

Algebraic independence of certain power series associated with d -adic expansion of real numbers

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1 Introduction.

Let $\omega > 0$ and let d be an integer greater than 1. The number ω is expressed as a d -adic expansion

$$\omega = \sum_{i=-l}^{\infty} \varepsilon_i d^{-i}, \quad l = \max\{\lfloor \log_d \omega \rfloor, 0\}, \quad \varepsilon_i \in \{0, 1, \dots, d-1\},$$

where $[x]$ denotes the largest integer not exceeding the real number x . For those ω having two ways of expression such as $2 = 1.9999\dots$ (10-adic), we adopt only the left-hand side expression. Then this expansion is uniquely determined. Let

$$a_k = [\omega d^k] \quad (k = 0, 1, 2, \dots).$$

It is clear that

$$a_k = \sum_{i=-l}^k \varepsilon_i d^{k-i},$$

namely the integer a_k is expressed as the d -adic number $\varepsilon_{-l}\varepsilon_{-l+1}\dots\varepsilon_{k-1}\varepsilon_k$. Hence we see that the sequence $\{a_k\}_{k \geq 0}$ satisfies the recurrence formula

$$a_0 = [\omega], \quad a_k = da_{k-1} + \varepsilon_k \quad (k = 1, 2, 3, \dots).$$

The author [3] proved that the number $\sum_{k=0}^{\infty} \alpha^{a_k}$ is transcendental for any algebraic number α with $0 < |\alpha| < 1$. In this paper we prove the following algebraic independence result. Let $\omega_1, \dots, \omega_m > 0$. Define

$$f_{id}(z) = \sum_{k=0}^{\infty} z^{[\omega_i d^k]} \quad (i = 1, \dots, m; d = 2, 3, 4, \dots). \quad (1)$$

In what follows, $\mathbb{Q}$ and $\mathbb{R}$ denote the sets of rational and real numbers, respectively.

Theorem 1. *If the numbers $\omega_1, \dots, \omega_m$ are linearly independent over $\mathbb{Q}$, then the numbers $f_{id}(\alpha)$ ($i = 1, \dots, m$; $d = 2, 3, 4, \dots$) are algebraically independent for any algebraic number α with $0 < |\alpha| < 1$.*

Corollary 1. *If the numbers $\omega_1, \dots, \omega_m$ are linearly independent over $\mathbb{Q}$, then the functions $f_{id}(z)$ ($i = 1, \dots, m$; $d = 2, 3, 4, \dots$) are algebraically independent over the field $\mathbb{C}(z)$ of rational functions.*

EXAMPLE. Let

$$\begin{aligned} f_{1,d}(z) &= \sum_{k=0}^{\infty} z^{d^k}, & f_{2,d}(z) &= \sum_{k=0}^{\infty} z^{\lfloor \sqrt{2}d^k \rfloor}, \\ f_{3,d}(z) &= \sum_{k=0}^{\infty} z^{\lfloor \sqrt{3}d^k \rfloor}, & f_{4,d}(z) &= \sum_{k=0}^{\infty} z^{\lfloor \pi d^k \rfloor} \quad (d = 2, 3, 4, \dots). \end{aligned}$$

For example we have

$$\begin{aligned} f_{2,10}(z) &= z + z^{14} + z^{141} + z^{1414} + z^{14142} + z^{141421} + \dots, \\ f_{3,10}(z) &= z + z^{17} + z^{173} + z^{1732} + z^{17320} + z^{173205} + \dots, \end{aligned}$$

and

$$f_{4,10}(z) = z^3 + z^{31} + z^{314} + z^{3141} + z^{31415} + z^{314159} + \dots$$

Then by Theorem 1 the numbers $f_{i,d}(\alpha)$ ($i = 1, \dots, 4$; $d = 2, 3, 4, \dots$) are algebraically independent for any algebraic number α with $0 < |\alpha| < 1$ since the numbers $1, \sqrt{2}, \sqrt{3}$, and π are linearly independent over $\mathbb{Q}$.

