

Diophantine problem of algebraic varieties and Hodge theory

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Dedicated to Professor Shoshichi Kobayashi on his sixtieth birthday.

§ 1. INTRODUCTION

In this article, we shall discuss some recent results related to Diophantine problems from point of view of algebraic geometry. Diophantine problems are concerned with polynomials with integral coefficients and most of these problems have their origins in the theory of algebraic curves. For example, P. Samuel's lecture note([S]) provides us a good reference. In the study of curves, the notion of genus is the most fundamental and indicates some complexity of polynomials which defines the curve. For higher dimensional algebraic varieties we have the notion of Kodaira dimension proposed by Iitaka([I]). The variety is said to be of general type if the Kodaira dimension attains the dimension of the variety.

First, note that the birational automorphisms group is finite if the variety is of general type. This was shown by H. Matsumura([Mat]), which is considered as a generalization of a theorem of Hurwitz.

Socond, Kobayashi-Ochiai ([KO])proved finiteness of the set of the generically surjective meromorphic maps from a compact complex manifold onto a variety of general type. In positive characteristic case Deschamps([DM1]) shows finiteness of the separable dominant rational maps from a variety onto a variety of general type. This is de Franchis' theorem in the theory of curves.

Third, fixing an algebraic variety X , we consider the set of separable dominant rational maps from X onto varieties of general type up to birational equivalence. We have the following question;

Iitaka's conjecture based on Severi's theorem.

Is the set finite?

Thanks to Kobayashi-Ochiai's theorem([KO]) or Deschamps'([DM1]), it suffices to show finiteness of birational equivalence classes of the varieties of general type which are images of the given variety X by separable dominant rational maps. In the case of characteristic 0, Deschamps and Mengaud([DM3]) have shown finiteness if X are surfaces of general type with the condition $q > 0$ or $p_g \geq 2$ and the author([M2]) shows it restricting image varieties to such varieties that can be birationally embedded by the m -th multicanonical maps for any given m . To accomplish it, it is enough to show that there exist a minimal model for a variety of general type and an upper bound for the indexes of the canonical divisorial sheaf. K.Ueno suggested to the author that a variety of general type can be replaced by

a polarized non uniruled variety in the problems above. We can consider some variation of the conjecture above. The author showed finiteness of isomorphism classes of varieties with ample canonical divisors which are dominated by surjective morphisms from a fixed variety([M1]) and Deschamps and Mengaud proved finiteness of birational classes of surfaces of general type which are dominated by surjective morphisms from a fixed variety([DM2]).

A famous conjecture by Mordell([Mor]) looks like a typical Diophantine problem. It is now Faltings theorem([Fal]). Bombieri and Noguchi conjecture that Faltings theorem should be generalized;

Mordell, Bombieri and Noguchi conjecture(resp. analogue).

Does there exist no variety of general type(resp. with exceptions) over a number field(resp. a function field) which has a dense set of rational points?

Manin([Ma]) and Grauert([Gra]) proved Mordell conjecture analogue over function fields. MBN conjecture analogue has an exception that is a birationally isotrivial variety. On the other hand, Lang([L1]) conjectured that the conjectures above hold if the notion of a variety of general type over a number field be replaced by that of a hyperbolic manifold considered as a complex manifold. Noguchi([N1],[N2],[N3]) proved Lang conjecture analogue over function fields completely.

Shafarevich conjecture which Faltings proved([Pa], [Ar], [Fal]) has classical models in number theory, i.e., Hermite's theorem and Minkowski's theorem([ZP]);

- (1) finiteness of the number of extensions L/K of K with fixed degree and fixed ramification,
- (2) there exists no extension unramified over $\mathbb{Q}$.

A generalization of Shafarevich conjecture.

Is the set of birational equivalence classes of varieties of general type with a fixed set of bad reductions and a fixed pluricanonical function ?

Its analogue over function fields has some exceptions, say, isotrivial varieties, isotrivial factors, We can study this problem replacing varieties of general type by polarized abelian varieties of dimension g or K3 surfaces([S],[SZ]) or polarized non uniruled varieties of a fixed dimension or polarized variations of Hodge structures([pe]), or Hodge Modules.

§ 2. RESULTS

Let X be a given variety of characteristic 0 and m any fixed number. We denote by $\mathcal{E}(X)$ a set of the birational equivalence classes of smooth varieties onto which there exists a dominant rational map from X . We define subsets of $\mathcal{E}(X)$, respectively:

- (1) $\mathcal{E}_m(X)$ consists of varieties of general type such that the m -th pluricanonical mappings are birational,

- (2) $\mathcal{E}_{pol \geq \ell}(X)$ consists of varieties of general type with the i -th pluri-genus P_{ai+b} polynomials for all $i \geq \ell$ and fixed a, b ,
- (3) $\mathcal{E}_{nef}(X)$ consists of varieties of general type with the dualizing sheaf ω_X nef,
- (4) $\mathcal{E}_{abun}(X)$ consists of varieties with the dualizing sheaf semi-ample.

Theorem 1 ([M2]).

- (1) For all m , $\mathcal{E}_m(X)$ is a finite set,
- (2) There exists an m such that $\mathcal{E}_{nef}(X) \subset \mathcal{E}_m(X)$,
- (3) There exists an m such that $\mathcal{E}_{pol \geq \ell}(X) \subset \mathcal{E}_m(X)$,
- (4) $\mathcal{E}_{abun}(X)$ is at most countable.

Theorem 2 ([M4], [M5]). Let K be a function field over $\mathbb{C}$ and X a smooth variety of general type over K . Let $\pi : \mathbb{P}(\Omega_X) \rightarrow X$ denote the projective bundle over X and $\mathcal{O}_P(1)$ the fundamental sheaf.

Suppose that one of equivalent conditions

- (1) $\mathcal{O}_P(1)$ is big and semiample,
- (2) $\mathcal{O}_P(1)$ is nef and $\mathcal{O}_P(\alpha) \otimes \pi^* \omega_X^{-1}$ is nef and big for some $\alpha > 0$.

Assume that the set of K -rational points is dense in X . Then X is isotrivial.

Proofs of Theorem 1 and 2 are reduced to show local birational triviality and boundedness of parametrizing family.

From geometric point of view, we interpret the situation above. Let $\Phi : \mathcal{X} \rightarrow \mathcal{S}$ be a surjective morphism between non singular complex varieties with the generic fibre a variety of general type which is isomorphic to X , where the rational function field of $\mathcal{S}$ is K . We may assume that $\mathcal{S}$ is a curve without generality.

Lemma 2.1. ([M4])

Assume that the set of (rational) sections $(C_\lambda)_{\lambda \in \Lambda}$ with the intersection number of the section C_λ and a canonical divisor K_X bounded is dense in $\mathcal{X}$. Then $\mathcal{X}$ is isotrivial.

We propose a generalization of the problem above.

Conjecture 2.2. Let X be a non singular variety and g a fixed number. Consider the set $(C_\lambda)_{\lambda \in \Lambda}$ of genus g embeded in X . We ask if the intersection number of (K_X, C_λ) has an upper bound independent of λ .

We can replace curves of genus g by higher dimensional subvarieties of a fixed multicanonical function P_m .

REMARK.

- (1) If X is a surface, Bogomolov and Miyaoka and Umezue (unpublished) using Miyaoka and Sakai inequality showed Conjecture 2.2.
- (2) If Ω_X is ample, Conjecture 2.2 holds.
- (3) S. Kobayashi showed if Ω_X is ample, X is hyperbolic ([Kob2]).

