数理解析研究所講究録
第 1570 巻 2007 年 104-111

CONSTANTS OF REVERSE HÖLDER'S INEQUALITY
(ヘルダーの不等式に纏わる定数)

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ABSTRACT. Hölder's inequality is considered as an estimation of the arithmetic mean to the power mean for positive numbers. The generalized Kantorovich constant $K(h,p)$ is used in a reverse Hölder's inequality where h represents the bound of the ratio for given positive numbers. On the other hand, the Specht ratio $S = S(h)$ was introduced as the ratio of the arithmetic mean to the geometric mean. It is a special case of ratios $\{S(h,r,s); -1 \leq r < s \leq 1\}$ among power means. In this note, we give an interpretation to $S(h,s,r)$ for $r < s$ and investigate several useful properties of them, one of which is the inversion formula $S(h,r,s) = S(h,s,r)^{-1}$. Another is a clear relation: $S(h,r,s) = K(h^r, \frac{s}{r})^{\frac{1}{s}}$. By these properties, one can understand the context of a masterly formula $S = e^{K'(1)} = e^{-K'(0)}$ due to Furuta. Moreover we give the some reverse inequalities by using the Specht ratio $S(h)$ and the generalized Kantorovich constant $K(h,p)$.

1. INTRODUCTION

This note is a short survey related to estimations represented to a reverse Hölder's inequality ([3]).

Let $a_1, \ldots, a_n$ be positive real numbers and $(w_1, \ldots, w_n)$ be a weight. Then, Hölder's inequality is equivalent to

$$(1) \qquad (\sum_{i=1}^{n} w_i a_i^p)^{\frac{1}{p}} \leq \sum_{i=1}^{n} w_i a_i \qquad (0 \leq p \leq 1).$$

The following Kantorovich inequality is studied as one of reverse Hölder's inequalities:

$$(2) \qquad \sum_{i=1}^{n} w_i a_i \leq \frac{(M+m)^2}{4Mm} (\sum_{i=1}^{n} w_i a_i^{-1})^{-1}$$

where $0 < m \leq a_i \leq M$. The estimation $\frac{(M+m)^2}{4Mm}$ is called the Kantorovich constant. This constant represents an estimation of the arithmetic mean by the harmonic mean. Furuta continuously generalized Ky Fan's result associated with Hölder-McCarthy and Kantorovich inequalities in [6, Theorem 1.5] : If a positive operator A on a Hilbert space H satisfies $0 < m \leq A \leq M$ for some $m < M$ and $x \in H$ is a unit vector, then

$$(3) \qquad \langle Ax, x\rangle^p \leq \langle A^p x, x\rangle \leq K_{\pm}(m, M, p)\langle Ax, x\rangle^p$$

2000 *Mathematics Subject Classification.* 41A44, 47N40 and 47A63.
Key words and phrases. Hölder inequality, Specht ratio, Kantorovich constant and power mean Hölder-McCarthy inequality.

for $p > 1$ and $p < 0$ respectively, where

$$K_+(m, M, p) = \frac{(p-1)^{p-1}}{p^p} \frac{(M^p - m^p)^p}{(M-m)(mM^p - Mm^p)^{p-1}} \quad \text{for} \quad p > 1,$$

and

$$K_-(m, M, p) = \frac{(mM^p - Mm^p)}{(p-1)(M-m)} \left(\frac{(p-1)(M^p - m^p)}{p(mM^p - Mm^p)} \right)^p \quad \text{for} \quad p < 0.$$

Furuta [7] proposed to reformulate the constants $K_\pm(m, M, p)$ as follows, cf. [6, Corollary 1.2] : For a given $h > 0$, put

$$K(h,p) = \frac{1}{h-1} \frac{h^p - h}{p-1} \left(\frac{p-1}{h^p - h} \frac{h^p - 1}{p} \right)^p \tag{4}$$

for all $p \in \mathbb{R}$. Following him, we call it the generalized Kantorovich constant. It is easily checked that if we take $h = \frac{M}{m}$, then $K(h,p) = K_+(m, M, p)$ for $p > 1$ and $K(h,p) = K_-(m, M, p)$ for $p < 0$. This formula (4) says that it can be defined for all $p \in \mathbb{R}$, and it has the symmetric property $K(h,p) = K(h, 1-p)$, that is, $K(p) = K(h,p)$ is a symmetric function with respect to $p = \frac{1}{2}$. The inequality (3) implies that

$$\sum_{i=1}^n w_i a_i \ \leq \ K(h,p)^{-\frac{1}{p}} (\sum_{i=1}^n w_i a_i^p)^{\frac{1}{p}} \qquad (0 \leq p \leq 1). \tag{5}$$

as a reverse inequality of (1).