Theorem 1 is proved by using the method developed from that of Nishioka used for proving the following:

Theorem 2 (Nishioka [2, Theorem 1]). *Let*

$$f_d(z) = \sum_{k=0}^{\infty} \sigma_{dk} z^{d^k} \quad (d = 2, 3, 4, \dots),$$

where the σ_{dk} ($k = 0, 1, 2, \dots$) are in a finite set of nonzero algebraic numbers for every d . Then the numbers $f_d(\alpha)$ ($d = 2, 3, 4, \dots$) are algebraically independent for any algebraic number α with $0 < |\alpha| < 1$.

We further obtain the following, which includes both Theorems 1 and 2.

Theorem 3. *Let $\omega_1, \dots, \omega_m > 0$. Define*

$$f_{id}(z) = \sum_{k=0}^{\infty} \sigma_{idk} z^{\lfloor \omega_i d^k \rfloor} \quad (i = 1, \dots, m; d = 2, 3, 4, \dots),$$

where the σ_{idk} ($k = 0, 1, 2, \dots$) are in a finite set of nonzero algebraic numbers for every i and for every d . If the numbers $\omega_1, \dots, \omega_m$ are linearly independent over $\mathbb{Q}$, then the numbers $f_{id}(\alpha)$ ($i = 1, \dots, m$; $d = 2, 3, 4, \dots$) are algebraically independent for any algebraic number α with $0 < |\alpha| < 1$.

Theorem 3 implies the following result, which also includes Theorem 1.

Theorem 4. Let $\omega_1, \dots, \omega_m > 0$ and $\eta_1, \dots, \eta_m \in \mathbb{R}$. Define

$$f_{id}(z) = \sum_{k=0}^{\infty} z^{[\omega_i d^k + \eta_i]} \quad (i = 1, \dots, m; d = 2, 3, 4, \dots).$$

If the numbers $\omega_1, \dots, \omega_m$ are linearly independent over $\mathbb{Q}$, then the numbers $f_{id}(\alpha)$ ($i = 1, \dots, m$; $d = 2, 3, 4, \dots$) are algebraically independent for any algebraic number α with $0 < |\alpha| < 1$.

2 Lemmas.

We prepare the notation for stating the lemmas. For any algebraic number α , we denote by $[\alpha]$ the maximum of the absolute values of the conjugates of α and by $\text{den}(\alpha)$ the smallest positive integer such that $\text{den}(\alpha) \cdot \alpha$ is an algebraic integer and define

$$\|\alpha\| = \max\{[\alpha], \text{den}(\alpha)\}.$$

If $\Omega = (\omega_{ij})$ is an $n \times n$ matrix with nonnegative integer entries and if $z = (z_1, \dots, z_n)$ is a point of $\mathbb{C}^n$ with $\mathbb{C}$ the set of complex numbers, we define the transformation $\Omega : \mathbb{C}^n \rightarrow \mathbb{C}^n$ by

$$\Omega z = \left(\prod_{j=1}^n z_j^{\omega_{1j}}, \prod_{j=1}^n z_j^{\omega_{2j}}, \dots, \prod_{j=1}^n z_j^{\omega_{nj}} \right).$$

Let $\{\Omega^{(k)}\}_{k \geq 0}$ be a sequence of $n \times n$ matrices with nonnegative integer entries. We put

$$\Omega^{(k)} = (\omega_{ij}^{(k)}) \quad \text{and} \quad \Omega^{(k)} z = (z_1^{(k)}, \dots, z_n^{(k)}).$$

In what follows, $\mathbb{N}$ and $\mathbb{N}_0$ denote the sets of positive and nonnegative integers, respectively. For $\lambda = (\lambda_1, \dots, \lambda_n) \in (\mathbb{N}_0)^n$, we define $z^\lambda = z_1^{\lambda_1} \dots z_n^{\lambda_n}$ and $|\lambda| = \lambda_1 + \dots + \lambda_n$. Let K be an algebraic number field. Let $\{f_1^{(k)}(z)\}_{k \geq 0}, \dots, \{f_m^{(k)}(z)\}_{k \geq 0}$ be sequences of power series in $K[[z_1, \dots, z_n]]$. Let $\chi = (z_1, \dots, z_n)$ be the maximal ideal generated by $z_1, \dots, z_n$ in the ring $K[[z_1, \dots, z_n]]$. In what follows, $c_1, c_2, \dots$ denote positive constants independent of k .

