

Inflationary Cosmology in Scalar-Tensor Theories

by

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*“One must still have chaos in oneself to be able to give birth to
a dancing star.”*

– Friedrich Nietzsche,
Thus Spoke Zarathustra.

Abstract

It is an exciting period for research in gravity and cosmology. Since the detection of the *Cosmic Microwave Background*, cosmology has achieved an impressive level of precision. The accuracy is such that we can discuss the origins of our universe. For instance, from the CMB temperature fluctuations we have robust evidence that *inflation*, an epoch of accelerated expansion, took place in the earliest stages of our universe. In a remarkable simple way, inflation sets the universe in a hot, homogeneous and isotropic plasma state with tiny fluctuations. These are the required initial conditions for the *standard model of cosmology* to explain the current structure of our visible universe in the context of General Relativity. However, this is not the end of the story.

There is strong evidence that there is more than General Relativity to gravity from both observational and theoretical point of views. Observationally, on cosmological scales we are aware of *dark matter*, *dark energy* and *inflation*. The last two are often modeled by gravitating scalar fields; but what are their fundamental origins? On the theoretical side, we believe that there should be a unifying framework, e.g. *string theory* and *quantum gravity*. These latter approaches generally involve higher dimensional spaces in which our 4D world is embedded. Interestingly, when the dimensionality is reduced the resulting effective theory is often General Relativity plus a non-minimally coupled scalar field. These general class of theories which involve gravity and a scalar field are called *scalar-tensor theories*.

To discern what theory of gravity rules our universe, we need to test and compare predictions coming from several theories of gravity. In this regard, the very early universe and specially inflation are the window to probe new physics, in particular involving gravity, due to the extremely high energy densities that took place. Furthermore, the recent detection of gravitational waves has re-sparked the hope to detect *gravitational waves* produced during inflation. If they were detected, we would have access to the value of the energy scale during inflation. Thus, with ongoing and future surveys, which will provide new data, it is worth to deepen our knowledge in theories of gravity and test its predictions.

It is the purpose of this thesis to focus on the study of scalar-tensor theories in the inflationary epoch. We will study theoretical aspects of scalar-tensor theories, providing interpretations and concrete models. In particular, we will focus on the role of transformations of the metric within scalar-tensor theories. We will see that they are a crucial element to understand scalar-tensor theories. They are meaningful not only mathematically but physically as well. In

this respect, the first result of this thesis is a proof of the *physical invariance* of scalar-tensor theories under a *general class of metric transformations*. This proof is relevant in two ways. First, one can work in the simplest form of the action without changing the predictions of the model. Second, we can map apparently different theories that share the same predictions. Furthermore, this result can be applied not only in the context of scalar-tensor theories but it provides a potential generalization to vector-tensor theories and other theories involving metric transformations.

As a second result, we present a class of models in which by using metric transformations and an additional scalar field we obtain interesting features in the CMB temperature power spectrum. For example, we provide a model that could *enhance the formation of primordial black holes*, perhaps leading to binary black hole mergers, or explain the *large scale suppression* of the CMB. The great achievement of this result is its simple and intuitive derivation, yet allowing for a wide range of applications. Moreover, these class of models are general enough so that they provide a new approach to study inflationary models. The results of this thesis are based on two publications in the Journal of Cosmology and Astroparticle Physics (JCAP).

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Notation and units

Physical constants

$\hbar$	Reduced Planck constant: $\hbar \approx 1.05 \cdot 10^{-34} \text{ m}^2 \text{ kg s}^{-1} \approx 6.58 \cdot 10^{-16} \text{ eVs}$
e	Electron (elementary) charge: $1\text{C} = 6.24 \cdot 10^{18} e$
G	Newton (gravitational) constant: $G \approx 6.67 \cdot 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$
k_B	Boltzmann constant: $k_B \approx 1.38 \cdot 10^{-23} \text{ m}^2 \text{ kg s}^{-2} \text{ K}^{-1}$ also $k_B \approx 8.62 \cdot 10^{-5} \text{ eV K}^{-1}$
k_e	Coloumb (electrostatic) constant: $k_e = \frac{\alpha \hbar c}{e^2} \approx 8.99 \cdot 10^9 \text{ m}^3 \text{ kg s}^{-2} \text{ C}^{-2}$
m_e	Electron mass: $m_e \approx 9.11 \cdot 10^{-31} \text{ kg}$
m_p	Proton mass: $m_p \approx 1.67 \cdot 10^{-27} \text{ kg}$
eV	Electron-Volt: $1 \text{ eV} = 1.6 \cdot 10^{-19} \text{ J}$
c	Speed of light. $c \approx 3 \cdot 10^8 \text{ m/s}$

Reduced Planck units

l_{pl}	Planck length. $l_{pl} = \left(\frac{8\pi G \hbar}{c^3}\right)^{1/2} \approx 8 \cdot 10^{-35} \text{ m}$
M_{pl}	Planck mass. $M_{pl} = \left(\frac{c \hbar}{8\pi G}\right)^{1/2} \approx 4 \cdot 10^{-6} \text{ gr}$
t_{pl}	Planck time. $t_{pl} = \left(\frac{8\pi G \hbar}{c^5}\right)^{1/2} \approx 2.5 \cdot 10^{-43} \text{ s}$
E_{pl}	Planck energy. $E_{pl} = \left(\frac{c^5 \hbar}{8\pi G}\right)^{1/2} \approx 2.4 \cdot 10^{18} \text{ GeV}$
T_{pl}	Planck temperature. $T_{pl} = \left(\frac{c^5 \hbar}{8\pi G k_B^2}\right)^{1/2} \approx 2.8 \cdot 10^{31} \text{ K}$

Cosmological parameters and relevant scales

pc	Parsec. Typical distance between stars in a galaxy. $\text{pc} \approx 3.26 \text{ ly}$ also $\text{pc} \approx 3 \cdot 10^{16} \text{ m}$
H	Hubble parameter. Value today: $H_0 = (67.27 \pm 0.66) \text{ km s}^{-1} \text{ Mpc}^{-1}$
$c H^{-1}$	Hubble radius. Value today: $c/H_0 \approx 10^{26} \text{ m}$ or $c/H_0 \approx 3 \text{ Gpc}$
H^{-1}	Hubble time. Value today: $1/H_0 \approx 4 \times 10^{17} \text{ s}$ or $1/H_0 \approx 12 \text{ Gyr}$

h	Ratio of Hubble parameter. $h = \frac{H_0}{100 \text{ km/s/Mpc}}$
T_0	Temperature of CMB today. $T_0 \approx 2.73\text{K}$
ρ_c	Critical energy density. $\rho_c \approx 2 \cdot 10^{-29} h^2 \text{ g/cm}^3$
Ω_m	Matter energy density. Value today: $\Omega_{m,0} = 0.3156 \pm 0.0091$
Ω_Λ	Dark energy density. Value today: $\Omega_{\Lambda,0} = 0.6879 \pm 0.0087$
Ω_b	Baryon energy density. Value today: $\Omega_{b,0} h^2 = 0.02225 \pm 0.00016$
Ω_{rad}	Matter energy density. Value today: $\Omega_{\text{rad},0} \approx 10^{-5}$
Ω_K	Curvature energy density. Value today: $\Omega_K = -0.040^{+0.038}_{-0.041}$

Formulas

$[a, b]$	Index antisymmetrization. $T_{[ab]} = \frac{1}{2} (T_{ab} - T_{ba})$
(a, b)	Index symmetrization. $T_{(ab)} = \frac{1}{2} (T_{ab} + T_{ba})$
∂_μ	Partial derivative. $\partial_\mu = \frac{\partial}{\partial x^\mu}$.
∇_μ	Covariant derivative. $\nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma_{\mu\nu}^\alpha A_\alpha$ and $\nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma_{\mu\alpha}^\nu A^\alpha$
$\square$	D'Alembertian. $\square = g^{\mu\nu} \nabla_\mu \nabla_\nu$
$\Gamma_{\mu\nu}^\sigma$	Christoffel symbols. $\Gamma_{\mu\nu}^\sigma = \frac{1}{2} g^{\sigma\alpha} (2\partial_{(\mu} g_{\nu)\alpha} - \partial_\alpha g_{\mu\nu})$
$R^\alpha{}_{\mu\beta\nu}$	Riemann Tensor. $R^\alpha{}_{\mu\beta\nu} = 2\partial_{[\beta} \Gamma_{\nu]\mu}^\alpha + 2\Gamma_{\mu[\nu}^\rho \Gamma_{\beta]\rho}^\alpha$
$R_{\mu\nu}$	Ricci tensor. $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$
R	Ricci scalar. $R = g^{\mu\nu} R_{\mu\nu}$
$G_{\mu\nu}$	Einstein tensor. $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$
X	Kinetic term for a scalar field. $X = -g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi$

Abbreviations

GR	General Relativity.
FLRW	Friedmann-Lemaître-Robertson-Walker.
CMB	Cosmic Microwave Background.

LSS Last Scattering Surface

Λ CDM Standard model of cosmology: it includes the cosmological constant Λ and Cold Dark Matter.

dS de Sitter.

WKB Wentzel-Kramers-Brillouin

PBH Primordial Black Hole

e.o.m. Equations of motion

d.o.f. Degrees of freedom

UV Ultra-Violet.

IR Infra-Red.

EW Electro-Weak.

GUT Grand Unified Theories.

C.L. Confidence Level

CHAPTER 1

Introduction

Contents

1.1 Gravity and Cosmology	2
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It is incredible that we can ask questions such as “*What was the origin of our universe?*” and that we are actually able to give a pretty good answer.¹ Not only this, we can even test it with experiments. Answering this question and many more are possible thanks to huge theoretical advances in gravity and particle physics as well as technological developments providing precise experiments to probe these theories. In this thesis, we will approach deep theoretical aspects of theories of the very early universe, discussing possible observational consequences. To introduce the topic, we will first briefly explain the current status of gravity and cosmology (see [1] for a review). We will start by giving a taste of the big picture of gravity and cosmology in Sec. 1.1. We will briefly discuss the tests of General Relativity (GR) and why we do think GR should be modified. In Sec. 1.2 we will introduce some of the alternatives theories of gravity and we will then focus on scalar-tensor theories. Once they are introduced we will have all the necessary background to explain the purpose of this thesis. Last, Sec. 1.4 contains the organization of the thesis.

¹Obviously it is not the final answer, but it is a remarkable one.

1.1 Gravity and Cosmology

We are all familiar with the fact that gravity is the dominant force at large distances.² By large we mean our solar system and beyond. The gravitational constant is well measured from 0.1 mm, roughly the size of an Amoeba, to 10^{14} m, roughly the boundary of our solar system (see Ref. [2, 3] for a detailed review on the measurements and Ref. [4] for a discussion on the scales in which we can probe gravity). However, we know that our universe is roughly as big as 10^{26} m and that it was as small as 10^{-30} m in the past. *Does the same gravitational law apply along this huge span of scales?* Actually, at galactic and cosmological scales there remain some open questions. They are related to the so-called *dark matter*, *dark energy* and *inflation*.

We are aware of the existence of dark matter for several reasons (see Ref. [5] for a review on this topic). Mainly, we find solid evidence of dark matter in galaxies to explain rotation curves. It also explains the formation of large scale structure in our universe. We do not “see” it since, as far as we know, it only interacts with us gravitationally. Dark energy is another interesting story. It is known that the universe is expanding and that it is doing so in an accelerated way. Evidence of the accelerated expansion comes from studies of supernovae and primordial radiation (the cosmic microwave background). See Refs. [6, 7] for a review. It turns out that the expansion is driven by something very close to, if not, vacuum energy density³ (also called cosmological constant). We call it dark energy.

More interestingly, it is not the first time that our universe undergoes an accelerated expansion; or at least it is widely believed so.⁴ This leads to the introduction of inflation [13–16], on which this thesis is based. Inflation is a period of accelerated expansion previous to the hot big bang where (i) the exponential expansion smooths inhomogeneities and anisotropies and (ii) *primordial fluctuations*, i.e. the seeds of our universe, originate from quantum fluctuations [17, 18]. In other words, inflation sets up the initial conditions of our universe, that is a *hot*, *homogeneous* and *isotropic* plasma with *tiny fluctuations* on the order of $1 : 10^5$. The subsequent evolution of the universe filled with such hot plasma is known as the hot big bang (we will review it in more detail in Chap. 2). We know inflation occurred, or something very similar to it, because we receive the first radiation that escaped from the primordial

²Luckily the electromagnetic force can be screened and matter is on average neutral. Take the ratio of the electrostatic and gravitational force between two electrons. Electrostatic force is 10^{45} time stronger!

³Interestingly, with our current knowledge of particle physics we fail in spectacular fashion to explain the current value of the cosmological constant. This is called the cosmological constant problem. See Ref. [8, 9] for a review.

⁴See Refs. [10–12] for a review on alternatives to inflation.

plasma. It is the so-called Cosmic Microwave Background (CMB).⁵ Just like astronomers know what is going on inside the sun by looking at the seismic oscillations on the surface, we know what was going on inside the CMB sphere by studying its tiny fluctuations.⁶

To summarize, we have a model that describes very well what we observe in our local universe from its earliest stages until today, the so-called *standard model of cosmology*. However, as the reader might begin to realize it is not fully satisfactory from a fundamental point of view, as it leaves many crucial questions unanswered. What are dark matter and dark energy? What drove inflation? What was going on when the universe was extremely tiny, around the Planck scale? Is there a theory of quantum gravity? Honestly, we do not know the answers to these questions yet and this is the reason why we must study *modifications of gravity* and confront it with observations. Before jumping into modified gravity in a broad sense let me briefly review what are our observables.

1.1.1 What do we measure?

This is an important question. One should know the predictions of a theory that can be tested. For example, we measure the expansion of the universe through the red-shifting of photons. Note that in this case the observable is the red-shift rather than the expansion itself (see Chapter 3 for a concrete example). Thus, as stringy as our universe might be, if we cannot detect it then it does not matter. For this reason, let me introduce what measurements are used to probe gravity theories. For an extensive review see Refs. [19, 20].

Einstein's General Relativity's predictions on solar system scales are impressive.⁷ Thus, stringent tests of gravity are placed on solar system scales. Some examples of these experiments are:

- (i) In the Laboratory one uses *torsion experiments* like the Eöt-Wash experiment [19].
- (ii) In the solar system we have the *bending of light* measured by the Cassini probe and Lunar Laser Ranging [19].

⁵It is in the microwave band due to the expansion of the universe. As the universe expands the wavelength of radiation is stretched, red-shifted.

⁶Inside means prior to. I actually heard this analogy from a talk given by Scott Douglas (from UBC) and I found it quite illuminating. The study of the inner structure of the sun by seismic oscillations is called Helioseismology.

⁷For example, the perihelion advance of Mercury and the bending of light around the Sun. Actually, GPS systems are so precise that their clocks advance with respect to ours and this is predicted by General Relativity.

- (iii) Astrophysical sources like *binary pulsars* and *dwarf galaxies* also place bounds [21–23].

It should be noted that these experiments, however, only tell us that *today* and at *solar system scales* gravity is very close to that predicted by Einstein’s general relativity. In other words, they probe the weak-field limit of gravity, i.e. non-relativistic regime.⁸

We will be more interested in tests of gravity in the strong field regime and at large scales, i.e. when full general relativistic effects are required. For example,

- (i) *Compact objects* like Neutron Stars and Black Holes [4].
- (ii) High redshift supernovae [24, 25].
- (iii) Large Scale Structure, e.g. galaxy and redshift surveys [26–28].
- (iv) The *very early universe* with the Cosmic Microwave Background [29].

In this thesis we will focus on the very early universe, in particular on *inflation*. The very early universe is the perfect playground⁹ to study strong gravity and high energy physics as the temperature of the universe gets as hot¹⁰ as $T \sim 10^{28} K$ and the density as large as $\rho \sim 10^{86} g/cm^3$. As we previously said, the window to the early universe and inflation is the Cosmic Microwave Background. The CMB is our observable. Remarkably, from correlations of the temperature fluctuations in the CMB we can prove the predictions of inflation. Let me make a brief note. It is often said that there is a “zoo” of inflationary models.¹¹ This does not undermine, however, the predictions of inflation as a paradigm.¹² Whether we will know the exact model of inflation, that is another challenge.

Recently, a new window to explore the universe has been opened with the detection of *gravitational waves* from a binary black hole merger [34]. This confirms the tensor structure of gravity and with more detections it will be a crucial test for theories of gravity [35]. Polarization of the CMB radiation gives access to gravitational waves created during inflation with the so-called B-modes [36].

⁸In this regime one uses the perturbative Parametrized Post Newtonian (PPN) formalism which includes deviations from General Relativity [19].

⁹Or as Zel’dovich said [30]: “an accelerator for poor people.”

¹⁰The exact value depends on the model one considers.

¹¹See Ref. [31] for an approximate idea of the size of the “zoo”.

¹²See [32, 33] for a recent controversy in that point.

1.2 Alternatives to General Relativity

We have seen that an important question to pose is: “*How do we know that General Relativity is the theory of gravity?*” To answer this question we should contrast GR’s predictions with experiments and compare them with predictions from other theories of gravity [19, 20, 35, 37]. This is in fact a very good reason to study alternative theories of gravity. Another good reason is that just like Newtonian gravity could not explain the advance of the perihelion of Mercury which led to GR, the fact that we need dark matter and dark energy suggests that on cosmological scales there might be a more fundamental theory of gravity. Furthermore, the CMB data shows that the simplest model of inflation within general relativity is almost excluded [38].

Is there any indication that gravity should be modified from the theoretical point of view? As it turns out, there is plenty of motivation (for an extensive review see Ref. [39]). Let me present some examples. First GR is non-renormalizable [40–43] and thus quantum corrections yield extra curvature invariant terms. This led to $f(R)$ theories, i.e. an arbitrary function of the curvature scalar (see [44, 45] for a review), which are used to model dark energy and inflation. It turns out that one of the favored models of inflation falls within this category [14, 46]. Another way of seeking a renormalizable theory is for example to break Lorentz invariance, i.e. treat time and space in a different way, as in Hořava-Lifshitz gravity [47, 48]. These theories¹³ are related to Ultra-Violet (UV) corrections of GR, i.e. high energies.

Second, in general relativity the graviton is massless, i.e. the interaction range of the gravitational force is infinite. However, if the graviton were massive the interaction would decay much faster with distance (like a Yukawa-type exponential suppression). At large distances gravity would be weaker and this could explain the current acceleration of the universe (dark energy) [49]. These type of modifications are in the Infra-Red (IR) regime of GR. In a similar direction, one could consider that there are two gravity sectors with two different metrics, the so-called bigravity [50].¹⁴ One interesting feature is that gravitons could oscillate between sectors [53].

Another interesting approach comes from symmetry arguments. What is beyond the Planck scale? Well, if at high energies one has conformal symmetry [54–56] then there is no privileged scale and the question is meaningless. We could say that just like the spontaneous symmetry breaking of the Higgs field, a similar process happened in the very early universe. Actually inflation is quite close to a de Sitter universe which is a maximally symmetric spacetime

¹³ $f(R)$ theories are also used to model dark energy and thus they could be regarded as IR modification in that case.

¹⁴It can also be motivated from brane-world models [51, 52].

(see Appendix I).

Among all the alternatives of gravity, there is one class of theories which are particularly attractive. Historically, it started with Dirac's large number hypothesis [57], i.e. that universal constants should depend on time, and it was followed by Jordan [58], Brans and Dicke [59].¹⁵ The latter was motivated by Mach's principle but failed to include it into General Relativity. These classes of theories are called *scalar-tensor theories* and, as the name says, they involve adding an extra scalar to General Relativity. With the detection of the *Higgs field* we are sure that scalar fields are present in our universe [61–63].

It was later realized that scalar-tensor theories are quite natural from higher dimensional theories [64], e.g. string theory, and that $f(R)$ theories can be mapped into a subclass of scalar-tensor theories [44, 45]. Scalar tensor theories have several applications. For example, they are interesting from a renormalization point of view [65, 66]. Most inflationary models are based on scalar fields [31] and they are used as an alternative to explain dark energy [67, 68]. Moreover, as we will see, they considerably helped to improve our understanding of gravity. For all these reasons, we consider scalar-tensor theories a promising alternative to GR and are the main subject of this thesis.

