Delta}(\vec{k})\right\}_{j=m_k+1,\cdots,f}\right]\,,
\end{aligned}
\tag{5.42}
$$

with substituting eqs. (5.39). Here, $S_{\mathrm{H}}^{<}=S_{\mathrm{H,kin}}^{<}+S_{\mathrm{H,int}}^{<}$. $S_{\mathrm{H,kin}}^{<}$ and $S_{\mathrm{H,int}}^{<}$ denote kinetic terms and non-linear interaction terms of IR modes, respectively. Because $S_{\mathrm{H,int}}^{<}$ and Γ' are independent of IR π fields in the absence of derivative interactions, one can put these terms outside of the integral over $\pi_j^{<c}(\vec{k})$ or $\pi_j^{<\Delta}(\vec{k})$. Using

$$
\begin{aligned}
S_{\mathrm{H,kin}}^{<} &= \int_{k<\epsilon a\left(t_{f-1}\right)H}\frac{\mathrm{d}^3k}{(2\pi)^3}\sum_{j=m_k+1}^{f-1}\Bigg[\pi_j^{<\Delta}(\vec{k})\left(\phi_{j+1}^{<c}(-\vec{k})-\phi_j^{<c}(-\vec{k})-\frac{\pi_j^{<c}(-\vec{k})}{Z_\phi a^3\left(t_j\right)}\Delta t\right)\\
&+\pi_j^{<c}(\vec{k})\left(\phi_{j+1}^{<\Delta}(-\vec{k})-\phi_j^{<\Delta}(-\vec{k})\right)-Z_\phi\,a^3(t_j)\left(\frac{k}{a\left(t_j\right)}\right)^2\phi_j^{<c}(\vec{k})\phi_j^{<\Delta}(-\vec{k})\Delta t\Bigg]+\mathcal{O}(\Delta t)\,,
\end{aligned}
\tag{5.43}
$$

and neglecting the $\mathcal{O}(\Delta t)$ terms which vanishes after taking the limit $\Delta t\to 0$, one obtains

$$
\begin{aligned}
&\left\langle z\left(\boldsymbol{\phi}_f'^{<}\cdots\boldsymbol{\phi}_1'^{<}\right)\right|\left.z\left(\boldsymbol{\phi}_f^{<}\cdots\boldsymbol{\phi}_1^{<}\right)\right\rangle\\
&=\mathcal{N}_1\,e^{i\left(S_{\mathrm{H,int}}^{<}+S_{\mathrm{kin}}^{<}+\Gamma'\right)}P\left[\boldsymbol{\xi}_\phi\,;\left\{\phi_j^{<c}(\vec{k})\right\}_{j=m_k+1,\cdots,f}\,,\,\left\{\phi_j^{<\Delta}(\vec{k})\right\}_{j=m_k+1,\cdots,f}\right]\,,
\end{aligned}
\tag{5.44}
$$

$$
\begin{aligned}
S_{\mathrm{kin}}^{<} &\coloneqq \int_{k<\epsilon a\left(t_{f-1}\right)H}\frac{\mathrm{d}^3k}{(2\pi)^3}Z_\phi\sum_{j=m_k+1}^{f-1}a^3\left(t_j\right)\\
&\times\left[\left(\frac{\phi_{j+1}^{<c}(\vec{k})-\phi_j^{<c}(\vec{k})}{\Delta t}\right)\left(\frac{\phi_{j+1}^{<\Delta}(-\vec{k})-\phi_j^{<\Delta}(-\vec{k})}{\Delta t}\right)-\left(\frac{k}{a\left(t_j\right)}\right)^2\phi_j^{<c}(\vec{k})\phi_j^{<\Delta}(-\vec{k})\right]\Delta t
\end{aligned}
\tag{5.45}
$$

with substituting eqs. (5.39) and

$$\xi_\phi(\vec{k}) = \phi^{<c}_{m_k+1}(\vec{k}) - \tilde{\mu}(t_{m_k}, \vec{k})\,. \tag{5.46}$$

Here, Z_ϕ denotes a wave function renormalization and $\mathcal{N}_1$ is a proportionality constant. From now on, let us discuss the quantum coherence of the "histories". Because $S^{<}_{\mathrm{H,int}}$, $S^{<}_{\mathrm{kin}}$, and $P[\xi_\phi]$ are real, the quantum coherence of different histories between $\boldsymbol{\phi}^{<}_f \cdots \boldsymbol{\phi}^{<}_1$ and $\boldsymbol{\phi}'^{<}_f \cdots \boldsymbol{\phi}'^{<}_1$ is given by

$$\begin{aligned}
&\left|\left\langle z\left(\boldsymbol{\phi}'^{<}_f \cdots \boldsymbol{\phi}'^{<}_1\right)\right| z\left(\boldsymbol{\phi}^{<}_f \cdots \boldsymbol{\phi}^{<}_1\right)\right\rangle\right| \\
&= \mathcal{N}_1\, e^{-\mathrm{Im}\Gamma'} P\left[\boldsymbol{\xi}_\phi\,;\left\{\phi^{<c}_j(\vec{k})\right\}_{j=m_k+1,\cdots,f}\,,\ \left\{\phi^{<\Delta}_j(\vec{k})\right\}_{j=m_k+1,\cdots,f}\right]\Bigg|_{\text{eqs.(5.39) and (5.46)}}\,.
\end{aligned} \tag{5.47}$$

Especially when $\{\boldsymbol{\phi}^{<}_j = \boldsymbol{\phi}'^{<}_j\}_{j=1\cdots f}$, $\mathrm{Im}\Gamma'$ which appears in eq. (5.47) vanishes, and hence the norm of the history $\boldsymbol{\phi}^{<}_f \cdots \boldsymbol{\phi}^{<}_1$ is given by

$$\left\|\left|z\left(\boldsymbol{\phi}^{<}_f \cdots \boldsymbol{\phi}^{<}_1\right)\right\rangle\right\|^2 = \mathcal{N}_1\, P\left[\boldsymbol{\xi}_\phi\,;\left\{\phi^{<}_j(\vec{k})\right\}_{j=m_k+1,\cdots,f}\,,\ 0\right]\Bigg|_{\text{eq.(5.46)}}\,. \tag{5.48}$$

From eqs. (5.47) and (5.48), one can obtain the following expression:

$$\frac{\left|\left\langle z\left(\boldsymbol{\phi}'^{<}_f \cdots \boldsymbol{\phi}'^{<}_1\right)\right| z\left(\boldsymbol{\phi}^{<}_f \cdots \boldsymbol{\phi}^{<}_1\right)\right\rangle\right|}{\sqrt{\left\|\left|z\left(\boldsymbol{\phi}^{<}_f \cdots \boldsymbol{\phi}^{<}_1\right)\right\rangle\right\|^2 \left\|\left|z\left(\boldsymbol{\phi}'^{<}_f \cdots \boldsymbol{\phi}'^{<}_1\right)\right\rangle\right\|^2}} = e^{-\mathrm{Im}\Gamma'}\mathcal{C}[P]\Big|_{\text{eqs.(5.39) and (5.46)}}\,, \tag{5.49}$$

$$\mathcal{C}[P] \coloneqq \frac{P\left[\boldsymbol{\xi}_\phi\,;\left\{\phi^{<c}_j(\vec{k})\right\}_{j=m_k+1,\cdots,f}\,,\ \left\{\phi^{<\Delta}_j(\vec{k})\right\}_{j=m_k+1,\cdots,f}\right]}{\sqrt{P\left[\boldsymbol{\xi}_\phi\,;\left\{\phi^{<}_j(\vec{k})\right\}_{j=m_k+1,\cdots,f}\,,\ 0\right] P\left[\boldsymbol{\xi}_\phi\,;\left\{\phi'^{<}_j(\vec{k})\right\}_{j=m_k+1,\cdots,f}\,,\ 0\right]}}\,. \tag{5.50}$$

This shows that the quantum coherence between different histories is given by the $\phi^{<\Delta}$-dependence of $\mathrm{Im}\,\Gamma'$ and of the noise weight function $P[\xi_\phi]$. It is well known that the quantum decoherence of IR fields efficiently occur during inflation [47, 48]. In order to see the efficient decoherence of histories in our language, let us evaluate only Γ' and $P[\xi_\phi]$ in $\lambda\phi^4$ theory as a toy model, adopting the Gaussian approximation with respect to IR-Δ fields for simplicity. Combined with the definition of $P[\xi_\phi]$ which is given by eq. (5.41), $i\Gamma'$ and $P[\xi_\phi]$ can be calculated by evaluating an influence functional $i\Gamma$ by writing down Feynman diagrams after taking the $\Delta t \to 0$ limit. Strictly speaking both $i\Gamma'$ and $P[\xi_\phi]$ can contribute to the decoherence of the history at the same order, here we will only evaluate Γ' by writing diagrams which are lowest-order in IR-Δ fields for simplicity to estimate how efficiently decoherence occurs. Γ' can be calculated by evaluating Feynman diagrams which does not contain bilinear vertex with an IR π field. Then, $\mathrm{Im}\,\Gamma'$ can be evaluated at leading order as

$$\mathrm{Im}\,\Gamma' \approx \int \frac{\mathrm{d}^3k}{(2\pi)^3}\left[\int_{t_k}^{t_f}\mathrm{d}t\,\left|\phi^{<\Delta}\left(t,\vec{k}\right)\right|^2 V_1\left(t;\vec{k}\right) + \left|\phi^{<\Delta}\left(t_k,\vec{k}\right)\right|^2 V_2\left(t_k;\vec{k}\right)\right], \tag{5.51}$$

with

$$V_1\left(t;\vec{k}\right)\sim\frac{\lambda^2H^2a^3(t)}{\epsilon^3}\,,\qquad V_2\left(t_k,\vec{k}\right)\sim\frac{\lambda^2Ha^3(t_k)}{\epsilon^3}\,. \tag{5.52}$$

This implies that the coherence between different histories $\boldsymbol{\phi}^<$ and $\boldsymbol{\phi}'^<$ which is expressed by the left-hand side of eq. (5.49) is remained when $\left|\boldsymbol{\phi}^<-\boldsymbol{\phi}'^<\right|$ is sufficiently small such that

$$\left|\phi^<\left(t,\vec{k}\right)-\phi'^<\left(t,\vec{k}\right)\right|^2\lesssim\Delta_{\rm dec}\left(t\right):=\frac{\epsilon^3}{\lambda^2H^2}\frac{1}{a^3(t)}\,, \tag{5.53}$$

after $t>t_k$. Here we have neglected unimportant numerical factors.[9] It is the point that the suppression factor $\mathrm{Im}\,\Gamma'$ becomes large at a late time even if the non-linear coupling λ is much smaller than unity. This shows the efficient decoherence of IR fields of light scalar fields during inflation.

5.1.3 Quantum decoherence and origin of IR secular effects

Coarse-grained histories

In the previous subsection 5.1.2, we have seen the decoherence of histories of IR fields during inflation by investigating the inner product of $\left\{\left|z\left(\boldsymbol{\phi}_f^<\cdots\boldsymbol{\phi}_1^<\right)\right\rangle\right\}$. However, this orthogonality is not exact: coherence exists allowed by (5.53). Therefore in order to make an approximately orthogonal basis, it is necessary to consider coarse-grained histories. Let us define the coarse-grained state by

$$\left|\bar{z}_\delta\left(\boldsymbol{\phi}_f^<\cdots\boldsymbol{\phi}_1^<\right)\right\rangle:=\prod_{i=1}^{f-1}\int\mathrm{d}\boldsymbol{\phi}_i'^<\,W_\delta\left[\boldsymbol{\phi}'^<;\,\boldsymbol{\phi}^<\right]\left|z\left(\boldsymbol{\phi}_f^<\,\boldsymbol{\phi}_{f-1}'^<\cdots\boldsymbol{\phi}_1'^<\right)\right\rangle\,, \tag{5.54}$$

where $\mathrm{d}\boldsymbol{\phi}_i^<:=\prod_{k<\epsilon a(t_i)H}\mathrm{d}\phi_i^<(\vec{k})$, and $W_\delta[\boldsymbol{\phi}'^<;\,\boldsymbol{\phi}^<]$ denotes a window function which specifies the coarse-graining of the history centered at some reference history $\boldsymbol{\phi}_{f-1}^<\cdots\boldsymbol{\phi}_1^<$ with a coarse-graining scale $\delta\Delta_{\rm dec}(t,k)$ for each IR mode $\vec{k}$. For example, a Gaussian window function may be given by

$$W_\delta\left[\boldsymbol{\phi}'^<;\,\boldsymbol{\phi}^<\right]=C\exp\left[-\frac{1}{2}\int_{k<\epsilon a(t_{f-1})H}\frac{\mathrm{d}^3k}{(2\pi)^3}\sum_{i=m_k+1}^{f-1}\delta^2\Delta_{\rm dec}^{-1}\left(t_i\right)\left|\phi_i(\vec{k})-\phi_i'(\vec{k})\right|^2\Delta t\right]\,, \tag{5.55}$$

where C denotes a normalization factor which ensures the following normalization condition

$$\prod_{i=1}^{f-1}\int\mathrm{d}\boldsymbol{\phi}_i^<\,W_\delta\left[\boldsymbol{\phi}'^<;\,\boldsymbol{\phi}^<\right]=1\,. \tag{5.56}$$

[9] Decoherence scale will depend on the field value $\phi^{<c}$ in general, although this does not change discussion below.