- (4) *Peternell-Campana and Kawamata proved that a hyperbolic manifold of general type has an ample canonical divisor.*

Question 2.3. *Let X be a non singular complex variety with a canonical divisor ample and C a fixed curve with genus ≥ 2 . Has the intersection number (K_X, C_λ) with $j_\lambda : C \hookrightarrow C_\lambda$ an embedding an upper bound $\leq \dim X \cdot (2g(C) - 2)$?*

Yoga of generalization insists that higher dimensional Shafarevich conjecture should hold. In any case Minkowski's theorem should be studied to the effect that there exists no smooth complex variety X with some exceptions such that $f : X \rightarrow Y$ is smooth with a canonical divisor K_X semiample and Y is complete with $\text{var}(f) > 0$. Faltings([Fal2]) gave an example non rigid family of abelian varieties with relative dimension 8 over a curve without isotrivial factors fixing ramification locus. M.Saito classifies non rigid families of abelian varieties with no isotrivial factors([S],[SZ]). A check point to a generalization of Shafarevich conjecture over function fields consists in

THE PROBLEM OF LOCAL TRIVIALITY:.

Let $\Phi : X \rightarrow S$ be a proper surjective smooth morphism with all fibres of ample canonical divisors between non singular quasi-projective varieties defined over $\mathbb{C}$. Assume $S = C_1 \times C_2$ where C_1, C_2 are curves. If $\Phi_1 : X|_{\Phi^{-1}(C_1)} \rightarrow C_1$ has no isotrivial factors, $\Phi_2 : X|_{\Phi^{-1}(C_2)} \rightarrow C_2$ is locally isotrivial.

§ 3. METHODS

We restrict ourselves to analogues over function fields over the complex number field. In these problems the following tools are essential, which come from Hodge theory to prove local birational triviality and boundedness of degrees.

We recall Viehweg's Definitions([V2],[V3],[V4],[M3]).

Definition 3. Let Y/S be a scheme and T a functor of the category of coherent sheaves over Y/S to that of coherent sheaves over X/S . Let $f : X \rightarrow S$ be a morphism, $\mathcal{F}$ a coherent $\mathcal{O}_Y$ -Module, L an invertible sheaf over X and U an open subset of X with $\text{depth}_{X \setminus U}(\mathcal{O}_X) \geq 2$. $\mathcal{F}$ is said to be f -weakly positive (with respect to T , L and U), if for any $\alpha > 0$ there exists $\beta_0 > 0$ such that for any $\beta \geq \beta_0$ the following canonical homomorphism over U are generically surjective

$$f^* f_* T(\mathcal{F}^{\otimes \alpha \beta} \otimes L^{\otimes \beta}) \longrightarrow T(\mathcal{F}^{\otimes \alpha \beta} \otimes L^{\otimes \beta}).$$

If $U = X, T = \text{id}$, $\mathcal{F}$ is said to be f -pseudo-effective. If $U = X, T = \text{id}$ and the canonical homomorphisms are surjective, $\mathcal{F}$ is said to be f -numerically effective(f -nef) or f -semipositive.

Remark

Let $f : M \rightarrow C$ be a surjective morphism with connected fibres from a Kähler manifold M onto a curve C . Hinted from Griffiths's work, Fujita([F1]) found the semipositivity of $f_* \omega_{M/C}$. Kawamata([Kaw2]) and Viehweg([V2]) generalized it. Kollár([Kol1],[Kol2]) found another approach([EV1],[M6]).

Theorem 4. Let $f : X \rightarrow S$ be a surjection with connected fibres between non singular varieties over $\mathbb{C}$. Then

- (1) $f_*(\omega_{X/S}^{\otimes m})$ is weakly positive for all $m > 0$.
- (2) $\omega_{X/S}$ is pseudo-effective with respect to f_* .

Definition 5. Let $(\text{Varieties})/k$ be the category of geometrically irreducible, reduced proper algebraic schemes over the ground field k restricting the morphisms to surjections. Let $X/S, Y/T$ be surjective morphisms of varieties. We consider X/S and X'/S' to be equivalent if there exists a birational map over S' between $X_{S'}/S'$ and X'/S' where S' is generically finite extension of S . Moreover X/S and Y/T are looked to be equivalent if main parts of $X_T/S \times T$ and $Y_S/S \times T$ are equivalent. We denote by $\text{var}(X/S)$ the minimal dimension of a base variety T such that Y/T is equivalent to X/S .

Itaka-Viehweg Conjecture 6.

(1)

$$\max_{m>0} \kappa(\det f_*(\omega_{X/S}^{\otimes m})) \geq \text{var}(X/S)$$

(2) for a general point $s \in S$

$$\kappa(\omega_{X/S}) - \kappa(\omega_{X_s}) \geq \text{var}(X/S)$$

Remark.

- (1) If the generic fibre of X/S is birationally equivalent to a variety with a canonical divisor semiample, Kawamata([Kaw3]) proves it.
- (2) If X/S has a variety of general type as the generic fibre, the conjecture above is essentially shown by Kollár and Viehweg([Kol3],[V4],[V5],[V6]).

§ 4. HODGE THEORY

Definition. Let X be a normal variety. Let $f : X \rightarrow S$ be a smooth morphism of varieties and D a divisor on X . Assume D has only simple normal crossings. If for each component C of D , the restriction $f : C \rightarrow S$ is smooth, then D is said to have only f -simple normal crossings.

Letting X be a variety, we denote by $[D]$ the integral part of $D \in \text{Div}(X) \otimes \mathbb{Q}$, $\{D\} = D - [D]$ and $\lceil D \rceil = -[-D]$.

Theorem 0([M6]). Let X be a non singular variety and $f : X \rightarrow S$ a projective smooth morphism of analytic varieties. Let D be a $\mathbb{Q}$ -divisor on X and D f -numerically equivalent to zero. Let A, B be divisors on X . Let D decompose itself into $D' + D''$ without common component. Assume $\{D\}_{\text{red}} + A + B$ has only f -simple normal crossings. Then Hodge-Deligne spectral sequence degenerates at E_1 :

$$E_1^{a,b} = R^b f_* \Omega_{X/S}^a (A + \{D'\}_{\text{red}}, B + \{D''\}_{\text{red}})(\lceil D' \rceil + \lceil D'' \rceil) \Rightarrow$$

$$R^{a+b}f_*\Omega_{X/S}^*(A + \{D'\}_{red}, B + \{D''\}_{red})([D'] + [D'']).$$

We denote by $\mathbf{Q}_{(p)}$ the subring of $\mathbf{Q}$ consisting of the irreducible fractionals without the denominator multiple of p .

THEOREM A(CF. DELIGNE-ILLUSIE([DI])). *Let k be a perfect field of characteristic $p > 0$, $S = \text{Spec } k$, $\tilde{S} = \text{Spec}(W_2(k))$. Let X be a proper smooth scheme over k of pure dimension d . Let D be a $\mathbf{Q}_{(p)}$ -divisor on X . Let $A+B$ be a divisor on X . Suppose $\{D\}_{red} + A + B$ has only normal crossings. Assume there exist liftings $\tilde{X}$ and $\tilde{D}$, $\tilde{A}$, $\tilde{B}$ such that $\tilde{X} \rightarrow \tilde{S}$ is flat and $\{\tilde{D}\}_{red} + \tilde{A} + \tilde{B}$ has only normal crossings on $\tilde{X}$ relative to $\tilde{S}$. Then*

- (1) *Let C be a component of $\{D\}$. If D is numerically equivalent zero, the next sequence is exact for all a, b*

$$H^{b-1}(C, \Omega_C^a(A, B + \{D\}_{red} - C)([D])) \rightarrow$$

$$H^b(X, \Omega_X^a(A, B + \{D\}_{red})(-C + [D])) \rightarrow$$

$$H^b(X, \Omega_X^a(A, B + \{D\}_{red} - C)([D])) \rightarrow 0.$$

- (2) *If D is ample, the next cohomologies vanish for $a + b > \max(d_X, 2d_X - p)$*

$$H^b(X, \Omega_X^a(A, B + \{D\}_{red})([D])) = 0.$$

COROLLARY B.