Lemma 1 (cf. Nishioka [2, Theorem 2]). Assume that

$$f_i^{(k)}(z) \rightarrow f_i(z) \quad \text{as } k \rightarrow \infty$$

with respect to the topology defined by χ for any i ($1 \leq i \leq m$). Suppose that all the $f_i^{(k)}(\mathbf{z})$ ($k \geq 0$), $f_i(\mathbf{z})$ ($1 \leq i \leq m$) converge in the n -polydisc $\{\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n \mid |z_j| < r \ (1 \leq j \leq n)\}$. If $\alpha = (\alpha_1, \dots, \alpha_n)$ is a point of K^n with $0 < |\alpha_j| < \min\{1, r\}$ ($1 \leq j \leq n$) and if the following three properties are satisfied, then the values $f_1^{(0)}(\alpha), \dots, f_m^{(0)}(\alpha)$ are algebraically independent.

(I) There exists a sequence $\{\rho_k\}_{k \geq 0}$ of positive numbers such that

$$\lim_{k \rightarrow \infty} \rho_k = \infty, \quad \omega_{ij}^{(k)} \leq c_1 \rho_k, \quad \log |\alpha_j^{(k)}| \leq -c_2 \rho_k.$$

(II) If we put

$$f_i^{(0)}(\alpha) = f_i^{(k)}(\Omega^{(k)}\alpha) + b_i^{(k)} \quad (1 \leq i \leq m),$$

then $b_i^{(k)} \in K$ and

$$\log \|b_i^{(k)}\| \leq c_3 \rho_k \quad (1 \leq i \leq m).$$

(III) For any power series $F(\mathbf{z})$ represented as a polynomial in $z_1, \dots, z_n, f_1(\mathbf{z}), \dots, f_m(\mathbf{z})$ with complex coefficients of the form

$$F(\mathbf{z}) = \sum_{\lambda, \mu = (\mu_1, \dots, \mu_m)} a_{\lambda, \mu} \mathbf{z}^\lambda f_1(\mathbf{z})^{\mu_1} \dots f_m(\mathbf{z})^{\mu_m},$$

where $a_{\lambda, \mu}$ are not all zero, there exists a $\lambda_0 \in (\mathbb{N}_0)^n$ such that if k is sufficiently large, then

$$|F(\Omega^{(k)}\alpha)| \geq c_4 |(\Omega^{(k)}\alpha)^{\lambda_0}|.$$

Although Theorem 2 of Nishioka [2] requires the assumption that the coefficients of $f_i^{(k)}(\mathbf{z})$ are in a finite set $S \subset K$ for all i and k , it can be weakened as in Lemma 1, which is proved by the almost same way as in the proof of Theorem 2 of Nishioka [2].

Lemma 2 (Nishioka [2]). Let $f(\mathbf{z}) = \sum_{\lambda_1, \dots, \lambda_n} c_{\lambda_1, \dots, \lambda_n} z_1^{\lambda_1} \dots z_n^{\lambda_n} \in \mathbb{C}[[z_1, \dots, z_n]]$ converge around the origin. If $\mathbf{z}$ is sufficiently close to the origin, then

$$\sum_{\lambda \geq H} |c_{\lambda_1, \dots, \lambda_n}| \cdot |z_1|^{\lambda_1} \dots |z_n|^{\lambda_n} \leq \gamma^{H+1} \max_{1 \leq i \leq n} |z_i|^H,$$

where γ is a positive constant depending on $f(\mathbf{z})$.