Below, we will choose this Gaussian window function. If we choose a coarse-graining scale much larger than the decoherence scale $\Delta_{\rm dec}$ by taking $(0<)\delta \ll 1$, we have

$$\begin{aligned}
&\left\langle \bar{z}_\delta\left(\boldsymbol{\phi}_f^{<-}\,\boldsymbol{\phi}_{f-1}^{<}\cdots\boldsymbol{\phi}_1^{<}\right)\right|\bar{z}_\delta\left(\boldsymbol{\phi}_f^{<+}\,\boldsymbol{\phi}_{f-1}^{<}\cdots\boldsymbol{\phi}_1^{<}\right)\big\rangle \\
&=\prod_{i=1}^{f-1}\int \mathrm{d}\boldsymbol{\phi}_i^{<+}\mathrm{d}\boldsymbol{\phi}_i^{<-}\left\langle z\left(\boldsymbol{\phi}_f^{<-}\,\boldsymbol{\phi}_{f-1}^{<-}\cdots\boldsymbol{\phi}_1^{<-}\right)\right| z\left(\boldsymbol{\phi}_f^{<+}\,\boldsymbol{\phi}_{f-1}^{<+}\cdots\boldsymbol{\phi}_1^{<+}\right)\big\rangle \\
&\times W_\delta\left[\boldsymbol{\phi}^{<+};\,\boldsymbol{\phi}^{<}\right]W_\delta\left[\boldsymbol{\phi}^{<-};\,\boldsymbol{\phi}^{<}\right] \\
&\simeq\prod_{i=1}^{f-1}\int \mathrm{d}\boldsymbol{\phi}_i^{<+}\mathrm{d}\boldsymbol{\phi}_i^{<-}\left\langle z\left(\boldsymbol{\phi}_f^{<-}\,\boldsymbol{\phi}_{f-1}^{<-}\cdots\boldsymbol{\phi}_1^{<-}\right)\right| z\left(\boldsymbol{\phi}_f^{<+}\,\boldsymbol{\phi}_{f-1}^{<+}\cdots\boldsymbol{\phi}_1^{<+}\right)\big\rangle \\
&\times W_{\sqrt{2}\delta}\left[\frac{\boldsymbol{\phi}^{<+}+\boldsymbol{\phi}^{<-}}{2};\,\boldsymbol{\phi}^{<}\right].
\end{aligned}\tag{5.57}$$

Here we used

$$W_\delta\left[\boldsymbol{\phi}^{<+};\,\boldsymbol{\phi}^{<}\right]W_\delta\left[\boldsymbol{\phi}^{<-};\,\boldsymbol{\phi}^{<}\right]=W_{\sqrt{2}\delta}\left[\frac{\boldsymbol{\phi}^{<+}+\boldsymbol{\phi}^{<-}}{2};\,\boldsymbol{\phi}^{<}\right]W_{\delta/\sqrt{2}}\left[\boldsymbol{\phi}^{<+}-\boldsymbol{\phi}^{<-};\,0\right], \tag{5.58}$$

and neglect the factor $W_{\delta/\sqrt{2}}\left[\boldsymbol{\phi}^{<\Delta};\,0\right]$. This approximation will be valid as long as $(0<)\delta \ll 1$. Note that eq. (5.58) is exact when we choose Gaussian window function of the form (5.55), but it will be generically possible to neglect the window function with respect to $\phi^{<+}-\phi^{<-}$ fields in the final expression of eq. (5.57) when taking the coarse-graining scale to be much larger than the decoherence scale. Eq. (5.57) allows us to express

$$\begin{aligned}
&\prod_{i=1}^{f-1}\int \mathrm{d}\boldsymbol{\phi}_i^{<}\left\langle \bar{z}_\delta\left(\boldsymbol{\phi}_f^{<-}\,\boldsymbol{\phi}_{f-1}^{<}\cdots\boldsymbol{\phi}_1^{<}\right)\right|\bar{z}_\delta\left(\boldsymbol{\phi}_f^{<+}\,\boldsymbol{\phi}_{f-1}^{<}\cdots\boldsymbol{\phi}_1^{<}\right)\big\rangle \\
&\simeq C'\prod_{i=1}^{f-1}\int \mathrm{d}\boldsymbol{\phi}_i^{<+}\mathrm{d}\boldsymbol{\phi}_i^{<-}\left\langle z\left(\boldsymbol{\phi}_f^{<-}\,\boldsymbol{\phi}_{f-1}^{<-}\cdots\boldsymbol{\phi}_1^{<-}\right)\right| z\left(\boldsymbol{\phi}_f^{<+}\,\boldsymbol{\phi}_{f-1}^{<+}\cdots\boldsymbol{\phi}_1^{<+}\right)\big\rangle .
\end{aligned}\tag{5.59}$$

Here, $C'\neq 1$ is an unimportant field-independent numeric factor which is given by[10]

$$C' \coloneqq \prod_{i=i}^{f-1}\int \mathrm{d}\boldsymbol{\phi}_i^{<}\,W_{\sqrt{2}\delta}\left[\frac{\boldsymbol{\phi}^{<+}+\boldsymbol{\phi}^{<-}}{2};\,\boldsymbol{\phi}^{<}\right]. \tag{5.60}$$

Eq. (5.59) shows that the contributions from various histories $\boldsymbol{\phi}_{f-1}^{<}\cdots\boldsymbol{\phi}_1^{<}$ including quantum coherence can be consistently treated by considering the single-sum of various coarse-grained histories. This reflects the fact that set of coarse-grained histories $\left\{\left|\bar{z}_\delta\left(\boldsymbol{\phi}_f^{<}\cdots\boldsymbol{\phi}_1^{<}\right)\right\rangle\right\}$ constitutes an approximately orthogonal basis so that one no longer need to take into account the quantum coherence between different coarse-grained histories. In other words, the nature of quantum coherence can be took into account by the procedure of coarse-graining.

[10]If the integrand of the right-hand side of eq. (5.60) is $W_\delta\left[\frac{\boldsymbol{\phi}^{<+}+\boldsymbol{\phi}^{<-}}{2};\,\boldsymbol{\phi}^{<}\right]$ instead of $W_{\sqrt{2}\delta}\left[\frac{\boldsymbol{\phi}^{<+}+\boldsymbol{\phi}^{<-}}{2};\,\boldsymbol{\phi}^{<}\right]$, $C'=1$ thanks to the normalization condition eq. (5.56).

Origin of infrared secular effects

From now on, using the essential properties of coarse-grained histories (5.59), we will show that large IR secular effects which appears in evaluating correlation functions can be regarded as an increase of a classical statistical variance. In order to see this, let us firstly consider the expectation value of IR fields. Below we formally write IR fields by $\hat{O}\left[\phi^{<},\pi^{<}\right]$ which can be written as $\hat{O}=\hat{O}^{<}(t_f)\otimes\hat{I}^{>}(t_f)$, where $\hat{I}^{>}(t_f)$ denotes an identity operator acting on UV Hilbert space $\mathcal{H}^{>}(t_f)$. With this notation, the expectation value of IR operator $\hat{O}$ for $|\Psi(t_f)\rangle$ is given by

$$\begin{aligned}
\langle\Psi\left(t_f\right)|\,\hat{O}\,|\Psi\left(t_f\right)\rangle &= \int \mathrm{d}\boldsymbol{\phi}_f^{<+}\mathrm{d}\boldsymbol{\phi}_f^{<-}\ \left\langle\boldsymbol{\phi}_f^{<-}\right|\hat{O}^{<}\left|\boldsymbol{\phi}_f^{<+}\right\rangle \\
&\qquad \times\prod_{i=1}^{f-1}\int \mathrm{d}\boldsymbol{\phi}_i^{<+}\mathrm{d}\boldsymbol{\phi}_i^{<-}\left\langle z\left(\boldsymbol{\phi}_f^{<-}\,\boldsymbol{\phi}_{f-1}^{<-}\cdots\boldsymbol{\phi}_1^{<-}\right)\right|z\left(\boldsymbol{\phi}_f^{<+}\,\boldsymbol{\phi}_{f-1}^{<+}\cdots\boldsymbol{\phi}_1^{<+}\right)\right\rangle \\
&\simeq C'^{-1}\int \mathrm{d}\boldsymbol{\phi}_f^{<+}\mathrm{d}\boldsymbol{\phi}_f^{<-}\left\langle\boldsymbol{\phi}_f^{<-}\right|\hat{O}^{<}\left|\boldsymbol{\phi}_f^{<+}\right\rangle \\
&\qquad \times\prod_{i=1}^{f-1}\int \mathrm{d}\boldsymbol{\phi}_i^{<}\ \left\langle\bar{z}_\delta\left(\boldsymbol{\phi}_f^{<-}\,\boldsymbol{\phi}_{f-1}^{<}\cdots\boldsymbol{\phi}_1^{<}\right)\right|\bar{z}_\delta\left(\boldsymbol{\phi}_f^{<+}\,\boldsymbol{\phi}_{f-1}^{<}\cdots\boldsymbol{\phi}_1^{<}\right)\right\rangle\,. \qquad (5.61)
\end{aligned}$$

Here we used the approximation (5.59) in the third line which is valid when $(0<)\delta\ll 1$. The expression (5.61) can be simplified by defining the quantum state corresponding to the coarse-grained history $\boldsymbol{\phi}_{f-1}^{<}\cdots\boldsymbol{\phi}_1^{<}$ by

$$\left|\bar{\Psi}_\delta\left(\boldsymbol{\phi}_{f-1}^{<}\cdots\boldsymbol{\phi}_1^{<}\right)\right\rangle \coloneqq \int \mathrm{d}\boldsymbol{\phi}_f^{<}\left|\boldsymbol{\phi}_f^{<}\right\rangle\left|\bar{z}_\delta\left(\boldsymbol{\phi}_f^{<}\cdots\boldsymbol{\phi}_1^{<}\right)\right\rangle\,, \qquad (5.62)$$

and define the expectation value of IR fields $\hat{O}$ for this state by $O_\delta\left(\boldsymbol{\phi}_{f-1}^{<}\cdots\boldsymbol{\phi}_1^{<}\right)$:

$$O_\delta\left(\boldsymbol{\phi}_{f-1}^{<}\cdots\boldsymbol{\phi}_1^{<}\right)\coloneqq\frac{\left\langle\bar{\Psi}_\delta\left(\boldsymbol{\phi}_{f-1}^{<}\cdots\boldsymbol{\phi}_1^{<}\right)\right|\hat{O}\left|\bar{\Psi}_\delta\left(\boldsymbol{\phi}_f^{<}\cdots\boldsymbol{\phi}_1^{<}\right)\right\rangle}{\left\|\left|\bar{\Psi}_\delta\left(\boldsymbol{\phi}_{f-1}^{<}\cdots\boldsymbol{\phi}_1^{<}\right)\right\rangle\right\|^2}\,. \qquad (5.63)$$

Next, we define the probability for each coarse-grained history $\boldsymbol{\phi}_{f-1}^{<}\cdots\boldsymbol{\phi}_1^{<}$ by

$$p_\delta\left(\boldsymbol{\phi}_{f-1}^{<}\cdots\boldsymbol{\phi}_1^{<}\right)\coloneqq C'^{-1}\left\|\left|\bar{\Psi}_\delta\left(\boldsymbol{\phi}_{f-1}^{<}\cdots\boldsymbol{\phi}_1^{<}\right)\right\rangle\right\|^2=C'^{-1}\int\mathrm{d}\boldsymbol{\phi}_f^{<}\left\|\left|\bar{z}_\delta\left(\boldsymbol{\phi}_f^{<}\cdots\boldsymbol{\phi}_1^{<}\right)\right\rangle\right\|^2\,, \qquad (5.64)$$

where $p_\delta\left(\boldsymbol{\phi}_{f-1}^{<}\cdots\boldsymbol{\phi}_1^{<}\right)$ satisfies the following approximate normalization condition

$$\prod_{i=1}^{f-1}\int\mathrm{d}\boldsymbol{\phi}_i^{<}p_\delta\left(\boldsymbol{\phi}_{f-1}^{<}\cdots\boldsymbol{\phi}_1^{<}\right)\simeq\left\||\Psi\left(t_f\right)\rangle\right\|^2=1\,. \qquad (5.65)$$

Then, from the definition of $\left|\bar{\Psi}_\delta\right\rangle$, eq. (5.63) can be rewritten as

$$\begin{aligned}
&O_\delta\left(\boldsymbol{\phi}_{f-1}^{<}\cdots\boldsymbol{\phi}_1^{<}\right)p_\delta\left(\boldsymbol{\phi}_{f-1}^{<}\cdots\boldsymbol{\phi}_1^{<}\right) \\
&=C'^{-1}\int\mathrm{d}\boldsymbol{\phi}_f^{<+}\mathrm{d}\boldsymbol{\phi}_f^{<-}\left\langle\boldsymbol{\phi}_f^{<-}\right|\hat{O}^{<}\left|\boldsymbol{\phi}_f^{<+}\right\rangle\left\langle\bar{z}_\delta\left(\boldsymbol{\phi}_f^{<-}\,\boldsymbol{\phi}_{f-1}^{<}\cdots\boldsymbol{\phi}_1^{<}\right)\right|\bar{z}_\delta\left(\boldsymbol{\phi}_f^{<+}\,\boldsymbol{\phi}_{f-1}^{<}\cdots\boldsymbol{\phi}_1^{<}\right)\right\rangle\,. \qquad (5.66)
\end{aligned}$$

Using eq. (5.66), eq. (5.61) reduces to

$$\langle\Psi\left(t_f\right)|\,\hat{O}\,|\Psi\left(t_f\right)\rangle \simeq \prod_{i=1}^{f-1}\int \mathrm{d}\boldsymbol{\phi}_i^{<} O_\delta\left(\boldsymbol{\phi}_{f-1}^{<}\cdots\boldsymbol{\phi}_1^{<}\right) p_\delta\left(\boldsymbol{\phi}_{f-1}^{<}\cdots\boldsymbol{\phi}_1^{<}\right) . \tag{5.67}$$

Now let us consider the physical meaning of this result. As we have seen in previous chapter, large IR secular effects appear when evaluating IR loop corrections to expectation values of IR fields $\hat{O}$ for $|\Psi(t_f)\rangle$ perturbatively in the coupling constant. On the other hand, IR field is treated as an external field when evaluating $O_\delta\big(\boldsymbol{\phi}_{f-1}^{<}\cdots\boldsymbol{\phi}_1^{<}\big)$, and hence it does not contain any large IR loop effects. Therefore, large IR secular effects appear when performing the path integral of IR fields. Eq. (5.67) shows that, one can perform IR path integral by assigning the probability to each coarse-grained path $\big(\boldsymbol{\phi}_{f-1}^{<}\cdots\boldsymbol{\phi}_1^{<}\big)$ and taking the classical statistical ensemble average. This means that IR secular effects can be understood as effects of taking classical ensemble average. Interestingly, such structure of decoherent histories of IR fields are encoded in UV Hilbert space $\mathcal{H}^{<}(t_f)$ which only contains the modes which are inside the Hubble scale during inflation.