1.3 Scalar-tensor theories

Our current understanding of the gravitational force says that gravity is related to the geometry of space and time and is described by General Relativity. The energy distribution of matter (e.g. momentum, mass and pressure) determines the geometry and at the same time matter lives in that geometry. Geometry is described in terms of a metric tensor which essentially represents the measures of length and time, i.e. using rulers and clocks.¹⁶

It is further assumed that gravity follows the *strong equivalence principle*, that is gravity affects all matter fields equally in any point of space and time. This assumption can be relaxed to the *weak equivalence principle* which states that gravity affects all matter fields equally. Both principles are consistent with current experiments.

Let us now see where do scalar-tensor theories originate from a theoretical point of view. The quest for a theory that rules them all, i.e. a unifying framework like string theory, has led the scientific community to consider that our space-time may actually not be 4 dimensional (3 spatial dimensions and 1

¹⁵See [60] for a review on Nordström's early attempt to use a scalar field to describe gravity.

¹⁶An alternative interpretation of scalar-tensor theories is that those rulers and clocks, i.e. our units, depend on space and time [69, 70].

dimension of time). Just like habitants of a (spatially) 2D world, flatland [71], are unaware of our 3D space. For example, we are talking about Kaluza-Klein or Braneworld models [72–76] where a wrapped 5th dimension is present.

It is particularly interesting that in higher dimensional theories after reducing the dimensionality down to our 4D world, scalar fields naturally appear everywhere. For example, a scalar field can be regarded as a coordinate of the extra dimensions. Furthermore, most of these scalar fields do not interact with gravity and/or matter in the usual way, i.e. they are non-minimally coupled. One can have an idea as follows. Imagine that we live in a 4D brane inside a 5D spacetime. In the 5D spacetime there might be a gravitational potential well which depends on the extra coordinate. Thus, the location of our 4D brane in that extra coordinate, call it a scalar field, will be important and play a role in the dynamics. From our point of view, we can expect that gravity in our brane interacts in a non-trivial way with the scalar field. For example, the gravitational constant could depend on time (see [77] for a review on variation of universal constants). Thus, in scalar-tensor theories only the weak equivalence principle holds. For the interested reader two books dedicated on scalar-tensor theories of gravity are Refs. [64] and [78].

It should be noted that now that we opened the door to non-minimal couplings, the theory space has expanded a great deal. We may ask then *“Which theory is the one that best describes our universe?”* There are two different ways to approach this question, usually referred to as top-down and bottom-up. Let me pose them into two questions respectively:

1. What kind of couplings do we obtain starting from a more fundamental theory, if any? What are its predictions?
2. What is the most general Lagrangian for a non-minimally coupled field? How much do observations constrain the form of the Lagrangian?

In fact the former is clearly more relevant, as it gives definite predictions. The second approach becomes useful when we ignore high energy physics that might come into play. This is used in Horndeski theories [79] and in the so-called Effective Field Theory (EFT) [80]. In this thesis, we will focus on the applications of scalar-tensor theories to cosmology and its consequences in a generic way. Rather than deriving an effective theory from a more fundamental theory, we will assume that such non-minimal couplings are present and study them in a general way.

1.3.1 Metric transformations

At the beginning of section 1.3, we have discussed that gravity is geometry and geometry is described by a metric. Scalar-tensor theories can be thought as follows. First assume that there is a dominant field that drives the evolution of the universe, e.g. the inflaton during inflation. This dominant field determines the metric. All other subdominant fields play in that metric. However, there is no reason to assume that the metric given by the dominant field is the same metric that affects matter. This is exactly the point of scalar-tensor theories. Relations among these metrics are called *metric transformations*. Following this line of thought Bekenstein gave a general form of such transformations [81]. To give an example of a metric transformation, take flat space and do a time-dependent conformal transformation, i.e. a global rescaling of the metric. We obtain an expanding universe!

Metric transformations as a non-trivial coupling to matter have been used in the late time acceleration of the universe (see for example Refs. [82, 83]) and it has attracted a great deal of attention recently [84–90]. In this case, however, the presence of the scalar field introduces time (or space) dependence of the gravitational constant and an extra force, a so-called fifth force, which are severely constrained on solar system scales [37, 39, 64, 90–94]. For example, the time variation of the gravitational constant is constrained to be [95]

$$\frac{\dot{G}}{G} < 6 \cdot 10^{-13} \text{yr}^{-1}. \quad (1.1)$$

Consequently, considerable effort has been placed on how to avoid such constraints by screening mechanisms [82, 96–104].

However, during inflation the situation is quite different. The inflaton field completely dominates the universe and apparently matter has nothing to say until radiation domination. Then whether matter coupled in a different way becomes a question of interpretation. For instance, the matter universe could avoid¹⁷ the initial singularity [105–108]. These kind of couplings could be constrained if they were present during the hot big bang [109–111]. However, if after inflation one recovers General Relativity it seems hard to find any signature from the matter coupling, i.e. from non-minimal couplings. In Chapter 4 we show that even if that happens, the matter coupling could be relevant during inflation and we also present some interesting examples.

On the theoretical side, metric transformations connect different scalar-tensor theories and are a crucial element to understand scalar-tensor theories (see Refs. [59, 112–116]). In particular, metric transformations leave physical

¹⁷Actually, there is no way to avoid singularities by a metric transformation. However, they can be postponed to past or future infinity.

observables invariant. In other words, one can work with the metric one pleases and obtain the same predictions. Thus, its a powerful tool in two ways: (i) two apparently different theories might be *mathematically related* and (ii) these two different theories are *physically equivalent*. We will show this in Chapter 3.

1.4 Outline of the thesis

This thesis focuses on interpretations of metric transformations in cosmology and applications to the very early universe in the context of scalar-tensor theories. At the end of each chapter a summary is provided where we will further discuss extensions and issues.

In Chapter 2 we give a broad review of the current status of cosmology and in particular we focus on inflation. We will mainly follow two great books in cosmology: S. Dodelson [36] and V. Mukhanov [30]. We also benefited from lecture notes by M. Sasaki [117, 118], D. Baumann [119, 120] and Cambridge DAMTP public lecture notes [121, 122]. We will review the notion of an expanding universe in General Relativity and explain the standard model of cosmology; the components of our universe and their observational evidence. Next, we will introduce inflation and its predictions, paying special attention to understand the connection between inflation and CMB observations.

We will dedicate Chapter 3 to understanding theoretical aspects of scalar-tensor theories. First, we will explain what non-minimal couplings are and we will present examples of how they naturally appear in higher dimensional theories. Then, we will introduce the notion of frames. We will focus on the couplings to matter and introduce two types of metric transformations, *conformal* and *disformal transformations*. Afterwards, we will review the *observable invariance* for the case of conformal transformations with simple examples. Next, we will present the work “cosmological disformal invariance” [123] published in JCAP where we proved the observable invariance for the case of disformal transformations in collaboration with M. Sasaki and A. Naruko. We will provide a thorough study with interpretations and explicit formulas.

In Chapter 4 we will focus on the applications to inflation of what we have seen in Chapter 3. This chapter is basically the work “conformal dependence of inflation” [124] published in JCAP in collaboration with M. Sasaki. We will start by reviewing the *notion of frames* in an explicit example. Then, we will

review a particular inflationary model, called *power-law inflation* [125], and its predictions. The advantage of considering that particular model is that all the calculations can be done analytically and we do not have to rely on any numerical computation (except for the last section). Next, we will present the main results of our work. We will show two different examples of how matter fields could experience *super-inflation* or a *bounce* (in their frame) during inflation. Furthermore, we will see how this can leave interesting observational effects if the matter field is a curvaton [126–128]. Concretely, we will see how we can obtain a blue tilt spectrum which could potentially create primordial black holes (and binary black holes which would merge and emit gravitational waves). We will also provide an explanation for the large scale suppression of the CMB power spectrum with another example. Matter experiences a bounce, i.e. the universe contracts and expands, which renders the spectrum blue at large scales. Lastly, in Chapter 5 we will summarize this thesis and provide further discussions and extensions of this work.

We provide complementary explanations and formulas in the appendices. They are organized as follows:

- Appendix A gives detailed formulas of metric transformations.
- Appendix B introduces the Horndeski Lagrangian.
- Appendix C studies a singular metric transformation and mimetic gravity.
- Appendix D provides examples of the cosmological disformal invariance.
- In appendix E we review cosmological perturbation theory applied to inflation.
- Appendix F presents the δN formalism.
- Appendix G is a review of the curvaton model.
- Appendix H gives a taste of the Hamiltonian formalism of GR.
- Appendix I discusses in more detail de Sitter spacetime.

Predictions from inflationary theory: a review

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Our everyday experience tells us that the world is quite inhomogeneous. When we look at the night sky, we see the stars of our own galaxy, mainly grouped along the milky way.¹ Perhaps, with a cheap telescope one can see the galaxy Andromeda in a clear night sky, which is about 780 kpc (2.5×10^6 light-years or 2×10^{19} km) away. What is really amazing is that when we look further away, coarse graining beyond 100 Mpc with redshift surveys, the universe appears to be *Homogeneous and Isotropic*. By extension of the Copernican principle, which says that the earth is not in the center of the solar system, the *Cosmological principle* assumes that our visible universe is homogeneous and isotropic in scales larger than $O(100)$ Mpc. In other words, there is no privileged location or direction in our universe.

Another strong piece of evidence is the CMB. It is formed by the first radiation that escaped the early universe soup of matter and radiation. The CMB is an homogeneous and isotropic black body radiation with temperature around 2.73 K. This corresponds to thermal radiation with a Planck spectrum

¹That depends whether you leave near a big city or not. In Beijing you might actually see a homogeneous and isotropic dust cloud instead.

that peaks at a frequency of 160 GHz (or a wavelength of 2 mm), i.e. microwave frequency band. How do we know that the CMB is what it is? First, the fact that it is so cold already tells us that it does not come from any star or galaxy. Second, if the universe expands radiation red-shifts, i.e. the temperature and the peak frequency decrease. This means that this radiation comes from far far away in time and space. To fully understand this we need to review some of the current knowledge of General Relativity and Particle Physics. See the books Refs. [30, 36] for more thorough explanations.

2.1 Standard model of Cosmology

What we currently know about our universe is the following:

1. It is *homogeneous and isotropic* on scales larger than 100 Mpc.
2. It is expanding according to the Hubble law² [129]. Actually it is an *accelerated expansion, today* [24, 25]. The responsible for the acceleration is unknown and it is called *Dark Energy* (DE). Furthermore, it corresponds to $\sim 70\%$ of the energy density of the universe today.
3. There is matter that we do not see but which interact with us gravitationally. It explains why the rotation curves are flat at large radius and also how large scale structures formed [36]. This matter is called *Dark Matter* (DM) and adds up to $\sim 25\%$ of the total energy density of the universe.
4. It is composed by baryons, radiation (photons and neutrinos), dark matter and dark energy. Most of the baryons are contained in galaxies and they are $\sim 5\%$ of the matter energy density.
5. It appears to be very close to a spatially *flat* universe.
6. We see the CMB with fluctuations $\delta T/T \sim 10^{-5}$.

Huge advances in Cosmology came with Einstein's theory of gravitation, GR, and technological advances which enabled us to explore into deep space. Let us see what does GR tell us about cosmology. General Relativity says that gravity is described by a curved space-time. Matter³ bends space-time

²The Hubble law says that galaxies are moving apart at a rate which depends on the relative distance between them. Formally, $\frac{dL}{dt} = HL$ where L is the distance and H is the Hubble parameter.

³By matter I mean any field.

and at the same time lives in that bent space-time.⁴ In fact, any form of energy gravitates. Mathematically speaking⁵ we have

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \quad (2.1)$$

where $g_{\mu\nu}$ is the metric of the space-time, $G_{\mu\nu}$ is the Einstein tensor, Λ is the cosmological constant, G is Newton's constant and $T_{\mu\nu}$ is the energy-momentum tensor of the matter contents. As we said, our universe is homogeneous and isotropic on large scales. The simplest content which can give rise to such a universe is a perfect fluid. Its energy-momentum tensor is given by

$$T_{\mu\nu} = (\rho + p/c^2) u_\mu u_\nu + p g_{\mu\nu}, \quad (2.2)$$

where ρ is the energy density, p is the pressure and u_μ is the 4-velocity of the fluid. The pressure is related to the energy density by what is called the equation of state, i.e.

$$\omega = \frac{p}{\rho c^2}. \quad (2.3)$$

In fact the properties of a perfect fluid are well described within Statistical Physics in terms of distribution functions. The distribution function $f(k)$ gives us the probability to find a particle with a given momentum k . Thus we can define the number density of particles as

$$n = \frac{g}{(2\pi\hbar)^3} \int d^3k f(k), \quad (2.4)$$

the energy density as

$$\rho = \frac{g}{(2\pi\hbar)^3} \int d^3k E(k) f(k), \quad (2.5)$$

and the isotropic pressure as

$$p = \frac{g}{(2\pi\hbar)^3} \int d^3k \frac{k^2}{3E(k)} f(k), \quad (2.6)$$

where $\hbar$ is the reduced Planck constant, g is the number of internal degrees of freedom (e.g. spin), the factor $1/3$ comes from isotropy and the number of dimensions and $E(k) = \sqrt{m^2 c^4 + k^2 c^2}$. Let us study what are the dynamics of the universe in the presence of matter, radiation and a cosmological constant. From now on I will set $c = \hbar = 1$ for simplicity.

⁴One can expect a quite non-linear theory.

⁵This equations of motion can be derived from the Einstein-Hilbert action, i.e.

$$S = \int d^4x \sqrt{-g} \{R + \mathcal{L}_m\}$$

where g is the determinant of the metric, $\mathcal{L}_m$ is the matter Lagrangian and R is the Ricci scalar.

2.1.1 An expanding universe

The usual approach to introduce an expanding universe is to start from an expanding shell of pressureless matter, a.k.a. dust, in Newtonian gravity and study its equation of motion. The result matches that from GR since the fluid has no pressure.⁶ However, a fluid with non-vanishing pressure has a different solution than in Newtonian gravity. Since we already introduced the Einstein Equations, let us work directly in that case.

First of all, the metric of an homogeneous and isotropic spacetime is given in general by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric, that is

$$ds^2 = -dt^2 + a(t)^2 d\sigma_{(K)}^2, \quad (2.7)$$

where a is the scale factor and $d\sigma_{(K)}^2$ is the metric of a homogeneous and isotropic 3D-space with constant curvature K . The scale factor $a(t)$ tells us how the physical distances evolve with time and $K = \{0, +1, -1\}$ for a flat, closed or open universe respectively. In polar coordinates $d\sigma_{(K)}^2$ reads

$$d\sigma_{(K)}^2 = \frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2, \quad (2.8)$$

with $d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi$ being the 2-sphere. The Einstein equations in that case reduce to the so-called Friedman equations:

$$H^2 \equiv \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3} \rho - \frac{K}{a^2} \quad (2.9)$$

and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3p). \quad (2.10)$$

A complementary equation, which can be derived from the other two as a consequence of the Bianchi identity,⁷ is the conservation of energy. For an expanding universe it is given by

$$\dot{\rho} + 3H(\rho + p) = 0. \quad (2.11)$$

This system of equations can be solved once an equation of state is specified, i.e. once $p(\rho)$ is given.

We can extract two important facts from this equations. First, H is the relevant length scale in an expanding universe. Take for example de Sitter space, where H is constant. In de Sitter space there is a cosmological horizon which is given by $1/H$ (see Appendix I). Thus, $1/H(t)$ which called the *Hubble*

⁶As we saw any form of energy (that includes pressure) gravitates in GR.

⁷The Bianchi identity says that $\nabla_\mu G^{\mu\nu} = 0$ and therefore it follows that $\nabla_\mu T^{\mu\nu} = 0$.

radius is the causal patch at a given time t . In other words, two particles separated over a distance $1/H(t)$ could not talk to each other at time t . Due to the time dependence of the Hubble radius we can define an integrated horizon, which is called the *Hubble horizon*. Consider the distance traveled by a photon, which is the maximum distance traveled by any known particle, that is ($ds^2 = 0$)

$$dr = \frac{dt}{a(t)} \equiv d\eta. \quad (2.12)$$

η is known as *conformal time* and it gives an idea of the *comoving distance* traveled by a photon. It is called comoving since it does not scale with the expansion. The relation between comoving distances and physical distances is the scale factor, i.e. $L_{\text{phys}} = a(t)L_{\text{com}}$. In the same way, $1/(aH) \equiv 1/\mathcal{H}$ is called the *comoving Hubble radius*. We can see the relation between the comoving hubble radius and the comoving horizon by rewriting Eq. (2.12) as

$$\eta = \int \frac{d \ln a}{aH}. \quad (2.13)$$

These notions will be important to discuss the causal structure of the universe and the problems with the big bang theory. The current value of H_0 is

$$H_0 = (67.27 \pm 0.66) \text{ km s}^{-1} \text{ Mpc}^{-1}. \quad (2.14)$$

This gives us an idea of the scales in our universe. For example, the current Hubble length is $c/H_0 \approx 10^{26} \text{ m}$ or $c/H_0 \approx 3 \text{ Gpc}$. The Hubble time is $1/H_0 \approx 4 \times 10^{17} \text{ s}$ or $1/H_0 \approx 12 \text{ Gyr}$. This is very roughly the size and age of our current universe.

The other fact can be seen from the second Friedmann equation (2.10). If $\rho + 3p > 0$ then the universe always decelerates, i.e. $\ddot{a} < 0$ or $\frac{d}{dt}(aH) < 0$. Using this fact and Eq. (2.13) we see that η is finite in the past. This has important consequences for our cosmological model and it is the reason why we need inflation. We will see this in detail in sections 2.1.2 and 2.2.

Let us present the most relevant cases for our universe, that is dust, radiation and cosmological constant. We will see that the universe evolves differently in each case.

Dust (or pressureless matter): In this case the equation of state is given by $\omega_{\text{m}} = 0$ (see Eqs. (2.6) and (2.6) with $m \gg k$, or just $p = 0$). The conservation of the energy can be integrated and yields

$$\rho_{\text{m}} = \rho_{\text{m},0} \left(\frac{a_0}{a} \right)^3, \quad (2.15)$$

where ρ_0 is the energy density at a_0 . It is important to note that the energy density of dust is proportional to a^{-3} . With this solution one finds that the scale factor evolves as

$$a \propto t^{2/3}. \quad (2.16)$$

It should be noted that at $t = 0$ we encounter a singularity. This is not important to us now, but we will go back to it later, in section 2.3

Radiation: For a perfect fluid made of radiation we have that $\omega_{\text{rad}} = 1/3$ (see Eqs. (2.6) and (2.6) with $k \gg m$) and therefore

$$\rho_{\text{rad}} = \rho_{\text{rad},0} \left(\frac{a_0}{a} \right)^4. \quad (2.17)$$

The energy density of radiation decays as a^{-4} . Actually, we can use this fact and the Stefan-Boltzmann relation, that is

$$\rho_{\text{rad}} = \sigma T^4, \quad (2.18)$$

to deduce that the temperature scales as

$$T = T_0 \frac{a_0}{a}. \quad (2.19)$$

This means that we can associate the temperature of radiation to given time. Lastly, the scale factor is given by

$$a \propto t^{1/2}. \quad (2.20)$$

Again, it should be noted that at $t = 0$ we encounter a singularity.

Cosmological constant: The last example is the vacuum energy density whose equation of state is $\omega_{\Lambda} = -1$. In that case the energy density is constant over all space and time, as one would expect from vacuum energy density, say

$$\rho_{\Lambda} = \rho_{\Lambda,0}. \quad (2.21)$$

In that case the universe expands exponentially, i.e. H is constant, and

$$a \propto e^{Ht}. \quad (2.22)$$

This is in fact the de Sitter space.