The following lemma is originally due to Masser [1] and improved by Nishioka [2].

Lemma 3 (Masser [1], Nishioka [2]). Let $b_1 > \dots > b_n \geq 2$ be pairwise multiplicatively independent integers. Let $\theta = \log b_1$ and $\theta_j = \theta / \log b_j$ ($1 \leq j \leq n$). Suppose that for each α in a finite set A we are given real numbers $\lambda_{1\alpha}, \dots, \lambda_{n\alpha}$, not all zero, and define the sequence

$$S_\alpha(k) = \sum_{j=1}^n \lambda_{j\alpha} b_j^{[\theta_j k]} \quad (k = 0, 1, 2, \dots).$$

If $\{k_l\}_{l \geq 1}$ is an increasing sequence of positive integers with $\{k_{l+1} - k_l\}_{l \geq 1}$ bounded, then there exists a positive number δ such that

$$R(\delta) = \{k_l \mid \min_{\alpha \in A} |S_\alpha(k_l)| \geq \delta b_1^{k_l}\} = \{m_l\}_{l \geq 1}, \quad m_l < m_{l+1},$$

is an infinite set and $\{m_{l+1} - m_l\}_{l \geq 1}$ is bounded.

Using Lemma 3, we have the following:

Lemma 4. Let $b_1, \dots, b_n$ be integers as in Lemma 3 and let $\theta_1, \dots, \theta_n$ be defined in Lemma 3. Let $\omega_1, \dots, \omega_m > 0$ be linearly independent over $\mathbb{Q}$. Then there exist an infinite set Λ of positive integers, a sequence $\{\delta(l)\}_{l \geq 1}$ of positive numbers, and a total order $\succ$ in $(\mathbb{N}_0)^{mn}$ such that if $\lambda = (\lambda_{ij}) \succ \mu = (\mu_{ij})$ with $|\lambda| = \lambda_{11} + \dots + \lambda_{mn}$, $|\mu| = \mu_{11} + \dots + \mu_{mn} \leq l$, then

$$\sum_{i=1}^m \sum_{j=1}^n \lambda_{ij} [\omega_i b_j^{\lfloor \theta_j q \rfloor}] - \sum_{i=1}^m \sum_{j=1}^n \mu_{ij} [\omega_i b_j^{\lfloor \theta_j q \rfloor}] \geq \delta(l) b_1^q$$

for all sufficiently large $q \in \Lambda$. Moreover, any subset S of $(\mathbb{N}_0)^{mn}$ has the minimal element with respect to the total order $\succ$.

Lemma 5 (Nishioka [2]). Let d be an integer greater than 1 and let

$$f_l(z) = \sum_{h=0}^{\infty} s_h^{(l)} z^{d^h} \quad (l = 1, 2, \dots),$$

where the coefficients $s_h^{(l)}$ are nonzero complex numbers. Then $f_l(z)$ ($l = 1, 2, \dots$) are algebraically independent over $\mathbb{C}(z)$.

3 Proof of Theorems 1 and 4.

Proof of Theorem 1. Let

$$D = \{d \in \mathbb{N} \mid d \neq a^n \ (a, n \in \mathbb{N}, n \geq 2)\}.$$

Then

$$\mathbb{N} \setminus \{1\} = \bigcup_{d \in D} \{d, d^2, \dots\},$$

which is a disjoint union since any two distinct elements of D are multiplicatively independent by the definition of D . Let $d_1 > \dots > d_n$ be elements of D and let $\mathbf{z} = (z_{11}, \dots, z_{m1}, \dots, z_{1n}, \dots, z_{mn})$, where $z_{11}, \dots, z_{m1}, \dots, z_{1n}, \dots, z_{mn}$ are distinct variables. For any i ($1 \leq i \leq m$) and for any $d_j \in D$ ($1 \leq j \leq n$), we define the sequence $\{r_k^{(i,j)}\}_{k \geq 0}$ by

$$r_0^{(i,j)} = 1, \quad r_k^{(i,j)} = [\omega_i d_j^k] \quad (k \geq 1) \quad (2)$$

and define

$$f_{ijl0}(z) = \sum_{h=0}^{\infty} \alpha^{r_{lh}^{(i,j)} - d_j^{lh}} z_{ij}^{d_j^{lh}} \quad (1 \leq i \leq m, 1 \leq j \leq n, 1 \leq l \leq t).$$