Probability of coarse-grained histories

We defined probability for coarse-grained histories in eq. (5.64). From now on we will show that this probability can be in fact calculable using stochastic formalism. From the definition eq. (5.64), we can obtain $p_\delta\big(\boldsymbol{\phi}_{f-1}^{<}\cdots\boldsymbol{\phi}_1^{<}\big)$ once we obtain the expression for $\big\||\bar{z}_\delta\big(\boldsymbol{\phi}_f^{<}\cdots\boldsymbol{\phi}_1^{<}\big)\rangle\big\|^2$. Substituting eq. (5.38) into eq. (5.57) and using $S_{\mathrm{H}}^{<} = S_{\mathrm{H,kin}}^{<} + S_{\mathrm{H,int}}^{<}$, where $S_{\mathrm{H,kin}}^{<}$ is defined by eq. (5.43), one obtains

$$\begin{aligned}
&\big\||\bar{z}_\delta\left(\boldsymbol{\phi}_f^{<}\cdots\boldsymbol{\phi}_1^{<}\right)\rangle\big\|^2 \\
&\simeq \int \mathcal{D}\phi^{<c}\mathcal{D}\phi^{<\Delta}\int\mathcal{D}\pi^{<c}\mathcal{D}\pi^{<\Delta} e^{i\left(S_{\mathrm{H,kin}}^{<}+S_{\mathrm{H,int}}^{<}\right)} \prod_{k<\epsilon a(t_f)H} \delta\left(\phi_f^{<\Delta}\big(\vec{k}\big)\right)\delta\left(\phi_f^{<c}\big(\vec{k}\big)-\phi_f^{<}\big(\vec{k}\big)\right) \\
&\times \prod_{k<\epsilon a(t_f)H}\int \frac{\mathrm{d}\pi_{m_k}^{<\Delta}\big(\vec{k}\big)}{2\pi} W_{\sqrt{2}\delta}\left[\boldsymbol{\phi}^{<c};\,\boldsymbol{\phi}^{<}\right] e^{i\int_{k<\epsilon a(t_f)H}\frac{\mathrm{d}^3k}{(2\pi)^3}\pi_{m_k}^{<\Delta}(\vec{k})\phi_{m_k+1}^{<c}(-\vec{k})}\; e^{i\Gamma}\Big|_{\mathrm{Eq.(5.39a)}} ,
\end{aligned} \tag{5.68}$$

where

$$\mathcal{D}\phi^{<c}\mathcal{D}\phi^{<\Delta} := \prod_{k<\epsilon a\left(t_f\right)H}\prod_{i=m_k+1}^{f}\mathrm{d}\phi_i^{<c}\big(\vec{k}\big)\mathrm{d}\phi_i^{<\Delta} = \prod_{i=1}^{f}\mathrm{d}\boldsymbol{\phi}_i^{<c}\mathrm{d}\boldsymbol{\phi}_i^{<\Delta} , \tag{5.69}$$

$$\mathcal{D}\pi^{<c}\mathcal{D}\pi^{<\Delta} := \prod_{k<\epsilon a\left(t_{f-1}\right)H}\prod_{i=m_k+1}^{f-1}\frac{\mathrm{d}\pi_i^{<c}\big(\vec{k}\big)}{2\pi}\frac{\mathrm{d}\pi_i^{<\Delta}\big(\vec{k}\big)}{2\pi} . \tag{5.70}$$

Let us decompose Γ into $\Gamma_{(\mathrm{d})}+\Gamma_{(\mathrm{s})}$, where $\Gamma_{(\mathrm{d})}$ denotes terms linear in IR-Δ fields and $\Gamma_{(\mathrm{s})}$ contains remaining pieces which are non-linear in IR-Δ fields. We also decompose

$S^{<}_{\mathrm{H,int}}$ into two parts as $S^{<}_{\mathrm{H,int}} = S^{<}_{\mathrm{H,int(d)}} + S^{<}_{\mathrm{H,int(d)}}$ in a parallel manner. Next, we formally express γ and $S^{<}_{\mathrm{H,int}}$ as

$$
\begin{aligned}
&S^{<}_{\mathrm{H,int(d)}} + \Gamma_{(\mathrm{d})}\Big|_{\mathrm{Eq.(5.39a)}} \\
&= -\int_{k<\epsilon a(t_f)H} \frac{\mathrm{d}^3k}{(2\pi)^3}\Bigg\{\pi^{<\Delta}_{m_k}\big(\vec{k}\big)\mu_1\big(t_{m_k},-\vec{k}\big) + a^3\left(t_{m_k}\right)\phi^{<\Delta}_{m_k+1}\big(\vec{k}\big)\mu_2\big(t_{m_k},-\vec{k}\big) \\
&+ \sum_{j=m_k+1}^{f}\left[a^3\left(t_j\right)\phi^{<\Delta}_j\big(\vec{k}\big)\mu_2\big(t_j,-\vec{k}\big)\right]\Delta t\Bigg\}, \qquad (5.71\mathrm{a}) \\
&e^{i\left(\Gamma_{(\mathrm{s})}+S^{<}_{\mathrm{H,int(s)}}\right)}\Big|_{\mathrm{Eq.(5.39a)}} \\
&= \int \mathcal{D}\tilde{\xi}_\pi \prod_{k<\epsilon a(t_f)H}\int \mathrm{d}\xi_\pi\big(\vec{k}\big)\int \mathrm{d}\xi_\phi\big(\vec{k}\big)\,P[\xi;\phi^{<c}] \\
&\times e^{i\int_{k<\epsilon a(t_f)H}\frac{\mathrm{d}^3k}{(2\pi)^3}\left(\phi^{<\Delta}_{m_k+1}(\vec{k})\xi_\pi(-\vec{k})-\pi^{<\Delta}_{m_k}(\vec{k})\xi_\phi(-\vec{k})\right)} e^{i\int_{k<\epsilon a(t_{f-1})H}\frac{\mathrm{d}^3k}{(2\pi)^3}\sum_{j=m_k+2}^{f}\phi^{<\Delta}_j(\vec{k})\tilde{\xi}_\pi(t_{j-1},-\vec{k})\Delta t} \\
&\qquad (5.71\mathrm{b})
\end{aligned}
$$

where

$$
\mathcal{D}\tilde{\xi}_\pi \coloneqq \prod_{k<\epsilon a(t_{f-1})H}\prod_{i=m_k+1}^{f-1}\mathrm{d}\tilde{\xi}_\pi\big(t_i,\vec{k}\big)\,. \qquad (5.72)
$$

Here $P[\xi;\phi^{<c}]$ is a functional of $\tilde{\xi}_\pi$, ξ_π, ξ_ϕ, and $\phi^{<c}$, although we simply write $P[\xi;\phi^{<c}]$ for notational simplicity. Finally we rewrite eq. (5.43) into the following form:

$$
\begin{aligned}
&S^{<}_{\mathrm{H,kin}} \\
&= \int_{k<\epsilon a\left(t_{f-1}\right)H}\frac{\mathrm{d}^3k}{(2\pi)^3}\Bigg\{\sum_{j=m_k+1}^{f-1}\Bigg[\pi^{<\Delta}_j\big(\vec{k}\big)\left(\phi^{<c}_{j+1}\big(-\vec{k}\big)-\phi^{<c}_j\big(-\vec{k}\big)-\frac{\pi^{<c}_j\big(-\vec{k}\big)}{Z_\phi a^3\left(t_j\right)}\Delta t\right) \\
&- Z_\phi\,a^3(t_j)\left(\frac{k}{a\left(t_j\right)}\right)^2\phi^{<c}_j\big(\vec{k}\big)\phi^{<\Delta}_j\big(-\vec{k}\big)\Delta t\Bigg] - \sum_{i=m_k+2}^{f-1}\left[\phi^{<\Delta}_i\big(\vec{k}\big)\left(\pi^{<c}_i\big(-\vec{k}\big)-\pi^{<c}_{i-1}\big(-\vec{k}\big)\right)\right] \\
&+ \phi^{<\Delta}_f\big(\vec{k}\big)\pi^{<c}_{f-1}\big(-\vec{k}\big) - \phi^{<\Delta}_{m_k+1}\big(\vec{k}\big)\pi^{<c}_{m_k+1}\big(-\vec{k}\big) + \mathcal{O}(\Delta t)\Bigg\}, \qquad (5.73)
\end{aligned}
$$

with the understanding that second terms in (5.73) which takes the form of $\sum_{i=m_k+2}^{f-1}[\cdots]$ vanish for any mode $\vec{k}$ satisfying $\epsilon a(t_{f-2})H \leq k < \epsilon a(t_{f-1})H$. We will use this notation throughout this chapter. Substituting eqs. (5.71) and (5.73) into eq. (5.68), one obtains

$$
\begin{aligned}
&\left\| \left| \bar{z}_\delta \left(\boldsymbol{\phi}_f^{<} \cdots \boldsymbol{\phi}_1^{<} \right) \right\rangle \right\|^2 \\
&\simeq \mathcal{N} \int \mathcal{D}\phi^{<c} \mathcal{D}\phi^{<\Delta} \int \mathcal{D}\pi^{<c} \mathcal{D}\pi^{<\Delta} \prod_{k<\epsilon a(t_f)H} \delta\left(\phi_f^{<\Delta}(\vec{k})\right) \delta\left(\phi_f^{<c}(\vec{k}) - \phi_f^{<}(\vec{k})\right) \\
&\times \int \mathcal{D}\tilde{\xi}_\pi \prod_{k<\epsilon a(t_f)H} \int \frac{\mathrm{d}\pi_{m_k}^{<\Delta}(\vec{k})}{2\pi} \int \mathrm{d}\xi_\pi(\vec{k}) \int \mathrm{d}\xi_\phi(\vec{k}) \\
&\times W_{\sqrt{2}\delta}\left[\boldsymbol{\phi}^{<c};\, \boldsymbol{\phi}^{<}\right] P\left[\xi; \phi^{<c}\right] \exp\left[i \int_{k<\epsilon a(t_f)H} \frac{\mathrm{d}^3 k}{(2\pi)^3} L(\vec{k}) \right],
\end{aligned} \tag{5.74}
$$

where $L(\vec{k})$ is given by

$$
\begin{aligned}
L(\vec{k}) \coloneqq \Bigg[&\sum_{j=m_k+1}^{f-1} \pi_j^{<\Delta}(\vec{k}) L_\phi(t_j, -\vec{k}) \Delta t - \sum_{i=m_k+2}^{f-1} \phi_i^{<\Delta}(-\vec{k}) L_\pi(t_i, \vec{k}) \Delta t + \phi_f^{<\Delta}(\vec{k}) \pi_{f-1}^{<c}(-\vec{k}) \\
&+ \pi_{m_k}^{<\Delta}(\vec{k}) L_\phi^{(\mathrm{ini})}(-\vec{k}) - \phi_{m_k+1}^{<\Delta}(\vec{k}) L_\pi^{(\mathrm{ini})}(-\vec{k}) + \mathcal{O}(\Delta t) \Bigg],
\end{aligned} \tag{5.75}
$$

with

$$
L_\phi(t_j, \vec{k}) \coloneqq \frac{\phi_{j+1}^{<c}(\vec{k}) - \phi_j^{<c}(\vec{k})}{\Delta t} - \frac{\pi_j^{<c}(\vec{k})}{Z_\phi a^3(t_j)}, \tag{5.76a}
$$

$$
L_\pi(t_i, \vec{k}) \coloneqq \frac{\pi_i^{<c}(\vec{k}) - \pi_{i-1}^{<c}(\vec{k})}{\Delta t} - \tilde{\xi}_\pi(t_{i-1}, \vec{k}) + a^3(t_i)\mu_2(t_i, \vec{k}) + Z_\phi\, a^3(t_i) \left(\frac{k}{a(t_i)} \right)^2 \phi_i^{<c}(\vec{k}), \tag{5.76b}
$$

$$
L_\phi^{(\mathrm{ini})}(\vec{k}) \coloneqq \phi_{m_k+1}^{<c}(\vec{k}) - \mu_1(t_{m_k}, \vec{k}) - \xi_\phi(\vec{k}), \tag{5.76c}
$$

$$
L_\pi^{(\mathrm{ini})}(\vec{k})\Big|_{k<\epsilon a(t_{f-1})H} \coloneqq \pi_{m_k+1}^{<c}(\vec{k}) + a^3(t_{m_k})\, \mu_2(t_{m_k}, \vec{k}) - \xi_\pi(\vec{k}), \tag{5.76d}
$$

$$
L_\pi^{(\mathrm{ini})}(\vec{k})\Big|_{\epsilon a(t_{f-1})H \le k < \epsilon a(t_f)H} \coloneqq a^3(t_{m_k})\, \mu_2(t_{m_k}, \vec{k}) - \xi_\pi(\vec{k}). \tag{5.76e}
$$

Substituting eq. (5.74) into eq. (5.64) combined with eqs. (5.75) and (5.76), one obtains

$$
\begin{aligned}
&C' p_\delta\left(\boldsymbol{\phi}_{f-1}^{<} \cdots \boldsymbol{\phi}_1^{<}\right) \\
&\simeq \mathcal{N} \int \mathcal{D}\phi^{<c} \int \mathcal{D}\pi^{<c} \prod_{k<\epsilon a(t_f)H} W_{\sqrt{2}\delta}\left[\boldsymbol{\phi}^{<c};\, \boldsymbol{\phi}^{<}\right] \\
&\times \int \mathcal{D}\tilde{\xi}_\pi \left[\prod_{k<\epsilon a(t_{f-1})H} \prod_{j=m_k+1}^{f-1} \delta\left(L_\phi(t_j, \vec{k})\right) \right] \left[\prod_{k<\epsilon a(t_{f-2})H} \prod_{i=m_k+2}^{f-1} \delta\left(L_\pi(t_i, \vec{k})\right) \right] \\
&\times \prod_{k<\epsilon a(t_f)H} \left[\int \mathrm{d}\xi_\pi(\vec{k}) \int \mathrm{d}\xi_\phi(\vec{k}) \delta\left(L_\phi^{(\mathrm{ini})}(\vec{k})\right) \right] \prod_{k<\epsilon a(t_{f-1})H} \left[\delta\left(L_\pi^{(\mathrm{ini})}(\vec{k})\right) \right] P[\xi; \phi^{<c}].
\end{aligned} \tag{5.77}
$$