Let k be a perfect field of characteristic $p > 0$, $S = \text{Spec } k$, $\tilde{S} = \text{Spec}(W_2(k))$. Let X, Y be a projective smooth schemes over k of pure dimension. Let $f : X \rightarrow Y$ be a surjective morphism whose restriction is smooth over Y° with $Y \setminus Y^\circ$ a normal crossing divisor. Assume there exists a lifting $\tilde{X}$ such that $\tilde{X} \rightarrow \tilde{S}$ is flat. Let L be an ample invertible sheaf over Y . Then

- (1) $(f_*\omega_{X/Y})^{\otimes a} \otimes L^{\otimes b}$ is generically generated by global sections for arbitrarily a and $b > \dim Y$.
- (2) If $\omega_{X/Y}$ is f -semiample, $f_*\omega_{X/Y}^{\otimes \ell}$ is weakly positive for $\ell > 0$.

COROLLARY C.

Let k be a perfect field of characteristic $p > 0$, $S = \text{Spec } k$, $\tilde{S} = \text{Spec}(W_2(k))$. Let X be a variety over k . Let $\mathcal{E}_{am}^p(X)$ be the set of smooth k -varieties $\{V\}$ onto which there exists a dominant separable rational map from X such that V have flat liftings over $\text{Spec}(W_2(k))$ and ample canonical divisors with a fixed pluricanonical polynomial P_m .

Then $\mathcal{E}_{am}^p(X)$ is finite.

Definition of Arithmetic variety.

An arithmetic variety $\mathcal{X}$ is a projective flat regular scheme X over $\text{Spec } \mathbb{Z}$ the base change to $\text{Spec } \mathbb{C}$ of whose generic fibre has a Kähler metric compatible with complex conjugate action.

Problem D

For an arithmetic variety, we want some vanishing theorems.

These Hodge theoretical cosequences through covering trick give us tools to attack Diophantine problems, say, vanishings and weak positivity of direct image sheaves of powers of dualizing sheaves.

Next we explain a covering technique to apply Hodge theory to algebraic geometry.

Definition 1([GM]).

Let X be a scheme and M an abelian group. We denote $\text{Spec } \mathcal{O}_X[M]$ by $D(M)$. An X -group $D(M)$ is called a diagonalizable group.

Note that for any variable X -scheme T one has

$$D(M)(T) = \text{Hom}_{\text{groups}}(M, \Gamma(T, \mathcal{O}_T^*)).$$

Let $\mathbb{Z}_{((n_i)_{i \in I})}$ denote $\bigoplus_{i \in I} \mathbb{Z}/n_i$. We denote an X -group $D(\mathbb{Z}_{((n_i)_{i \in I})})$ by $\mu_{((n_i)_{i \in I})}$. The μ_n is the usual group of n -th root of unity.

We define a Kummer covering and a generalized Kummer covering according to Grothendieck- Murre ([GM]). Let X be a scheme and $\mathbf{a} = (a_i)_{i \in I}$ a finite set of sections of $\mathcal{O}_X$. Let $\mathbf{n} = (n_i)_{i \in I}$ be a set of positive integers. We denote $\mathcal{A}_{\mathbf{n}}^{\mathbf{a}} = \mathcal{O}_X[(T_i)_{i \in I}] / ((T_i^{n_i} - a_i)_{i \in I})$ and $Z_{\mathbf{n}}^{\mathbf{a}} = \text{Spec } \mathcal{A}_{\mathbf{n}}^{\mathbf{a}}$. We give the action of μ_n on $Z_{\mathbf{n}}^{\mathbf{a}}$ with $\phi : \mathcal{A}_{\mathbf{n}}^{\mathbf{a}} \rightarrow \mathcal{A}(\mu_n) \otimes \mathcal{A}_{\mathbf{n}}^{\mathbf{a}}$ given $\phi(t_i) = u_i \otimes t_i (i \in I)$.

Definition 2.

Assuming the a_i are regular, a couple (Y, G) consisting of an X -scheme and an X -group with the action on Y is said to be a Kummer covering of X relative to the sections $\mathbf{a}$, if (Y, G) is isomorphic to a couple $(Z_{\mathbf{n}}^{\mathbf{a}}, \mu_n)$ for a suitable set of integers $\mathbf{n}$, with every n_i prime to the residue characteristics of X .

Lemma ([GM], lemma 1.2.4).

Let $\mathbf{a} = (a_i)_{i \in I}$ and $\mathbf{b} = (b_i)_{i \in I}$ be sets of regular sections on X . Assume that there exist a set of $g_i \in \Gamma(X, \mathcal{O}_X^*)$ for $i \in I$ such that $a_i = g_i b_i$. Then there exists an etale surjective morphism $e : U \rightarrow X$ such that one has a μ_n -isomorphism

$$(Z_{\mathbf{n}}^{\mathbf{b}})_U \cong (Z_{\mathbf{n}}^{\mathbf{a}})_U.$$

Here $U = \text{Spec } \mathcal{O}_X[(V_i)_{i \in I}] / ((V_i^{n_i} - g_i)_{i \in I})$. The isomorphism $\mathcal{A}_{\mathbf{n}}^{\mathbf{b}} \rightarrow \mathcal{A}_{\mathbf{n}}^{\mathbf{a}}$ is given by

$$t'_i \mapsto v_i t_i.$$

We will define the generalized Kummer covering due to Grothendieck-Murre ([GM]). Let L be a factor group of μ_n :

$$\mathbb{Z}_{\mathbf{n}} \rightarrow L \rightarrow 0.$$

$D(L) \subset D(\mathbf{Z}_n) = \mu_n$ is a diagonal subgroup. The kernel N of $\mathbf{Z}_n \rightarrow L$ determines the quotient $\mu_n/K = D(N)$.

We construct the quotient space Z_n^a/K . Identifying $\mathcal{A}(\mu_n) \cong \mathcal{O}_X[\mathbf{Z}_n]$, one has

$$\mathbf{u}^\alpha = \prod_{i \in I} u_i^{\alpha_i}.$$

By the homomorphism $\phi : \mathcal{O}_X[\mathbf{Z}_n] \rightarrow \mathcal{O}_X[L]$, putting $\mathbf{u}^\alpha = \prod_{i \in I} u_i^{\alpha_i}$ and $u'_i = \phi(u_i)$, we have $N = \{\alpha \in \mathbf{Z}_n : \prod_{i \in I} (u'_i)^{\alpha_i} = 1\}$. The N is said to be the orthogonal subgroup of μ_n to $K = D(L)$. Let $\mathcal{B}$ denote the $\mathcal{O}_X$ -subalgebra $\bigoplus_{\alpha \in N} t^\alpha \mathcal{O}_X$ of $\mathcal{A}_n^a$. Note that the $\mathcal{O}_X$ -subalgebra B is identified with the kernel of

$$p_1 : \mathcal{A}_n^a \rightarrow \mathcal{O}_X[L] \bigotimes_{\mathcal{O}_X} \mathcal{A}_n^a$$

and

$$p_2 : \mathcal{A}_n^a \rightarrow \mathcal{O}_X[L] \bigotimes_{\mathcal{O}_X} \mathcal{A}_n^a$$

with $p_1(t^\alpha) = \mathbf{u}'^\alpha \otimes t^\alpha$, $p_2(t^\alpha) = 1 \otimes t^\alpha$.

Lemma 3 ([GM], Prop.1.3.2).

- (1) $\mu_n/K \cong D(N)$
- (2) $Z_n^a/K = \text{Spec } \mathcal{B}$.

Definition 4.