For reasons that will be clear below, the first friedmann equation (2.9) is usually recast as

$$1 = \Omega_{\text{rad}} + \Omega_{\text{m}} + \Omega_K + \Omega_\Lambda \quad (2.23)$$

where we took into account all the species in the universe,

$$\Omega_i = \frac{8\pi G}{3H^2} \rho_i \equiv \frac{\rho_i}{\rho_c} \quad (2.24)$$

and $\rho_c \equiv \frac{3H^2}{8\pi G} \approx 10^{-29} \text{ g/cm}^3$ is the critical energy density of the universe, i.e. the energy density above which the universe recollapses and below which the universe expands in an accelerated way. We have seen how the universe evolves depending on the dominant species at a given time. Observation tells us that we are now living in an accelerated expanding universe. Tracing back in time we had an epoch of matter domination where large scale structures and galaxies formed. Before that we had an epoch of radiation domination. As we go back in time, the universe shrinks and the temperature increases to values at which particle physics becomes relevant. In the next section, we start from the early universe and study the evolution as the universe expands and the temperature drops. For simplicity, from the next section and hereafter we will work in *reduced Planck units*, that is $\hbar = c = k_B = 1$ and $M_{\text{pl}}^{-2} = 8\pi G$. We can recover units later by looking at the Notation and units section, which can be found below the table of contents.

Before moving into the next section, let me review how do we know that the universe is expanding. Consider the energy of a photon with momentum k_μ measured by a comoving observer with 4-velocity u_μ , that is

$$\mathcal{E} = -k^\mu u_\mu, \quad (2.25)$$

where $u_\mu = (-a(\eta), \vec{0})$, $k^\mu = (E, \vec{k})$, $|\vec{k}| = E = h\nu$ and ν is the frequency of the photon. If we compare the energy of the photon which was emitted at η_{emit} and we compare it to the one observed at η_{obs} we obtain

$$\frac{\mathcal{E}_{\text{emit}}}{\mathcal{E}_{\text{obs}}} = \frac{a(\eta_{\text{obs}})}{a(\eta_{\text{emit}})}. \quad (2.26)$$

The ratio of energies of the photon is proportional to the amount of expansion of the universe. We define the redshift z as

$$1 + z \equiv \frac{a(\eta_{\text{obs}})}{a(\eta_{\text{emit}})} = \frac{\nu_{\text{emit}}}{\nu_{\text{obs}}} = \frac{\lambda_{\text{obs}}}{\lambda_{\text{emit}}}. \quad (2.27)$$

Note that if the universe expands, i.e. $a(\eta_{\text{obs}}) > a(\eta_{\text{emit}})$, then the wavelength moves to the red side of the spectrum, that is $\lambda_{\text{obs}} > \lambda_{\text{emit}}$.

2.1.2 Hot big bang

With the current knowledge of particle physics we can know what occurred in the early universe. Particles interact at a certain rate. Let Γ be the *rate of interaction* and let H be the *rate of expansion* universe. Thus, when $\Gamma \gg H$ particles have enough time⁸ to interact within a Hubble time. On the other hand, if $H \gtrsim \Gamma$ the particles no longer have time to interact and *decouple*. The interaction rate is given by

$$\Gamma \equiv n\sigma v, \quad (2.28)$$

where n is the number density, σ is the cross-section and v is the average particle velocity. The cross section depends on the interaction we are considering, i.e. strong, electroweak, compton, etc. To be more precise, consider a radiation dominated universe where $\rho_{\text{rad}} \propto T^4$ and therefore $H \propto T^2$. Then by dimensional analysis we know that $n \propto a^{-3} \propto T^3$ and σ is usually given by $\sigma \propto a^2 \propto T^{-2}$, i.e. proportional to the area. We can write the ratio of rates as

$$\frac{\Gamma}{H} \sim \frac{T_{\Gamma}}{T}, \quad (2.29)$$

where T_{Γ} is the temperature given by the theoretical parameters. That means that below the temperature T_{Γ} the particles decouple. On the other hand, the distribution function for particles in thermal equilibrium is given by

$$f(E) = \frac{1}{e^{E/T} \pm 1}, \quad (2.30)$$

where $+$ is for fermions and $-$ for bosons. It is important to note that when $T \ll m$ the particle behaves non-relativistically and we have an exponential suppression, i.e.

$$f(E) \sim e^{-m/T}. \quad (2.31)$$

Thus relativistic particles dominate the energy density of the primordial universe. This fact is important when computing the total energy density which is

$$\rho_{\text{rad}} = \frac{\pi^2}{30} g_*(T) T^4, \quad (2.32)$$

where $g_*(T)$ is the total number of relativistic degrees of freedom. At early times where all particles are relativistic $g_* = 106.75$. Nowadays we have photons and neutrinos, and so $g_* = 3.38$.

Let us briefly sketch the evolution (see Mukhanov's book [30] for more details).

⁸The time scale is given by the inverse of the rate. $t_H \equiv H^{-1}$ and $t_{\Gamma} \equiv \Gamma^{-1}$.

1. Leptogenesis [130] and Baryogenesis [131, 132]. It is the origin of the asymmetry between matter and anti-matter. It is still an open question.
2. Electroweak phase transition. Fields receive mass through the Higgs mechanism below $T \sim 125 \text{ GeV}$ (around the Higgs mass) which roughly corresponds to $z \sim 10^{15}$.
3. QCD phase transition. Strong interaction between quarks and gluons dominates below $T \sim 150 \text{ MeV}$ and the quarks get confined in baryons and mesons. This occurs around $z \sim 10^{12}$.
4. Dark Matter freeze-out. Dark matter is weakly interacting and decouples. The exact values depends on the dark matter particle model parameters.
5. Neutrino decoupling. The weak interaction decoupling is not efficient below $T \sim 1 \text{ MeV}$ and neutrino decouples.
6. Electron-positron annihilation. Below $T \sim 0.5 \text{ MeV}$ (the mass of the electron) a photon cannot create an electron-positron pair. The energy of the remaining pairs is transferred into photons. This occurs around $z \sim 10^9$.
7. Nucleosynthesis. Time where the light elements were formed at $z \sim 4 \cdot 10^8$ and $T \sim 100 \text{ keV}$
8. Matter-radiation equality. The universe goes from a radiation dominated stage to matter domination. Using cosmological parameters this corresponds to $z \sim 3 \cdot 10^3$ and $T \sim 0.75 \text{ eV}$.
9. Recombination, Photon decoupling (CMB). The scattering between photons and electrons becomes inefficient. The photons will then barely interact with matter and neutral hydrogen forms. This occurs around $T \sim 0.2 \text{ eV}$ and $z \sim 10^3$.
10. Dark energy- matter equality. Time when dark energy dominates the evolution of the universe, that is $T \sim 0.3 \text{ meV}$ and $z \sim 0.4$. Note that it is quite close to today, i.e. $z \sim 0$. Why it started dominating so close to our time? This is called the *coincidence problem*.

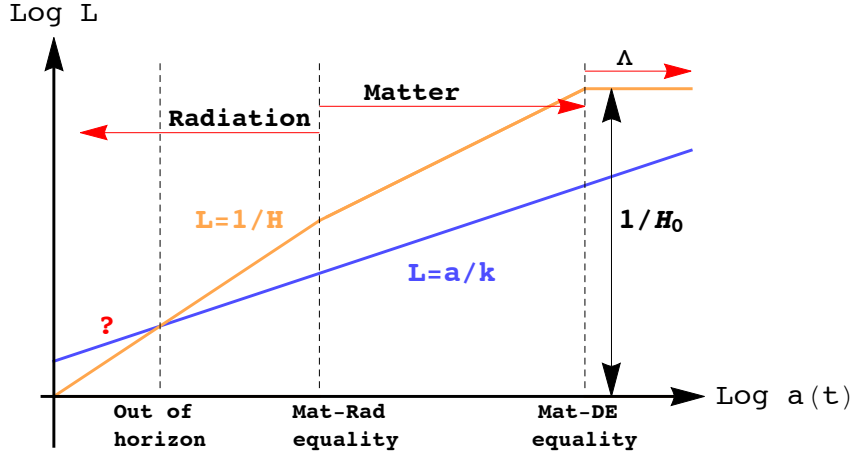


Figure 2.1: As we go back in time a given length scale L , in blue, exits the horizon scale, in orange. How were the initial conditions set to be so homogeneous if they were out of causal contact?

2.1.3 Shortcomings of the hot big bang

The big bang model does an incredible job in explaining the universe we see today from a very hot and dense plasma at early times. However, there are important facts that it fails to explain. Concretely, why is the universe so homogeneous on large scales? The primordial plasma was in thermal equilibrium over a size much larger than the causal patch at that time. Why was it in equilibrium without causal contact? The EW phase transition gave rise to many topological defects. Why don't we see them today? Let me briefly present these problems with some numbers.

Horizon (or homogeneity) problem: As we have anticipated in section 2.1.1, the fact that radiation and matter satisfy the strong energy condition, i.e. $\rho + 3p > 0$, implies that the conformal time is finite in the past. For example, for $a \propto t^p$ with $p < 1$ we have

$$\eta = \int \frac{dt}{a} \propto t^{1-p}, \quad (2.33)$$

which is finite as t decrease. This actually means that a photon emitted at early times is only able to travel a finite fraction of the space-time, which is tiny compared to the actual size we observe. For example, we can compare the comoving distance traveled by a photon since the Last Scattering Surface (LSS). We find that for a matter dominated universe

$$\frac{\eta_{LSS}}{\eta_0} \propto \left(\frac{a}{a_0} \right)^{1/2} = (1 + z_{LSS})^{-1/2} \quad (2.34)$$

Now using the fact that $z_{LSS} \sim 10^{-3}$ we see that the comoving distance traveled by a photon since the LSS covers only 10^{-3} sr of the total CMB sphere. Of course, if we compare it to earlier times it is worse. For example, comparing the comoving Hubble radius of our current universe with the comoving Hubble radius at the Planck time, recalling that it is given by $l_c = (aH)^{-1}$ we find

$$\frac{l_0}{l_{pl}} = \frac{a_{pl}H_{pl}}{a_0H_0} \sim 10^{28}, \quad (2.35)$$

where I used that $a_{pl}/a_0 = T_0/T_{pl} \sim 10^{-32}$ and $H_{pl}/H_0 = E_{pl}/H_0 \sim 10^{60}$. Thus, our current volume of the universe is made of 10^{84} causally disconnected Planck regions. We can see this fact in Fig 2.1. A given physical scale L was out of the Hubble radius at early enough times. Why do we see such an homogeneous universe then? This is called the *horizon (or homogeneity) problem*.

Flatness (or entropy) problem: We can do a similar estimate on the total energy density fraction of the universe at, say, the planck time. Using previous estimates and that $\Omega - 1 = \Omega_K \propto (aH)^{-2}$ we find

$$\Omega_{pl} - 1 = (\Omega_0 - 1) \left(\frac{a_0H_0}{a_{pl}H_{pl}} \right)^2 \leq 10^{-56}. \quad (2.36)$$

Recall that from observations we have that nowadays

$$|\Omega_{K,0}| < 10^{-3}. \quad (2.37)$$

Note how precisely the universe must be flat at very early times. It must be flat up to 56 digits of precision! This is what is called the *flatness problem*.

It is also called entropy problem since the same bound can be derived by computing the entropy of the universe now and in the past. The entropy is dominated by radiation and it is proportional to the total amount of radiation (mainly number of photons nowadays), say N_{rad} . Moreover in an adiabatic universe entropy is conserved and we can compute its value as

$$N_{\text{rad}} = n_{\text{rad}} \left(\frac{a(t)}{\sqrt{|K|}} \right)^3 \sim \left(\frac{T_0}{H_0} \right)^3 |1 - \Omega_0|^{-3/2} > \left(\frac{T_0}{H_0} \right)^3 \sim 10^{84} \quad (2.38)$$

where I used that $n_{\text{rad}} \propto a^{-3}$, $T_0 \sim 10^{-4}\text{eV}$ and $H_0 \sim 10^{-32}\text{eV}$.

Monopole problem: I will only briefly mention this problem. As we have seen, the history of the universe contains various phase transitions, e.g. Grand Unification Theory (GUT) phase transition, EW phase transition,

etc. As a consequence, one expects plenty of topological defects [133]. For example, assume that there is at least one monopole created per Hubble volume at the GUT scale, i.e. 10^{16}GeV . Then, nowadays there should be of the order of $(l_{\text{GUT}}/l_0)^3 \sim 10^{75}$ monopoles. This is a huge number. Roughly that is 10^{-3} monopoles/ m^3 . Compare it to the baryon density $\rho_b \approx 10^{-1}$ nucleon/ m^3 . The problem is that we do not see any effect of topological defects and they are constrained to be less than 10^{-30} per nucleon. This is called the *monopole problem*.

We have seen the successes and shortcomings of the hot big bang model. In next section we will see how we can explain the initial conditions of the big bang with a simple mechanism.

2.2 Inflation

The biggest problem of the big bang model is the horizon problem. Recall that it was due to the fact that the comoving hubble radius $1/(aH)$ increases with time or equivalently $\ddot{a} < 0$. What if there was a period before the big bang where the comoving hubble radius decrease with time, i.e. it was bigger at very early times? In other words, could have there been a phase of accelerated expansion in the very early universe? This should not sound so surprising, since we observe an accelerated expansion today. Thus, a small patch of the universe, which was homogeneous and isotropic, would experience an extremely fast expansion up to the big bang.

2.2.1 Alleviating the big bang problems

Let us review the big bang shortcomings assuming that previously there was an accelerated expansion.

Horizon problem: Let us assume that there is a field which violates the strong energy condition $\rho + 3p < 0$ and that $a \propto t^p$ with $p > 1$. The conformal time is then

$$\eta = \int \frac{dt}{a} \propto -t^{-(p-1)}. \quad (2.39)$$

We can see that as $t \rightarrow 0$, $\eta \rightarrow -\infty$. The comoving horizon was much bigger in the very early universe and thus our current hubble patch was in causal contact much before the big bang, as we can see in Fig. 2.2. A given physical length scale which was out of the horizon before the big bang was in fact inside the horizon at a much earlier time during inflation.

How much did the universe had to expand, so it would set the initial conditions for the big bang? Well, we can have an idea by again computing the ratio of comoving hubble radius between some initial time and today, i.e.

$$\frac{l_0}{l_{ini}} = \frac{a_{ini}H_{ini}}{a_{end}H_{end}} \frac{a_{end}H_{end}}{a_0H_0}, \quad (2.40)$$

where the subindex *ini* refers to initial time of inflation and *end* to the end of inflation. Note that now we have a contribution from a epoch before the big bang. If we take as a rough estimate that $a_{end}H_{end}/a_0H_0 \sim 10^{28}$ then we see that

$$\frac{a_{end}}{a_{ini}} > 10^{28} \frac{H_{ini}}{H_{end}}. \quad (2.41)$$

If we further assume that the expansion rate was almost constant at that time, i.e. almost a cosmological constant (see Sec. 2.1.1), we find that the scale factor ratio between the some initial time and the end of inflation is

$$\frac{a_{end}}{a_{ini}} > e^{64}. \quad (2.42)$$

This means that the accelerated expansion lasted for 64 Hubble times, since $a \propto e^{Ht}$, or as it is usually called 64 *e-folds*. The total number of e-folds is formally defined as the Hubble times,⁹ or the logarithm of the expansion factor, counted backwards in time from the end of inflation, i.e.

$$N(t) = \ln \left(\frac{a_{end}}{a(t)} \right) = \int_t^{t_{end}} H(t) dt. \quad (2.43)$$

The number of e-folds is also useful to compute the perturbations during inflation (see Appendix F). This huge expansion does not only solve the horizon problem but the flatness and monopole problems as well.

Flatness problem: By the same rule we find that

$$\Omega_{ini} - 1 = (\Omega_0 - 1) \left(\frac{l_{ini}}{l_0} \frac{l_{end}}{l_{ini}} \right)^2, \quad (2.44)$$

which implies

$$\Omega_{ini} - 1 \gtrsim \Omega_0 - 1. \quad (2.45)$$

We see that the extreme 10^{-56} fine tuning we needed before has been replaced by a much weaker bound.

⁹In one Hubble time the universe expands by an amount of $a \propto e^H$

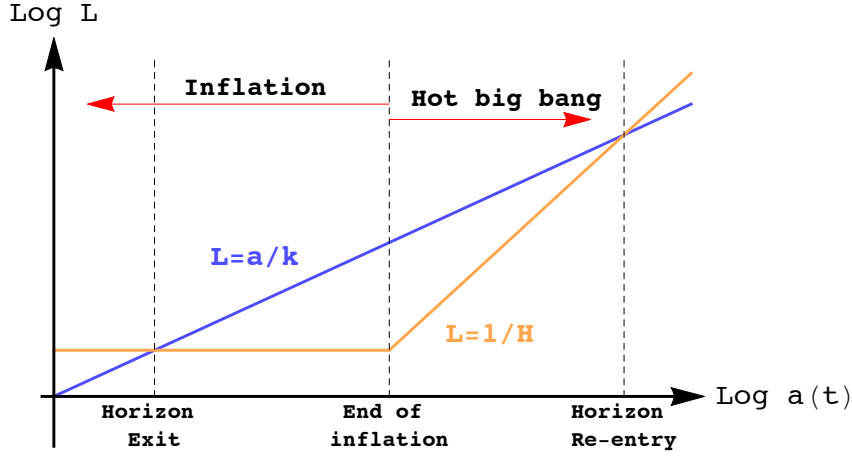


Figure 2.2: Continuation of Fig. 2.1. The horizon problem is solved by an epoch of accelerated expansion. See how the physical length, in blue, enters the horizon, in orange, as we go back in time.

Monopole problem: A simple reasoning tells that since the universe underwent an exponential expansion, the number density of monopoles was washed away by a factor greater than 10^{84} .

We have just seen that an *accelerated expansion* of the universe is able to alleviate the big bang problems. It should be noted that we say alleviate since they are not a solution but offer a more plausible explanation. In the next section we will see how this expansion could have been possible.

2.2.2 Slow roll inflation and quantum fluctuations

What was responsible for inflation? First of all we could say it was a cosmological constant but this would face several issues. For example, how would the accelerated expansion end? Furthermore, a cosmological constant leads to de Sitter space which due to the fact that it is maximally symmetric¹⁰ makes the notion of an observer not well defined. Therefore, we must depart from a strict cosmological constant. We will see in this section that in fact inflation's predictions are robust.

What could easily preserve homogeneity and isotropy and violate the strong energy condition? As a first guess, a scalar field. First of all, mainly because of its simplicity. Secondly, we now know that the Standard Model contains the Higgs scalar field. Most importantly, observational data agrees well with a scalar field. Perhaps inflation was driven by something else. Nev-

¹⁰The de Sitter isometry group is $SO(4,1)$.

ertheless, a scalar field captures the essential point. To understand it, we need to use some quantum field theory in curved space time.