Letting $\alpha = (\alpha, \dots, \alpha, \dots, \alpha, \dots, \alpha)$, we have

$$f_{ijl0}(\alpha) = \sum_{h=0}^{\infty} \alpha^{r_{lh}^{(i,j)}} = \alpha + \sum_{h=1}^{\infty} \alpha^{[\omega_i d_j^{lh}]} = f_{id_j^l}(\alpha) - \alpha^{[\omega_i]} + \alpha,$$

where f_{id} is defined by (1). Hence it suffices to prove the algebraic independency of the values $f_{ijl0}(\alpha)$ ($1 \leq i \leq m, 1 \leq j \leq n, 1 \leq l \leq t$). For the purpose we apply Lemma 1.

Put $b_j = d_j^{t!}$, $\theta = \log b_1$, and $\theta_j = \theta / \log b_j$ ($1 \leq j \leq n$). Noting that

$$0 \leq r_{lh+t![\theta_j q]}^{(i,j)} - r_{t![\theta_j q]}^{(i,j)} d_j^{lh} \leq d_j^{lh} - 1 \quad (1 \leq i \leq m),$$

we put

$$\begin{aligned} \Sigma_q &= \left(\alpha^{r_{lh+t![\theta_j q]}^{(i,j)} - r_{t![\theta_j q]}^{(i,j)} d_j^{lh}} \right)_{1 \leq i \leq m, 1 \leq j \leq n, 1 \leq l \leq t, h \geq 0} \\ &\in \prod_{h=0}^{\infty} \prod_{j=1}^n \prod_{l=1}^t \{1, \alpha, \dots, \alpha^{d_j^{lh}-1}\}^m \end{aligned}$$

for any $q \in \Lambda$ with the Λ defined in Lemma 4. Since the right-hand side is a compact set, there exists a converging subsequence $\{\Sigma_{q_k}\}_{k \geq 1}$ of $\{\Sigma_q\}_{q \in \Lambda}$, where q_1 will be chosen sufficiently large. Let

$$\lim_{k \rightarrow \infty} \Sigma_{q_k} = \left(\alpha^{s_h^{(i,j,l)}} \right)_{1 \leq i \leq m, 1 \leq j \leq n, 1 \leq l \leq t, h \geq 0}$$

and define

$$\begin{aligned} f_{ijlk}(z) &= \sum_{h=0}^{\infty} \alpha^{r_{lh+t![\theta_j q_k]}^{(i,j)} - r_{t![\theta_j q_k]}^{(i,j)} d_j^{lh}} z_{ij}^{d_j^{lh}} \\ &\quad (1 \leq i \leq m, 1 \leq j \leq n, 1 \leq l \leq t, k \geq 1) \end{aligned}$$

and

$$f_{ijl}(z) = \sum_{h=0}^{\infty} \alpha^{s_h^{(i,j,l)}} z_{ij}^{d_j^{lh}} \quad (1 \leq i \leq m, 1 \leq j \leq n, 1 \leq l \leq t).$$

Then

$$\lim_{k \rightarrow \infty} f_{ijlk}(z) = f_{ijl}(z).$$

Define the $mn \times mn$ matrix

$$\Omega^{(k)} = \text{diag} \left([\omega_1 b_1^{[\theta_1 q_k]}], \dots, [\omega_m b_1^{[\theta_1 q_k]}], \dots, [\omega_1 b_n^{[\theta_n q_k]}], \dots, [\omega_m b_n^{[\theta_n q_k]}] \right).$$