Given the regular sections $\mathbf{a} = (a_i)_{i \in I}$, a couple (Y, G) is called a generalized Kummer covering of X relative to a set of $\mathbf{a}$ if there exists a set of positive integers $\mathbf{n} = (n_i)_{i \in I}$ each n_i prime to the residue characteristics of X and a diagonalizable subgroup K of X such that the couple (Y, G) is isomorphic to the couple $(Z_n^a/K, D(N))$.

We remark that there exists the canonical morphism

$$(u, \phi) : (Z_n^a, \mu_n) \rightarrow (Z_n^a/K, D(N)).$$

Let L_1 be a quotient of $\mathbf{Z}_n$ and L a quotient of L_1 . Let $K = D(L)$, $K_1 = D(L_1)$ and $N = \ker(\mathbf{Z}_n \rightarrow L)$, $N_1 = \ker(\mathbf{Z}_n \rightarrow L_1)$. Then the canonical morphism

$$(Z_n^a/K, \mu_n/K = D(N)) \rightarrow (Z_n^a/K_1, \mu_n/K_1 = D(N_1))$$

induces the isomorphism

$$\begin{aligned} & ((Z_n^a/K)/(K_1/K), (\mu_n/K)/(K_1/K) = D(N_1)) \\ & \cong (Z_n^a/K_1, \mu_n/K_1 = D(N_1)). \end{aligned}$$

Lemma 5([GM], Prop.1.3.5). Let $\mathbf{a} = (a_i)_{i \in I}$, $\mathbf{b} = (b_i)_{i \in I}$ be sets of sections and $\mathbf{n} = (n_i)_{i \in I}$, $\mathbf{m} = (m_i)_{i \in I}$ sets of positive integers with $m_i = n_i q_i$ where $q_i \in \mathbb{Z}$, and $a_i = g_i^{n_i} b_i$ with $g_i \in \Gamma(X, \mathcal{O}_X^*)$ for $i \in I$. Let L be a quotient of $\mathbb{Z}_{\mathbf{n}}$ and $K = D(L)$. Then the canonical morphism

$$(Z_{\mathbf{m}}^{\mathbf{b}}, \mu_{\mathbf{m}}) \rightarrow (Z_{\mathbf{n}}^{\mathbf{a}}, \mu_{\mathbf{n}})$$

induces the isomorphism

$$(Z_{\mathbf{m}}^{\mathbf{b}}/K', D(N')) \rightarrow (Z_{\mathbf{n}}^{\mathbf{a}}/K, D(N)).$$

Here $\phi_{\mathbf{n}, \mathbf{m}} : \mu_{\mathbf{m}} \rightarrow \mu_{\mathbf{n}}$ is given by

$$\mathbb{Z}_{\mathbf{n}} \rightarrow \mathbb{Z}_{\mathbf{m}}$$

$$(\alpha_i) \mapsto (q_i \alpha_i).$$

$$N' = \{(q_i \alpha_i) : (\alpha_i) \in \mathbb{Z}_{\mathbf{n}}\}, K' = D(\mathbb{Z}_{\mathbf{m}}/N').$$

Note that the construction and the morphisms above are compatible with arbitrary base change.

Lemma 6([GM], Lemma 1.3.10).

Let $\mathbf{a} = (a_i)$ be a set of regular sections on X and let a_i be decomposed into $a_i = \prod_{\lambda \in J_i} a_{i\lambda}$. Putting $\mathbf{b} = (a_{i\lambda})$, $\mathbf{n}' = (n_{i\lambda})$ with $n'_{i,\lambda} = n_i$, one has the canonical X -morphism:

$$(Z_{\mathbf{n}'}^{\mathbf{b}}, \mu_{\mathbf{n}'}) \rightarrow (Z_{\mathbf{n}}^{\mathbf{a}}, \mu_{\mathbf{n}})$$

which is given by the $\mathcal{O}_X$ -Algebra homomorphism and the group homomorphism

$$\mathcal{A}_{\mathbf{n}}^{\mathbf{a}} \rightarrow \mathcal{A}_{\mathbf{n}'}^{\mathbf{b}}$$

such that

$$t_i \mapsto \prod_{\lambda \in J_i} t_{i\lambda}$$

$$\psi_{\mathbf{n}, \mathbf{n}'} : \mathbb{Z}_{\mathbf{n}} \rightarrow \mathbb{Z}_{\mathbf{n}'}$$

$$(\alpha_i) \mapsto (\beta_{i\lambda})$$

such that $\beta_{i\lambda} = \alpha_i$. Let $\mathbb{Z}_{\mathbf{n}} = \{(\alpha_i) : 0 \leq \alpha_i < n_i\}$. Putting $M = \psi_{\mathbf{n}, \mathbf{n}'}(\mathbb{Z}_{\mathbf{n}})$, one has the induced isomorphism

$$(Z_{\mathbf{n}'}^{\mathbf{b}}/D(\mathbb{Z}_{\mathbf{n}'}/M), \mu_{\mathbf{n}'}/D(\mathbb{Z}_{\mathbf{n}'}/M) = D(M)) \rightarrow (Z_{\mathbf{n}}^{\mathbf{a}}, \mu_{\mathbf{n}}).$$

Given each subgroup N of $\mathbb{Z}_{\mathbf{n}}$, putting $N' = \psi_{\mathbf{n}, \mathbf{n}'}(N)$, $K = D(\mathbb{Z}_{\mathbf{n}}/N)$, $K' = D(\mathbb{Z}_{\mathbf{n}'}/N')$, one has, moreover, the canonical morphism

$$(Z_{\mathbf{n}'}^{\mathbf{b}}, \mu_{\mathbf{n}'}) \rightarrow (Z_{\mathbf{n}}^{\mathbf{a}}/K, \mu_{\mathbf{n}}/K)$$

which induces the isomorphism

$$(Z_{n'}^b/K', \mu_{n'}/K' = D(N')) \rightarrow (Z_n^a/K, \mu_n/K = D(N)).$$

Lemma 7([GM], Prop.1.8.5).

Let $\mathbf{D} = (D_i)$ be a set of divisors on a locally noetherian normal scheme X and (Y, G) a generalized Kummer covering of X relative to $\mathbf{D}$. Then

- (1) if (D_i) has normal crossings, Y is normal,
- (2) if (D_i) are regular divisors with normal crossings, a Kummer covering (Z_n^D, μ_n) is regular over the points of $\bigcup_{i \in I} \text{supp } D_i$.
- (3) if (D_i) are regular divisors with normal crossings a generalized Kummer covering (Y, G) is regular over the regular points of $\bigcup_{i \in I} \text{supp } D_i$.

Example 1. Let X be a locally noetherian normal scheme and a a section of $\mathcal{O}_X$ such that $\text{div } a$ has normal crossings. The Kummer covering $Z_3^{a^2} = \text{Spec } \mathcal{O}_X[T]/(T^3 - a^2)$ is not normal. The map

$$\mathcal{O}_X[T]/(T^3 - a^2) \rightarrow \mathcal{O}_X[U]/(U^3 - a)$$

$$T \mapsto U^2$$

and the isomorphism

$$\mathbb{Z}_3 \rightarrow \mathbb{Z}_3$$

$$\alpha \mapsto 2\alpha$$

give a birational morphism

$$(Z_3^a, \mu_3) \rightarrow (Z_3^{a^2}, \mu_3).$$

The normalization of $\text{Spec } \mathcal{O}_X[T]/(T^3 - a^2)$ is $\text{Spec } \mathcal{O}_X[U]/(U^3 - a)$.