Eliminating $\delta\big(L_\phi(t_j,\vec{k})\big)$ and $\delta\big(L_\phi^{(\mathrm{ini})}(\vec{k})\big)$ by performing $\int \mathcal{D}\pi^{<c}$ and $\int \mathrm{d}\phi^{<c}_{m_k+1}(\vec{k})$ respectively, eq. (5.77) reduces to

$$\begin{aligned}
&C' p_\delta\left(\boldsymbol{\phi}^{<}_{f-1}\cdots\boldsymbol{\phi}^{<}_{1}\right)\\
&\simeq \mathcal{N}\int\mathcal{D}\phi^{<c}\int\mathcal{D}\tilde{\xi}_\pi\int\mathrm{d}\boldsymbol{\xi}_\pi\int\mathrm{d}\boldsymbol{\xi}_\phi\, W_{\sqrt{2}\delta}\left[\boldsymbol{\phi}^{<c};\,\boldsymbol{\phi}^{<}\right]P[\xi;\phi^{<c}]\\
&\times\left[\prod_{k<\epsilon a(t_{f-2})H}\prod_{i=m_k+2}^{f-1}\delta\left(L_\pi(t_i,\vec{k})\right)\right]\prod_{k<\epsilon a(t_{f-1})H}\left[\delta\left(L_\pi^{(\mathrm{ini})}(\vec{k})\right)\right]\Bigg|_{L_\phi=L_\phi^{(\mathrm{ini})}=0}\,,
\end{aligned}\tag{5.78}$$

where $\int\mathrm{d}\boldsymbol{\xi}_\pi := \prod_{k<\epsilon a(t_f)H}\int\mathrm{d}\xi_\pi(\vec{k})$ and $\int\mathrm{d}\boldsymbol{\xi}_\phi := \prod_{k<\epsilon a(t_f)H}\int\mathrm{d}\xi_\phi(\vec{k})$. Here we also redefine the meaning of $\mathcal{D}\phi^{<c}$ as

$$\mathcal{D}\phi^{<c} := \prod_{k<\epsilon a(t_{f-1})H}\prod_{i=m_k+2}^{f}\mathrm{d}\phi_i^{<c}(\vec{k})\,. \tag{5.79}$$

From this expression we can understand that the probability assigned to the coarse-grained history can be evaluated by solving stochastic equations and deriving probability assigned to the trajectory of $\phi^{<c}$ field which is specified by the window function $W_{\sqrt{2}\delta}\left[\boldsymbol{\phi}^{<c};\,\boldsymbol{\phi}^{<}\right]$: it is broadened around the reference trajectory $\boldsymbol{\phi}^{<}_{f-1}\cdots\boldsymbol{\phi}^{<}_{1}$. Below we explain the structure of the stochastic equations:

1. *Time evolution of of $\phi^{<c}$ field and $\pi^{<c}$ field.*

   Time evolution of $\phi_j^{<c}(\vec{k})$ field and $\pi_i^{<c}(\vec{k})$ field are given by stochastic effective EOMs $\left\{L_\phi(t_j,\vec{k})=0\right\}_{j=m_k+1\cdots f-1}$ and $\left\{L_\pi(t_i,\vec{k})=0\right\}_{i=m_k+2\cdots f-1}$, respectively. These two EOMs are coupled to each other and describe stochastic evolution because of the existence of the stochastic noise $\left\{\xi_\pi(t_i,\vec{k})\right\}_{i=m_k+2\cdots f-1}$ whose probability distribution is given by $P[\xi;\phi^{<c}]$.

2. *Initial conditions for $\phi^{<c}$ and $\pi^{<c}$.*

   Field values of $\phi^{<c}_{m_k+1}(\vec{k})$ and $\pi^{<c}_{m_k+1}(\vec{k})$ are related to the value of $\xi_\phi(\vec{k})$ and $\xi_\pi(\vec{k})$ through equations $L_\phi^{(\mathrm{ini})}=0$ and $L_\pi^{(\mathrm{ini})}=0$, respectively. $\phi^{<c}_{m_k+1}(\vec{k})$ and $\pi^{<c}_{m_k+1}(\vec{k})$ have fluctuations because $\xi_\phi(\vec{k})$ and $\xi_\pi(\vec{k})$ are stochastic variables whose probability distributions are give by $P[\xi;\phi^{<c}]$.[11]

Eq. (5.78) shows that the norm of the history which is disfavored by EOM is indeed suppressed. This suppression comes from the rapid oscillation of the phase factor appearing in the first line of the r.h.s of eq. (5.44). This suppression is not manifest when considering the norm of fine-grained state $\left|z\left(\boldsymbol{\phi}^{<}_{f}\cdots\boldsymbol{\phi}^{<}_{1}\right)\right\rangle$ given in eq. (5.48). By considering the coarse-grained history, one can relate it to the stochastic formalism and discuss the probability.

[11] This implies that the bilinear vertexes plays the role of recovering correctly the information of fluctuations at the UV $\to$ IR transition time t_{m_k+1} for each mode $\vec{k}$.

Continuous limit

Now let us discuss the continuous limit of the expression eq. (5.78). As we already explained, effective EOMs are $L_\phi = L_\pi = L_\phi^{(\text{ini})} = L_\pi^{(\text{ini})} = 0$, where $L_\phi^{(\text{ini})} = L_\pi^{(\text{ini})} = 0$ gives rise to the initial conditions for set of first-order differential equations. After taking the limit $\Delta t \to 0$, and referring to the UV to IR transition time t_{m_k} and t_{m_k+1} as t_k and t'_k, respectively, these effective EOMs are set of Langevin equations of the form

$$\dot{\phi}^{<c}\left(t, \vec{k}\right) = \frac{\pi^{<c}\left(t, \vec{k}\right)}{Z_\phi a^3(t)} + \left[\mu_1\left(t_k, \vec{k}\right) + \xi_\phi(\vec{k})\right]\delta\left(t - t_k\right), \tag{5.80a}$$

$$\begin{aligned}\dot{\pi}^{<c}\left(t, \vec{k}\right) = &-a^3(t)\mu_2\left(t, \vec{k}\right) - Z_\phi a^3(t)\left(\frac{k}{a(t)}\right)^2\phi^{<c}\left(t, \vec{k}\right) + \tilde{\xi}_\pi\left(t, \vec{k}\right) \\ &+ \left[-a^3\left(t_k\right)\mu_2\left(t_k, \vec{k}\right) + \xi_\pi(\vec{k})\right]\delta\left(t - t_k\right),\end{aligned} \tag{5.80b}$$

with boundary conditions $\phi^{<c}\left(t_k, \vec{k}\right) = \pi^{<c}\left(t_k, \vec{k}\right) = 0$. One can understand the correctness of these boundary conditions in the continuous limit by integrating eq. (5.80) over t in the vicinity of $t = t_k$ to get

$$\phi^{<c}\left(t'_k, \vec{k}\right) = \mu_1\left(t_k, \vec{k}\right) + \xi_\phi(\vec{k}), \tag{5.81a}$$

$$\pi^{<c}\left(t'_k, \vec{k}\right) = -a^3\left(t_k\right)\mu_2\left(t_k, \vec{k}\right) + \xi_\pi(\vec{k}). \tag{5.81b}$$

Eqs. (5.81) give initial condition which are indeed equivalent to $L_\phi^{(\text{ini})} = 0$ and $L_\pi^{(\text{ini})} = 0$. In this way one can take the continuous limit of EOMs. Then, introducing new variables $v^{<c}\left(t, \vec{k}\right) := \pi^{<c}\left(t, \vec{k}\right)/a^3(t)$, $\tilde{\xi}_v\left(t, \vec{k}\right) := \tilde{\xi}_\pi\left(t, \vec{k}\right)/a^3(t)$, and $\xi_v\left(\vec{k}\right) := \xi_\pi\left(\vec{k}\right)/a^3(t)$, the continuous limit of eq. (5.78) can be written as

$$\begin{aligned}&C' p_\delta\left(\boldsymbol{\phi}_{f-1}^{<} \cdots \boldsymbol{\phi}_1^{<}\right) \\ &\simeq \mathcal{N}_2 \int \mathcal{D}\phi^{<c} \int \mathcal{D}\tilde{\xi}_v \int \mathrm{d}\boldsymbol{\xi}_v \int \mathrm{d}\boldsymbol{\xi}_\phi\, W_{\sqrt{2}\delta}\left[\boldsymbol{\phi}^{<c};\, \boldsymbol{\phi}^{<}\right] P[\xi; \phi^{<c}] \\ &\times \prod_{k<\epsilon a(t_f)H} \prod_{t=t_k}^{t_f} \delta\left(\phi^{<c}\left(t, \vec{k}\right) - \mathcal{S}\left(t, \vec{k}\right)\right)\delta\left(v^{<c}\left(t'_k, \vec{k}\right) + \mu_2\left(t_k, \vec{k}\right) - \xi_v(\vec{k})\right)\Bigg|_{L_\phi = L_\phi^{(\text{ini})} = 0}.\end{aligned} \tag{5.82}$$

Here, $\mathcal{N}_2$ denotes a proportionality constant and $\mathcal{S}\left(t, \vec{k}\right)$ denotes the solution of EOMs eq. (5.80) with above mentioned boundary conditions, neglecting the second term of the right-hand side of eq. (5.80b) which is suppressed by some positive power of ϵ:

$$\mathcal{S}\left(t,\vec{k}\right) \coloneqq \int_{t_k}^{t} \mathrm{d}t'\, G\left(t,t'\right)\left[-\mu_2\left(t',\vec{k}\right)+\xi_v\left(t',\vec{k}\right)\right] \\ - G\left(t,t_k\right)\left(\xi_v\left(\vec{k}\right)-\mu_2\left(t_k,\vec{k}\right)\right)+\mu_1\left(t_k,\vec{k}\right)+\xi_\phi\left(\vec{k}\right), \tag{5.83a}$$

$$Z_\phi G(t,t') \coloneqq \frac{1}{3H}\left(1-\left(\frac{a(t')}{a(t)}\right)^3\right). \tag{5.83b}$$

At leading order in the coupling constant λ, $\mu_2\left(t,\vec{k}\right)$ is simply given by the slope of $\lambda\phi^4$ potential:

$$\mu_2\left(t,\vec{k}\right)=\frac{\lambda}{3!}\prod_{i=1}^{3}\int\frac{\mathrm{d}^3k_i}{(2\pi)^3}\Theta\left(\epsilon aH-k_i\right)\phi\left(t,\vec{k}_i\right)\,(2\pi)^3\delta^{(3)}\left(\vec{k}-\sum_{j=1}^{3}\vec{k}_j\right)+\mathcal{O}(\lambda^2), \tag{5.84}$$

where higher-order corrections come from the UV loop corrections, and we choose a renormalization scheme so that the effective mass correction vanishes exactly as we have done in the previous chapter. Similarly we have $Z_\phi = 1+\mathcal{O}(\lambda^2)$ so that below we will take $Z_\phi = 1$.

Real-space representation

Of course it is also possible to get the real-space representation of eq. (5.82) by considering the Fourier transformation:

$$C' p_\delta\left(\boldsymbol{\phi}^<_{f-1}\cdots\boldsymbol{\phi}^<_1\right) \\ \simeq \mathcal{N}_2\int\mathcal{D}\phi^{<c}\int\mathcal{D}\tilde{\xi}^<_v\int\mathrm{d}\boldsymbol{\xi}^<_v\int\mathrm{d}\boldsymbol{\xi}^<_\phi\, W_{\sqrt{2}\delta}\left[\boldsymbol{\phi}^{<c};\,\boldsymbol{\phi}^<\right]P[\xi^<;\phi^{<c}] \\ \times\prod_{\vec{x}}\prod_{t=t_0}^{t_f}\delta\left(\phi^{<c}\left(t,\vec{x}\right)-\mathcal{S}^<\left(t,\vec{x}\right)\right)\delta\left(v^{<c}\left(\vec{x};t\right)+\mu_2^<\left(\vec{x};t\right)-\xi_v^<(\vec{x};t)\right)\Big|_{L_\phi=L_\phi^{(\mathrm{ini})}=0}, \tag{5.85}$$

where

$$\mathcal{S}^<\left(t,\vec{x}\right) \coloneqq \int\frac{\mathrm{d}^3k}{(2\pi)^3}\,\mathcal{S}\left(t,\vec{k}\right)\Theta\left(\epsilon aH-k\right)e^{i\vec{k}\cdot\vec{x}} \\ = \int_{t_0}^{t}\mathrm{d}t'\,G\left(t,t'\right)\left[-\mu_2^<\left(t',\vec{x}\right)+\xi_v^<\left(t',\vec{x}\right)\right]-{\xi'}_v^<(\vec{x};t)+{\mu'}_2^<(\vec{x};t)+\mu_1^<(\vec{x};t)+\xi_\phi^<(\vec{x};t), \tag{5.86}$$

with defining $A^<(t,\vec{x}) = (2\pi)^{-3}\int\mathrm{d}^3k\,A\left(t,\vec{k}\right)\Theta(\epsilon aH-k)\exp\left[i\vec{k}\cdot\vec{x}\right]$ and $A^<(\vec{x};t) = (2\pi)^{-3}\int\mathrm{d}^3k\,A\left(t_k,\vec{k}\right)\Theta(\epsilon aH-k)\exp\left[i\vec{k}\cdot\vec{x}\right]$ for an arbitrary quantity A. We also introduced the following notation:

$$\xi'^{<}_{v}(\vec{x};t) \coloneqq \int \frac{\mathrm{d}^3k}{(2\pi)^3}\, G\left(t,t_k\right) \xi_v\big(\vec{k}\big)\Theta\left(\epsilon a H - k\right)\, e^{i\vec{k}\cdot\vec{x}}\,, \tag{5.87}$$

$$\mu'^{<}_{2}(\vec{x};t) \coloneqq \int \frac{\mathrm{d}^3k}{(2\pi)^3}\, G\left(t,t_k\right) \mu_2\big(t_k,\vec{k}\big)\Theta\left(\epsilon a H - k\right)\, e^{i\vec{k}\cdot\vec{x}}\,. \tag{5.88}$$