We assert first that $\{\Omega^{(k)}\}_{k \geq 1}$, $\alpha = (\alpha, \dots, \alpha, \dots, \alpha, \dots, \alpha)$, and $\rho_k = b_1^{q_k}$ ($k \geq 1$) satisfy the assumptions (I) and (II) of Lemma 1. Since $b_1 > \dots > b_n$, we have

$$b_1^{q_k-1} \leq b_j^{-1} b_1^{q_k} < b_j^{[\theta_j q_k]} \leq b_1^{q_k}$$

and so

$$\frac{1}{2} \left(\min_{1 \leq i \leq m} \omega_i \right) b_1^{q_k-1} \leq \left(\min_{1 \leq i \leq m} \omega_i \right) b_1^{q_k-1} - 1 < [\omega_i b_j^{[\theta_j q_k]}] \leq b_1^{q_k} \max_{1 \leq i \leq m} \omega_i$$

for any i ($1 \leq i \leq m$), j ($1 \leq j \leq n$), and for all $k \geq 1$, if q_1 is sufficiently large. Hence the assumption (I) is satisfied.

Let $K = \mathbb{Q}(\alpha)$. Then $f_{ijkl}(z) \in K[[z]]$ ($1 \leq i \leq m$, $1 \leq j \leq n$, $1 \leq l \leq t$, $k \geq 0$) and

$$f_{ijkl}(\Omega^{(k)} \alpha) = \sum_{h=0}^{\infty} \alpha^{r_{lh+t}^{(i,j)}} = f_{ijl0}(\alpha) - \sum_{h=0}^{(t!/l)[\theta_j q_k]-1} \alpha^{r_{lh}^{(i,j)}} \\ (1 \leq i \leq m, 1 \leq j \leq n, 1 \leq l \leq t, k \geq 1).$$

Since $r_{l(k+1)}^{(i,j)} > r_{lk}^{(i,j)}$ ($1 \leq i \leq m$, $1 \leq j \leq n$, $1 \leq l \leq t$) for all sufficiently large k by the definition, there is a positive constant C such that $\max_{0 \leq h \leq k-1} r_{lh}^{(i,j)} \leq C r_{lk}^{(i,j)}$ ($1 \leq i \leq m$, $1 \leq j \leq n$, $1 \leq l \leq t$) for all $k \geq 1$. Hence

$$\log \left\| - \sum_{h=0}^{(t!/l)[\theta_j q_k]-1} \alpha^{r_{lh}^{(i,j)}} \right\| \leq \log(t!/l)[\theta_j q_k] + \left(\max_{0 \leq h \leq (t!/l)[\theta_j q_k]-1} r_{lh}^{(i,j)} \right) \log \|\alpha\| \\ \leq \left(1 + C \left(\max_{1 \leq i \leq m} \omega_i \right) \log \|\alpha\| \right) \rho_k,$$

and the assumption (II) is satisfied.

Therefore, if the assumption (III) is also satisfied, the proof is completed. Noting that $z_{11}, \dots, z_{m1}, \dots, z_{1n}, \dots, z_{mn}$ are distinct variables, we see by Lemma 5 that the functions $f_{ijl}(z)$ ($1 \leq i \leq m$, $1 \leq j \leq n$, $1 \leq l \leq t$) are algebraically independent over $\mathbb{C}(z_{11}, \dots, z_{m1}, \dots, z_{1n}, \dots, z_{mn})$. Let

$$F(z) = \sum_{\mu=(\mu_{ij}), \nu=(\nu_{ijl})} a_{\mu, \nu} z^{\mu} f_{111}^{\nu_{111}} \dots f_{mnt}^{\nu_{mnt}} = \sum_{\lambda=(\lambda_{ij}) \in (\mathbb{N}_0)^{mn}} c_{\lambda} z^{\lambda},$$

where the coefficients $a_{\mu, \nu}$ are not all zero, and let $\lambda_0 = (\lambda_{ij}^{(0)})$ be the minimal element in $(\mathbb{N}_0)^{mn}$ with respect to the total order $\succ$ defined in Lemma 4 among λ with $c_{\lambda} \neq 0$. Let