Example 2. Let X be a locally noetherian normal scheme and a a section of $\mathcal{O}_X$ such that $\text{div } a$ has normal crossings. The Kummer covering $Z_n^{a^n} = \text{Spec } \mathcal{O}_X[T]/(T^n - a^n)$ is not irreducible. A morphism

$$(Z_n^a = \text{Spec } \mathcal{O}_X[U]/(U^n - a), \mu_n) \rightarrow (\text{Spec } \mathcal{O}_X[T]/(T^n - a^n), \mu_n)$$

defined by

$$\mathcal{O}_X[T]/(T^n - a^n) \rightarrow \mathcal{O}_X[U]/(U^n - a)$$

$$T \mapsto U^n$$

and

$$\mathbb{Z}_n \rightarrow \mathbb{Z}_n$$

$$1 \mapsto n.$$

This induces an isomorphism

$$(\text{Spec } \mathcal{O}_X[U^n]/(U^n - a) = X, 0) \cong (\text{Spec } \mathcal{O}_X[T]/(T^n - a^n)/\mu_n, 0).$$

Proposition 1. Let X be a locally noetherian normal scheme and $a = \prod_{i \in I} a_i^{k_i}$ a section of $\mathcal{O}_X$. Assume that $\sum_{i \in I} \text{div}(a_i)$ has normal crossings. Let n be a positive integer such that the greatest common number $(n, k_i)_{i \in I} = 1$. Consider a Kummer covering $Z_n^a = \text{Spec } \mathcal{O}_X[T]/(T^n - a)$ and take a map

$$\mathcal{O}_X[T]/(T^n - a) \rightarrow \mathcal{O}_X[(U_i)_{i \in I}]/((U_i^n - a_i)_{i \in I})$$

$$T \mapsto \prod_{i \in I} U_i^{k_i}$$

and a homomorphism

$$\mathbb{Z}_n \rightarrow \prod_{i \in I} \mathbb{Z}_n$$

$$\alpha \mapsto (\alpha k_i),$$

whose image we denote by M . This gives a birational morphism

$$(\text{Spec } \mathcal{O}_X[U_i]/((U_i^n - a_i))/D(\prod_{i \in I} \mathbb{Z}_n/M), D(M)) \rightarrow (\text{Spec } \mathcal{O}_X[T]/(T^n - a), \mu_n).$$

Hence the normalization of $\text{Spec } \mathcal{O}_X[T]/(T^n - a)$ is identified with

$$\text{Spec } \mathcal{O}_X[U_i]/((U_i^n - a_i))/D(\prod_{i \in I} \mathbb{Z}_n/M) = \text{Spec } \bigoplus_{\alpha \in M} \mathfrak{u}^\alpha \mathcal{O}_X$$

proof: One has an isomorphism

$$\bigoplus_{\alpha} t^\alpha \mathcal{O}_X \rightarrow \bigoplus_{\alpha} \prod_{i \in I} u_i^{\alpha k_i} \mathcal{O}_X$$

$$t \mapsto \prod u_i^{k_i}.$$

The integral closure of $\bigoplus_{\alpha} \prod_{i \in I} u_i^{\alpha k_i} \mathcal{O}_X$ is $\bigoplus_{\alpha \in M} \mathfrak{u}^\alpha \mathcal{O}_X$. Hence this induces a birational morphism

$$(\text{Spec } \bigoplus_{\alpha \in M} \mathfrak{u}^\alpha \mathcal{O}_X, D(M)) \rightarrow (\text{Spec } \mathcal{O}_X[T]/(T^n - a), \mu_n).$$

Corollary 8. Assume $e = (n, k_i)_{i \in I} > 1$. One has the isomorphism

$$\begin{aligned} & (\text{Spec } \mathcal{O}_X[T]/(T^n - a)/D(\mathbf{Z}_n/e\mathbf{Z}_n), D(e\mathbf{Z}_n)) \cong \\ & (\text{Spec } \mathcal{O}_X[U]/(U^{n/e} - \prod_{i \in I} a_i^{k_i/e}), D(\mathbf{Z}_{n/e})) \end{aligned}$$

Theorem 9 ([EV], [K2]). Let X is a locally noetherian regular scheme. Let $\mathcal{L}$ be an invertible sheaf over X . Assume that $\mathcal{L}^{\otimes n}$ is represented by a divisor $D = \sum \nu_i C_i$ where (C_i) are regular divisors with normal crossings. The local Kummer covering (Z_n^D, μ_n) is well defined globally. It has quotient singularities over the non regular points on $\text{supp } D$. Let $f : Z_n^D \rightarrow X$ be a structure morphism and $\delta : Y \rightarrow Z_n^D$ an arbitrary resolution of singularities. Let Δ be a divisor such that $\Delta + \sum_i C_i$ has normal crossings. Let $J \subset I$ and $\ell \in \mathbf{Z}$. One has, then,

$$\begin{aligned} & (f \circ \delta)_*(\Omega_Y^a \langle (f \circ \delta)^*(\sum_{i \in J} C_i + \Delta) \rangle ((f \circ \delta)^* \frac{\ell}{n} D)) = \\ & \bigoplus_{0 \leq k < n} \Omega_X^a \langle \{ \frac{k}{n} D \} + \{ \frac{\ell}{n} D \} + \sum_{i \in J} C_i + \Delta \rangle ([\frac{k}{n} D] - \mathcal{L}^{\otimes k} + [\frac{\ell}{n} D] + (-[\frac{\ell}{n} D] + \ell \text{div } t)) \end{aligned}$$

. The Galois group $\text{Gal}(R(Y)/R(X)) = \text{Gal}(R(Z_n^D)/R(X)) = \mu_n$ acts naturally on

$$(f \circ \delta)_*(\Omega_Y^a \langle (f \circ \delta)^*(\sum_{i \in J} C_i + \Delta) \rangle ((f \circ \delta)^* \frac{\ell}{n} D))$$

and its invariant part is a direct factor, i.e.,

$$\begin{aligned} & H^0(\mu_n, (f \circ \delta)_*(\Omega_Y^a \langle (f \circ \delta)^*(\sum_{i \in J} C_i + \Delta) \rangle ((f \circ \delta)^* \frac{\ell}{n} D))) = \\ & \Omega_X^a \langle \{ \frac{\ell}{n} D \} + \sum_{i \in J} C_i + \Delta \rangle ([\frac{\ell}{n} D]). \end{aligned}$$

proof: see the proof of Theorem 17.

Corollary 10 (Kawamata-Esnault-Viehweg covering).

Let X is a locally noetherian regular scheme. Let $(\mathcal{L}_j)_{j \in K}$ be invertible sheaves over X . Assume that $\mathcal{L}_j^{\otimes n}$ are represented by divisors $D_j = \sum_i \nu_{ji} C_i$ with $(C_i)_{i \in I}$ regular divisors with normal crossings. Let $\mathbf{D} = (C_i)_{i \in I}$, $\mathbf{n} = (n)_{i \in I}$. The local Kummer covering $(Z_{(\mathbf{n})_{j \in K}}^{(\mathbf{D}_j)_{j \in K}}, \mu_{(\mathbf{n})_{j \in K}})$ is well defined globally. It has quotient singularities over the non regular points on $\text{supp } D_j$ for each j . Let $f : Z_{(\mathbf{n})_{j \in K}}^{(\mathbf{D}_j)_{j \in K}} \rightarrow X$ be a structure morphism and $\delta : Y \rightarrow Z_{(\mathbf{n})_{j \in K}}^{(\mathbf{D}_j)_{j \in K}}$ an arbitrary resolution of singularities. Let $g = f \circ \delta$. Let Δ be a divisor such that $\Delta + \sum_j D_j$ has normal crossings. Let $J \subset I$.