Consistency with previous chapter

Let us comment on the consistency of results in this chapter with those obtained in previous chapter 4. Firstly we note that a set of Langevin equations (5.80) with boundary conditions $\phi^{<c}\Big(t_k,\vec{k}\Big) = \pi^{<c}\Big(t_k,\vec{k}\Big) = 0$ are certainly equivalent to effective EOMs eqs. (4.38) obtained in previous chapter with the same boundary conditions, with the understanding that

$$\xi_\phi\big(\vec{k}\big) = \int_{t_k-0}^{t'_k} \mathrm{d}t\, \xi_\phi\left(t,\vec{k}\right)\,, \quad \xi_\pi\big(\vec{k}\big) = \int_{t_k-0}^{t'_k} \mathrm{d}t\, \xi_\pi\left(t,\vec{k}\right)\,, \tag{5.89}$$

$$\tilde{\xi}_\pi\left(t,\vec{k}\right) = a^3(t)\, \xi_v\left(t,\vec{k}\right) \quad \text{for} \quad t > t_k\,. \tag{5.90}$$

Here, the left-hand side quantities are introduced in this chapter and the right-hand side quantities are defined in the previous chapter. In order to see the consistency of the contents developed in this chapter with those of the previous chapter at the level of path integral expression, it will be useful to firstly rewrite (5.77) into the following form:

$$\begin{aligned} C' p_\delta\left(\boldsymbol{\phi}^{<}_{f-1}\cdots\boldsymbol{\phi}^{<}_{1}\right) \simeq \mathcal{N}\int \mathcal{D}\tilde{\xi}_\pi \prod_{k<\epsilon a(t_f)H} \int \mathrm{d}\xi_\pi\big(\vec{k}\big)\int \mathrm{d}\xi_\phi\big(\vec{k}\big) \\ \times\, P\left[\xi_\pi,\xi_\phi;\phi^{<c}\right] W_{\sqrt{2}\delta}\left[\boldsymbol{\phi}^{<c};\,\boldsymbol{\phi}^{<}\right]\Big|_{L_\phi=L_\pi=L^{(\mathrm{ini})}_\phi=L^{(\mathrm{ini})}_\pi=0}\,, \end{aligned} \tag{5.91}$$

by performing the integration over $\phi^{<c}$ and $\pi^{<c}$ fields to eliminate all the delta functionals in the integrand of eq. (5.77). Then, the consistency of the contents developed in this chapter with those of the previous chapter will become clear by noticing that the path integral for IR modes $Z^<[0]$ is given by

$$\begin{aligned} Z^<[0] &\coloneqq \left\||\Psi(t_f)\rangle\right\|^2 \\ &= \int \mathcal{D}\phi^{<c}\mathcal{D}\phi^{<\Delta}\int \mathcal{D}\pi^{<c}\mathcal{D}\pi^{<\Delta} e^{i\left(S^<_{\mathrm{H,kin}}+S^<_{\mathrm{H,int}}\right)} \prod_{k<\epsilon a(t_f)H} \delta\left(\phi^{<\Delta}_f\big(\vec{k}\big)\right) \\ &\times \prod_{k<\epsilon a(t_f)H}\int \frac{\mathrm{d}\pi^{<\Delta}_{m_k}\big(\vec{k}\big)}{2\pi} e^{i\int_{k<\epsilon a(t_f)H}\frac{\mathrm{d}^3k}{(2\pi)^3}\pi^{<\Delta}_{m_k}(\vec{k})\phi^{<c}_{m_k+1}(-\vec{k})}\, e^{i\Gamma}\Big|_{\mathrm{Eq.}(5.39\mathrm{a})}\,. \end{aligned} \tag{5.92}$$

This is the same expression as the right-hand side of eq. (5.68) with eliminating the window function $W_{\sqrt{2}\delta}[\boldsymbol{\phi}^{<c};\boldsymbol{\phi}^{<}]$ which plays the role of picking up one particular coarse-grained history. Therefore, performing the same calculations for obtaining eq. (5.91) from eq. (5.68), we get

$$Z^{<}[0] = \mathcal{N} \int \mathcal{D}\tilde{\xi}_\pi \prod_{k<\epsilon a(t_f)H} \int \mathrm{d}\xi_\pi\left(\vec{k}\right) \int \mathrm{d}\xi_\phi\left(\vec{k}\right)\, P\left[\xi_\pi, \xi_\phi; \phi^{<c}\right]\Big|_{L_\phi = L_\pi = L_\phi^{(\mathrm{ini})} = L_\pi^{(\mathrm{ini})} = 0}\,. \tag{5.93}$$

As we explained above, effective EOMs $L_\phi = L_\pi = L_\phi^{(\mathrm{ini})} = L_\pi^{(\mathrm{ini})} = 0$ are equivalent to those obtained in the previous chapter. Therefore, the expression (5.93) is equivalent to eq. (4.34).

5.2 On the infrared effects on observed correlation functions

We have investigated the structure of quantum states and identified the origin of large IR effects when evaluating correlation functions. In sec. 5.2.1, we will discuss the quantum state which is recognized by a local observer. Main spirit is the same as the conventional stochastic approach to inflation in the literature: we will not observe a whole set of decohered histories, rather we will observe one of them as long as each histories are distinguishable. Next, in sec. 5.2.2, we will evaluate the amplitudes of quantum fluctuations of deep-IR fields in our observed local universe.

5.2.1 Stochastic picture and the quantum state for local observer

In sec 5.1.3, we discussed the coarse-grained history of IR fields $\boldsymbol{\phi}^{<}_{f-1} \cdots \boldsymbol{\phi}^{<}_{1}$ and probability assigned to it. This history is defined by inserting projection operators in all IR modes, the corresponding quantum state $\left|\bar{\Psi}_\delta\left(\boldsymbol{\phi}^{<}_{f-1} \cdots \boldsymbol{\phi}^{<}_{1}\right)\right\rangle$ which is defined in eq. (5.62) expresses the quantum state where IR fields takes some particular values

$$\phi^{<}(t, \vec{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3}\, \phi\left(t, \vec{k}\right) \Theta\left(\epsilon a H - k\right) e^{i\vec{k}\cdot\vec{x}} \tag{5.94}$$

for entire spacetime region $t_0 < t < t_f$ and $\forall \vec{x}$. On the other hand, what we are really want to know is the expression of quantum state which local observers, such as us, may recognize. Then from causality, any local observer cannot distinguish different histories as long as the history inside the causal past of the observer is unchanged. This is because, an expectation value of any local operator on each history is unchanged as long as the causal-past history is fixed. For example, let us consider two different coarse-grained histories $\boldsymbol{\phi}^{<}_{f-1} \cdots \boldsymbol{\phi}^{<}_{1}$ and $\boldsymbol{\phi}'^{<}_{f-1} \cdots \boldsymbol{\phi}'^{<}_{1}$ which coincide with each other within the causal past of us. That is, referring to an intersection between the causal past of us and a constant-t hypersurface Σ_t as $\Sigma_O(t)$, the following condition is satisfied:

$$\left\{\phi^{<}(t, \vec{x}) = \phi'^{<}(t, \vec{x})\right\}_{\forall \vec{x} \in \Sigma_O(t)} \quad \text{for} \quad t_0 < \forall t < t_f\,. \tag{5.95}$$

Then, an expectation value of IR fields $\hat{O}$ evaluated for each history is, if $\hat{O}$ is an arbitrary product of local operators $\prod_{i=1}^{n} \hat{O}_i(\vec{x}_i)$ with $\vec{x}_1, \cdots \vec{x}_n \in \Sigma_O(t_f)$, coincide with each other:[12]

$$O_\delta\left(\boldsymbol{\phi}_{f-1}^{<} \cdots \boldsymbol{\phi}_1^{<}\right) = O_\delta\left(\boldsymbol{\phi}_{f-1}'^{<} \cdots \boldsymbol{\phi}_1'^{<}\right) . \tag{5.96}$$

Here, as is defined in eq. (5.63), $O_\delta\left(\boldsymbol{\phi}_{f-1}^{<} \cdots \boldsymbol{\phi}_1^{<}\right)$ denotes an expectation value evaluated for a coarse-grained history $\boldsymbol{\phi}_{f-1}^{<} \cdots \boldsymbol{\phi}_1^{<}$. Eq. (5.96) implies that no local observer can perform measurement which approximately picks up only a coarse-grained history $\boldsymbol{\phi}_{f-1}^{<} \cdots \boldsymbol{\phi}_1^{<}$ among many histories which give the same history of the causal past of him: rather he will observe a superposed state of such histories.

Taking into account this point, it is more appropriate to consider the following quantum state corresponding to a coarse-grained history within our causal past:

$$\left|\bar{\Psi}_\delta\left(\boldsymbol{\phi}_{f-1}^{<} \cdots \boldsymbol{\phi}_1^{<}; \Sigma_O\right)\right\rangle \coloneqq \int \mathcal{D}\phi_{\notin \Sigma_O}^{<} \left|\bar{\Psi}_\delta\left(\boldsymbol{\phi}_{f-1}^{<} \cdots \boldsymbol{\phi}_1^{<}\right)\right\rangle , \tag{5.97}$$

where

$$\mathcal{D}\phi_{\notin \Sigma_O}^{<} \coloneqq \prod_{t_0 < t < t_f} \prod_{\vec{x} \notin \Sigma_O(t)} \mathrm{d}\phi^{<}(t, \vec{x}) . \tag{5.98}$$

This state $\left|\bar{\Psi}_\delta\left(\boldsymbol{\phi}_{f-1}^{<} \cdots \boldsymbol{\phi}_1^{<}; \Sigma_O\right)\right\rangle$ contains large number of histories of modes satisfying medium-IR field which is defined by

$$\phi^{(\mathrm{m})}(t, \vec{x}) \coloneqq \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \phi\left(t, \vec{k}\right) \Theta_{(\mathrm{m})}(t, k) e^{i\vec{k}\cdot\vec{x}} , \tag{5.99}$$

with

$$\Theta_{(\mathrm{m})}(t, k) \coloneqq \Theta\left(\epsilon a(t) H - k\right) \Theta\left(k - p_{50}\right) , \tag{5.100}$$

and hence medium-IR fields behave as random variables.[13] Next let us focus on the coarse-grained histories of deep-IR fields $\phi^{\ll}(t, \vec{x})$ which is define by

$$\phi^{\ll}(t, \vec{x}) \coloneqq \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \phi\left(t, \vec{k}\right) \Theta\left(L_t^{-1} - k\right) e^{i\vec{k}\cdot\vec{x}} , \tag{5.101}$$

where L_t is defined by

$$L_t^{-1} \coloneqq \begin{cases} \epsilon a(t_*) H = p_{50} & \text{for} \quad t \geq t_* \\ \epsilon a(t) H & \text{for} \quad t \leq t_* . \end{cases} \tag{5.102}$$

This approximately coincides with the comoving size of causal past for $t \gtrsim t_* + H^{-1} \ln \epsilon^{-1}$ and is ϵ^{-1} times larger than it for $t \lesssim t_*$, and hence deep-IR field is almost spatially

[12]Causality is manifest in eq. (5.86) which gives a solution of effective EOMs.

[13]Strictly speaking the behavior of modes satisfying $L_t^{-1} \leq k \ll \epsilon a H$ is a little bit subtle: number of histories of such modes are not enough for discussing the statistical quantities. Nevertheless we will also refer to these modes as medium modes in this study for simplicity.

constant inside the causal past. This means that deep-IR field has effectively one degrees of freedom when considering the dynamics within our causal past. Therefore the state $\left|\bar{\Psi}_\delta\left(\boldsymbol{\phi}^<_{f-1}\cdots\boldsymbol{\phi}^<_1;\Sigma_O\right)\right\rangle$ contains one coarse-grained history of a deep-IR field. Note that . Let us note that the modes in the vicinity of the boundary of medium modes and deep IR modes have the intermediate properties between the medium modes and deep-IR modes, strictly speaking. Properties of these modes are subtle and difficult to capture, but this will not be problematic because we are interested in the dynamics of field in the IR limit which causes the large IR effects. It will be sufficient to understand the dynamics of modes $k \ll L_t^{-1}$ for the discussion of IR secular effects.

Actually this state $\left|\bar{\Psi}_\delta\left(\boldsymbol{\phi}^<_{f-1}\cdots\boldsymbol{\phi}^<_1;\Sigma_O\right)\right\rangle$ will approximately express a quantum state corresponding to a stochastic picture of inflation where a deep-IR field $\phi^{\ll}(t,\vec{x})$ takes some spatially-constant value at each time. There is however an important difference between conventional stochastic picture of inflation and the state $\left|\bar{\Psi}_\delta\left(\boldsymbol{\phi}^<_{f-1}\cdots\boldsymbol{\phi}^<_1;\Sigma_O\right)\right\rangle$: in the latter state the effect of quantum decoherence is appropriately taken into account by considering the coarse-grained histories. Such coarse-graining has not been considered in the conventional stochastic approach to inflation in the literature, and hence the non-zero coherence has been neglected by hand.

Then what is the appropriate quantum state for evaluating observables thorough the cosmological observations? In this thesis, adopting the spirit of previous studies [7, 27], we will regard the following state $\left|\bar{\Psi}_\delta\big(\boldsymbol{\phi}^{\ll}(T);\Sigma_O\big)\right\rangle$ as a quantum state resembling physical state recognized by us:

$$\left|\bar{\Psi}_\delta\big(\boldsymbol{\phi}^{\ll}(T);\Sigma_O\big)\right\rangle := \prod_{\vec{x}\in\Sigma_O}\prod_{t<T}\int \mathrm{d}\phi^<(t,\vec{x}) \prod_{t=T}^{t_f}\int \mathrm{d}\phi^{(\mathrm{m})}(t,\vec{x}) \left|\bar{\Psi}_\delta\left(\boldsymbol{\phi}^<_{f-1}\cdots\boldsymbol{\phi}^<_1;\Sigma_O\right)\right\rangle\,, \tag{5.103}$$

where T denotes the time satisfying $T \simeq t_f - H^{-1}$.[14] That is, we assume that what we observe thorough the cosmological observation will be n-point correlation functions of curvature perturbations which are evaluated for this state. Plausibility of this assumption is the following. In ordinary quantum theories, we have to consider the projected state when we measure only one realization of some observable operator: for example, if the measurement some observable $\hat{A}$ and its outcome is a_1, we will then recognize the projected state which is orthogonal to any states where the measurement outcome of $\hat{A}$ differs from a_1. As we mentioned above, deep-IR field in our causal past will correspond to such degrees of freedom. This motivates us to consider the state where some particular value of deep-IR field is realized in the vicinity of the final time t_f. This epoch is denoted by $T \le t < t_f$ in eq. (5.103). However except for this epoch $t < T$ we perform the integration over deep-IR field because we only perform the measurement once. On the other hand when predicting the statistical quantities of other d.o.f. of which we can observe many realizations, it is unnecessary to consider projected quantum state. In our case such d.o.f. corresponds to medium-IR fields. This is the reason why do not perform the projection of medium-IR fields, which is reflected by performing the integration over medium-IR fields in eq. (5.103) for $T \le t < t_f$.