On the other hand, the Specht ratio is introduced in [10] as the ratio of the arithmetic mean to the geometric mean, that is, it is the best constant $S(h)$ satisfying the reverse inequality

$$\frac{a_1 + \cdots + a_n}{n} \leq S(h)(a_1 \cdots a_n)^{\frac{1}{n}} \tag{6}$$

for all $0 < m \leq a_1, \ldots, a_n \leq M$, where $h = \frac{M}{m}$ for some $m < M$. Following Specht [10], it is exactly given by

$$S(h) = \frac{h^{\frac{1}{h-1}}}{e \log h^{\frac{1}{h-1}}}, \tag{7}$$

see also [2]. It is also expressed as a constant enjoying that if $0 < m \leq a, b \leq M$, then

$$(1-t)a + tb \leq S(h) a^{1-t} b^t \tag{8}$$

for all $t \in [0, 1]$, see also [11].

By the way, we recognize the importance of the family of power means $M_{r,t}$ ($r \in \mathbb{R}$). The mean of 1 and $x \geq 0$ by $M_{r,t}$ with weight $\{1-t, \ t\}$ ($t \in [0, 1]$) is defined by

$$M_{r,t}(x) = (1 - t + tx^r)^{\frac{1}{r}}.$$

From this point of view, one could understand that Specht discussed the ratio among power means in the following general setting: If $-1 \leq r < s \leq 1$, then $M_{r,t}(x) \leq M_{s,t}(x)$ and

$$\frac{M_{s,t}(x)}{M_{r,t}(x)} \leq \left(\frac{s-r}{r} \frac{h^s - 1}{h^s - h^r} \right)^{\frac{1}{r}} \left(\frac{r}{s-r} \frac{h^s - h^r}{h^r - 1} \right)^{\frac{1}{s}} = S(h, r, s) \tag{9}$$

for $\frac{1}{h} \leq x \leq h$. We note that $S(h,r,s)$ is the best constant for upper bounds of $\frac{M_{s,t}}{M_{r,t}}$. Since $M_{0,t}(x) = x^t$, (8) is the special case $r=0$ and $s=1$ in (9). In other words, $S(h,0,1)$ is the Specht ratio $S(h)$, i.e., $\lim_{r\to+0} S(h,r,1) = S(h)$.

The most crucial result on the the generalized Kantorovich constant and Specht ratio is the following formula due to Furuta:

$$S = e^{K'(1)} = e^{-K'(0)}, \tag{10}$$

where $S = S(h)$ and $K(p) = K(h,p)$ for a fixed $h > 1$. In the below, this formula (10) is called the Furuta formula (on the generalized Kantorovich constant).

Motivated by the Furuta formula, we investigate several useful properties of $S(h,r,s)$ and $K(h,p)$ in this note. For this, we give an interpretation to $S(h,s,r)$ for $r < s$. Consequently we have the inversion formula $S(h,r,s) = S(h,s,r)^{-1}$. On a relationship of $S(h,s,r)$ to the generalized Kantorovich constant $K(h,p)$, we get

$$S(h,r,s) = K(h^r, \frac{s}{r})^{\frac{1}{s}}$$

for all $r,s \in \mathbb{R}$ with $rs \neq 0$. By these properties, one can understand the context of the Furuta formula (10). As a consequence, we have the following result:

The Furuta formulas

$$(\mathrm{F0}) : S = e^{-K'(0)} \quad \text{and} \quad (\mathrm{F1}) : S = e^{K'(1)}$$

are equivalent to the Yamazaki-Yanagida formulas [13]

$$(\mathrm{K0}) : \lim_{p\to+0} K(h^p, \frac{1}{p}) = S \quad \text{and} \quad (\mathrm{K1}) : \lim_{p\to+0} K(h^p, \frac{p+1}{p}) = S,$$

respectively. From this result we see that (5) implies (6) by $p \to 0$.