Then

$$g_*(\Omega_Y^a \langle g^*(\sum_{i \in J} C_i + \Delta) \rangle (g^* \sum_j \frac{\ell_j}{n} D_j)) =$$

$$\bigoplus_{j \in K} \bigoplus_{0 \leq k_j < n} \Omega_X^a \langle \{ \frac{\sum_{j \in K} k_j D_j}{n} \} + \sum_{i \in J} C_i + \Delta \rangle ($$

$$[\frac{\sum_{j \in K} k_j D_j}{n}] - \bigotimes_{j \in K} \mathcal{L}_j^{\otimes k_j} + [\frac{\sum_j \ell_j D_j}{n}] + (-[\frac{\sum_j \ell_j D_j}{n}] + \sum_j \ell_j \mathbf{div} t_j)).$$

The Galois group $\text{Gal}(R(Y)/R(X)) = \mu_{(n), i \in K}$ acts naturally on

$$g_* \Omega_Y^a \langle g^*(\sum_{i \in J} C_i + \Delta) \rangle (g^* \frac{\sum_j \ell_j D_j}{n})$$

and its invariant part is a direct factor, i.e.,

$$H^0(\mu_{(n), j}, g_*(\Omega_Y^a \langle g^*(\sum_{i \in J} C_i + \Delta) \rangle (g^* \frac{\sum_j \ell_j D_j}{n})) =$$

$$\Omega_X^a \langle \{ \frac{\sum_j \ell_j D_j}{n} \} + \sum_{i \in J} C_i + \Delta \rangle ([\frac{\sum_j \ell_j D_j}{n}]).$$

Let X_{et} denote the site of the category of etale schemes over X endowed with the etale topology.

Intuitive Definition 11. Let $(X_i)_{i \in I}$ be a set of schemes and the $e_{ij} : X_{ij} \rightarrow X_i$ etale morphisms such that $\phi_{ij} : X_{ij} \cong X_{ji}$ are isomorphisms. We identify X_{ij} of X_{et} and X_{ji} of X_{et} for all $i, j \in I$ and obtain a new site. This site is said to be a scheme in etale topology.

Remark. One can replace the etale topology by arbitrary Grothendieck topology. This process of enlarging the notion of shemes enables us to take polynomial roots of divisors

This forms in fact a **Gerbe**([GM]). We denote by X_{ijk} the scheme defined by the universal property such that $T \rightarrow X_{ij}, T \rightarrow X_{ik}$ are X_i -morphisms, $T \rightarrow X_{jk}, T \rightarrow X_{ji}$ are X_j -morphisms and $T \rightarrow X_{ki}, T \rightarrow X_{kj}$ are X_k -morphisms respectively.

Definition 12. Let X be a scheme in etale topology. A sheaf $\mathcal{F}$ over X is defined to be a functor satisfying an exact sequence

$$\mathcal{F}(X) \longrightarrow \Pi_i \mathcal{F}(X_i) \rightrightarrows \Pi_{ij} \mathcal{F}(X_{ij}).$$

The cohomology groups are calculated by Čech cohomologies.

We can say a scheme in etale topology is regular or normal and so on if it is of local property as you can easily imagine.

Theorem 13(Hodge-Kodaira-Deligne). Let X be a complete non singular variety in etale topology over the complex number field and D a divisor which is numerically equivalent to zero. The natural maps

$$H^b(X, \Omega_X^a(D)) \rightarrow H^b(X, \Omega_X^{a+1}(D))$$

are killed and the Hodge spectral sequence degenerates at E_1 . Furthermore, let (Δ_i) be a set of regular divisors with normal crossings on X with $\Delta = \Sigma \Delta_i$. Then the natural maps

$$H^b(X, \Omega_X^a(\Delta)(D)) \rightarrow H^b(X, \Omega_X^{a+1}(\Delta)(D))$$

are killed and the Deligne spectral sequence degenerates at E_1

$$E_1^{ab} = H^b(X, \Omega_X^a(\Delta)(D)) \Rightarrow H^{a+b}(X, \Omega_X^\bullet(\Delta)(D)).$$

Definition-Proposition 14. Let X be a scheme in etale topology, $\mathbf{n} = (n_i)_{i \in I}$ and $\mathbf{D} = (D_i)$ a set of divisors on X satisfied the same conditions as in local Kummer coverings. Then $(Z_{\mathbf{n}}^{\mathbf{D}}, \mu_{\mathbf{n}})$ is defined globally as a scheme in etale topology. $(Z_{\mathbf{n}}^{\mathbf{D}}, \mu_{\mathbf{n}})$ is said to be a Kawamata covering if the (D_i) are a set of regular divisors with normal crossings. It is said to be an Esnault-Viehweg covering if the $(\text{supp } D_i)$ are a set of divisors with normal crossings and $I = \{1\}$. It is said to be a Kawamata-Esnault-Viehweg covering if the $(\text{supp } D_i)$ are a set of divisors with normal crossings.

Theorem 15. Let X be a locally noetherian regular scheme in etale topology and $\mathbf{D} = (D_i)$ a set of regular divisors with normal crossings. Then a Kawamata covering $(Z_{\mathbf{n}}^{\mathbf{D}}, \mu_{\mathbf{n}})$ is a regular scheme in etale topology.

Theorem 16. Let X be a locally noetherian regular scheme in etale topology, $\mathbf{n} = (n_i)_{i \in I}$ and $\mathbf{D} = (D_i)_{i \in I}$ a set of regular divisors with normal crossings on X . Let $\pi : Z = Z_{\mathbf{n}}^{\mathbf{D}} \rightarrow X$ be the structure morphism of a Kawamata covering $(Z_{\mathbf{n}}^{\mathbf{D}}, \mu_{\mathbf{n}})$. Let Δ be a divisor such that $\Delta + \Sigma D_i$ has normal crossings. Let $\ell_i \in \mathbb{Z}$ and $J \subset I$. Then

$$\begin{aligned} & \pi_*(\Omega_Z^a(\pi^*(\Sigma_{i \in J} D_i + \Delta))(\pi^*(\Sigma_{i \in I} \frac{\ell_i}{n_i} D_i))) = \\ & \bigoplus_{(k_i) \in \mathbb{Z}_{\mathbf{n}}} \Omega_X^a(\Sigma_{i \in I} \{\frac{k_i}{n_i}\} D_i + \Sigma_{i \in I} \{\frac{\ell_i}{n_i}\} D_i + \Sigma_{i \in J} D_i + \Delta)(\Sigma_{i \in I} -k_i \text{div } t_i + \\ & \quad \Sigma_{i \in I} [\frac{\ell_i}{n_i}] D_i + \{\frac{\ell_i}{n_i}\} n_i \text{div } t_i). \end{aligned}$$

The Galois group $\text{Gal}(R(Z_{\mathbf{n}}^{\mathbf{D}})/R(X)) = \mu_{\mathbf{n}}$ acts naturally on

$$\pi_*(\Omega_Z^a(\pi^*(\Sigma_{i \in J} D_i + \Delta))(\pi^*(\Sigma_{i \in I} \frac{\ell_i}{n_i} D_i)))$$

, the invariant part of which is a direct factor

$$H^0(\mu_n, \pi_*(\Omega_Z^a \langle \pi^*(\sum_{i \in J} D_i + \Delta) \rangle (\pi^*(\sum_{i \in I} \frac{\ell_i}{n_i} D_i)))) = \\ \Omega_X^a \langle \sum_{i \in J} D_i + \sum_{i \in I} \{ \frac{\ell_i}{n_i} \} D_i + \Delta \rangle (\sum_{i \in I} [\frac{\ell_i}{n_i}] D_i).$$

proof: The problem is a local question and so X can be seen an affine regular scheme in etale topology. Let $D_i = \text{div } z_i$ such that the z_i are a part of regular parameters for X . Let $t_i^{n_i} = z_i$. One has as $\mathcal{O}_X$ -modules

$$\mathcal{O}_Z = \bigoplus_{(k_i) \in \mathbb{Z}_n} \prod_{i \in I} t_i^{k_i} \mathcal{O}_X$$

and

(1) for $i \in J$

$$\frac{dt_i}{t_i} = \frac{dz_i}{z_i}$$

(2) for $i \in I \setminus J$

$$dt_i = t_i \frac{dz_i}{z_i}$$

(3) as $\mathcal{O}_X$ -modules

$$\mathcal{O}_Z(\pi^* \sum_{i \in I} \frac{\ell_i}{n_i} D_i) = \bigoplus_{(k_i) \in \mathbb{Z}_n} \prod_{i \in I} t_i^{k_i} \cdot \prod_{i \in I} t_i^{-\ell_i} \mathcal{O}_X$$