[14] As we will see below specific choice of T does not change the quantitative result below. We thus use T which satisfies $T \simeq t_f - H^{-1}$ which represents a measurement time.

We are now interested in how the quantum coherence of deep-IR field can contribute to observables, namely the correlation functions of curvature perturbations for the state $\left|\bar{\Psi}_\delta\big(\boldsymbol{\phi}^{\ll}(T);\Sigma_O\big)\right\rangle$. After specifying the model, effects of deep-IR fields to curvature perturbations will be evaluated once we get the 2-point correlation functions of deep-IR fields. Therefore we move on to evaluate the amplitude of quantum fluctuations of deep-IR field $\phi^{\ll}(t,\vec{x})$ in our local universe.

5.2.2 Quantum fluctuation of deep-IR field in local universe

For later convenience let us refer to the expectation value of IR field $\hat{F}$ evaluated for $\left|\bar{\Psi}_\delta\left(\boldsymbol{\phi}^{\ll}(T);\Sigma_O\right)\right\rangle$ by

$$\langle F\rangle_{\boldsymbol{\phi}^{\ll},\Sigma_O} \coloneqq \frac{\left\langle\bar{\Psi}_\delta\left(\boldsymbol{\phi}^{\ll}(T);\Sigma_O\right)\right|\hat{F}\left|\bar{\Psi}_\delta\left(\boldsymbol{\phi}^{\ll}(T);\Sigma_O\right)\right\rangle}{p_{\boldsymbol{\phi}^{\ll},\Sigma_O}}\,, \tag{5.104}$$

$$p_{\boldsymbol{\phi}^{\ll},\Sigma_O} \coloneqq \left\|\left|\bar{\Psi}_\delta\left(\boldsymbol{\phi}^{\ll}(T);\Sigma_O\right)\right\rangle\right\|^2 . \tag{5.105}$$

Here, we also assume that when $\hat{F}$ are products of non-commuting IR operators, they are ordered which are calculable by using c-field in the Schwinger-Keldysh formalism.[15] From eq. (5.63) and the definition of $\left|\bar{\Psi}_\delta\left(\boldsymbol{\phi}^{\ll}(T);\Sigma_O\right)\right\rangle$, correlation function of $\phi^{\ll}(x)$ can be expressed as

$$\begin{aligned}
&\langle\phi^{\ll}(x_1)\cdots\phi^{\ll}(x_n)\rangle_{\boldsymbol{\phi}^{\ll},\Sigma_O}\\
&\simeq p^{-1}_{\boldsymbol{\phi}^{\ll},\Sigma_O}\prod_t\prod_{\vec{x}\in\Sigma_O}\int \mathrm{d}\phi_c^{<}(x)\int\mathrm{d}\xi_\phi^{<}(x)\int\mathrm{d}\tilde{\xi}_v^{<}(x)\int\mathrm{d}\xi_v^{<}(x)\\
&\qquad\times\phi_c^{\ll}(x_1)\cdots\phi_c^{\ll}(x_n)W_{\sqrt{2}\delta}\left[\phi_c^{\ll}(x);\,\phi^{\ll}(x)\right]P[\xi^{<}(x);\phi_c^{<}(x)]\\
&\qquad\times\delta\left(\phi_c^{<}(x)-\mathcal{S}^{<}(x)\right)\delta\left(v_c^{<}(\vec{x};t)+\mu_2^{<}(\vec{x};t)-\xi_v^{<}(\vec{x};t)\right)\big|_{L_\phi=L_\phi^{(\mathrm{ini})}=0}\,,
\end{aligned}\tag{5.106}$$

where x and $x_1\cdots x_n$ express spacetime argument; *e.g.*, x stands for $(t,\vec{x})$. We also simply write noise probability distribution functions after integrating over variables which live outside the causal past as $P[\xi^{<}(x);\phi_c^{<}(x)]$. Note that we lower the index "c" of fields like $\phi^{<c}\to\phi_c^{<}$ for notational simplicity. We adopt this notation below. Window function in the integrand is

$$W_{\sqrt{2}\delta}\left[\phi_c^{\ll}(x);\,\phi^{\ll}(x)\right]=C\exp\left[-\frac{\lambda^2H^2\delta^2}{\epsilon^3}\int_T^{t_f}\mathrm{d}t\,a^3\int_{\vec{x}\in\Sigma_O(t)}\mathrm{d}^3x\,\left(\phi_c^{\ll}(x)-\phi^{\ll}(x)\right)^2\right]. \tag{5.107}$$

We are now interested in the amplitude of deep-IR field fluctuations. We thus would like to perform the integration over medium-IR fields in eq. (5.106). This procedure is not necessary for the dynamics before t_* because medium-IR field is absent for $t<t_*$: $\phi^{\ll}(t<t_*,\vec{x})=\phi^{<}(t<t_*,\vec{x})$ and $\phi^{(\mathrm{m})}(t<t_*,\vec{x})=0$ by definition of the medium-IR

[15] As was explained in the previous chapter, this is the ordering which is calculable using stochastic formalism. However this is not a bad news because the operator ordering is not important thanks to the squeezing.

fields. We thus need to know how the medium-IR modes contribute to the dynamics of a deep-IR field for $t \geq t_*$. The formal solution of deep-IR field is

$$\begin{aligned}\mathcal{S}^{\ll}(x) &= \int \frac{\mathrm{d}^3 k}{(2\pi)^3}\,\mathcal{S}\Big(t,\vec{k}\Big)\Theta\left(L_t^{-1}-k\right)e^{i\vec{k}\cdot\vec{x}} \\ &= \int_{t_0}^{t}\mathrm{d}t'\,G\left(t,t'\right)\Big[-\mu_2^{\ll}\left(t',\vec{x}\right)+\tilde{\xi}_v^{\ll}\left(t',\vec{x}\right)\Big]-\xi'^{\ll}_v(\vec{x};t)+\mu'^{\ll}_2(\vec{x};t)+\mu_1^{\ll}\left(\vec{x};t\right)+\xi_\phi^{\ll}(\vec{x};t)\,.\end{aligned} \tag{5.108}$$

Medium-IR fields can in principle affect deep-IR dynamics through the correlation between deep-IR noise and medium-IR noise. However at leading order deep-IR noise and medium-IR noise are not correlated, and hence the contribution from medium modes to the deep-IR noises $\xi_v^{\ll}$ or $\tilde{\xi}_v^{\ll}$ will be small. Another chance for medium-IR field to contribute to deep-IR dynamics is through the non-linear potential term $\mu_2(t',\vec{x})$ on the right-hand side of eq. (5.108). This becomes manifest by writing $\mu_2(t,\vec{k})$ by

$$\begin{aligned}\mu_2\left(t,\vec{k}\right) &\simeq \frac{\lambda}{3!}\prod_{i=1}^{3}\int\frac{\mathrm{d}^3k_i}{(2\pi)^3}\Theta\left(\epsilon aH-k_i\right)\phi\left(t,\vec{k}_i\right)\,(2\pi)^3\delta^{(3)}\left(\vec{k}-\sum_{j=1}^{3}\vec{k}_j\right) \\ &\simeq \frac{\lambda}{3!}\prod_{i=1}^{3}\int\frac{\mathrm{d}^3k_i}{(2\pi)^3}\Theta\left(L_t^{-1}-k_i\right)\phi\left(t,\vec{k}_i\right)\,(2\pi)^3\delta^{(3)}\left(\vec{k}-\sum_{j=1}^{3}\vec{k}_j\right) \\ &\quad+\frac{\lambda}{3!}\left[3\eta_1\left(t,\vec{k}\right)+\eta_2\left(t_2,\vec{k}\right)\right]\,,\end{aligned} \tag{5.109}$$

where

$$\begin{aligned}\eta_1\left(t,\vec{k}\right) &:= \int\frac{\mathrm{d}^3q}{(2\pi)^3}\Theta\left(L_t^{-1}-q\right)\phi\left(t,\vec{q}\right) \\ &\quad\times\prod_{i=1}^{2}\int\frac{\mathrm{d}^3k_i}{(2\pi)^3}\Theta_{(\mathrm{m})}\left(t,k_i\right)\phi\left(t,\vec{k}_i\right)\,(2\pi)^3\delta^{(3)}\left(\vec{k}-\vec{q}-\sum_{j=1}^{2}\vec{k}_j\right)\,,\end{aligned} \tag{5.110a}$$

$$\eta_2\left(t,\vec{k}\right) := \prod_{i=1}^{3}\int\frac{\mathrm{d}^3k_i}{(2\pi)^3}\Theta_{(\mathrm{m})}\left(t,k_i\right)\phi\left(t,\vec{k}_i\right)\,(2\pi)^3\delta^{(3)}\left(\vec{k}-\sum_{j=1}^{3}\vec{k}_j\right)\,. \tag{5.110b}$$

In eq. (5.109) we neglected the UV-loop corrections in the first line, and medium-medium-deep contributions in the second line which is suppressed owing to the momentum conservation. After performing straightforward calculation, we get

$$\left|\frac{\langle\eta_i^2(t,\vec{x};\varepsilon)\rangle_{(\mathrm{m})}-\Big(\langle\eta_i(t,\vec{x};\varepsilon)\rangle_{(\mathrm{m})}\Big)^2}{\Big(\langle\eta_i(t,\vec{x};\varepsilon)\rangle_{(\mathrm{m})}\Big)^2}\right|\sim\mathcal{O}(\varepsilon^3)\,, \tag{5.111}$$

for $i=1,2$. Here, $<\cdots>_{(\mathrm{m})}$ denotes a statistically averaged quantity regarding the medium-IR modes, and we defined $\eta_i(t,\vec{x};\varepsilon)$ by

$$\eta_i(t,\vec{x};\varepsilon) := \int\frac{\mathrm{d}^3k}{(2\pi)^3}\Theta\left(\varepsilon L_t^{-1}-k\right)\eta_i\left(t,\vec{k}\right)e^{i\vec{k}\cdot\vec{x}}\,,\quad\text{for}\quad i=1,2 \tag{5.112}$$

and hence $\eta_i(t,\vec{x};\varepsilon=1)=\eta_i^{\ll}(t,\vec{x})$. Eq. (5.111) means that the effects of fluctuations of medium-IR fields to deep-IR dynamics can be took into account in a good approximation by simply performing the following replacement:

$$\mu_2(t',\vec{x}) \to \langle \mu_2(t',\vec{x})\rangle_{(\mathrm{m})} \ . \tag{5.113}$$

This means that medium-IR can modify the deterministic part of deep-IR dynamics, but cannot modify the stochastic properties. This is an expected result: effects of short-scale fluctuations to large-scale dynamics will be treated by taking the expectation value with regards to the short-scale fluctuations in a good approximation.[16] We thus conclude that, the integration over medium-IR modes in eq. (5.106) gives rise to

$$\begin{aligned}
&\langle \phi^{\ll}(x_1)\cdots\phi^{\ll}(x_n)\rangle_{\phi^{\ll},\Sigma_O} \\
&\simeq p^{-1}_{\phi^{\ll},\Sigma_O}\prod_{\vec{x}\in\Sigma_O}\left[\prod_t\int \mathrm{d}\phi_c^{\ll}(x)\int \mathrm{d}\tilde{\xi}_v^{\ll}(x)\right]\left[\prod_{t\le t_*}\int \mathrm{d}\xi_\phi^{\ll}(\vec{x};t)\int \mathrm{d}\xi_v^{\ll}(\vec{x};t)\right] \\
&\quad\times \phi_c^{\ll}(x_1)\cdots\phi_c^{\ll}(x_n) W_{\sqrt{2}\delta}\left[\phi_c^{\ll}(x);\,\phi^{\ll}(x)\right] P[\xi^{\ll}(x);\phi_c^{\ll}(x)] \\
&\quad\times \left[\prod_{t>t_*}\delta\left(\phi_c^{\ll}(x)-\mathcal{S}^{\ll}(x)\big|_{\mu_2\to\langle\mu_2\rangle_{(\mathrm{m})}}\right)\right] \\
&\quad\times \left.\left[\prod_{t\le t_*}\delta\left(\phi_c^{\ll}(x)-\mathcal{S}^{\ll}(x)\right)\delta\left(v_c^{\ll}(\vec{x};t)+\mu_2^{\ll}(\vec{x};t)-\xi_v^{\ll}(\vec{x};t)\right)\right]\right|_{L_\phi=L_\phi^{(\mathrm{ini})}=0} \ .
\end{aligned} \tag{5.114}$$

Here we used the fact that $\xi_\phi^{\ll}(x)$ and $\xi_v^{\ll}$ variables does not exist for $t>t_*$ because their Fourier components are time independent and the upper bound of the momentum integral is given by L_t^{-1} which is constant for $t>t_*$.[17] It should be noted here that the above replacement (5.113) will lead to $\mathcal{O}(1)$ error regarding the dynamics of modes in the vicinity of L_t^{-1}, strictly speaking: left-hand side of eq. (5.111) is not suppressed in case of $\varepsilon\sim\mathcal{O}(1)$. However this will not be a drawback because it will be sufficient to understand the dynamics of modes $k\sim L_t^{-1}$ in order to understand the suppression of large IR secular effects, as we have mentioned above. If the more precise treatment of the dynamics of deep-IR fields including the modes with $k\ll L_t^{-1}$ is in order, one has to numerically simulate a set of Langevin equations. Our attitude is that we should firstly clarify how much deep-IR fluctuations can contribute to observables, by adopting several approximations including the one employed here.

Before analyzing the expression (5.114) more in detail, let us introduce an useful notation. Because deep-IR field is almost spatially constant inside the causal past, we will below abbreviate the spatial argument of deep-IR variables inside the causal past: for example we refer to $\phi_c^{\ll}(t,\vec{x})$ and $\xi_\phi^{\ll}(\vec{x};t)$ with $\vec{x}\in\Sigma_O(t)$ simply as $\phi_c^{\ll}(t)$ and $\xi_\phi^{\ll}(;t)$,

[16] This approximation will be also expected to be broken down when the hierarchy of the scales are absent. This situation is realized in our case when $\varepsilon\sim\mathcal{O}(1)$.