Slowly rolling scalar field: A spatially homogeneous and isotropic scalar field which only depends on time, i.e. $\phi(t)$, is described as a perfect fluid with energy density and pressure respectively given by

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) \quad (2.46)$$

and

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi), \quad (2.47)$$

where $\dot{\phi}$ is the time derivative of the field and $V(\phi)$ its potential.¹¹ We readily see that in order to have an almost accelerated expansion, i.e. $\rho \approx -p$, the field must be rolling very slowly, that is

$$\dot{\phi}^2 \ll V \sim 3H^2 M_{pl}^2, \quad (2.48)$$

where in the last step I used Friedmann equation (2.9). Note that the expansion must last for at least 64 e-folds (see Eq. (2.43)), i.e.

$$N(\phi) = \int_{\phi}^{\phi_{end}} \frac{H}{\dot{\phi}} d\phi \approx \int_{\phi}^{\phi_{end}} \frac{V}{V_{\phi}} d\phi \approx \int_{\phi}^{\phi_{end}} \frac{d\phi}{\sqrt{2\varepsilon}}. \quad (2.49)$$

How constant is the expansion rate is given by

$$\varepsilon \equiv \frac{-\dot{H}}{H^2} = \frac{1}{2} \frac{\dot{\phi}^2}{H^2 M_{pl}^2} \ll 1. \quad (2.50)$$

In addition, as we want this expansion to last long enough we have to require

$$\eta \equiv \frac{-\dot{\varepsilon}}{H\varepsilon} \ll 1. \quad (2.51)$$

Equations (2.50) and (2.51) also imply for a canonical scalar field that $\delta \equiv \ddot{\phi}/(H\dot{\phi}) \ll 1$, where we used the conservation of energy (2.11), i.e.

$$\ddot{\phi} + 3H\dot{\phi} + V_{\phi} = 0, \quad (2.52)$$

¹¹This equations can be derived from a canonical scalar field Lagrangian, i.e.

$$\mathcal{L}(\phi) = -\frac{1}{2}\nabla_{\mu}\phi\nabla^{\mu}\phi - V(\phi)$$

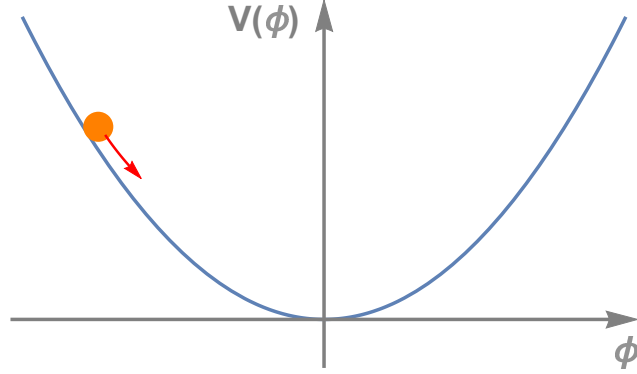


Figure 2.3: A scalar field is slowly rolling along slope of its potential. Inflation ends before reaching the minimum when $\varepsilon, \eta \sim 1$. Then reheating occurs during the oscillations around the bottom of the potential.

where a subindex ϕ refers to a partial derivative with respect to ϕ . Furthermore, we can translate these conditions into restrictions on the potential of a canonical scalar field, that is

$$\varepsilon \approx \varepsilon_V \equiv \frac{1}{2} \frac{V_\phi^2 M_{pl}^2}{V^2} \ll 1 \quad (2.53)$$

and

$$\eta_V \equiv \frac{V_{\phi\phi} M_{pl}^2}{V} \ll 1, \quad (2.54)$$

In particular, it should be noted that $\eta \approx -2\eta_V + 4\varepsilon_V \ll 1$ implies $\eta_V \ll 1$ which means that the potential must be flat enough.

We have seen that a slowly-rolling scalar field can drive an accelerated expansion of the universe. Inflation will stop when $\varepsilon \sim 1$ or $\eta \sim 1$. The fact that inflation must end and give rise to the hot big bang is known as *graceful exit*. The inflaton is thought to transmit its energy to the standard model through a stage which is called *re-heating* [30]. There might have been also a stage of *pre-heating* [134].

Quantum fluctuations: The most impressive prediction is that the structure we see today can be explained from quantum fluctuations during inflation. Let us briefly see how did the quantum fluctuations originate and provide the seeds to the big bang.

Quantum field theory in curved space-time is applied on top of a background, i.e. a leading order solution not affected by quantum fluctuations. In our case it is a FLRW metric Eq (2.7) with a slowly rolling spatially homogeneous scalar field, $\phi(t)$. Once we have our background metric we can study quantum fluctuations of the inflaton, that is $\phi = \phi(t) + \delta\phi(t, x^i)$. In fact,

we should consider the fluctuations of the spatial metric as well, but for our purposes we will neglect them for the moment. In fact, we can always choose a coordinate system where those fluctuations are set to zero. See Appendix E for a more rigorous explanation and derivation.

The expectation value of a quantum scalar field in an expanding universe with comoving wave number k which satisfies $k \gg aH$, i.e. a comoving scale well within the comoving hubble radius, is given by

$$\left| \langle \delta\phi | \vec{k} \rangle \right|^2 \equiv |\delta\phi_k|^2 \quad (2.55)$$

where

$$\delta\phi_k \approx \frac{1}{a^{3/2}\sqrt{2\omega_k}} e^{-i\omega_k t} \quad ; \quad \omega_k = \frac{k}{a} \gg H. \quad (2.56)$$

Eq. (2.56) is the lowest energy state with positive frequency which is de Sitter invariant. It is also called Euclidean or Bunch-Davies vacuum. As the universe expands there is a moment when $k < aH$, that is when the original comoving scale is much greater than the comoving hubble radius. In cosmology, the time when $k \sim aH$ is called *horizon crossing*. After a mode exits the horizon, the quantum fluctuations $\delta\phi_k$ stop oscillating and become constant. This is regarded as “classicalization” of the quantum fluctuations. However, what is going on is that the fluctuations get so squeezed that they look like classical, i.e. the phase space (position and momentum) of the system gets confined in a narrow range. This is to say that the uncertainty (Heisenberg) principle actually is so narrow that it behaves like a classical variable. See Refs [135–138] for a detailed study. Now, since the mode freezes after horizon crossing, we can use Eq (2.56) at the time $k \sim aH$, which yields

$$\delta\phi_k \approx \frac{H}{\sqrt{2k^3}} \quad , \quad \frac{k}{a} \ll H. \quad (2.57)$$

Thus, the mean square amplitude in unit of logarithmic interval of k ,¹² which is called the *power spectrum*¹³ is given by

$$\mathcal{P}_{\delta\phi}(k) = \frac{4\pi k^3}{(2\pi)^3} |\delta\phi_k|^2 \approx \left(\frac{H}{2\pi} \right)_{H=k/a}^2. \quad (2.58)$$

Actually, the quantity that is relevant for observations is what is called the

¹²The power spectrum is defined as

$$\langle \delta\phi(x)\delta\phi(x) \rangle = \int d\ln k \, P_{\delta\phi}(k).$$

¹³The power spectrum gives us an idea of the amplitude of fluctuations at each wavelength.

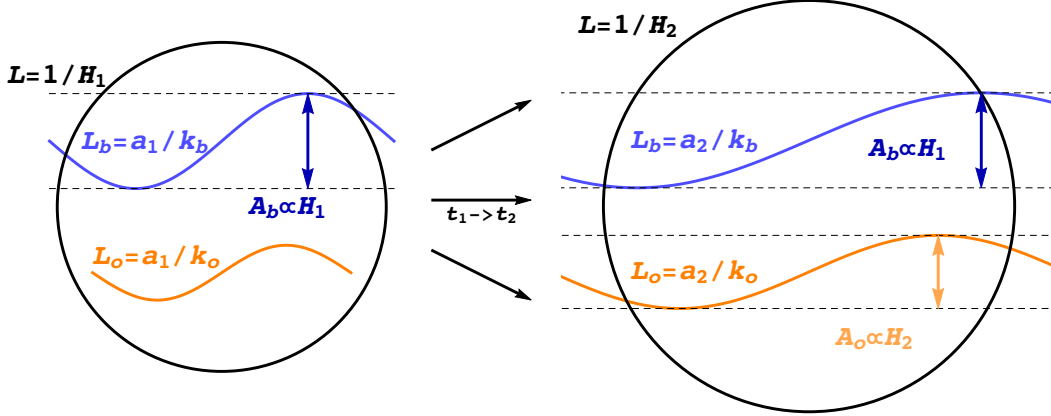


Figure 2.4: As the quantum fluctuations are stretched the horizon radius increases. The amplitude of the fluctuations is proportional to H . On the right: the blue mode exits the horizon and freezes. On the left: the orange mode exits the horizon some time later. Since the amplitude of fluctuations are proportional to H the shorter wavelength orange mode has a smaller amplitude. Thus we have a red tilt. Additionally, note that the wavelength redshifted, i.e. increased. Obviously this picture is not at scale.

curvature perturbation on the comoving hypersurface $\mathcal{R}_c$. This quantity is non-linearly conserved in scales larger than the horizon if perturbations are adiabatic [139]. This fact makes it useful for later computations. The comoving curvature perturbation is related to the inflaton perturbation by

$$\mathcal{R}_c = -\frac{H}{\dot{\phi}}\delta\phi. \quad (2.59)$$

See Appendix E for a derivation of this relation. Equations (2.59) and (2.58) lead us to

$$\mathcal{P}_{\mathcal{R}_c}(k) = \frac{4\pi k^3}{(2\pi)^3} |\mathcal{R}_{c,k}|^2 \approx \left(\frac{H^2}{2\pi\dot{\phi}} \right)^2_{H=k/a} = \frac{H^2}{(2\pi)^2 \varepsilon M_{pl}^2} \Big|_{H=k/a}. \quad (2.60)$$

This power spectrum is the one we will use to compare with observations in section 2.2.3. The tilt n_s of the power spectrum is given by

$$n_s - 1 \equiv \frac{d\mathcal{P}_{\mathcal{R}_c}(k)}{d \ln k} = \frac{d\mathcal{P}_{\mathcal{R}_c}(k)}{H dt} = -2\varepsilon - \eta. \quad (2.61)$$

The greatest success of inflation is encoded in the equation (2.60) we just derived. Note that under the slow-roll approximation $\mathcal{P}_{\mathcal{R}_c}(k)$ is *almost scale invariant* and has a *red tilt*, i.e. the amplitude for large scales is larger than small scales.

Before ending this section it is interesting to note that inflation also generates gravitational waves, or more correctly tensor modes.¹⁴ Tensor modes behave completely like a massless scalar field in our case. Thus, we can use our previous results Eq. (2.58) and find

$$\mathcal{P}_T(k) = 2 \times \frac{4\pi k^3}{(2\pi)^3} |h_k|^2 \approx 2 \left(\frac{H}{\pi M_{pl}} \right)_{H=k/a}^2, \quad (2.62)$$

where I defined h as the amplitude of the tensor modes. Note the extra factor 2 comes from the fact that tensor modes have 2 polarizations. The spectral index is also *red* and given by

$$n_T \equiv \frac{d\mathcal{P}_T(k)}{d \ln k} = -2\varepsilon. \quad (2.63)$$

Nevertheless, observations are able to constrain the contribution from tensor modes to the power spectrum with what is called the *tensor-to-scalar ratio*, that is

$$r \equiv \frac{\mathcal{P}_T}{\mathcal{P}_{\mathcal{R}_c}} = 16\varepsilon. \quad (2.64)$$

For inflation theorists, the spectral index n_s and tensor-to-scalar ratio r are their doors to observations. In the next section, we will see how we compare our theoretical predictions.

Predictions of inflation: Let us summarize what are the main predictions of inflation.

1. Primordial fluctuations are *gaussian*, since they have a quantum origin.
2. They are *adiabatic*, e.g. if there is only a single inflaton field. This also roughly means that the fluctuations initially affect all matter fields equally.
3. Slow-roll implies that the power spectrum is *almost scale invariant* and has a *red tilt*.
4. *Tensor fluctuations* are also generated. However, they turn out to be really hard to detect.

¹⁴Tensor modes are defined as the traceless and transverse part of the spatial metric.

2.2.3 Connection to observations

The amount of work behind observational cosmology is remarkable; in the sense that a great deal of physics from many areas are used to give an incredibly good prediction. Let us briefly go through the evolution of perturbations. Throughout the process, we have to bear in mind that our observations are made using photons. Thus, the information of the early universe is in the evolution of the distribution function using Boltzmann equations in an expanding universe. Einstein and Boltzmann equations tell us that in the very early universe, well before recombination, the gravitational potential was constant in time. Furthermore, in the case of adiabatic perturbations the gravitational potential acts equally for any field. We have seen in Sec. 2.2 that the initial seeds for the gravitational potential were set by inflation. The relation between the curvature perturbation $\mathcal{R}_c$, which is constant on super-horizon scales, and the Newtonian potential ($-\Phi$) is given by (see Appendix E)

$$\mathcal{R}_c = \Phi + \frac{2}{3} \frac{\Phi + \dot{\Phi}/H}{1 + \omega}. \quad (2.65)$$

Note that for a matter dominated universe where $\omega_m = 0$ on super-horizon scales, i.e. $\dot{\Phi} = 0$, then $\mathcal{R}_c = \frac{5}{3}\Phi$. On the other hand, for a radiation dominated universe, i.e. $\omega_{\text{rad}} = 1/3$, we have $\mathcal{R}_c = \frac{3}{2}\Phi$. The Newtonian potential Φ is the one directly related to the temperature fluctuations in the CMB, which are of the order of $\delta T/T \sim 10^{-5}$.

Now the question is: how does one go from the initial conditions set by inflation Eq. (2.60) to the anisotropies we see today in the CMB and the galaxy distribution on large scales? The answer is that one has to consider the evolution of each fluctuation for every field and its interactions in a radiation-matter dominated universe up to recombination and then until today. What we see today is a snapshot of recombination, or the LSS. We will not deal with the derivation here but will outline the important points.

After inflation and re-heating our universe is thermalized and in a deceleration phase. The comoving Hubble radius starts to decrease and modes which were outside the horizon start to enter again. This is the *horizon re-entry*. On top of that, baryons and photons are tightly coupled,¹⁵ while dark matter only interacts gravitationally. Neutrinos interact only through weak interactions.

¹⁵Electrons interact with photons exchanging momentum, i.e.

$$e^-(\vec{p}_1) + \gamma(\vec{k}_1) \rightarrow e^-(\vec{p}_2) + \gamma(\vec{k}_2).$$

This process is called *Compton scattering*. Since nucleons are much more massive than electrons they barely contribute to the exchange. Tight coupling implies that baryons and photons behave like a single fluid, i.e. they have the same velocity. In that case, only monopole and dipole contributions are relevant.

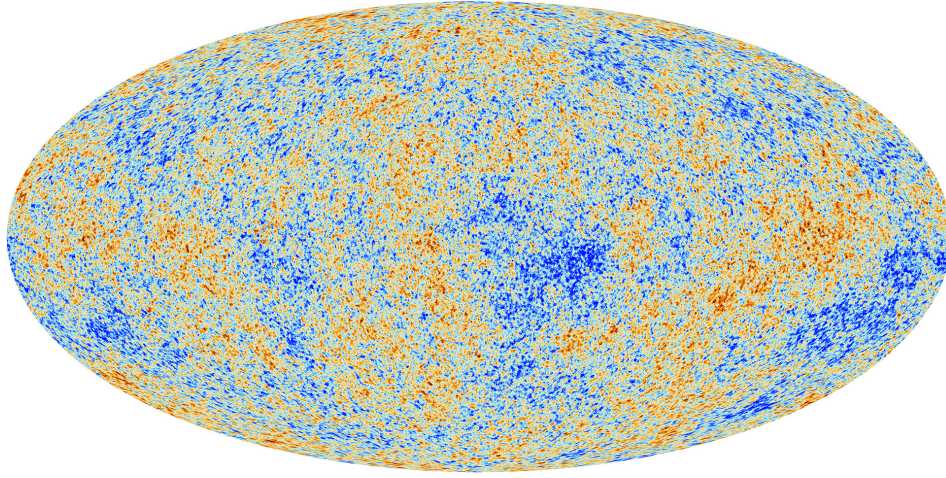


Figure 2.5: Map of the CMB fluctuations with the background removed from Planck 2015.

As the modes enter the horizon, they start to oscillate and we see its last behavior at the time of recombination. That is the peaks on the power spectrum are the modes which peaked at recombination. On the other hand, dark matter clusters under the effect of gravity and baryons fall into such gravitational potential.¹⁶ The growth rate of dark matter fluctuations is roughly proportional to the scale factor itself and the function which captures this evolution is called the *Growth Function*. Furthermore, the baryon sound speed depends on the ratio of density in the universe. Thus, the photon-baryon fluid oscillates at a frequency related to c_s after they enter the horizon. This creates the *Baryon Acoustic Oscillations* peaks.¹⁷ On scales small enough we find the *Silk Damping* due to the fact that the photon mean free path is finite and that recombination is not instantaneous. After recombination, the CMB photons will travel to us in a matter dominated and a late stage of dark energy dominated universe. The main effects imprinted in the CMB can be roughly classified into:

1. *Sachs-Wolfe effect* (monopole): due to the fact that photons have to climb up the gravitational potential.
2. *Dopper Shift* (dipole): Our galaxy is moving with respect to the CMB rest frame.

¹⁶There is no exact one-to-one correspondence between the number of galaxies and the number of dark matter halos. The relation is given in terms of a *bias factor*.

¹⁷They can be used as a standard ruler in cosmology as its position in time and space is very well measured [7, 140].

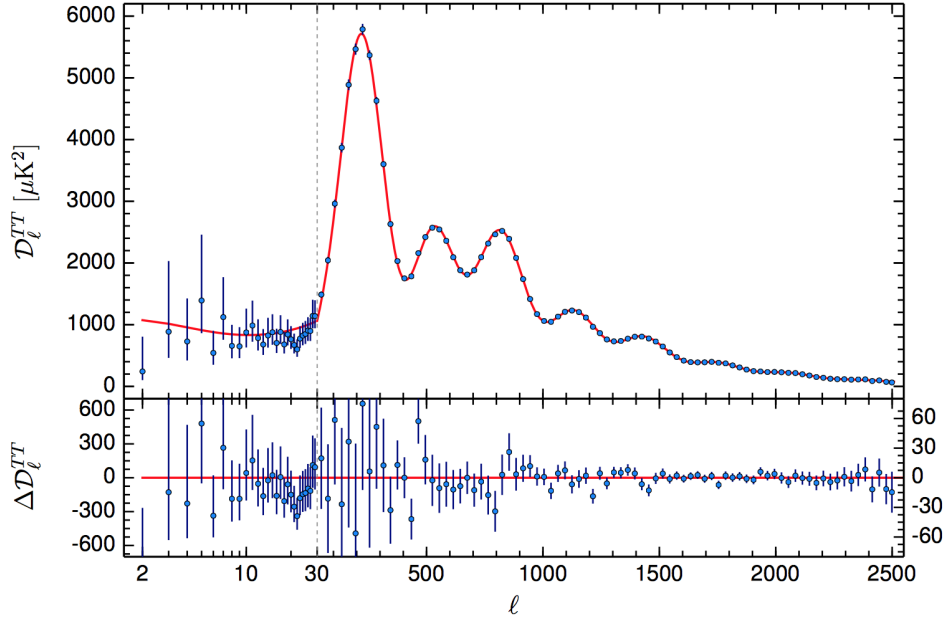


Figure 2.6: Planck 2015 temperature power spectrum. The position of the peaks confirm inflation and the standard model of cosmology.

3. *Integrated Sachs-Wolfe effect*: contribution from the dark energy domination stage.
4. *Cosmic variance*: For very large modes, i.e. close to the size of our causal universe, the number of samples is obviously very scarce. Thus, there is a fundamental uncertainty in the knowledge of those modes.