(4)

$$t_i^{-\ell_i} = z_i^{-[\frac{\ell_i}{n_i}]} t_i^{-\{\frac{\ell_i}{n_i}\} n_i}.$$

Corollary 17. Let X be a locally noetherian regular scheme in etale topology and $\mathbf{D} = (C_i)_{i \in I}$ a set of regular divisors with normal crossings and $D = \sum_{i \in I} \nu_i C_i$, $\mathbf{n} = (n)_{i \in I}$. Let Y be the normalization of an Esnault-Viehweg covering Z_n^D and $\eta : Y \rightarrow X$ the structure morphism. Let Δ be a divisor such that $\Delta + \sum_{i \in I} C_i$ has normal crossings. Let $J \subset I$ and $\ell_i \in \mathbb{Z}$. Let $M = \text{im}(\mathbb{Z}_n \rightarrow \mathbb{Z}_n (1 \mapsto (\nu_i)))$. Then one has

$$\mathcal{O}_Y = \bigoplus_{(k_i) \in M} \prod_{i \in I} t_i^{k_i} \mathcal{O}_X$$

and

$$\eta_* \Omega_Y^a \langle \eta^*(\sum_{i \in J} C_i + \Delta) \rangle (\eta^*(\frac{\ell}{n} D)) =$$

$$\begin{aligned}
& \bigoplus_{0 \leq k < n} \Omega_X^a \langle \{ \frac{k}{n} D \} + \{ \frac{\ell}{n} D \} + \sum_{i \in J} C_i + \Delta \rangle (([\frac{k}{n} D] - k \operatorname{div} t) + \\
& \quad [\frac{\ell}{n} D] + (\ell \operatorname{div} t - [\frac{\ell}{n} D])) = \\
& \bigoplus_{(k_i) \in M} \Omega_X^a \langle \sum_{i \in I} \{ \frac{k_i}{n} \} C_i + \sum_{i \in I} \{ \frac{\ell_i}{n} \} C_i + \sum_{i \in J} C_i + \Delta \rangle (\sum_{i \in I} ([\frac{k_i}{n}] C_i - k_i \operatorname{div} u_i) + \\
& \quad [\frac{\ell}{n} D] + \sum_{i \in I} (-[\frac{\ell_i}{n}] C_i + \ell_i \operatorname{div} u_i)).
\end{aligned}$$

The Galois group $\operatorname{Gal}(R(Y)/R(X)) = \mu_n$ acts naturally on

$$\eta_*(\Omega_Y^a \langle \eta^*(\sum_{i \in J} C_i + \Delta) \rangle (\eta^*(\frac{\ell}{n} D)))$$

, the invariant part of which is a direct factor

$$\begin{aligned}
H^0(\mu_n, \eta_*(\Omega_Y^a \langle \eta^*(\sum_{i \in J} C_i + \Delta) \rangle (\eta^*(\frac{\ell}{n} D)))) = \\
\Omega_X^a \langle \sum_{i \in J} C_i + \{ \frac{\ell}{n} D \} + \Delta \rangle ([\frac{\ell}{n} D]).
\end{aligned}$$

proof: Let Z be a Kawamata covering Z_n^D and $\pi : Z \rightarrow X$ the structure morphism. Then letting $D = \operatorname{div} d, C_i = \operatorname{div} c_i$,

$$\mathcal{A}(Z_n^D) = \mathcal{O}_X[T]/(T^n - d) \rightarrow \mathcal{A}(Z) = \mathcal{O}_X[(U_i)_{i \in I}]/((U_i^n - c_i)_{i \in I})$$

$$T \rightarrow \prod_{i \in I} U_i^{\nu_i},$$

one has the canonical morphism $f : Z \rightarrow Z_n^D$ which factors $\pi : Z \rightarrow X$ and Let $M = \operatorname{im}(\mathbf{Z}_n \rightarrow \mathbf{Z}_n(1 \mapsto (\nu_i)))$. Then the structure sheaf of the normalization of the Esnault-Viehweg covering relative to D is

$$\mathcal{O}_Y = \bigoplus_{(k_i) \in M} \prod_{i \in I} u_i^{k_i} \mathcal{O}_X.$$

Making account of

(1)

$$\begin{aligned}
\mathcal{O}_X(\frac{\ell}{n} D) &= \mathcal{O}_X(\{ \frac{\ell}{n} \} \operatorname{div}(t^n) + [\frac{\ell}{n} D]) \\
&= t^{-\{ \frac{\ell}{n} \} n} \mathcal{O}_X([\frac{\ell}{n} D])
\end{aligned}$$

(2)

$$\frac{k\nu}{n} = [\frac{k\nu}{n}] + \{ \frac{k\nu}{n} \}$$

(3)

$$M = (\{\frac{k\nu_i}{n}\}_n)_{i \in I}, k_i = k\nu_i$$

(4)

$$u_i^{\{\frac{k\nu_i}{n}\}_n} \mathcal{O}_X = \mathcal{O}_X(-\{\frac{k\nu_i}{n}\} \operatorname{div}(u_i^n)) = \\ \mathcal{O}_X([\frac{k\nu_i}{n}]C_i - \frac{k\nu_i}{n} \operatorname{div}(u_i^n)).$$

Thus

$$\mathcal{O}_X(\sum_{i \in I}([\frac{k\nu_i}{n}]C_i - \frac{k\nu_i}{n} \operatorname{div}(u_i^n))) = \mathcal{O}_X([\frac{k}{n} D] - k \operatorname{div}(t)).$$

Hence one has

$$\eta_* \Omega_Y^a \langle \eta^*(\sum_{i \in J} C_i + \Delta) \rangle (\eta^*(\frac{\ell}{n} D)) = \\ \bigoplus_{(k_i) \in M} \Omega_X^a \langle \sum_{i \in I} \{\frac{k_i}{n}\} C_i + \{\frac{\ell}{n} D\} + \sum_{i \in J} C_i + \Delta \rangle (\sum_{i \in I} ([\frac{k_i}{n}]C_i - k_i \operatorname{div} u_i) + \\ ([\frac{\ell}{n} D] + \sum_i (-[\frac{\ell\nu_i}{n}] C_i + \ell\nu_i \operatorname{div} u_i)).$$

Corollary 18. Let X be a locally noetherian regular scheme in etale topology, $\mathbf{D} = (C_i)_i = (\operatorname{div} c_i)_i$ a set of regular divisors with normal crossings on X , $\mathbf{n} = (n_i)_{i \in I}$ and $D_j = \operatorname{div} d_j = \sum_i \nu_{ji} C_i$. Let $\pi : Z_{(n)_j}^{(D_j)_j} \rightarrow X$ be the structure morphism of a Kawamata-Esnault-Viehweg covering $(Z_{(n)_j}^{(D)_j}, \mu_{(n)_j})$. Let Δ be a divisor such that $\Delta + \sum_i C_i$ has normal crossings. Let Y be the normalization of Kawamata-Esnault-Viehweg covering which factors $Z_{\mathbf{D}}^{\mathbf{n}} \rightarrow X$ and $\eta : Y \rightarrow X$ the structure morphism. Let $Z = Z_{\mathbf{D}}^{\mathbf{n}}$ be a Kawamata covering. Let

$$M = \operatorname{im}(Z_{(n)_j} \rightarrow Z_{(n)_i} (k_j \mapsto (m_i = \sum_j k_j \nu_{ji})))_i.$$