Moreover we give the some reverse inequalities by using t $S(h)$ and $K(h,p)$.

2. Fundamental properties of $S(h)$, $S(h,r,s)$ and $K(h,p)$

Firstly, we mention some properties of this Specht ratio $S(h)$:

Lemma 1. *Let $h > 0$ be given. Then*
(1) $S(h) = S(\frac{1}{h})$.
(2) $L(1,\frac{1}{h}) \leq S(h) \leq L(1,h)$ *for $h \geq 1$ where the logarithmic mean $L(s,t)$ is defined by $L(s,t) := \frac{t-s}{\log t - \log s}$ for $0 < s,t,\ s \neq t$.*
(3) $\lim_{h\to1} S(h) = 1$.

Secondary, we state some important properties of $K(h,p)$ and $S(h,r,s)$ which will be needed in the below.

Lemma 2. *Let $h > 0$ be given. Then*
(0) *$K(h,p)$ is defined for all $p \in \mathbb{R}$.*
(1) $K(h,p) = K(\frac{1}{h},p)$ *for all $p \in \mathbb{R}$.*
(2) $K(h,p) = K(h,1-p)$ *for all $p \in \mathbb{R}$.*
(3) $K(h,0) = K(h,1) = 1$ *and* $K(1,p) = 1$ *for all $p \in \mathbb{R}$,*
where $K(h,0) = \lim_{p\to0} K(h,p)$, $K(h,1) = \lim_{p\to0} K(h,1+p)$ *and* $K(1,p) = \lim_{h\to1} K(h,p)$.

The property (1) in Lemma 2 is imagined by that in Lemma 1.

Related to a result of Mond and Pečarić [9], the following relationship was presented in our seminar talk about five years ago, which is implicitly appeared in [12, Remark 2].

Lemma 3. *Let $h > 0$ and $r, s \in \mathbb{R}$. Then*

$$S(h, r, s) = K(h^r, \frac{s}{r})^{\frac{1}{s}} \quad \textit{if } rs \neq 0,$$

$$S(h, 0, s) = S(h^s) \textit{ and } S(h, r, 0) = S(h^r)^{-1}.$$

By the above lemma, one could recognize that Lemma 2 (0) is quite meaningful. As a corollary, we have the following variant of the Yamazaki-Yanagida formula [13]:

Corollary 4. *For $h > 0$,*

$$\text{(K0)} \qquad \lim_{r \to 0} K(h^r, \frac{1}{r}) = S(h).$$

Proof. The continuity of $S(h, r, s)$ and Lemma 3 imply that

$$S(h) = \lim_{r \to 0} S(h, r, 1) = \lim_{r \to 0} K(h^r, \frac{1}{r}).$$

□

Lemma 5. (Inversion formula) *Let $h > 0$ and $r, s \in \mathbb{R}$. Then*

$$S(h, r, s) = S(h, s, r)^{-1}.$$

Consequently, if $rs \neq 0$, then

$$K(h^r, \frac{s}{r})^{\frac{1}{s}} = K(h^s, \frac{r}{s})^{-\frac{1}{r}}.$$

In particular, if $r \neq 0$, then

$$K(h^r, \frac{1}{r}) = K(h, r)^{-\frac{1}{r}}.$$

Incidentally, since $M_{r,t}(x) \leq M_{s,t}(x)$ for $r < s$, $S(h, s, r)$ for $r < s$ might be defined by the lower bound of

$$S(h, s, r) M_{s,t}(x) \leq M_{r,t}(x).$$

It is rephrased by

$$\frac{M_{s,t}(x)}{M_{r,t}(x)} \leq S(h, s, r)^{-1}.$$

Hence the inversion formula could be expected.

3. Equivalent relation between Furuta and Yamazaki-Yanagida formulas

First of all, we cite the representation of the Specht ratio by the limit of the generalized Kantorovich constant due to Yamazaki and Yanagida [13].