All of this is captured in what is called the *transfer function*. Since all these peaks depend on the cosmological parameters, we can constrain our cosmological model from the CMB. The current constraints are from the latest Planck 2015 mission (Temperature and polarization 68% confidence level):

1. Hubble parameter today: $H_0 = 67.27 \pm 0.66 \text{ km/s/Mpc}$
2. Baryon density parameter today: $\Omega_b h^2 = 0.02225 \pm 0.00016$. Note that $h = H_0 / (100 \text{ km/s/Mpc})$.
3. Dark energy density parameter today: $\Omega_\Lambda = 0.6879 \pm 0.0087$
4. Matter density parameter today: $\Omega_m = 0.3156 \pm 0.0091$
5. Spatial curvature density parameter today: $\Omega_K = -0.040^{+0.038}_{-0.041}$
6. Matter-Radiation equality redshift: $z_{eq} = 3395 \pm 33$

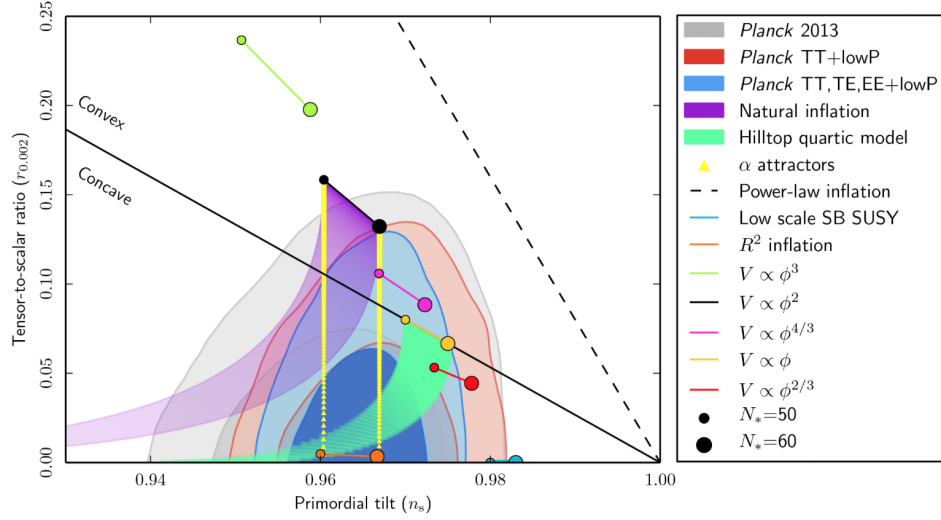


Figure 2.7: Tensor-to-scalar ratio r vs. primordial tilt n_s from Planck [38]. Note how a simple $m^2\phi^2$ potential, black line, is out of the 2σ contour! Also see how R^2 just lies on the sweet spot of the data.

7. Recombination redshift: $z_{rec} = 1090.06 \pm 0.30$
8. Dark energy equation of state: $\omega_\Lambda = -1.55^{+0.58}_{-0.48}$. In fact combining supernova Type Ia data we have a tighter constraint: $\omega_\Lambda = -1.006 \pm 0.045$.
9. Effective number of relativistic species: $N_{eff} = 2.99^{+0.41}_{-0.39}$
10. Sum of neutrino masses: $\sum m_\nu < 0.492$

Constraints on inflation: What kind of information can we obtain from the CMB about inflation? As we said before, CMB places constraints on the spectral tilt of the power spectrum as well as the tensor-to-scalar ratio. Fortunately, that is not all. From the power spectrum we also have constraints on the running of the tilt. Even more, we can compute correlation functions between more than two points. These will give us information about *non-linear effects* during inflation.¹⁸ The three and four-point correlation function are respectively called *bispectrum*, also called *Non-Gaussianities*¹⁹, and *trispec-*

¹⁸One must be careful since gravity is non-linear and therefore non-linear effects are expected from the evolution after inflation as well.

¹⁹The name non-gaussianity comes from the fact that a gaussian random field is completely specified by the two-point function. Any departure of such is called non-gaussianity in cosmology.

trum. The current bounds are (temperature and polarization 95% confidence level):

1. Amplitude of power spectrum: $A_s = (2.207 \pm 0.074) \cdot 10^{-9}$
2. Spectral index: $n_s = 0.9645 \pm 0.0049$
3. Tensor-to-scalar ratio: $r < 0.0987$
4. Running of spectral index: $dn_s/d \ln k = -0.006 \pm 0.0014$
5. Local Non-Gaussianity: $f_{NL} = 0.8 \pm 5.0$
6. Local Trispectrum: $g_{NL} = (-9.0 \pm 7.7) \cdot 10^4$

Before ending this section, let us mention that similar conclusions can be extracted from the E-mode polarization and Weak lensing of the CMB [29]. This helps to get tighter constraints. Unfortunately, no B-mode polarization has been detected so far and thus we only have an upper bound on the amount of tensor modes.

2.3 Summary and discussion

We have seen that our universe is well described by the standard model of cosmology, a.k.a. Λ CDM, plus inflation. In section 2.1 we have seen that our universe is homogeneous and isotropic on cosmological scales. It is composed by matter (baryons and dark matter), radiation (neutrino and photons) and dark energy and its dynamics is well described by GR plus perfect fluid. Then in section 2.2 we have introduced inflation to provide the initial conditions for the hot big bang. We have seen that inflation is an epoch of accelerated expansion most likely driven by a scalar field which smooths out inhomogeneities and anisotropies. Furthermore, quantum fluctuations of the scalar field are amplified and stretched out of the horizon. The power spectrum of the fluctuations is almost scale invariant with a slight red tilt.

In section 2.2.3 we have seen that we can test the predictions of inflation with the CMB temperature fluctuations and we have given the best fit values for the standard model of cosmology. Note that these results are currently the best fits to data once a theoretical model is given. We use Λ CDM plus inflation since we have very strong evidence to believe in that model.²⁰ An interesting result from Planck data [144] is that it seems to *favor inflationary models with*

²⁰Note that despite the good agreement of the standard Λ CDM model with current data [141, 142], some tension appears from direct measurements of the time evolution of the Hubble parameter [143].

a *non-minimal coupling* (see R^2 model in Fig. 2.7), i.e. scalar-tensor theories of gravity [59, 64, 78] which includes the so-called $f(R)$ theory [145, 146].

Despite inflation's successes, it is not free from issues (although they do not change its predictions). In particular regarding the initial conditions and the initial singularity. For example, Ekpyrotic, Bouncing universes and quantum creation of the universe from nothing try to tackle these issues. Particularly interesting is the model of *open inflation* [147] where inflation occurs after a bubble is nucleated from vacuum decay [148]. Since the bubble is created by quantum tunneling it will be maximally symmetric and, therefore, the universe inside the bubble will be homogeneous and isotropic. This model is attractive because of its well defined initial conditions. Observationally, if Ω_K is found to be negative, open inflation would be a great candidate.²¹ Another interesting direction is the presence of other fields during inflation. They usually produce small oscillations in the power spectrum [149–151] and non-negligible amounts of non-gaussianity [152, 153]. Looking at these features one could obtain information of extra fields during inflation.

²¹Bubble nucleation always create an open universe inside the bubble.

From frame to frame: observable invariance

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In Chapters 1 and 2 we have argued that scalar fields are present in our universe and that in fact these scalar fields often non-minimally couple to matter and/or gravity. These class of theories are called Scalar-Tensor theories. Furthermore, we saw that in order to explain observational data, e.g. CMB, we must depart from simple GR and a canonical scalar field.

So far, we have not yet specified what non-minimal couplings are. To see this more clearly, let us start with the meaning of minimal coupling. A field is minimally coupled when its Lagrangian is of the simplest form. For example, the action of a massless scalar field ϕ minimally coupled to gravity is given by

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \{ M_{pl}^2 R - g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi \} , \quad (3.1)$$

where R is the Ricci Scalar, i.e. the scalar curvature of space-time, $g_{\mu\nu}$ is the metric, g is the metric determinant and ∇_μ the covariant derivative. Any

departure from Eq. (3.1) is called non-minimal coupling. To be more concrete, an easy extension¹ to the previous action could be

$$S = \frac{1}{2} \int d^4x \sqrt{-\tilde{g}} \left\{ \phi^2 \tilde{R} - \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \phi \tilde{\nabla}_\nu \phi \right\}. \quad (3.2)$$

Naively, we expect physics derived from Eqs (3.1) and (3.2) to be different. The truth is that we must be careful about what we mean by physics. This is important since, as we will see, two apparently different theories may give exactly the same predictions, the same observables. Actually we will see in more detail later that by doing a change in the metric, i.e. a *metric transformation*, the non-minimal coupling can be eliminated. Roughly speaking we could do the following:

$$\tilde{g}_{\mu\nu} \rightarrow M_{pl}^2 \phi^{-2} g_{\mu\nu} \quad \Rightarrow \quad \tilde{R} \rightarrow M_{pl}^{-2} \phi^2 R + \dots \quad \text{and} \quad \sqrt{-\tilde{g}} \rightarrow \phi^{-4} M_{pl}^4 \sqrt{-g}. \quad (3.3)$$

Then we can show that

$$\sqrt{-\tilde{g}} \phi^2 \tilde{R} \rightarrow \sqrt{-g} M_{pl}^2 R + \dots. \quad (3.4)$$

We removed the non-minimal coupling by working with another metric! The metric transformation we used is called *conformal transformation* and its generalization is called *disformal transformation*. We will see that metric transformations play a very important role in the Scalar-Tensor theories:

- (i) They simplify the form of the action.
- (ii) They relate apparently different theories.
- (iii) They maintain the causal structure.
- (iv) They do not modify physical predictions.

Actually, it was first realized by Brans and Dicke [59], that by means of a field dependent conformal transformation the non-minimal coupling can be absorbed and we are left with the usual Einstein-Hilbert action with a scalar field. Thus, we have *two different representations* of the same physics [113]. This led to the notions of frames, i.e. theories related by a metric transformation. If gravity is as simple as Einstein-Hilbert it is called *Einstein frame*. On the other hand, we can define the *Matter frame*, a.k.a. *Jordan frame*, where matter takes the simplest form possible.

Before going into details, there is an important point to be clarified. Note that the metric transformation in Eq. (3.3) is a *passive* metric transformation.

¹We will see that this naive extension to the Standard Model is actually well motivated and can be derived from fundamental physics.

We have replaced $\tilde{g}$ by $\tilde{g}(g)$. An *active* metric transformation would be a transformation $\tilde{g} \rightarrow g$. Obviously if we take the same functional form of the action and replace the metric, i.e.

$$\sqrt{-\tilde{g}} \phi^2 \tilde{R} \rightarrow \sqrt{-g} \phi^2 R, \quad (3.5)$$

we do not have the same theory unless $\tilde{g} = g$. For this reason, hereafter we always deal with a passive metric transformation, i.e.

$$\sqrt{-\tilde{g}} \phi^2 \tilde{R} \rightarrow \sqrt{-\tilde{g}(g)} \phi^2 \tilde{R}[\tilde{g}(g)] \rightarrow \sqrt{-g} M_{pl}^2 R + \dots \quad (3.6)$$

In this case, we have rewritten our original theory in terms of new variables. This might look like a trivial field redefinition, but as we will see it is not straightforward to prove.

So far we have heuristically argued that scalar fields often non-minimally couple in higher dimensional theories and that this non-minimal coupling can be removed by a metric transformation. In section 3.1 we explore further the meaning of such non-minimal couplings and metric transformations. Next, in Secs. 3.2 and 3.3 we prove the invariance of physics under a metric transformation. We will do so in detail, studying the gravity and matter actions. We work in reduced Planck units, i.e. $\hbar = c = 1$, and we will keep M_{pl} for illustrative purposes.

3.1 Non-minimal couplings and metric transformations

Let us spend some time to see where non-minimal coupling originate from. For example, it is interesting to see how they naturally appear in higher dimensional theories (see Ref. [64] for other examples). In this set up, one considers that we (matter) are restricted to live in a 4D brane embedded in a higher dimensional space. Here, for simplicity, we will take that gravity acts only on one extra dimension. The non-minimal coupling will appear after dimensional reduction. Note that we will not consider the full dynamics of the metric in the extra dimensions. We will present some examples to motivate and illustrate the form of scalar-tensor theories.

3.1.1 Conformal transformation

A well known example are dilatonic scalars, often present in string theory. Consider a 5D metric given by

$$ds_{5D}^2 = g_{\mu\nu}(x^\alpha, y) dx^\mu dx^\nu + dy^2, \quad (3.7)$$

where $g_{\mu\nu}$ is a 4D metric and y is the extra dimension. Let us assume that we are restricted to live in a 4D brane fixed at some $y = y_{\text{brane}}$. Our 4D effective line element can be written as

$$ds_{4D}^2 = \tilde{g}_{\mu\nu}(x^\alpha) dx^\mu dx^\nu, \quad (3.8)$$

where $\tilde{g}_{\mu\nu}$ is the metric for matter fields. What is the exact form of $\tilde{g}_{\mu\nu}$ depends on our initial set up. Nevertheless, it is interesting to note that a simple coupling $g_{\mu\nu}(x^\alpha, y = y_{\text{brane}}) = \tilde{g}_{\mu\nu}(x^\alpha)$, i.e. no dependence on the extra dimension, is not expected at all. In most cases, our effective metric takes the form

$$g_{\mu\nu}(x^\alpha, y = y_{\text{brane}}) = e^{y(x^\alpha)} \tilde{g}_{\mu\nu}(x^\alpha). \quad (3.9)$$

This is what is known as *conformal (or dilatonic) coupling*. As we advanced before, this kind of relation between the metrics $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ is known as a *conformal transformation*. Now, if we regard y as a scalar field, say ϕ , we can write this relation in a more general form, that is

$$g_{\mu\nu} = \Omega^2(\phi) \tilde{g}_{\mu\nu}. \quad (3.10)$$

Note that in this set up, there is no metric more physical than any other, as we may choose to work in the variables we find more appropriate. In other words, it is clear that there is no unique natural conformal or physical frame a priori.

Let us be more concrete and study a particular case. Let us assume that in Eq. (3.10) $\tilde{g}_{\mu\nu}$ is the metric that interact with matter fields. Then we expect that the action in terms of $g_{\mu\nu}$ takes the form

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} M_{pl}^2 R + \mathcal{L}(\phi, X) + \Omega^{-4}(\phi) \mathcal{L}_m(\phi, g, \psi, A, \dots) \right\}, \quad (3.11)$$

where $\mathcal{L}(\phi)$ is the Lagrangian of the scalar field, X is the kinetic term of the scalar field $X \equiv -\nabla_\mu \phi \nabla^\mu \phi$, Ω is the coupling due to the higher dimensions and $\mathcal{L}_m$ is the matter field Lagrangian. For example, ψ and A could be a fermion and a gauge field respectively. This form of the action is called the *Einstein (gravity) frame*, as gravity is the Einstein-Hilbert action. Note that ϕ is minimally coupled to $g_{\mu\nu}$. However, there is a non-minimal coupling

between ϕ and matter fields. What would be the form of the action if we decide work in terms of the metric $\tilde{g}_{\mu\nu}$? After the conformal transformation Eq. (3.10) we find

$$S = \int d^4x \sqrt{-\tilde{g}} \left\{ \frac{1}{2} M_{pl}^2 \Omega^2(\phi) \tilde{R} + \tilde{\mathcal{L}}(\phi, \tilde{X}) + \mathcal{L}_m(\tilde{g}, \psi, A, \dots) \right\}. \quad (3.12)$$

This time matter fields do not couple to ϕ but we paid the price of a non-minimal coupling between the scalar field ϕ and gravity. Note that the form of $\tilde{\mathcal{L}}$ is in general different than $\mathcal{L}$. This form of the action is called *Matter (Jordan) frame*. Some notation: from now on we will refer to the metric in the Einstein frame *gravity metric* and that from the Jordan frame *matter metric*.

What is the use of the frames (3.11) and (3.12)? Well, if one is interested in studying the gravitational dynamics, it is appropriate to work in the Einstein frame (3.11). In this case, we can apply well-known knowledge of GR such as the energy theorems [154]. For instance, see how the area of a black hole in the Jordan Frame decreases for a while during collapse in Refs. [155–157]. On the other hand, if we are interested in matter dynamics we should work in the Matter frame (3.12), simply because matter follows geodesics of the matter metric.

A note is in order. It is often assumed that matter *universally* couples to gravity. This assumption is experimentally consistent for baryons. However, it could be the case that there is a non-universal coupling with matter fields, i.e. each matter field couples in a different way. Then we could for example write the matter Lagrangian as

$$\sum_A \Omega_A(\phi) \mathcal{L}_A(\phi, Q_A), \quad (3.13)$$

where $Q_A = \{\psi, A, \dots\}$. In this case, there is no clear definition of a matter (Jordan) frame and at best we may define a Jordan frame for each field Q_A .

3.1.1.1 Transformation rules

We provide a detailed derivation of the transformation rules for a metric transformation in Appendix A. However, it is appropriate to show the relevant formulas as make use of them later. We will follow Refs. [113, 158]. Under a conformal transformation given by

$$g_{\mu\nu} = \Omega^2 \tilde{g}_{\mu\nu} \quad (3.14)$$

the Ricci scalar transforms as

$$R = \Omega^{-2} \left[\tilde{R} - (D-1) \left(2 \frac{\tilde{\square} \Omega}{\Omega} - (D-4) \tilde{g}^{\mu\nu} \frac{\tilde{\nabla}_\mu \Omega \tilde{\nabla}_\nu \Omega}{\Omega^2} \right) \right], \quad (3.15)$$

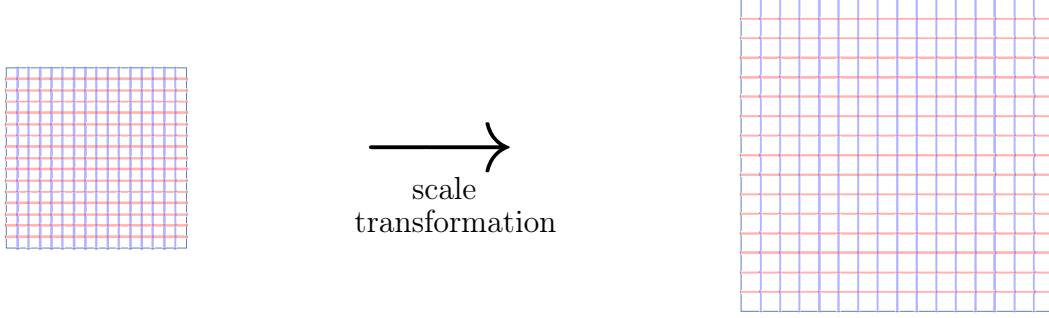


Figure 3.1: Scale transformation from $ds^2 = dx^2 + dy^2$ to $d\tilde{s}^2 = 4(dx^2 + dy^2)$.

where D is the number of dimensions. The matter fields also transform under a conformal transformation and if at short scales we neglect the dynamics of the scalar field, i.e. the scalar field is roughly constant, we find that they transform according to:

$$\begin{aligned}\chi &= \Omega^{-(D-2)/2} \tilde{\chi} && \text{for scalars,} \\ A_\mu &= \Omega^{-(D-4)/2} \tilde{A}_\mu && \text{for vectors,} \\ \psi &= \Omega^{-(D-1)/2} \tilde{\psi} && \text{for fermions.}\end{aligned}\tag{3.16}$$

For illustrative purposes let us study in detail the fermion coupled to a gauge field. Dirac fermion's action reads

$$S = \int d^4x \sqrt{-g} \left[-i\bar{\psi}\gamma^\mu D_\mu \psi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \right], \tag{3.17}$$

where m is the mass of the fermion, $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$, γ^μ are the gamma matrices, $D_\mu = \partial_\mu + ieA_\mu - \frac{1}{4}\omega_{ab\mu}\Sigma^{ab}$, $\Sigma^{ab} = \frac{1}{2}[\gamma^a, \gamma^b]$ and $\omega_{ab\mu} = e_{a\nu}\nabla_\mu e_b^\nu$. We now do the conformal transformation (3.14) to the action (3.17). We find that the factors Ω can be completely absorbed into new variables if we redefine $\gamma^\mu = \Omega^{-1}\tilde{\gamma}^\mu$, $\psi = \Omega^{-3/2}\tilde{\psi}$, $m = \Omega^{-1}\tilde{m}$ and we do not need to touch the gauge field A_μ as it is conformal invariant in 4 dimensions. Then the action becomes

$$S = \int d^4x \sqrt{-\tilde{g}} \left[-i\tilde{\psi}\tilde{\gamma}^\mu D_\mu \tilde{\psi} - \tilde{\mathbf{m}}\tilde{\psi}\tilde{\psi} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \right]. \tag{3.18}$$

The mass of the fermion is highlighted to emphasize the basic effect of a conformal transformation; on cosmological scales the mass of the fermion has a *spacetime dependence*. Likewise, one can check that the mass of scalar field becomes *spacetime dependent* in the same way.