Let ϕ

$$\mathcal{A}(Z_{(D_j)_j}^{(n)_j}) = \mathcal{O}_X[(T_j)_j] / ((T_j^n - d_j)_j) \rightarrow \mathcal{A}(Z) = \mathcal{O}_X[(U_i)_i] / ((U_i^n - c_i)_i)$$

be a $\mathcal{O}_X$ -homomorphism defined by $T_j \mapsto \prod_i U_i^{\nu_{ji}}$. Then

$$\mathcal{O}_Y = \bigoplus_{(m_i) \in M} \prod_i u_i^{m_i} \mathcal{O}_X$$

and

$$\eta_*(\Omega_Y^a \langle \eta^*(\sum_{i \in Q} C_i + \Delta) \rangle (\eta^*(\sum_j \frac{\ell_j}{n} D_j))) =$$

$$\begin{aligned}
& \bigoplus_j \bigoplus_{0 \leq k_j < n} \Omega^a \langle \{ \frac{\Sigma_j k_j D_j}{n} \} + \{ \frac{\Sigma_j \ell_j D_j}{n} \} + \Sigma_{i \in Q} C_i + \Delta \rangle ((\frac{\Sigma_j k_j D_j}{n} - \Sigma_j k_j \mathbf{div} t_j) + \\
& \quad [\frac{\Sigma_j \ell_j D_j}{n}] + (- [\frac{\Sigma_j \ell_j D_j}{n}] + \Sigma_j \ell_j \mathbf{div} t_j)) = \\
& \bigoplus_{(m_i) \in M} \Omega_X^a \langle \Sigma_i \{ \frac{m_i}{n} \} C_i + \Sigma_i \{ \frac{\Sigma_j \ell_j \nu_{ji}}{n} \} C_i + \Sigma_{i \in Q} C_i + \Delta \rangle (\\
& \quad \Sigma_i ([\frac{m_i}{n}] C_i - m_i \mathbf{div} u_i) + ([\frac{\Sigma_j \ell_j D_j}{n}] + \Sigma_i (- [\frac{\Sigma_j \ell_j \nu_{ji}}{n}] C_i + (\Sigma_j \ell_j \nu_{ji}) \mathbf{div} u_i))).
\end{aligned}$$

The Galois group $\text{Gal}(R(Z_{(n)}^{(D_j)_j})/R(X)) = \mu_{(n)_j}$ acts naturally on

$$\eta_*(\Omega_Y^a \langle \eta^*(\Sigma_{i \in Q} C_i + \Delta) \rangle (\eta^*(\frac{\Sigma_j \ell_j D_j}{n})))$$

, the invariant part of which is a direct factor

$$\begin{aligned}
& H^0(\mu_{(n)_j}, \eta_*(\Omega_Y^a \langle \eta^*(\Sigma_{i \in Q} C_i + \Delta) \rangle (\eta^*(\frac{\Sigma_j \ell_j D_j}{n})))) = \\
& \Omega_X^a \langle \Sigma_{i \in Q} C_i + \{ \frac{\Sigma_j \ell_j D_j}{n} \} + \Delta \rangle ([\frac{\Sigma_j \ell_j D_j}{n_i}]).
\end{aligned}$$

Theorem 19(Deligne-Illusie, [DI]). Let k be a field of characteristic $p > 0$, $S = \text{Spec } k$, $\tilde{S} = \text{Spec } W_2(k)$ and X an S -scheme in etale topology. Let D and $\tilde{D}$ be divisors which are numerically equivalent to zero on X and $\tilde{X}$, respectively. Associated to any flat $\tilde{S}$ -scheme in etale topology $\tilde{X}$ lifting X an isomorphism is determined canonically:

$$\phi_{\tilde{X}} : \bigoplus_{a < p} \Omega_{\tilde{X}'/S}^a(D')[-a] \cong \tau_{<p} F_* \Omega_{X/S}^\bullet(D)$$

in $D(X')$ such that $\mathcal{H}^a \phi_{\tilde{X}} = C^{-1}$ for $a < p$.

Corollary 20. Let k be a field of characteristic $p > 0$, $S = \text{Spec } k$, $\tilde{S} = \text{Spec } W_2(k)$ and X an S -scheme. Let $\mathbf{D} = (D_i)_{i \in I}$ be a set of smooth divisors with normal crossings on X . Let Δ be a divisor on X such that $\Delta + \Sigma_i D_i$ has normal crossings. Let (n_i) be a set of positive integer prime to p and the k_i integers.

(1) Assume that $\Sigma_{i \in I} \frac{k_i}{n_i} D_i$ is numerically equivalent to zero.

Associated to any flat $\tilde{S}$ -couple of scheme and relative divisor $(\tilde{X}, \tilde{D} + \tilde{\Delta})$ lifting $(X, D + \Delta)$, an isomorphism is determined canonically:

$$\phi_{(\tilde{X}, \tilde{D} + \tilde{\Delta})} : \bigoplus_{a < p} \Omega_{\tilde{X}'/S}^a \langle \Delta' + \Sigma_{i \in I} D'_i + \{ \Sigma_{i \in I} \frac{k_i}{n_i} D'_i \} \rangle ([\Sigma_{i \in I} \frac{k_i}{n_i} D'_i]) [-a] \cong$$

$$\tau_{<p} F_* \Omega_{X/S}^\bullet \langle \Delta + \sum_{i \in J} D_i + \{ \sum_{i \in I} \frac{k_i}{n_i} D_i \} \rangle ([\sum_{i \in I} \frac{k_i}{n_i} D_i])$$

in $D(X')$ such that $\mathcal{H}^a \phi_{\tilde{X}} = C^{-1}$ for $a < p$.

Corollary 21. Let $(E_j)_{j \in J}$ be a set of reduced divisors different from each other and any sum of D_i 's. Instead of (i), we assume that $\sum_{i \in I} \frac{k_i}{n_i} D_i + \sum_{j \in J} \frac{\ell_j}{n_j} E_j$ is numerically equivalent to zero and that $[\sum_{j \in J} \frac{\ell_j}{n_j} E_j] = 0$. Assume moreover that (E_j) has a lifting property. Then

$$\phi_{(\tilde{X}, \tilde{D} + \tilde{\Delta})} : \bigoplus_{a < p} \Omega_{X'/S}^a \langle \Delta' + \sum_{i \in J} D'_i + \{ \sum_{i \in I} \frac{k_i}{n_i} D'_i \} \rangle ([\sum_{i \in I} \frac{k_i}{n_i} D'_i])[-a] \cong$$

$$\tau_{<p} F_* \Omega_{X/S}^\bullet \langle \Delta + \sum_{i \in J} D_i + \{ \sum_{i \in I} \frac{k_i}{n_i} D_i \} \rangle ([\sum_{i \in I} \frac{k_i}{n_i} D_i]).$$

Further the Hodge-Deligne spectral sequence degenerates at E_1 for $a + b < p$

$$E_1^{ab} = H^b(X, \Omega_X^a \langle \Delta + \sum_{i \in J} D_i + \{ \sum_{i \in I} \frac{k_i}{n_i} D_i \} \rangle ([\sum_{i \in I} \frac{k_i}{n_i} D_i]) \Rightarrow$$

$$H^{a+b}(X, \Omega_X^\bullet \langle \Delta + \sum_{i \in J} D_i + \{ \sum_{i \in I} \frac{k_i}{n_i} D_i \} \rangle ([\sum_{i \in I} \frac{k_i}{n_i} D_i])).$$

Remark 22. We can take the coefficients of divisors in real numbers or adic numbers.

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