Theorem A. *The Specht ratio $S = S(h)$ and the generalized Kantorovich constant $K(h,p)$ are defined in (7) and (4), respectively, and take $h > 0$. Then*

$$(\mathrm{K0}): \lim_{p\to+0} K(h^p, \frac{1}{p}) = S \quad \textit{and} \quad (\mathrm{K1}): \lim_{p\to+0} K(h^p, \frac{p+1}{p}) = S.$$

Now we consider the Furuta formulas

$$(\mathrm{F0}): S = e^{-K'(0)} \quad \text{and} \quad (\mathrm{F1}): S = e^{K'(1)}.$$

Since $K(0) = K(h,0) = 1$ and $K(1) = K(h,1) = 1$ by Lemma 2 (3), they should be understood as

$$\log S = -\frac{K'(0)}{K(0)} \quad \text{and} \quad \log S = \frac{K'(1)}{K(1)},$$

respectively, where $K(p) = K(h,p)$ for a fixed $h > 0$. Therefore, if we put $f(p) = \log K(p)$, then

$$(\mathrm{F0}): \log S = -f'(0) \quad \text{and} \quad (\mathrm{F1}): \log S = f'(1).$$

By the way, since $f(0) = 0$, we have

$$-f'(0) = -\lim_{p\to 0} \frac{f(p) - f(0)}{p} = -\lim_{p\to 0} \frac{f(p)}{p} = \lim_{p\to 0} \frac{\log K(p)}{-p} = \lim_{p\to 0} \log K(p)^{-\frac{1}{p}}.$$

Moreover the inversion formula $K(h^p, \frac{1}{p}) = K(h,p)^{-\frac{1}{p}} = K(p)^{-\frac{1}{p}}$ implies that

$$-f'(0) = \log \lim_{p\to 0} K(h^p, \frac{1}{p}).$$

It says that (F0) is equivalent to (K0) in Theorem A.

Next we discuss the equivalence between (F1) and (K1) in Theorem A. Since $f(1) = 0$, we have

$$f'(1) = \lim_{p\to 0} \frac{f(p+1) - f(1)}{p} = \lim_{p\to 0} \frac{f(p+1)}{p} = \lim_{p\to 0} \frac{\log K(p+1)}{p} = \lim_{p\to 0} \log K(p+1)^{\frac{1}{p}}.$$

Using the symmetric property $K(h,p) = K(h,q)$ for $p + q = 1$ by Lemma 2 (2) and the inversion formula $K(h^r, \frac{1}{r}) = K(h,r)^{-\frac{1}{r}}$, we have

$$K(h^p, \frac{p+1}{p})^p = K(h^{p+1}, \frac{p}{p+1})^{-(p+1)} = K(h^{p+1}, \frac{1}{p+1})^{-(p+1)} = K(h, p+1).$$

Taking the power $\frac{1}{p}$ on both sides,

$$K(p+1)^{\frac{1}{p}} = K(h,p+1)^{\frac{1}{p}} = K(h^p, \frac{p+1}{p}).$$

Therefore it follows that

$$f'(1) = \log \lim_{p\to 0} K(h^p, \frac{p+1}{p}),$$

which means that (F1) is equivalent to (K1) in Theorem A.

Summing up the above argument, we have the following conclusion:

Theorem 6. *The Furuta formulas*

$$(\mathrm{F0}) : S = e^{-K'(0)} \quad \textit{and} \quad (\mathrm{F1}) : S = e^{K'(1)}$$

are equivalent to the Yamazaki-Yanagida formulas

$$(\mathrm{K0}) : \lim_{p\to+0} K(h^p, \frac{1}{p}) = S \quad \textit{and} \quad (\mathrm{K1}) : \lim_{p\to+0} K(h^p, \frac{p+1}{p}) = S,$$

respectively.

4. Some reverse inequalities by $S(h)$ and $K(h,p)$

The generalized Kantorovich constant $K(h,p)$ and the Specht ratio $S(h)$ appear in some reverse inequalities. In this section we note some examples.