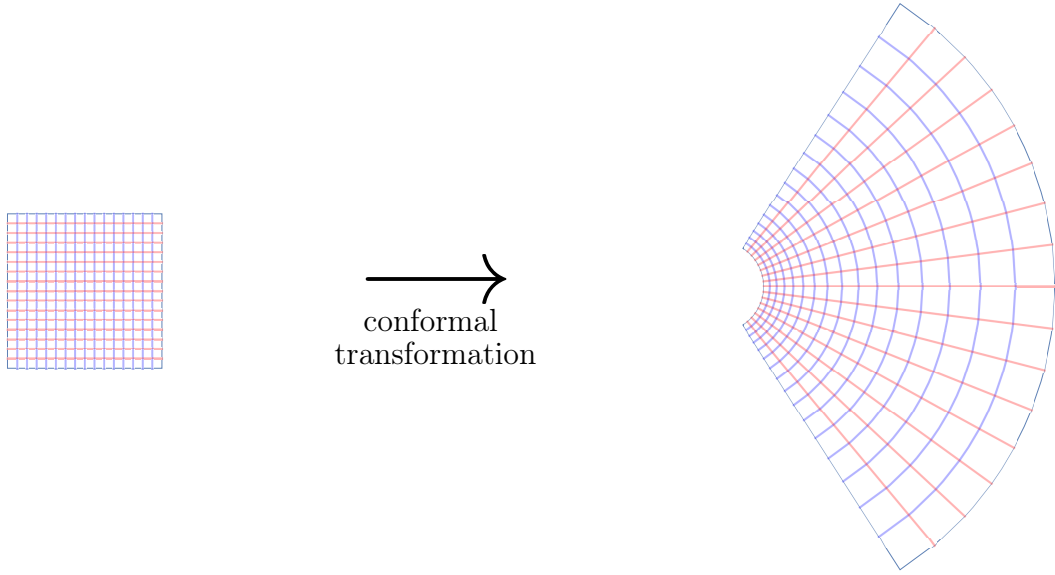


Figure 3.2: Conformal transformation from $ds^2 = dx^2 + dy^2$ to $d\tilde{s}^2 = e^{2x} ds^2$. Note how the 90° angles are kept.

3.1.1.2 Applications and interpretations

Conformal transformations are not exclusively used in cosmology. In fact, they are used in a much broader context [159]. For example, in GR they are used to discuss initial value formulation and asymptotic flatness and they are used to draw Penrose diagrams. They are also important in quantum field theory in curved spacetime [43]. For example, it is believed that conformal symmetry might be a key symmetry of the very early universe and inflation [159, 160]. On the particle physics side, they are constantly used in Conformal Field Theories (CFT) with growing interest in the ADS/CFT correspondence [161, 162]. As a fact of curiosity, they are even used in optics under the name of conformal mappings. See for an interesting application of conformal transformations to build an invisibility device in Ref. [163].

Before ending this subsection, let us see what is the meaning of a conformal transformation. Recall that a conformal transformation is given by

$$g_{\mu\nu} = \Omega^2 \tilde{g}_{\mu\nu} . \quad (3.19)$$

First we can see that a conformal transformation:

- (i) Is a global rescaling of the metric.
- (ii) Keeps the angles.

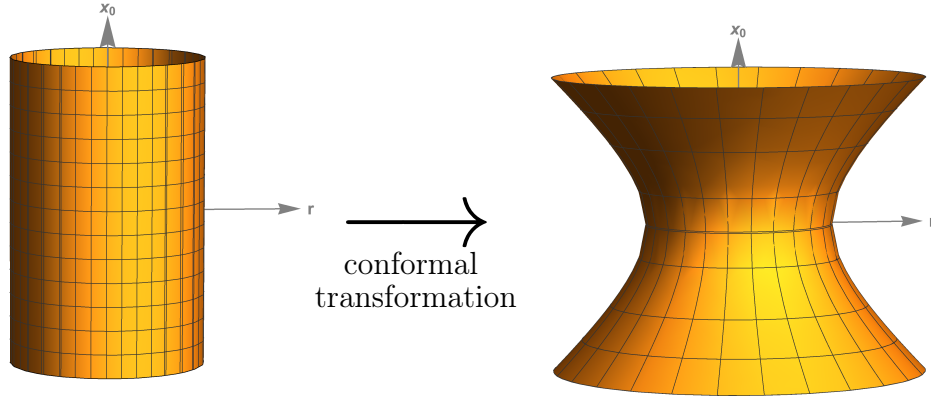


Figure 3.3: Conformal transformation from a static closed universe to a closed expanding universe. Note how the 90° angles are kept.

(iii) Does not modify the light cone, i.e. for a null vector field k_μ we have

$$g^{\mu\nu}k_\mu k_\nu = 0 \quad \Rightarrow \quad \tilde{g}^{\mu\nu}k_\mu k_\nu = 0.$$

As an illustrative example see Figs. 3.1, 3.2 and 3.3. In Fig. 3.1 we see a scale transformation of a grid which is defined by our metric, in our case a flat 2D space. A scale transformation is a constant conformal transformation. Fig. 3.2 shows a conformal transformation which depends on the exponential of one coordinate. See how the angles do not change. Actually this latter example shows us that Minkowsky space-time is connected to an expanding universe by a conformal transformation (see Fig. 3.3).

3.1.2 Disformal transformation

In the previous example, we considered that the position of our brane in the extra dimension is fixed. What would be our 4D effective metric if the brane was actually moving [164–166]? This is easily seen from the following. Take again the 5D line element as before, i.e.

$$ds_{5D}^2 = g_{\mu\nu}(x^\alpha, y)dx^\mu dx^\nu + dy^2, \quad (3.20)$$

and assume that the brane is moving in the y direction, that is to say $y(x^\alpha)$. Then, the effective line element in our brane is given by

$$ds_{4D}^2 = \left(g_{\mu\nu}(x^\alpha, y = y(x^\alpha)) + \frac{\partial y}{\partial x^\mu} \frac{\partial y}{\partial x^\nu} \right) dx^\mu dx^\nu = \bar{g}_{\mu\nu}(x^\alpha) dx^\mu dx^\nu. \quad (3.21)$$

This time, we can regard the effective metric as

$$\bar{g}_{\mu\nu}(x^\alpha) = g_{\mu\nu}(x^\alpha) + \nabla_\mu y \nabla_\nu y. \quad (3.22)$$

Again, if we consider this extra dimension as a scalar field ϕ we can write relation (3.22) in a general way given by

$$\bar{g}_{\mu\nu} = g_{\mu\nu} + B(\phi) \nabla_\mu \phi \nabla_\nu \phi. \quad (3.23)$$

This transformation is known as *disformal transformation*.² In fact, this coupling was first introduced by Bekenstein [81], although he relayed on completely different arguments.³ Let us proceed as in section 3.1.1 and consider that $\bar{g}_{\mu\nu}$ in Eq. (3.23) is the matter metric, i.e. the Einstein frame is given by

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} M_{pl}^2 R + \mathcal{L}(\phi, X) + F(\phi, X) \mathcal{L}_m(\phi, g, \psi, A, \dots) \right\}, \quad (3.24)$$

where $F(\phi, X) = \sqrt{1 - BX}$. What is the form of the matter frame? As one we may guess from the appearance of first derivatives of the scalar field in the transformation (3.23) we will end up with a higher order derivative theory. The form of the Jordan frame roughly reads

$$S = \int d^4x \sqrt{-\bar{g}} \left\{ \frac{1}{2} M_{pl}^2 \sqrt{1 + BX} \bar{R} + \frac{BM_{pl}^2}{\sqrt{1 + BX}} ((\bar{\square}\phi)^2 - \bar{\nabla}_\mu \bar{\nabla}_\nu \phi \bar{\nabla}^\mu \bar{\nabla}^\nu \phi) + \bar{\mathcal{L}}(\phi, \bar{X}) + \mathcal{L}_m(\bar{g}, \psi, A, \dots) \right\}. \quad (3.25)$$

These form of the Lagrangian falls into the recent rediscovered Horndeski theory [79, 167, 168], which is a general scalar-tensor theory with second order field equations of motion. Since it was first pointed out by Ref. [116] that disformal transformations connect Horndeski theories, they have received a lot of attention [169]. Furthermore, such transformations were useful to discover new scalar-tensor theories, which are known as beyond Horndeski or GLPV theory [170–173] (see also its further extensions [174–179]). It has been proven that these generalized higher derivative theories are closed under disformal transformations [116, 169, 176, 180, 181]. It should be noted that a disformal transformation is no longer just a simple field redefinition, because it involves derivatives of the scalar field.

²One could include a conformal transformation as well but we will keep it simple.

³Bekenstein looked for a general way in which matter could couple to a metric. He started from Finsler geometry and then he used arguments involving covariance, causal structure and the weak equivalence principle to reduce the Finslerian metric to a Riemannian disformal metric.

A note is in order. We started with an action in the Einstein frame form Eq. (3.24). We could have started for example in a more general set up inside Horndeski theories and we would have ended up with a more complicated form of the Lagrangian. The important point to note is that in higher order derivatives theories, the existence of an Einstein frame is not always true. However, one could find an Einstein frame perturbatively [116, 123, 180]. On the other hand, the matter frame is always present if there is a universal coupling.

3.1.2.1 Transformation rules

Let us show some relevant transformation rules of a disformal transformation. An extensive list of formulas can be found in Appendix A. Take a disformal transformation with a constant coefficient B , i.e.

$$g_{\mu\nu} = \bar{g}_{\mu\nu} - B \nabla_\mu \phi \nabla_\nu \phi. \quad (3.26)$$

Under this transformation the Ricci scalar transforms as

$$R = \bar{R} + \frac{B}{1 + B\bar{X}} ((\bar{\square}\phi)^2 - \bar{\nabla}_\mu \bar{\nabla}_\nu \phi \bar{\nabla}^\mu \bar{\nabla}^\nu \phi) + \frac{1}{\sqrt{-g}}(\text{total derivative}), \quad (3.27)$$

where the last term is a total derivative of $g_{\mu\nu}$ and can be dropped in Eq. (3.25). For the matter fields one finds [123]

$$\begin{aligned} \chi &= \bar{\chi} && \text{for scalars,} \\ \mathbb{A} &= \bar{\mathbb{A}} && \text{for vectors,} \\ \psi &= \bar{\psi} && \text{for fermions.} \end{aligned} \quad (3.28)$$

We will see this in more details in section 3.3. Note that conformal and disformal transformations commute as long as they are regular. See Appendix C for a study of singular metric transformations.

3.1.2.2 Applications and interpretations

In cosmology disformal transformations appear in brane world models and from massive gravity theories (see [49, 182–185] and references therein) and have been applied to inflation [186, 187], dark energy [99, 110, 188], varying speed of light models [189–192], atomic physics [193] and mimetic gravity [115]. They have also been studied in the language of differential forms [194, 195] and used in generalized vector field theories [196–198].

Interestingly, the famous Kerr-schild ansatz that was used to find the Kerr solution takes the form of a disformal transformation [199, 200]. Furthermore,

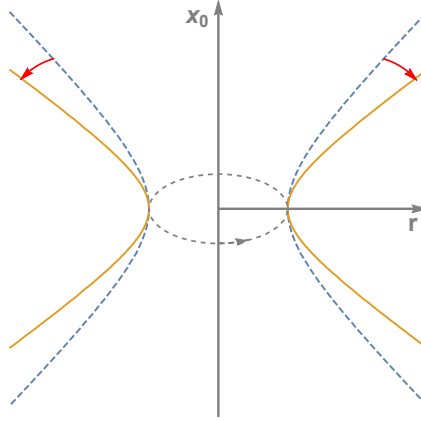


Figure 3.4: Disformal transformation of de Sitter space along the time direction. That is $ds^2 = -dt^2 + e^{-Ht}dx^2$ to $d\bar{s}^2 = -(1 + f(t))dt^2 + e^{-Ht}dx^2$. As we stretch the time direction, the de Sitter hyperboloid opens. Alternatively it can be understood as $-x_0^2 + x_1^2 + \dots = \ell^2$ to $-(x_0/\alpha)^2 + x_1^2 + \dots = \ell^2$. See Appendix I.

in K-essence theories, perturbations of the scalar field follow a metric which is disformally related to the original one [201, 202].

Before ending this subsection, let us roughly see the interpretations of a disformal transformation. Take a disformal transformation given by

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \nabla_\mu \phi \nabla_\nu \phi. \quad (3.29)$$

We can see that a disformal transformation:

- (i) Is a stretching of the metric in a particular direction, i.e. there is a *preferred frame*, given by the gradient of the scalar field. This is quite usual in cosmology where a scalar field can be used as a time coordinate.
- (ii) Keeps the same causal structure.
- (iii) If two massless fields couple to two different disformal metrics, they propagate at different speeds. For example, photons follow $\bar{g}_{\mu\nu}$ and gravitons $g_{\mu\nu}$.

For a graphical interpretation, take a look at Fig. 3.4. A disformal transformation changes the aperture of the de Sitter hyperboloid. See Appendix I for an introduction to de Sitter space.

Before ending this section, let us explain some notation. We will use $g_{\mu\nu}$ for the Einstein frame metric, $\tilde{g}_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ for a conformally and disformally related matter metrics respectively.

3.2 Conformal invariance

Conformal transformations are used in many areas of physics [203]. Regarding scalar-tensor theories, it has been proven that at a classical level they do not modify observable quantities. Many works in the literature have studied the observable invariance under conformal transformations in scalar tensor theories, see for example Refs. [70, 112–114, 204–210]. There is also some work following a different point of view within $f(R)$ theories; see Refs. [211–214]. On the other hand, whether the physical equivalence holds at the quantum level is still an open question (see Refs. [215, 216]). On that point, the final answer hopefully resides in the theory of quantum gravity.

In this section, we show an illustrative example [113, 158] of this equivalence and the difference in interpretations. For this purpose, we will consider the simplest case where matter minimally couples to an expanding background. In particular, we will focus on the redshift before and after a conformal transformation. Later we will argue that the same applies to cosmological perturbations.

Take the line element of a FLRW universe Eq. (2.7), that is

$$ds^2 = -dt^2 + a^2(t)d^2\sigma_{(K)}, \quad (3.30)$$

where $d^2\sigma_{(K)}$ is the line element of an homogeneous and isotropic 3 dimensional space, $K = \pm 1, 0$ and $a(t)$ is the scale factor. The universe evolves according to the first Friedman equation, i.e. Eq. (2.9)

$$\left(\frac{\dot{a}}{a}\right)^2 \equiv H^2 = \frac{\rho}{3M_{pl}^2} - \frac{K}{a^2}. \quad (3.31)$$

Since the universe is expanding, the wavelength of the photons is stretched and thus we observe a cosmological redshift (see Eq. (2.26)) which is given by

$$\mathcal{E}_{obs} = \frac{\mathcal{E}_{emit}}{1+z}, \quad (3.32)$$

where $\mathcal{E}_{emit}$ and $\mathcal{E}_{obs}$ respectively are the energy of the photon at the time of emission and observation. Our *interpretation* of the data is that such redshift is “proof” that the universe is expanding. However, we could give an alternative interpretation by going to a different frame.

Now, let us work in a peculiar frame by doing a conformal transformation, Eq. (3.14), with $\Omega = a$ and working with conformal time $d\eta = dt/a$. The line element of the resulting frame is given by

$$d\tilde{s}^2 = \Omega^{-2}ds^2 = -d\eta^2 + d\sigma_{(K)}^2. \quad (3.33)$$

Note that the time dependence completely disappeared and thus we ended up with a *static universe*. For obvious reasons, in a static universe photons do not redshift in time. Is this frame unphysical? Actually, we can show that we observe the same cosmological redshift and thus this frame is as physical as any other.

First, we have seen that under a conformal transformation the mass of a fermion, here we can consider an electron, gets rescaled according to Eq. (3.18), that is

$$\tilde{m} = \Omega m = \frac{m}{1+z}, \quad (3.34)$$

where we defined redshift as $1+z \equiv a^{-1}$. As a consequence of this time dependence in the electron mass, photons emitted earlier will have a different frequency than photons emitted today. Recall that the Bohr radius scales inversely proportional to the mass, that is $R_{\text{Bohr}} \propto m^{-1}$. Then, it is well known that the energy levels of the atom depend on $1/R_{\text{Bohr}}$. Therefore, the energy levels scale as $\propto m$, which upon using Eq. (3.34) we have that

$$\tilde{\mathcal{E}}_n = \frac{\mathcal{E}_n}{1+z}, \quad (3.35)$$

where $\tilde{\mathcal{E}}_n$ and $\mathcal{E}_n$ respectively are the energy levels in the static and Jordan (matter) frame. In this way, it is apparent that the frequency of photons depend on the time they were emitted. For example, a transition from the level n to the level n' is given by

$$\tilde{\mathcal{E}}_{nn'} = \frac{\mathcal{E}_{nn'}}{1+z}. \quad (3.36)$$

This exactly matches what we detect as the Hubble's law. We have seen that in these two very different frames, the observed redshift is exactly the same. How does the interpretation change? Let us go quickly through it. In the static frame, the universe has been in thermal equilibrium at a temperature $T = 2.73 \text{ K}$ since the earliest of times, for example at the Last Scattering Surface around a redshift $z > 10^3$. At this high redshift, the electron mass was 10^3 times smaller than today. After the LSS, the CMB photons have never redshifted and they kept the black body spectrum. One may wonder if the rate of scattering and interaction also follows the discussion in Chap. 2. We can check that the Thomson cross section, call it $\tilde{\sigma}_T$, is inversely proportional to the squared of the mass of the electron, i.e. $\tilde{\sigma}_T \propto m^{-2}$. Furthermore, the electron number density in the static frame, say $\tilde{n}_e$, is obviously constant. Now, if we compare the rate of scattering per unit of proper time in each frame we find that

$$\tilde{\sigma}_T \tilde{n}_e d\eta = \frac{\sigma_T n_e}{1+z} d\eta = \sigma_T n_e dt. \quad (3.37)$$

As we expected it remains unchanged! In the first equality we have used that the relation between static and matter frame quantities is respectively given by

$$\tilde{\sigma}_T = \sigma_T(1+z)^2 \quad \text{and} \quad \tilde{n}_e = \frac{n_e}{(1+z)^3}. \quad (3.38)$$

This short example showed us that observables, in our case the redshift, are frame independent. However, the way we interpret such observables could be very different in various frames.⁴ Note that this fact clearly illustrates that the metric itself should not be regarded as an observable, as we mentioned earlier in Chapter 1.

3.2.1 Cosmological perturbations

We have seen that background observables, such as the redshift, are invariant under a conformal transformation. Does the same apply to cosmological perturbations? It does and it is easy to understand why. The curvature perturbation, i.e. the responsible for the temperature fluctuations in the CMB, is the perturbation of the total expansion in the uniform- ϕ slicing (see Appendices E and F). On super-horizon scales the expansion factor is roughly given by the time derivative of the scale factor, which including perturbations is

$$a(t, x^i) = a(t) e^{\mathcal{R}_c(t, x^i)}, \quad (3.39)$$

where $\mathcal{R}_c$ is the comoving curvature perturbation. If we now do an arbitrary conformal transformation (3.14) we have

$$\tilde{a}(t, x^i) = \Omega^{-1}(\phi) a(t) e^{\mathcal{R}_c(t, x^i)}, \quad (3.40)$$

where ϕ is the inflaton field. We can also split $\tilde{a}$ into background and perturbations as

$$\tilde{a}(t, x^i) = \tilde{a}(t) e^{\tilde{\mathcal{R}}_c(t, x^i)}, \quad (3.41)$$

where $\tilde{\mathcal{R}}_c(t, x^i)$ is the curvature perturbation in the tilde frame. Recall that we are working in the uniform- ϕ slicing, i.e. $\phi = \phi(t)$ and $\delta\phi = 0$. Thus we can absorb the background value of Ω into $\tilde{a}$ and rewrite Eq. (3.40) as

$$\tilde{a}(t, x^i) = \tilde{a}(t) e^{\mathcal{R}_c(t, x^i)}. \quad (3.42)$$

In this way we conclude that $\mathcal{R}_c(t, x^i) = \tilde{\mathcal{R}}_c(t, x^i)$. Since we have proven that at the background observables are frame independent, we have that the

⁴For this reason, perhaps we should call them *representations* rather than frames [113].

curvature perturbation is also invariant under a conformal transformation. This was a rough explanation of the invariance, for more details see [114, 204, 205].