[17] As we have seen in eqs. (5.81) they simply express the amplitude of fluctuations of $\phi(t,\vec{k})$ or $v(t,\vec{k})$ at transition time t_k, respectively. Deep-IR fields does not have such contributions because L_t^{-1} is constant for $t\ge t_*$.

respectively. Using this notation, from now on we analyze (5.114). Because the temporal correlation length of noises will be of the order of Hubble time, we assume that the following relation holds:

$$P[\{\xi^{\ll}(t)\}_{t<t_f};\phi_c^{\ll}] \simeq P[\{\xi^{\ll}(t)\}_{t>t_*};\phi_c^{\ll}]P[\{\xi^{\ll}(t)\}_{t\leq t_*};\phi_c^{\ll}]\,. \tag{5.115}$$

Employing this approximation, eq. (5.114) can be rewritten as

$$\begin{aligned}&\langle\phi^{\ll}(t_1)\cdots\phi^{\ll}(t_n)\rangle_{\phi^{\ll},\Sigma_O}\\ &\simeq p^{-1}_{\phi^{\ll},\Sigma_O}\left[\prod_t\int \mathrm{d}\phi_c^{\ll}(t)\right]\left[\prod_{t>t_*}\int \mathrm{d}\tilde{\xi}_v^{\ll}(t)\delta\left(\phi_c^{\ll}(t)-\mathcal{S}^{\ll}(t)|_{\mu_2\to\langle\mu_2\rangle_{(\mathrm{m})}}\right)\right]\\ &\quad\times\phi_c^{\ll}(t_1)\cdots\phi_c^{\ll}(t_n)W_{\sqrt{2}\delta}\left[\phi_c^{\ll}(t);\,\phi^{\ll}(t)\right]P[\{\xi^{\ll}(t)\}_{t>t_*};\phi_c^{\ll}]D\left[\{\phi_c^{\ll}(t)\}_{t\leq t_*}\right]\,,\end{aligned} \tag{5.116}$$

where

$$\begin{aligned}D\left[\{\phi_c^{\ll}(t)\}_{t\leq t_*}\right]\coloneqq&\prod_{t\leq t_*}\int \mathrm{d}\xi_\phi^{\ll}(;t)\int \mathrm{d}\xi_v^{\ll}(;t)P[\{\xi^{\ll}(t)\}_{t\leq t_*};\phi_c^{\ll}]\\ &\times\delta\left(\phi_c^{\ll}(t)-\mathcal{S}^{\ll}(t)\right)\delta\left(v_c^{\ll}(;t)+\mu_2^{\ll}(;t)-\xi_v^{\ll}(;t)\right)\Big|_{L_\phi=L_\phi^{(\mathrm{ini})}=0}\,.\end{aligned} \tag{5.117}$$

$D[\{\phi_c^{\ll}(t)\}_{t\leq t_*}]$ describes a prior distribution of $\phi^{\ll}(t)$ for $t\leq t_*$. For this epoch $\phi^{\ll}(t)$ coincides with IR field and obeys a set of Langevin equations (5.80) which are at leading order given by

$$\dot{\phi}_c^{\ll}(x)=v_c^{\ll}(x)+\xi(x)\,,\quad \dot{v}_c^{\ll}(x)=-3Hv_c^{\ll}(x)-\frac{\lambda}{3!}\phi_c^{\ll}(x)^3\,, \tag{5.118a}$$

$$\xi(t,\vec{x})\coloneqq\int\frac{\mathrm{d}^3k}{(2\pi)^3}\xi_\phi(\vec{k})\Theta\left(L_t^{-1}-k\right)\delta\left(t-t_k\right)e^{i\vec{k}\cdot\vec{x}}\,, \tag{5.118b}$$

where noise correlation of $\xi(x)$ inside the causal past is given by

$$\begin{aligned}&\langle\xi(t_1,\vec{x}_1)\xi(t_2,\vec{x}_2)\rangle|_{\vec{x}_1\in\Sigma_O(t_1)\,,\,\vec{x}_2\in\Sigma_O(t_2)}\\ &=\left[\prod_{i=1}^2\int\frac{\mathrm{d}^3k_i}{(2\pi)^3}\Theta\left(L_{t_i}^{-1}-k_i\right)\delta\left(t_i-t_{k_i}\right)e^{i\vec{k}_i\cdot\vec{x}_i}\right]\left\langle\xi_\phi(\vec{k}_1)\xi_\phi(\vec{k}_2)\right\rangle\Big|_{\vec{x}_1\in\Sigma_O(t_1)\,,\,\vec{x}_2\in\Sigma_O(t_2)}\\ &\simeq\frac{H^3}{4\pi^2}\delta(t-t')\,.\end{aligned} \tag{5.119}$$

Here we used $\vec{x}_1\in\Sigma_O(t_1)\,,\,\vec{x}_2\in\Sigma_O(t_2)$ in the final line. Note that we recover a spatial argument of deep-IR fields for a while to make the discussion clearer. Below we will again abbreviate it. Therefore $D[\{\phi_c^{\ll}(t)\}_{t\leq t_*}]$ can be evaluated in a good approximation in principle by solving a Markovian dynamics (5.118) which corresponds to a standard stochastic formalism. In order to estimate the amplitude of deep-IR field fluctuations in our local universe, We also adopt the Markovian approximation as is done in eq. (5.52):

this leads to the following noise amplitude:

$$\left\langle \tilde{\xi}_v^{\ll}(t_1)\tilde{\xi}_v^{\ll}(t_2)\right\rangle \simeq \left[\prod_{i=1}^{2}\int\frac{\mathrm{d}^3k_i}{(2\pi)^3}\Theta\left(L_{t_i}^{-1}-k_i\right)\delta\left(t_i-t_{k_i}\right)\right]\left\langle \tilde{\xi}_v(t_1,\vec{k}_1)\tilde{\xi}_v(t_2,\vec{k}_2)\right\rangle$$
$$\sim \lambda^2 H^5\left(\frac{a(t_*)}{a(t_1)}\right)^3\delta(t_1-t_2)\,. \tag{5.120}$$

Here we neglected an unimportant proportionality constant. Note that when we adopt the following approximation

$$G(t,t')\sim\frac{1}{3H}\,, \tag{5.121}$$

which is equivalent to neglect the time-derivative of $v_c^{\ll}$ at the level of effective EOMs, $\mathcal{S}^{\ll}(t)|_{\mu_2\to\langle\mu_2\rangle_{(\mathrm{m})}}$ reduces to

$$\mathcal{S}^{\ll}(t)|_{\mu_2\to\langle\mu_2\rangle_{(\mathrm{m})}} \sim\frac{1}{3H}\int_{t_0}^{t}\mathrm{d}t'\,\left[-\langle\mu_2^{\ll}(t')\rangle_{(\mathrm{m})}+\tilde{\xi}_v^{\ll}(t')\right]$$
$$-\frac{1}{3H}\left(\xi_v^{\ll}(;t)-\mu_2^{\ll}(;t)\right)+\mu_1^{\ll}(;t)+\xi_\phi^{\ll}(;t)\,. \tag{5.122}$$

Using the fact that the terms in the second line are time-independent after t_*, the above expression can be rewritten as

$$\mathcal{S}^{\ll}(t)|_{\mu_2\to\langle\mu_2\rangle_{(\mathrm{m})}} \sim\frac{1}{3H}\int_{t_*}^{t}\mathrm{d}t'\,\left[-\langle\mu_2^{\ll}(t')\rangle_{(\mathrm{m})}+\tilde{\xi}_v^{\ll}(t')\right]+\phi_c^{\ll}(t_*)\,, \tag{5.123}$$

for $t>t_*$. Using eq. (5.123) and Markovian approximation, eq. (5.116) leads to

$$\left\langle e^{i\int_{t_*}J\phi^{\ll}\mathrm{d}t}\right\rangle_{\boldsymbol{\phi}^{\ll},\Sigma_O}$$
$$\sim p^{-1}_{\boldsymbol{\phi}^{\ll},\Sigma_O}\prod_{t>t_*}\int\mathrm{d}\phi_c^{\ll}(t)\int\mathrm{d}\tilde{\xi}_v^{\ll}(t)e^{i\int_{t_*}J\phi_c^{\ll}\mathrm{d}t}W_{\sqrt{2}\delta}\left[\phi_c^{\ll}(t);\,\phi^{\ll}(t)\right]P[\{\xi^{\ll}(t)\}_{t>t_*};\phi_c^{\ll}]$$
$$\times\,\delta\left(\phi_c^{\ll}(t)-\frac{1}{3H}\int_{t_*}^{t}\mathrm{d}t'\,\left[-\langle\mu_2^{\ll}(t')\rangle_{(\mathrm{m})}+\tilde{\xi}_v^{\ll}(t')\right]-\phi_c^{\ll}(t_*)\right)D\left[\phi_c^{\ll}(t_*)\right]\,, \tag{5.124}$$

where

$$D\left[\phi_c^{\ll}(t_*)\right]\coloneqq\left[\prod_t\int\mathrm{d}\phi_c^{\ll}(t)\right]D\left[\{\phi_c^{\ll}(t)\}_{t\le t_*}\right]\,. \tag{5.125}$$

In eq. (5.124), the factor $D\left[\phi_c^{\ll}(t_*)\right]$ expresses the initial prior distribution at $t=t_*$. Therefore what we have to do is to derive the distribution of $\phi^{\ll}(t)$ with the boundary condition at around the final time $t\approx t_f$ which is specified by the window function $W_{\sqrt{2}\delta}$ whose explicit form is given by

$$W_{\sqrt{2}\delta}\left[\phi_c^{\ll}(t);\,\phi^{\ll}(t)\right]=C\exp\left[-\frac{\lambda^2H^2\delta^2}{\epsilon^3}\int_T^{t_f}\mathrm{d}t\,a^3\int_{\vec{x}\in\Sigma_O(t)}\mathrm{d}^3x\,\left(\phi_c^{\ll}(x)-\phi^{\ll}(x)\right)^2\right]$$
$$\approx C\exp\left[-\frac{\lambda^2H^2\delta^2}{\epsilon^3}\int_T^{t_f}\mathrm{d}t\,\left(aL_t\right)^3\left(\phi_c^{\ll}(t)-\phi^{\ll}(t)\right)^2\right]\,. \tag{5.126}$$

The point is that this window function is almost equivalent to the delta function of the form

$$W_{\sqrt{2}\delta}\left[\phi_c^{\ll}(t);\, \phi^{\ll}(t)\right] \approx \prod_{T<t<t_f} \delta\left(\phi_c^{\ll}(t) - \phi^{\ll}(t)\right) , \tag{5.127}$$

because

$$\frac{\epsilon^3}{\lambda^2 H^2 \delta^2} \left(a(t_f) L_t\right)^{-3} H = \frac{\epsilon^3}{\lambda^2 \delta^2} \left(\frac{\epsilon a(t_*)}{a(t_f)}\right)^3 H^2 \ll H^2 . \tag{5.128}$$

Therefore the δ-dependence will be negligibly small. Then, in effect we can evaluate the distribution of $\phi^{\ll}$ by imposing the boundary condition $\phi_c^{\ll}(t_f) = \phi^{\ll}(t_f)$. For example, the distribution of $\phi_c^{\ll}(t)$ at $t = t_*$ can be evaluated using the Bayes's theorem as

$$P\left(\phi_c^{\ll}(t_*)|\phi^{\ll}(t_f)\right) = \frac{P(\phi_c^{\ll}(t_*))}{P(\phi^{\ll}(t_f))} P\left(\phi^{\ll}(t_f)|\phi_c^{\ll}(t_*)\right) , \tag{5.129}$$

where $P\left(\phi_c^{\ll}(t_*)|\phi^{\ll}(t_f)\right)$ and $P\left(\phi^{\ll}(t_f)|\phi_c^{\ll}(t_*)\right)$ denote conditional probabilities. $P(\phi_c^{\ll}(t_*))$ and $P(\phi^{\ll}(t_f))$ are prior distributions, and hence in our notation $P(\phi_c^{\ll}(t_*)) = D[\phi_c^{\ll}(t_*)]$. All the quantities on the right-hand side are calculable by solving stochastic equations. Now we evaluate $P\left(\phi^{\ll}(t_f)|\phi_c^{\ll}(t_*)\right)$. This is the probability distribution of $\phi^{\ll}$ at $t = t_f$ with the initial condition $\phi_c^{\ll}(t_*)$. During the epoch $t_* < t < t_f$ the formal solution is given by

$$\phi_c^{\ll}(t) = \frac{1}{3H} \int_{t_*}^{t} \mathrm{d}t' \left[- \left\langle \mu_2^{\ll}(t') \right\rangle_{(\mathrm{m})} + \tilde{\xi}_v^{\ll}(t') \right] + \phi_c^{\ll}(t_*) , \tag{5.130}$$

with noise correlations (5.120). The total amount of the variance generated by $\tilde{\xi}_v$ from t_* to t_f is estimated as

$$\int_{t_*}^{t_f} \mathrm{d}t\, \lambda^2 H^3 \left(\frac{a(t)}{a(t_*)}\right)^3 \simeq \frac{\lambda^2 H^2}{3} . \tag{5.131}$$

Therefore, suppose that the background trajectory predicts $\phi^{\ll}(t_f) = \mathcal{B}\left(\phi_c^{\ll}(t_*)\right)$, one will get

$$P\left(\phi^{\ll}(t_f)|\phi_c^{\ll}(t_*)\right) = C_2 \exp\left[\frac{-1}{2\sigma^2} \left(\phi^{\ll}(t_f) - \mathcal{B}\left(\phi_c^{\ll}(t_*)\right)\right)^2\right] , \tag{5.132}$$

where $\sigma^2 \sim \lambda^2 H^2$ and C_2 denotes normalization constant. Therefore, combined with eq. (5.129), one has

$$P\left(\phi_c^{\ll}(t_*)|\phi^{\ll}(t_f)\right) \sim C_2 \frac{P(\phi_c^{\ll}(t_*))}{P(\phi^{\ll}(t_f))} \exp\left[\frac{-1}{2\sigma^2} \left(\phi^{\ll}(t_f) - \mathcal{B}\left(\phi_c^{\ll}(t_*)\right)\right)^2\right] . \tag{5.133}$$

If we expand the above expression in terms of the perturbation of $\phi_c^{\ll}$ at $t = t_*$, which is given by

$$\delta\phi_c^{\ll}(t_*) \coloneqq \phi_c^{\ll}(t_*) - \mathcal{B}^{-1}\left(\phi^{\ll}(t_f)\right) , \tag{5.134}$$

eq. (5.133) can be rewritten as

$$P\left(\phi_c^{\ll}(t_*)|\phi^{\ll}(t_f)\right) \sim C_2 \frac{P(\phi_c^{\ll}(t_*))}{P(\phi^{\ll}(t_f))} \exp\left[\frac{-\left(\mathcal{B}'\right)^2}{2\sigma^2}\delta\phi_c^{\ll}(t_*)^2\right] , \quad \mathcal{B}' = \left.\frac{\partial\mathcal{B}(\phi)}{\partial\phi}\right|_{\phi=\mathcal{B}^{-1}\left(\phi^{\ll}(t_f)\right)} . \tag{5.135}$$

Here, we neglected the higher-order terms in $\delta\phi_c^{\ll}(t_*)$. Therefore, unless $\mathcal{B}'$ become much smaller than σ, the fluctuations of $\phi^{\ll}$ at $t = t_*$ will be suppressed to $\mathcal{O}(\lambda H)$, typically.