The reverse Hölder-McCarthy inequality (3) leads for $0 \le p \le 1$

$$\langle Ax, x\rangle \le K(h,p)^{-\frac{1}{p}} \langle A^p x, x\rangle^{\frac{1}{p}}. \tag{11}$$

Moreover since

$$\begin{aligned}\lim_{p\downarrow 0} \log\langle A^p x, x\rangle^{\frac{1}{p}} &= \lim_{p\downarrow 0} \log \frac{\langle A^p x, x\rangle}{p} = \lim_{p\downarrow 0} \frac{d\langle A^p x, x\rangle/dp}{\langle A^p x, x\rangle} \\ &= \lim_{p\downarrow 0} \frac{\langle A^p \log Ax, x\rangle}{\langle A^p x, x\rangle} = \langle(\log A)x, x\rangle\end{aligned}$$

and

$$\lim_{p\downarrow 0} K(h,p)^{-\frac{1}{p}} = \lim_{p\downarrow 0} K(h^p, \frac{1}{p}) = S(h)$$

by Lemma 5 (Inversion formula) and Yamazaki and Yanagida (K0), we have

$$\langle Ax, x\rangle \ \le \ S(h) \exp\langle(\log A)x, x\rangle. \tag{12}$$

In 2005, Bebiano, Lemos and Providência [1] showed the following norm inequality: For $A, B \ge 0$

$$\|A^{\frac{1+t}{2}} B^t A^{\frac{1+t}{2}}\| \ \le \ \|A^{\frac{1}{2}}(A^{\frac{s}{2}} B^s A^{\frac{s}{2}})^{\frac{t}{s}} A^{\frac{1}{2}}\| \tag{13}$$

for all $s \ge t \ge 0$. In [4], we gave a reverse inequality of (13) by using the generalized Kantorovich constant $K(h,p)$ as follows:

Corollary 7. *Let A and B be positive operators such that $0 < m \le B \le M$ for some scalars $0 < m < M$ and $h := \frac{M}{m} > 1$. Then*

$$\|A^{\frac{1}{2}}(A^{\frac{s}{2}} B^s A^{\frac{s}{2}})^{\frac{t}{s}} A^{\frac{1}{2}}\| \ \le \ K\left(h^t, \frac{s}{t}\right)^{\frac{t}{s}} \|A^{\frac{1+t}{2}} B^t A^{\frac{1+t}{2}}\| \tag{14}$$

for $s \ge t \ge 0$.

5. Concluding remarks

Concluding this note, we add to two remarks on the Yamazaki-Yanagida formulas (K0), (K1) and a comment on references of Kantorovich type inequalities for readers' convenience.

(i) Though a short proof of (K0) is given as Corollary 4, we cite a direct proof of it.

$$\begin{aligned} K(h^p, \frac{1}{p}) &= S(h,p,1) \\ &= \frac{p}{h^p-1}\frac{1}{(1-p)^{\frac{1}{-p}}}\frac{h-h^p}{1-p}\left(\frac{h-1}{h-h^p}\right)^{\frac{1}{p}} \\ &\to \frac{1}{\log h}\frac{1}{e}(h-1)h^{\frac{1}{h-1}} = S(h) \quad \text{as } p \to +0, \end{aligned}$$

where the convergence of the final term is assured by l'Hospital theorem as follows:

$$\lim_{p\to+0}\frac{\log(h-1)-\log(h-h^p)}{p} = \lim_{p\to+0}\frac{h^p\log h}{h-h^p} = \frac{\log h}{h-1} = \log h^{\frac{1}{h-1}}.$$

(ii) The equivalence between (K0) and (K1) is ensured by Theorem 6 because of the symmetric property $K(p) = K(1-p)$. However, we can show it by a direct computation, in which the symmetric property is used, of course. As a matter of fact, it follows from Lemma 2 (2) that

$$K(h^p, \frac{p+1}{p}) = K(h^p, 1-\frac{p+1}{p}) = K((\frac{1}{h})^{-p}, \frac{1}{-p}).$$

Therefore (K1) holds for h if and only if so does (K0) for $\frac{1}{h}$ by noting that $S(h) = S(\frac{1}{h})$; thus we have the equivalence between (K0) and (K1). We here want to remark that Lemma 2 (0) played an important role in the above discussion, and that we identified (K0) with

$$\lim_{p\to 0} K(h^p, \frac{1}{p}) = S$$

by virtue of Corollary 3.

(iii) Finally we mention that the paper [6] by Furuta is quite valuable in this field and that [5] and [8] are a suitable textbook for Kantorovich type inequalities.

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