Note however that if the conformal transformation involves a dependence on the kinetic term X then the definition of curvature perturbation changes from frame to frame. For example, consider that

$$\tilde{a}(t, x^i) = \Omega^{-1}(\phi, X) a(t) e^{\mathcal{R}_c(t, x^i)}, \quad (3.43)$$

where $X = \dot{\phi}^2/N^2$ in the uniform- ϕ slicing. Assuming that perturbative expansion holds, then we can write Eq. (3.43) as

$$\tilde{a}(t, x^i) = \tilde{a}(t) e^{\mathcal{R}_c(t, x^i) - \frac{d \ln \Omega}{dX} \delta X}, \quad (3.44)$$

This means that

$$\tilde{\mathcal{R}}_c(t, x^i) = \mathcal{R}_c(t, x^i) + 2\dot{\phi}^2 \frac{d \ln \Omega}{dX} A(t, x^i), \quad (3.45)$$

where $N(t, x^i) = 1 + A(t, x^i)$. See how the definition of the conserved curvature perturbation changes with the frame! This was pointed out by Refs. [173, 217]. Thus a transformation that depends on first derivatives of the field is clearly more involved. Then, what would happen in the case of a disformal transformation?

3.3 Cosmological disformal invariance

In the previous section we have shown the physical invariance under conformal transformations. Regarding disformal transformations, one might think that the proof straightforwardly follows. However, recall that a disformal transformation is a transformation involving derivatives of a field and we have seen that the dependence on the kinetic term for a conformal transformations changes the definition of curvature perturbation. Thus, the generalization from the conformal transformation case is not straightforward. In this section, we will see our work on the physical equivalence of two disformally related frames. We will study as well the effect of a disformal transformation on the matter fields. We closely follow our article in JCAP, Ref. [123].

There have been other works in this direction, e.g. Refs. [217–222] study the disformal invariance of cosmological perturbations at the linear level with a positive answer and Ref. [221] extends it to multi-field disformal transformations. The merit of our work is the non-linear proof of the invariance.

In what follows, we are first going to understand the meaning of a disformal transformation in cosmology in Sec. 3.3.1 and we will do so on the cosmological

background. We will show the disformal invariance of observables focusing on the background dynamics, the sound speed of waves and the causal structure. Then, we will generalize this result to include perturbations in Sec. 3.3.2. First, we deal with the gravity sector in Sec. 3.3.2.1 and then we will move on to discuss the effects of a disformal transformation to the matter fields in Sec. 3.3.2.2. Lastly we will consider the whole system of gravity and matter and discuss possible features.

3.3.1 Disformal transformations in cosmology

The general form of a pure disformal transformation is given by Eq. (3.23)

$$\bar{g}_{\mu\nu} = g_{\mu\nu} + B(\phi, X)\nabla_\mu\phi\nabla_\nu\phi, \quad (3.46)$$

where we generalized the function B to include the kinetic term. As we have seen, in cosmology it is often assumed that there is a scalar field which drives the evolution of the universe. This means that on cosmological scales $\nabla_\mu\phi$ is regular and time-like everywhere. The presence of such scalar field, implies the presence of a preferred time-slicing where the scalar field is homogeneous, called uniform- ϕ slicing (see Appendix D for the transformation in a general slicing). In other words, there is a preferred frame where the time direction is given by the gradient of the scalar field. In the language of the ADM 3+1 decomposition, the homogeneous ϕ slicing is given by $n_\mu = \nabla_\mu\phi/\sqrt{X}$. In that slicing we can regard ϕ as solely a function of time, i.e. $\phi = \phi(t)$. From now on, we make use of that property to simplify interpretations and discussions without loss of generality. In the multi-field case, similar logic applies when the adiabatic limit has been reached [223].

With that in mind, one easily realizes from Eq. (3.46) that only the lapse function is affected by a pure cosmological disformal transformation, i.e. the time-time component of the metric in the uniform- ϕ slicing. We can write this down explicitly by introducing the (3+1)-decomposition for each metric, that is

$$\begin{aligned} ds^2 &= -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \\ \rightarrow d\bar{s}^2 &= -\bar{N}^2 dt^2 + \bar{h}_{ij}(dx^i + \bar{N}^i dt)(dx^j + \bar{N}^j dt), \end{aligned} \quad (3.47)$$

where bar and unbarred quantities are related by (3.46) and they are given by

$$\bar{N}^2 = N^2\alpha^2, \quad \bar{N}^i = N^i, \quad \bar{h}_{ij} = h_{ij}, \quad (3.48)$$

where for the sake of simplicity we defined

$$\alpha^2 = 1 - B(\phi, X)X. \quad (3.49)$$

As simple as it looks, it should be noted that X reduces to $\dot{\phi}^2/N^2$. Since the lapse N has a spatial dependence in general, the disformal factor B and α depend on the spatial coordinates as well. This is the reason why the proof of disformal invariance is non-trivial. From (3.48) we infer that in order to preserve the Lorentzian signature of the metric one must require $\alpha^2 > 0$ [81], which is assumed throughout this paper. Due to the fact that just a particular component of the metric is modified, one may naively think that the causal structure and the propagation speed are altered as well. In fact, it is rather the opposite as shown later.

A note is in order. There is a particular choice of B that leads to a singular metric transformation (see Ref. [115]) and the resulting model is called Mimetic gravity [224, 225]. To be more concrete we can see that if

$$B(\phi, X) = \frac{1}{X} - b(\phi), \quad (3.50)$$

where $b(\phi)$ is an arbitrary function of ϕ with $b > 0$, then we are led to a degenerate metric with no lapse function, that is

$$d\bar{s}^2 = -b(\phi)(\partial_t\phi)^2 dt^2 + \bar{h}_{ij}(dx^i + \bar{N}^i dt)(dx^j + \bar{N}^j dt). \quad (3.51)$$

This degeneracy removes the Hamiltonian constraint, which would result from the variation of the lapse, and gives rise to the “mimetic” degree of freedom which could potentially explain dark matter (see Appendix C for more details on mimetic gravity). We will not consider this singular case in what follows.

We are now going to see the effects of a disformal transformation on a cosmological background, i.e. homogeneous and isotropic, in Sec. 3.3.1.1. Then we generalize our discussion to non-linear perturbation level in Sec. 3.3.1.2.

3.3.1.1 Cosmological background

The metric for a spatially homogeneous and isotropic spacetime is given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t) d^2\sigma_{(K)} \quad (3.52)$$

where we set $N = 1$, i.e. we work with cosmic proper time, and $d^2\sigma_{(K)}$ is the metric of a homogeneous and isotropic 3-space (see Eq. (2.8)). We use Eq. (3.46) to find that the metric in the disformally related frame is

$$d\bar{s}^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu = -\alpha^2(t) dt^2 + a^2(t) d^2\sigma_{(K)}. \quad (3.53)$$

Note that α is only a function of time on the cosmological background. This means that we can relate the cosmic proper time in the barred frame, i.e.

$\bar{N} = 1$, with the cosmic proper time in the unbarred frame, i.e. $N = 1$, by a time coordinate transformation, that is given by

$$d\bar{t} = \alpha(t)dt. \quad (3.54)$$

With this new time coordinate, the barred metric is expressed simply as

$$d\bar{s}^2 = -d\bar{t}^2 + a^2(t)d^2\sigma_{(K)}. \quad (3.55)$$

A note of warning. It should not be confused with time translations of the diffeomorphism invariance. A disformal transformation leads to a completely different metric. However, the effect of a disformal transformation can be removed by a proper time redefinition $d\bar{t} = \alpha dt$, if we work with the proper time of the original frame.

What we learned from the previous calculation is that a disformal transformation at the background level is a replacement of the proper cosmic time coordinate, i.e. $t \rightarrow \bar{t}$. At the same time, the scale factor has to be understood as a function of $\bar{t}$, i.e. $\bar{a}(\bar{t}) = a(t(\bar{t}))$. As a scalar function, the value of the scale factor does not change, but its functional form could be quite different. From this, we can anticipate that the form of the action would be a replacement $S[\bar{g}; t] \rightarrow S[g; \bar{t}]$, i.e. the change in the metric $\bar{g} \rightarrow g$ is absorbed by a change in the time coordinate $t \rightarrow \bar{t}$. This might be slightly confusing and we show the invariance of the action explicitly in Sec. 3.3.2.1.

At the practical level, we can conclude that a (cosmological) disformal transformation (3.48) is equivalent to a rescaling of the time coordinate (3.54). As such, we expect that all physical observables remain unchanged under a disformal transformation. Again, interpretations from frame to frame could greatly differ. Let us show that this reasoning is valid by studying the causal structure and the propagation speed between two disformally related frames, which led to some confusion in the literature. We will explicitly see the cosmological disformal invariance. Here it is important to note that no scalar function is modified by such a passive disformal transformation. Essentially, the functional form has apparently changed but not its value. We will use this fact below.

Wave propagation. Consider first a scalar wave in the barred frame of comoving wavenumber k and propagating with sound speed $\bar{c}_s$ in the cosmological background, that is

$$\left[\frac{1}{a^3} \frac{d}{d\bar{t}} \left(a^3 \frac{d}{d\bar{t}} \right) + \frac{\bar{c}_s^2 k^2}{a^2} \right] \bar{\phi}_k(t) = 0, \quad (3.56)$$

If we apply now apply the passive disformal transformation (3.48), we obtain

$$\left[\frac{1}{a^3 \alpha} \frac{d}{N dt} \left(\frac{a^3}{\alpha} \frac{d}{N dt} \right) + \frac{\bar{c}_s^2 k^2}{a^2} \right] \bar{\phi}_k(t) = \frac{1}{\alpha^2} \left[\frac{\alpha}{a^3} \frac{d}{dt} \left(\frac{a^3}{\alpha} \frac{d}{dt} \right) + \frac{\alpha^2 \bar{c}_s^2 k^2}{a^2} \right] \bar{\phi}_k(t) = 0, \quad (3.57)$$

where in the last step we set $N = 1$ so that we chose to work in the cosmic proper time of the unbarred frame t . In this form, we understand why sometimes a disformal transformation is confused as a change in the sound speed. Looking at Eq. (3.57), we see that apparently the sound speed has been modified to $c_{s,\text{ap}} = \alpha \bar{c}_s$. The subindex in $c_{s,\text{ap}}$ stands for apparent, as we will see it is not a physical change of the sound speed.

Now, recall that the effect of a disformal transformation can be removed by a time redefinition $d\bar{t} = \alpha dt$. Thus, the real sound velocity should be read off from the equation rewritten in terms of $\bar{t}$, that is

$$\left[\frac{1}{a^3} \frac{d}{d\bar{t}} \left(a^3 \frac{d}{d\bar{t}} \right) + \frac{\bar{c}_s^2 k^2}{a^2} \right] \phi_k(\bar{t}) = 0, \quad (3.58)$$

where $\phi_k(\bar{t}) = \bar{\phi}_k(t(\bar{t}))$ or equivalently $\bar{\phi}_k(t) = \phi_k(\bar{t}(t))$. Thus, it is clear that the physical sound velocity $\bar{c}_s$ is the same in both frames. In fact, this is similar to that pointed out by Ellis and Uzan in [226, 227]. We will make use of this result again in section 3.3.3.

Causal structure. We can use a similar logic to discuss the causal structure of two disformally related frames. For that purpose, it is useful to consider a scalar quantity which will not be affected by our transformation. Consider the norm of a vector k^μ in the barred frame, i.e.

$$k^2 \equiv \bar{g}_{\mu\nu} \bar{k}^\mu \bar{k}^\nu = -\bar{N}^2 (\bar{k}^t)^2 + a^2(t) \sigma_{(K),ij} \bar{k}^i \bar{k}^j, \quad (3.59)$$

where $k^2 > 0$, $= 0$ or < 0 for a spacelike, null or time-like vector, respectively and $\sigma_{(K),ij} dx^i dx^j = d^2 \sigma_{(K)}$.

Note that, since we know that k^2 is a scalar and that $\bar{N} = \alpha N$, the effect of the disformal transformation could be balanced if the vector components $\bar{k}^t$ change appropriately, similarly as we rescaled the time coordinate. With that in mind, the passive disformal transformation (3.48) yields

$$\begin{aligned} k^2 &= \bar{g}_{\mu\nu} \bar{k}^\mu \bar{k}^\nu \\ &= (g_{\mu\nu} + B(\phi, X) \nabla_\mu \phi \nabla_\nu \phi) \bar{k}^\mu \bar{k}^\nu = -\alpha^2 (\bar{k}^t)^2 + a^2(t) \sigma_{(K),ij} \bar{k}^i \bar{k}^j, \end{aligned} \quad (3.60)$$

where again we have set $N = 1$ so that we work in the proper time of the unbarred frame t . We can now easily identify the transformation rules for the vector components. They are given by

$$\bar{k}^t = \alpha^{-1} k^t \quad \text{and} \quad \bar{k}^i = k^i, \quad (3.61)$$

and the covariant components transform according to

$$\bar{k}_t = \alpha k_t \quad \text{and} \quad \bar{k}_i = k_i. \quad (3.62)$$

It should be noted that if we look at the sound speed of the new vector, we could wrongly conclude that the sound speed has been modified, i.e.

$$c_s^{-1} \equiv \frac{dk^t}{d|k|} = \alpha \frac{d\bar{k}^t}{d|k|} = \alpha c_{s,ap}^{-1}, \quad (3.63)$$

where $c_{s,ap}$ is the apparent sound speed, as in the case of a scalar wave propagation. Once again, we can redefine the time coordinate with Eq. (3.54) and rewrite k^2 as

$$k^2 = -(\bar{k}^t)^2 + g_{ij} \bar{k}^i \bar{k}^j, \quad (3.64)$$

where $\bar{k}^t = \alpha k^t$. In the proper time coordinate of the barred frame, we see that the physical sound speed is frame independent as well, namely

$$\bar{c}_s^{-1} \equiv \frac{d\bar{k}^t}{d|k|} = c_s^{-1}. \quad (3.65)$$

3.3.1.2 Non-linear considerations

Before going into the explicit proof of the disformal invariance at non-linear level, it is worth arguing why this might be the case. The argument goes as follows. First, assume that perturbation expansion is valid, i.e. there exist a homogeneous and isotropic background solution from which we can study its perturbations. Then consider the previous discussion in Sec. 3.3.1.1 and account for a spatial dependence of the scalar field and in turn a spatial dependence of the metric and the disformal factor.

Under the assumption that there exists a homogeneous ϕ time-slicing, i.e. the normal vector of constant ϕ hyper-surfaces is assumed to be time-like everywhere, our conclusion can be generalized by introducing a local Lorentz frame. Recall that in the homogeneous ϕ slicing, the only spatial dependence in the disformal factor α comes through the spatial dependence of the Lapse function. For this reason, in the local Lorentz frame we can reabsorb the effects of a disformal transformation into a *local* rescaling of the time coordinate. In this way, we generalized the disformal invariance of the background to include the effects of non-linear perturbations.

3.3.2 Action invariance

So far we have seen that a disformal transformation in cosmology can be understood as a rescaling of time, at least at the metric level. We gave an argument why we believe this invariance holds at non-linear level. Next, we explicitly show that our expectations hold at the action level at non-linear order in perturbation expansion. Afterwards, we turn into the transformation rules for matter fields. Briefly, we proceed as follows. We start from a given action in what we called the barred frame, i.e. the action in terms of $\bar{g}_{\mu\nu}$. Then we rewrite the action in terms of the disformally related metric $g_{\mu\nu}$ and prove that both actions are equivalent upon a rescaling of the time coordinate.

3.3.2.1 Scalar-Tensor Theories

Let us focus first on the gravity sector, that is gravity plus a scalar field. For clarity, we discuss only the Einstein-Hilbert action plus a canonical scalar field. At the end of the section, we generalize our results to non-minimal couplings.

So far our discussion has been based upon the existence of the uniform- ϕ slicing and we will continue to do so otherwise noted. In other words, we set the inflaton perturbations to zero, i.e. $\delta\phi = 0$. That ensures that the spatial dependence of the transformation is due to the spatial dependence in the metric perturbations only. Since we already sliced our space-time in a preferred time-slicing, it is appropriate to work within the (3+1)-decomposition of the unbarred/barred metrics. We use the following decomposition:

$$\begin{aligned} ds^2 &= -N^2(t, x)dt^2 + h_{ij}(t, x)dx^i dx^j, \\ d\bar{s}^2 &= -\bar{N}^2(t, x)dt^2 + h_{ij}(t, x)dx^i dx^j, \end{aligned} \quad (3.66)$$

where N , $\bar{N}$ and h_{ij} are respectively the unbarred/barred lapse and the spatial metric.⁵ To simplify more the discussion, we chose a particular set of spatial coordinates where the shift vector is set to be zero, i.e. $\bar{N}^i = N^i = 0$. One may include a non-vanishing shift without difficulties.

We proceed by expanding the action in the ADM decomposition, i.e.

$$\begin{aligned} S_g = \frac{M_{pl}^2}{2} \int d^3x dt \bar{N} \sqrt{h} \Big\{ & R^{(3)} + \bar{K}_{ij} \bar{K}^{ij} - \bar{K}^2 + M_{pl}^{-2} \bar{N}^{-2} (\partial_t \phi)^2 \\ & - 2M_{pl}^{-2} V(\phi) + 2\bar{\nabla}_\mu (\bar{n}^\mu \bar{\nabla}_\nu \bar{n}^\nu - \bar{n}^\nu \bar{\nabla}_\nu \bar{n}^\mu) \Big\}, \end{aligned} \quad (3.67)$$

⁵We are using the fact that a disformal transformation does not affect the spatial components. We use h_{ij} not to overload notation.

where ${}^{(3)}R$, $\bar{K}_{ij}$ and $\bar{n}_\mu dx^\mu = -\bar{N}dt$ respectively are the spatial Ricci scalar, the extrinsic curvature and the normal vector of the spatial hyper-surface. Since we want to see the explicit dependence of the action with respect to $\bar{N}$ and h_{ij} , we show the values of the extrinsic curvature and the gradient of the normal vector from Eq. (3.66), that is

$$K_{ij} = \frac{1}{2\bar{N}} \partial_t h_{ij}, \quad \bar{K} = h^{ij} \bar{K}_{ij}, \quad (3.68)$$

and

$$\bar{\nabla}_\mu \bar{n}_\nu = \delta_\mu^0 \delta_\nu^i \bar{N}_{,i} + \delta_\mu^i \delta_\nu^j \bar{K}_{ij}. \quad (3.69)$$

Thus, we find that the action is given by

$$S_g = \frac{M_{pl}^2}{2} \int d^3x dt \bar{N} \sqrt{h} \left\{ R^{(3)} + \bar{K}_{ij} \bar{K}^{ij} - \bar{K}^2 + M_{pl}^{-2} \bar{N}^{-2} (\partial_t \phi)^2 \right. \\ \left. - 2M_{pl}^{-2} V(\phi) + \frac{2}{\sqrt{h} \bar{N}} \partial_t (\sqrt{h} \bar{K}) - \frac{2}{\sqrt{h} \bar{N}} \partial_i (\sqrt{h} h^{ij} \partial_j \bar{N}) \right\}. \quad (3.70)$$

Note that the last two terms are total derivatives. We keep them as they will be necessary for the generalization to non-minimal couplings.