5.3 Summary and Implications

Now let us summarize what we found in this chapter 5. Firstly in order to investigate the theoretical consistency of stochastic approach, we studied the quantum coherence of the history of IR fields. We considered $\lambda\phi^4$ theory as a concrete example. In sec. 5.1, we firstly developed a method how to calculate the quantum coherence of the history of IR fields, by fully utilizing the UV-IR splitting method developed in the previous chapter 4, which allows us to integrating out UV fields. Then we estimated the non-zero quantum coherence of IR fields and found that it is possible to interpret IR secular effects appear as a consequence of taking a classical statistical ensemble average, with respect to a set of coarse-grained histories: see the discussion around eq. (5.67). This means that the stochastic approach is theoretically consistent only after performing the coarse-graining of histories. We proved that the decoherent structures of histories of IR fields are encoded in the Hilbert space of UV fields at the final time t_f in which only the modes which are inside the Hubble scale during inflation are contained. We also clarified that the probability assigned to the coarse-grained history can be calculable using our extended stochastic formalism developed in the previous chapter: see the discussion around eq. (5.78). This reveals how the decoherence and the stochastic description of IR fields during inflation are related to each other.

Taking into account this necessity of coarse-graining for maintaining the theoretical consistency, in sec. 5.2, we firstly defined a plausible quantum state in eq. (5.103) which resembles the state which should be used for predicting correlation functions of cosmological perturbations. We then evaluated the amplitude of quantum fluctuations of deep IR fields in our local universe. We found that deep-IR fluctuations will be typically suppressed by the coupling constant λ: this is represented by eq. (5.135).

Now let us discuss briefly the implication of this result on the issues of IR effects in models with multi-field inflation. While we do not treat the concrete inflationary model, this result implies that as long as the isocurvature modes remain light fields during inflation, the boundary conditions at around the final time t_f will suppress the deep-IR fluctuations of isocurvature modes. On the other hand, if the isocurvature modes becomes massive during inflation and eventually decay within the epoch $t_* < t < t_f$, the boundary conditions at around final time will not suppress the deep-IR fluctuations before they decay. This is encoded in the appearance of $\mathcal{B}'$ factor in eq. (5.135). Because $\mathcal{B}$ is the map between $\phi^{\ll}(t_f)$ and $\phi^{\ll}(t_*)$ which is given by the background trajectory, $\mathcal{B}'$ becomes significantly small when the mechanism which make various histories shrink into one history at around t_f works. In realistic inflationary model this can generally happen, so there might be chance for deep-IR fluctuations contributing to cosmological correlators. However, in this case the background field trajectory must be curved in field space. As is reported in [27], IR fluctuations of isocurvature modes are also affected by the residual *gauge* transformations when the background field trajectory is curved. Therefore, in order to clarify whether or not IR effects from isocurvature modes can be observationally relevant, we need to consider the specific model where the background field trajectory is curved and extend our method developed here to the case where the fluctuations of spacetime are also taken into account. We will study this aspect in our future work.

Chapter 6

Conclusions

In this thesis, we have studied the dynamics of quantum fluctuations of IR fields of a test scalar field on de Sitter background, in order to capture the essential properties of IR fluctuations of isocurvature modes in multi-field inflation.

In chapter 4 we showed that all the IR secular terms which appear in correlation functions can be recovered by the set of Langevin equations. Our extended stochastic formalism allows us for the first time to derive systematically the Langevin equations which describe IR effects beyond LO. We also performed the consistency check of our method in $\lambda\phi^4$ theory: we explicitly evaluated two-point function using our method and confirmed that it coincides with the one evaluated using perturbative QFT.

In chapter 5 we further developed our technique invented in chapter 4 to analyze the structure of quantum states. We constructed the method to calculate the degree of coherence of the histories of IR fields, which is analogue to the coherence discussed in the context of double slit experiment. We then estimated the decoherence rate in $\lambda\phi^4$ theory and confirmed the efficient decoherence at a late time. We proved that large IR effects appear when taking the ensemble average over the whole set of coarse-grained histories which are decohered to each other. We therefore succeeded in giving a good evidence of the theoretical consistency of stochastic approach to inflation, regarding the fluctuations of isocurvauture modes.

In the latter part of the chapter 5, we constructed the quantum state which will be recognized by a local observer, and estimated the amplitude of the fluctuations of deep-IR field. We showed that the amplitude of IR fluctuations are typically suppressed by the coupling constant unless ϕ becomes massive before the final time t_f and their fluctuations decay. Here, the final time t_f resembles the time when inflation ends. We expect that this result indicates that the IR fluctuation of isocurvature modes in our local universe will be also suppressed, unless they decay before the end of inflation.

This result motivates us for investigating what happens when considering the multi-field inflation models where the background field trajectories shrink and become one-dimensional at around the final time. In such scenario however it will become essentially important for treating multiple inflatons simultaneously with taking into account the space-time fluctuations in a consistent manner. We will study this aspect in our future work.

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Appendix A

Detail calculations in the extended stochastic formalism

In this appendix, we will show summarize several details associated with the chapter 4, such as detailed computational techniques or useful materials *etc.*

A.1 The products of mode functions

In many cases, the momenta of propagators are restricted to $k = \epsilon a H$ due to the bilinear vertex. In this case, the products of mode functions are evaluated as follows.

$$u_k(t)u_k^*(t')\big|_{k=\epsilon aH} = \frac{H^2}{2(\epsilon aH)^3}\left(1+\frac{\epsilon^2}{2}+\frac{i\epsilon^3}{3}+\mathcal{O}\left(\epsilon^4\right)\right)\left(1+i\epsilon\frac{a}{a(t')}\right)e^{-i\epsilon a/a(t')}\,, \tag{A.1}$$

$$\dot{\Phi}_k(t)\Phi_k^*(t')\Big|_{k=\epsilon aH} = \frac{-H^3\epsilon^2}{2(\epsilon aH)^3}e^{i\epsilon}\left(1+i\epsilon\frac{a}{a(t')}\right)e^{-i\epsilon a/a(t')}\,, \tag{A.2}$$

$$\mathrm{Re}\left[u_k(t)u_k^*(t')\big|_{k=\epsilon aH}\right] = \frac{H^2}{2(\epsilon aH)^3}f_R(\Delta t)\left(1+\mathcal{O}\left(\epsilon^2\right)\right)\,, \tag{A.3}$$

$$\mathrm{Re}\left[\dot{\Phi}_k(t)\Phi_k^*(t')\Big|_{k=\epsilon aH}\right] = \mathrm{Re}\left[u_k(t)u_k^*(t')\big|_{k=\epsilon aH}\right]\times\left(-H\epsilon^2\right)\left(1+\mathcal{O}(\epsilon)\right)\,, \tag{A.4}$$

$$\mathrm{Im}\left[u_k(t)u_k^*(t')\big|_{k=\epsilon aH}\right] = \frac{H^2}{2(\epsilon aH)^3}\left(f_I(\Delta t)+\frac{\epsilon^3}{3}f_R(\Delta t)\right)\,, \tag{A.5}$$

$$\mathrm{Im}\left[\dot{\Phi}_k(t)\Phi_k^*(t')\Big|_{k=\epsilon aH}\right] = -\frac{1}{2a^3\epsilon}\left[f_I(\Delta t)+\epsilon f_R(\Delta t)\right]\,. \tag{A.6}$$

Here, $f_R(\Delta t)$ and $f_I(\Delta t)$ are defined by

$$\begin{aligned}
\Delta t &\coloneqq t-t'\,,\\
f_R(\Delta t) &\coloneqq \epsilon e^{H\Delta t}\sin(\epsilon e^{H\Delta t})+\cos(\epsilon e^{H\Delta t})\,,\\
f_I(\Delta t) &\coloneqq \epsilon e^{H\Delta t}\cos(\epsilon e^{H\Delta t})-\sin(\epsilon e^{H\Delta t})\,,
\end{aligned} \tag{A.7}$$

which show rapid oscillations for $\Delta t \gg \frac{1}{H}\ln(1/\epsilon)$. On the other hand, they do not show rapid oscillations for $\Delta t \ll \frac{1}{H}\ln(1/\epsilon) \Leftrightarrow \epsilon a/a(t') \ll 1$, and approximately behave as

$$f_R(\Delta t) = 1 + \mathcal{O}\left(\epsilon^2\right) \,, \quad f_I(\Delta t) = -\frac{\epsilon^3}{3}\left(\frac{a}{a(t')}\right)^3 \,. \tag{A.8}$$

Therefore, for $\Delta t \ll \frac{1}{H}\ln(1/\epsilon)$, the products of mode functions can be evaluated approximately as follows:

$$\begin{aligned}
\mathrm{Re}\left[u_k(t)u_k^*(t')\big|_{k=\epsilon aH}\right] &= \frac{H^2}{2(\epsilon aH)^3}\left(1+\mathcal{O}\left(\epsilon^2\right)\right) \,, \\
\mathrm{Re}\left[\dot{\Phi}_k(t)\Phi_k^*(t')\Big|_{k=\epsilon aH}\right] &= -\frac{H^3}{2(\epsilon aH)^3}\times\epsilon^2\left(1+\mathcal{O}\left(\epsilon^2\right)\right) \,, \\
\mathrm{Im}\left[u_k(t)u_k^*(t')\big|_{k=\epsilon aH}\right] &= \frac{H^2}{2(\epsilon aH)^3}\frac{\epsilon^3}{3}\left(1-\left(\frac{a}{a(t')}\right)^3\right)\left(1+\mathcal{O}\left(\epsilon^2\right)\right) \,, \\
\mathrm{Im}\left[\dot{\Phi}_k(t)\Phi_k^*(t')\Big|_{k=\epsilon aH}\right] &= -\frac{1}{2a^3}\left(1+\mathcal{O}\left(\epsilon^2\right)\right) \,.
\end{aligned} \tag{A.9}$$

A.2 Derivation of LO stochastic dynamics

In this appendix, by using eq. (4.58), we derive LO stochastic EoM described by eq. (3.63) as an exercise. First of all, we should mention that the form of eq. (4.55) is different from eq. (3.63). From eqs. (4.56) and (4.57), one can see that the following approximation is valid up to LO:

$$G(T,t) \simeq \frac{-i}{3H}\frac{1}{a^3}\Theta\left(T-t\right) \,. \tag{A.10}$$

Then, the expression (4.56) is the formal solution of the following equation:

$$3H\dot{\phi}_c^< = -\mu_2 + \dot{\mu}_1 + 3H\mu_1 + \tilde{\xi} \,. \tag{A.11}$$

One may notice that the approximation (A.10) corresponds to neglecting the second order time derivative term of $\phi_c^<$ from the effective EoM of $\phi_c^<$.

Furthermore, up to LO, $\tilde{\xi}$ reduces to

$$\tilde{\xi} \simeq 3H\xi_\phi + \xi_v \,, \tag{A.12}$$

because there is no oscillatory function in the integrand on the right-hand side of eq. (4.56) up to LO, and as a result $\dot{\xi}_\phi$ term in $\tilde{\xi}$ cannot contribute to LO IR secular growth.[1] Combining eq. (A.11) and (A.12), eq. (4.55) reduces to

$$\dot{\phi}_c^< = -\frac{1}{3H}\left(\mu_2 - \dot{\mu}_1\right) + \mu_1 + \xi_\phi + \frac{1}{3H}\xi_v \,. \tag{A.13}$$

[1] As one can see from eq. (3.63), the noise correlation has no oscillatory feature because $\delta(t_1 - t_2)$ is replaced by an exponentially decaying function with respect to $|t_1 - t_2|$ after coarse-graining.

Let us consider $\lambda\phi^4$ theory for example. As we discussed in sec. 4.2.1, only 0-th order diagrams can contribute to LO IR secular growth. There is no 0-th order diagram which contribute to μ_2, so that we can approximate μ_2 by a pure IR vertex $\mu_2 \simeq V'(\phi_c^<) = \lambda\phi_\Delta^< \phi_c^{<3}/6$ at leading order. This pure IR vertex corresponds to $l = 1, n = 4, m = 3$ diagram, which satisfies $(n/2) + 2l - (m + 1) = 0$ indeed. Therefore, μ_2 reduces to $\lambda\phi_c^{<3}/6 = V'(\phi_c^<)$ and $\mu = 0$ in eq. (3.63) up to LO. 0-th order noise diagrams are shown in fig. 4.6. These diagrams correspond to $l = 0, n = 2, m = 0$ diagrams, which also satisfy $(n/2) + 2l - (m + 1) = 0$, and hence contribute to LO IR secular growth. However, by performing straightforward calculation, it turns out that among these diagrams, only the diagram without $\phi_\Delta^<$ fields has $\mathcal{O}(\epsilon^0)$ values and other diagrams are $\mathcal{O}(\epsilon)$ suppressed. Therefore, ξ_v terms on the right-hand side of eqs. (A.12) and (A.13) are negligible. One can show that the noise diagram in fig. 4.6 without $\phi_\Delta^<$ fields generates the noise correlation $\langle\xi_\phi\xi_\phi\rangle$ which is described by eq. (3.63).

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