$l = 2(|\lambda_0| + 1) \left(\left\lceil \frac{\max_{1 \leq i \leq m} \omega_i}{\min_{1 \leq i \leq m} \omega_i} \right\rceil + 1 \right) b_1$. If k is sufficiently large, then by Lemma 2

$$\begin{aligned} & \sum_{|\lambda| \geq l} |c_\lambda| \cdot |\alpha|^{\lambda_{11}[\omega_1 b_1^{[\theta_1 q_k]}]} \dots |\alpha|^{\lambda_{m1}[\omega_m b_1^{[\theta_1 q_k]}]} \dots |\alpha|^{\lambda_{1n}[\omega_1 b_n^{[\theta_n q_k]}]} \dots |\alpha|^{\lambda_{mn}[\omega_m b_n^{[\theta_n q_k]}]} \\ & \leq \gamma^{l+1} \left(|\alpha|^{\frac{1}{2}(\min_{1 \leq i \leq m} \omega_i) b_1^{q_k-1}} \right)^l \\ & \leq \gamma^{l+1} |\alpha|^{(\max_{1 \leq i \leq m} \omega_i) b_1^{q_k} (|\lambda_0| + 1)}. \end{aligned}$$

Since

$$\begin{aligned} & \lambda_{11}^{(0)}[\omega_1 b_1^{[\theta_1 q_k]}] + \dots + \lambda_{m1}^{(0)}[\omega_m b_1^{[\theta_1 q_k]}] + \dots + \lambda_{1n}^{(0)}[\omega_1 b_n^{[\theta_n q_k]}] + \dots + \lambda_{mn}^{(0)}[\omega_m b_n^{[\theta_n q_k]}] \\ & \leq |\lambda_0| \left(\max_{1 \leq i \leq m} \omega_i \right) b_1^{q_k}, \end{aligned}$$

we have

$$\frac{|\sum_{|\lambda| \geq l} c_\lambda (\Omega^{(k)} \alpha)^\lambda|}{|(\Omega^{(k)} \alpha)^{\lambda_0}|} \leq \gamma^{l+1} |\alpha|^{(\max_{1 \leq i \leq m} \omega_i) b_1^{q_k}}$$

if k is sufficiently large. If $|\lambda| < l$ and $\lambda \neq \lambda_0$, then by Lemma 4

$$\frac{|c_\lambda (\Omega^{(k)} \alpha)^\lambda|}{|(\Omega^{(k)} \alpha)^{\lambda_0}|} \leq |c_\lambda| \cdot |\alpha|^{\delta(l) b_1^{q_k}}$$

for all sufficiently large k . Therefore

$$|F(\Omega^{(k)} \alpha) / (\Omega^{(k)} \alpha)^{\lambda_0} - c_{\lambda_0}| \rightarrow 0 \quad (k \rightarrow \infty),$$

which implies (III), and the proof of the theorem is completed.

Proof of Theorem 4. Define

$$g_{id}(z) = \sum_{k=0}^{\infty} \alpha^{[\omega_i d^k + \eta_i] - [\omega_i d^k]} z^{[\omega_i d^k]} \quad (i = 1, \dots, m; d = 2, 3, 4, \dots).$$

Then

$$\alpha^{[\omega_i d^k + \eta_i] - [\omega_i d^k]} \in \{\alpha^{[\eta_i]}, \alpha^{[\eta_i] + 1}\},$$

since $0 \leq [\omega_i d^k + \eta_i] - [\omega_i d^k] - [\eta_i] \leq 1$ for any i, d , and for all k . By Theorem 3 the numbers $g_{id}(\alpha)$ ($i = 1, \dots, m; d = 2, 3, 4, \dots$) are algebraically independent, which implies the theorem.

References

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