We do a disformal transformation to the action (3.70), i.e. we write the barred quantities in terms of the unbarred ones, and we find that

$$S_g = \frac{M_{pl}^2}{2} \int d^3x dt \alpha N \sqrt{h} \left\{ R^{(3)} + \alpha^{-2} (K_{ij} K^{ij} - K^2 + M_{pl}^{-2} N^{-2} (\partial_t \phi)^2) \right. \\ \left. - 2M_{pl}^{-2} V(\phi) + \frac{2}{\sqrt{h} \alpha N} \partial_t (\sqrt{h} \alpha^{-1} K) - \frac{2}{\sqrt{h} \alpha N} \partial_i (\sqrt{h} h^{ij} \partial_j (\alpha N)) \right\}. \quad (3.71)$$

We remind the reader that $\bar{N}(t, x^i) = \alpha(\phi, X) N(t, x^i)$ from the transformation rules, Eq. (3.48). We also defined the extrinsic curvature in the unbarred frame as $K_{ij} = \frac{1}{2N} \partial_t h_{ij}$.

So far we have not used any perturbation expansion. Recall however that we showed in Sec. 3.3.1.1 that the disformal transformation can be absorbed in a time coordinate redefinition at the background. Therefore, we assume that perturbation expansion is valid and we split the lapse and disformal factor into background and perturbation as follows:

$$N(t, x^i) = e^{n(t, x^i)} \quad \text{and} \quad \alpha(t, x^i) = \alpha_0(t) e^{s(t, x^i)}, \quad (3.72)$$

where $\alpha_0(t)$ is the value of the disformal factor at the background. It should be noted that we cannot straightforwardly define the time coordinate since the disformal factor depends on the spatial coordinates through the lapse, i.e.

$$\alpha(\phi, X) = \alpha(t, N) = \alpha(t, x^i). \quad (3.73)$$

A similar expansion for the barred Lapse function yields

$$\bar{N}(t, x^i) = \alpha_0(t) e^{n(t, x^i) + s(t, x^i)} \equiv \alpha_0(t) e^{\bar{n}(t, x^i)}, \quad (3.74)$$

where in the last step we defined $\bar{n}(t, x^i) \equiv n(t, x^i) + s(t, x^i)$. If we look at Eq. (3.71) we can readily see that, under this perturbative expansion, we can absorb the background value of the disformal transformation into a time redefinition, that is given by

$$d\bar{t} = \alpha_0(t) dt. \quad (3.75)$$

This result implies that physics is invariant *at the background level* as long as perturbation theory is valid. Whether the same applies for perturbations is not straightforward. We dedicate next paragraph to study this case.

Implications for cosmological perturbations. Let us drop the total derivative terms in (3.71) for the moment and perform the time redefinition Eq. (3.75). We will recover the total derivatives at the end of this subsection. In this way, we find that the action including perturbations is given by

$$S_g = \frac{M_{pl}^2}{2} \int d^3x d\bar{t} \sqrt{h} e^{n(t, x^i) + s(t, x^i)} \left\{ R^{(3)} - 2M_{pl}^{-2} V(\phi) \right. \\ \left. + e^{-2[n(t, x^i) + s(t, x^i)]} \left(E_{ij} E^{ij} - E^2 + M_{pl}^{-2} (\partial_{\bar{t}} \phi)^2 \right) \right\}, \quad (3.76)$$

where we factored out the lapse from the extrinsic curvature and we defined $E_{ij} \equiv \frac{1}{2} \partial_{\bar{t}} h_{ij}$ and $E = h^{ij} E_{ij}$, following the notation of Maldacena [228]. Recall that as we did in section 3.3.1.1 one should understand $\phi(t) = \phi(t(\bar{t}))$ and $h_{ij}(t, x^i) = h_{ij}(t(\bar{t}), x^i)$. Note that in this way the action take the same form as the barred action (3.70) with $\bar{N} = 1$ at the background, except that in the present case the perturbations of the Lapse depend on t and not $\bar{t}$. In addition, we have the perturbations of the disformal factor as well. Nevertheless, we can find a solution to the lapse function by looking at the Hamiltonian constraint, i.e. the variation with respect to the lapse (see Appendix H).

It is known that in the Hamiltonian formalism, the Lapse function acts as a Lagrange multiplier, its variation yielding the so-called Hamiltonian constraint. Thus, we can redefine the Lapse $n(t, x^i)$ to absorb the disformal factor $s(t, x^i)$ and the dynamics of the system remain unchanged. This new lapse is what we previously defined as

$$\bar{n}(t, x^i) \equiv n(t, x^i) + s(t, x^i). \quad (3.77)$$

We can now formally solve the new lapse $\bar{n}$ in terms of the time $\bar{t}$. The variation with respect to the new lapse yields

$$R^{(3)} - e^{-2\bar{n}(t, x^i)} \left[E_{ij} E^{ij} - E^2 + M_{pl}^{-2} (\partial_{\bar{t}} \phi)^2 \right] - 2M_{pl}^{-2} V(\phi) = 0. \quad (3.78)$$

Note that it has the same solution as the barred frame (3.70) action with $\bar{N} = 1$ at the background level. The crucial difference is that the lapse depends on t while the time derivatives are with respect to $\bar{t}$. This leads us to conclude that the perturbed barred lapse is equal to the unbarred one with the replacement $t \rightarrow \bar{t}$, that is

$$\bar{n}(t, x^i) = n(\bar{t}, x^i). \quad (3.79)$$

Therefore, the action in the unbarred frame (3.76) is exactly the same as the action in the barred frame (3.67) but in terms of the redefined time $\bar{t}$. Thus, the cosmological disformal invariance holds for perturbations at full non-linear order as well.

Before finishing this subsection, let us briefly consider the generalization to include non-minimal couplings. The essential point is that from the disformal transformation, i.e. $\bar{N} = \alpha N$, the disformal factor α only appears next to the lapse N . Therefore, the background value of the disformal factor $\alpha_0(t)$ can always be removed by a rescaling of the time coordinate. At the perturbation level, the same logic as before applies, i.e. the Hamiltonian constraint tells us that $\bar{n}(t, x^i) = n(\bar{t}, x^i)$ and the time coordinate rescaling applies to perturbations as well. Let us show this with an example. Take the total derivative terms we dropped before from Eq. (3.71). If there is a non-minimal coupling present, the previously total derivative term will contribute as

$$S_g \supset \int d^3x dt M_{pl}^2 \alpha N \sqrt{h} G(\phi, X) \frac{1}{\sqrt{h} \alpha N} \partial_t (\sqrt{h} \alpha^{-1} K), \quad (3.80)$$

where $G(\phi, X)$ is the non-minimal coupling. We proceed as before and absorb the background value $\alpha_0(t)$ in a new time coordinate. After integrating by parts we find

$$S_g \supset - \int d^3x d\bar{t} M_{pl}^2 \sqrt{h} e^{-\bar{n}(t, x^i)} E \partial_{\bar{t}} G, \quad (3.81)$$

where the factor X inside G changes as well according to

$$X = \frac{1}{2\alpha^2 N^2} (\partial_t \phi)^2 = \frac{e^{-2\bar{n}(t, x^i)}}{2} (\partial_{\bar{t}} \phi)^2. \quad (3.82)$$

Once again, this action coincides with Eq. (3.67) in the presence of non-minimal coupling except that $\bar{n}(t, x^i)$ depends on t and not $\bar{t}$. Since all

the other terms take exactly the same form, we conclude that the solution $\bar{n}(t, x^i) = n(\bar{t}, x^i)$ also holds in the presence of non-minimal couplings.

To conclude this section, a disformal transformation in a cosmological space-time, in the homogeneous- ϕ slicing is equivalent to a rescaling of the time coordinate both at the background and perturbation level. This interpretation is valid even for higher orders in perturbative expansion. Thus, cosmological perturbations are invariant under a cosmological disformal transformation, as long as perturbation theory applies.

3.3.2.2 Matter Fields

We saw that for the gravitational action a disformal transformation can be understood as a rescaling of the time coordinate. For matter fields we expect that the same result applies. The question we address in this subsection is how are the matter fields affected by a disformal transformation. For example, in the case of a conformal transformation the matter fields rescale (see Eq. (3.16)). We deal with the scalar, vector and fermion field actions. We will decompose the action in 3 + 1-decomposition, given in Eq. (3.66), and then do a disformal transformation. We only deal at the background level as we discussed the generalization to include perturbations in Sec. 3.3.1.2.

Scalar field. Consider a minimally coupled scalar field χ with mass m , i.e.

$$S = -\frac{1}{2} \int d^4x \sqrt{-\bar{g}} \left(\bar{g}^{\mu\nu} \partial_\mu \chi \partial_\nu \chi + m^2 \chi^2 \right), \quad (3.83)$$

Let us explicitly expand this action in terms of the metric Eq. (3.66), yielding

$$S = \frac{1}{2} \int d^3x \, dt \, \bar{N} \sqrt{\bar{h}} \left\{ \bar{N}^{-2} (\partial_t \chi)^2 - \bar{h}^{ij} \partial_i \chi \partial_j \chi - m^2 \chi^2 \right\}. \quad (3.84)$$

We now do a cosmological disformal transformation by rescaling the lapse $\bar{N}(t) = \alpha(t) N(t)$, recall the transformation rules (3.48), which leads us to

$$S = \frac{1}{2} \int d^3x \, dt \, \alpha N \sqrt{h} \left\{ N^{-2} \alpha^{-2} (\partial_t \chi)^2 - h^{ij} \partial_i \chi \partial_j \chi - m^2 \chi^2 \right\}. \quad (3.85)$$

Note that the scalar field action in the present form (3.85) may lead to the conclusion that the sound speed has been modified, in a similar fashion as in Sec. 3.3.1.1. Nevertheless, we already know that is an artifact which can be removed by a time coordinate redefinition given by Eq. (3.54). In terms of the new time $\bar{t}$ the action reads

$$S = \frac{1}{2} \int d^3x \, d\bar{t} \, N \sqrt{h} \left\{ N^{-2} (\partial_{\bar{t}} \chi)^2 - h^{ij} \partial_i \chi \partial_j \chi - m^2 \chi^2 \right\}. \quad (3.86)$$

The action (3.86) looks as if we started in the unbarred frame and replaced $t \rightarrow \bar{t}$. Alternatively, if we call a solution of the scalar field in the barred frame $\bar{\chi}(t)$ the corresponding solution in the unbarred frame is $\chi(\bar{t}) = \bar{\chi}(t)$.

Electromagnetic field. Let us focus on the electromagnetic field, since photons are the ones that carry the information of the CMB. The electromagnetic field in 4D is known to be conformal invariant [229]. However, in general, it is not disformal invariant.

Take the action for the electromagnetic field A_μ in the barred frame, which is given by

$$S = -\frac{1}{4} \int d^4x \sqrt{-\bar{g}} \bar{g}^{\alpha\mu} \bar{g}^{\beta\nu} F_{\alpha\beta} F_{\mu\nu}, \quad (3.87)$$

where $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$. In the 3+1-decomposition (3.66) the action reads

$$S = \frac{1}{4} \int d^3x dt \bar{N} \sqrt{\bar{h}} \left(2\bar{N}^{-2} \bar{h}^{ij} F_{ti} F_{tj} - \bar{h}^{ij} \bar{h}^{kl} F_{ik} F_{jl} \right). \quad (3.88)$$

The disformal transformation Eq. (3.48) leads us to the unbarred frame action, i.e.

$$S = \frac{1}{4} \int d^3x dt \alpha N \sqrt{h} \left(2\alpha^{-2} N^{-2} h^{ij} F_{ti} F_{tj} - h^{ij} h^{kl} F_{ik} F_{jl} \right). \quad (3.89)$$

We readily see that the second term in the right hand side is unaffected. However, the first term, i.e. $F_{ti} = \partial_t A_i - \partial_i A_t$, contains a factor α . We already saw in Sec. 3.3.1.1 that the temporal component of a vector field is redefined as

$$A_{\bar{t}} = \alpha^{-1} A_t, \quad (3.90)$$

under the time coordinate redefinition Eq. (3.54). Thus once we absorb the disformal factor in the time coordinate and accordingly change the time component of the vector, we end up with

$$S = \frac{1}{4} \int d^3x d\bar{t} N \sqrt{h} \left(2N^{-2} h^{ij} F_{\bar{t}i} F_{\bar{t}j} - h^{ij} h^{kl} F_{ik} F_{jl} \right). \quad (3.91)$$

Note that we assumed that $\alpha(t)$ is homogeneous and isotropic and therefore it commutes with spatial derivatives, i.e. $\partial_i A_{\bar{t}} = \alpha^{-1} \partial_i A_t$. This action would be the action in the unbarred metric but for a rescaling of the time coordinate $t \rightarrow \bar{t}$.

In simpler fashion, the same conclusion can be drawn within the differential form approach. Take a 1-form and redefine the time coordinate. The change in the time component is given by

$$\begin{aligned}\mathbb{A} &= A_\mu dx^\mu = A_t dt + A_i dx^i \\ &= A_t \alpha^{-1} d\bar{t} + A_i dx^i = A_{\bar{t}} d\bar{t} + A_i dx^i.\end{aligned}\quad (3.92)$$

This shows us that in terms of differential forms, the 1-form $\mathbb{A}$ is invariant under cosmological disformal transformations.

Dirac field. To end this section, let us consider a fermion field, e.g. an electron. A fermion field Ψ with mass m and charge e in curved spacetime is described by [43]

$$S = -\frac{i}{2} \int d^4x \sqrt{-\bar{g}} \left(\bar{\Psi} \bar{\gamma}^\mu \bar{D}_\mu \Psi + m \bar{\Psi} \Psi \right), \quad (3.93)$$

where $\bar{\Psi} = \Psi^\dagger \gamma^0$ is the adjoint spinor, $\bar{D}_\mu = \partial_\mu + ieA_\mu + \bar{\Gamma}_\mu$, $\bar{\gamma}^\mu$ are the gamma matrices in curved spacetime and $\bar{\Gamma}_\mu$ is the spin connection. The gamma matrices and the spin connection are given in terms of the tetrad components $\bar{e}_{(a)}^\mu$, which are defined by

$$\eta_{ab} = \bar{e}_{(a)}^\mu \bar{e}_{(b)}^\nu \bar{g}_{\mu\nu}. \quad (3.94)$$

In this way, we have $\bar{\gamma}^\mu = \gamma^a \bar{e}_{(a)}^\mu$ and

$$\bar{\Gamma}_\mu = \frac{1}{8} [\gamma^a, \gamma^b] \bar{e}_{(a)}^\lambda \bar{\nabla}_\mu \bar{e}_{(b)\lambda}. \quad (3.95)$$

Let us expand the action in the (3+1)-decomposition, that is

$$S = -\frac{i}{2} \int d^3x dt \bar{N} \sqrt{\bar{h}} \left(\bar{\Psi} \gamma^0 \bar{N}^{-1} \bar{D}_t \Psi + \bar{\Psi} \bar{\gamma}^i \bar{D}_i \Psi + m \bar{\Psi} \Psi \right). \quad (3.96)$$

We find the tetrad components to be

$$\bar{e}_{(a)}^t = \delta_{(a)}^0 \bar{N}^{-1} \quad \text{and} \quad \bar{e}_{(a)}^i \bar{e}_{(b)}^j = \delta_{ab} \bar{h}^{ij}. \quad (3.97)$$

In particular, note that $\bar{\gamma}^t = \gamma^0 \bar{N}^{-1}$.

Now let us see the effects of a disformal transformation Eq. (3.48). First, the barred and unbarred tetrads are respectively related by

$$\bar{e}_{(a)}^t = e_{(a)}^t \alpha^{-1} \quad \text{and} \quad \bar{e}_{(a)}^i = e_{(a)}^i, \quad (3.98)$$

which in turn yields a similar transformation for the gamma matrices,

$$\bar{\gamma}^t = \gamma^t \alpha^{-1}, \quad \bar{\gamma}^i = \gamma^i. \quad (3.99)$$

Regarding the transformation of the spin connection $\bar{\Gamma}_\mu$, we have a contribution from the change in the tetrad as well as in the covariant derivative. The change in the latter is given by the change in the Christoffel symbols, which are defined by $\Gamma_{\mu\nu}^\lambda = \frac{1}{2} \bar{g}^{\lambda\sigma} (\partial_\mu \bar{g}_{\sigma\nu} + \partial_\nu \bar{g}_{\mu\sigma} - \partial_\sigma \bar{g}_{\mu\nu})$. Let us start by explicitly showing the non-vanishing Christoffel symbols for the metric (3.53). These are given by

$$\begin{aligned} \bar{\Gamma}_{tt}^t &= \frac{1}{2} \bar{N}^{-2} \partial_t \bar{N}^2, & \bar{\Gamma}_{tt}^i &= \frac{1}{2} \bar{h}^{ij} \partial_j \bar{N}^2, \\ \bar{\Gamma}_{it}^t &= \frac{1}{2} \bar{N}^{-2} \partial_i \bar{N}^2, & \bar{\Gamma}_{ij}^t &= \frac{-1}{2} \bar{N}^{-2} \partial_t \bar{h}_{ij}, \\ \bar{\Gamma}_{jt}^i &= \frac{1}{2} \bar{h}^{ik} \partial_t \bar{h}_{kj}, \end{aligned} \quad (3.100)$$

where the pure spatial component $\bar{\Gamma}_{jk}^i$ is not affected by a disformal transformation. We can now see how the spin connection transforms under a disformal transformation. First, take a look the time component $\bar{\Gamma}_t$, i.e.

$$\bar{\Gamma}_t = \frac{1}{8} [\gamma^a, \gamma^b] \bar{e}_{(a)}^\lambda \bar{\nabla}_t \bar{e}_{(b)\lambda}. \quad (3.101)$$

The component affected by a disformal transformation is the tetrad and the covariant derivative, which after expanding them we are led to

$$\bar{e}_{(a)}^\lambda \bar{\nabla}_t \bar{e}_{(b)\lambda} = \bar{e}_{(a)}^t \partial_t \bar{e}_{(b)t} + \bar{e}_{(a)}^i \partial_t \bar{e}_{(b)i} - \bar{e}_{(a)}^t \bar{\Gamma}_{tt}^t \bar{e}_{(b)t} - \bar{e}_{(a)}^i \bar{\Gamma}_{ti}^j \bar{e}_{(b)j} - \bar{e}_{(a)}^t \bar{\Gamma}_{tt}^i \bar{e}_{(b)i} - \bar{e}_{(a)}^i \bar{\Gamma}_{ti}^t \bar{e}_{(b)t}, \quad (3.102)$$

If we take a close look at equation (3.102) we note that the terms with a time derivative end up having no α dependence, these are the first four terms in the right hand side. The time derivatives of α are canceled among each other. On the contrary, the terms with a spatial derivative yield a common factor α .

Second, the spatial component $\bar{\Gamma}_j$ is given by

$$\bar{\Gamma}_j = \frac{1}{8} [\gamma^a, \gamma^b] \bar{e}_{(a)}^\lambda \bar{\nabla}_j \bar{e}_{(b)\lambda}. \quad (3.103)$$

Again, the only part affected by a disformal transformation is the tetrad and the covariant derivative, that is

$$\bar{e}_{(a)}^\lambda \bar{\nabla}_j \bar{e}_{(b)\lambda} = \bar{e}_{(a)}^t \partial_j \bar{e}_{(b)t} + \bar{e}_{(a)}^i \partial_j \bar{e}_{(b)i} - \bar{e}_{(a)}^t \bar{\Gamma}_{jt}^t \bar{e}_{(b)t} - \bar{e}_{(a)}^i \bar{\Gamma}_{ji}^k \bar{e}_{(b)k} - \bar{e}_{(a)}^i \bar{\Gamma}_{ij}^t \bar{e}_{(b)t} - \bar{e}_{(a)}^t \bar{\Gamma}_{jt}^i \bar{e}_{(b)i}. \quad (3.104)$$

This time the terms containing spatial derivatives yield no α dependence, these are the first four terms in the right hand side of Eq. (3.104). On the

other hand, the time derivative terms have a common factor α^{-1} . We can summarize the transformation rules for the spin connection as follows:

$$\bar{\Gamma}_t[\partial_t, \partial_i] = \Gamma_t[\partial_t, \alpha\partial_i] = \alpha\Gamma_t[\alpha^{-1}\partial_t, \partial_i] \quad \text{and} \quad \bar{\Gamma}_i[\partial_t, \partial_i] = \Gamma_i[\alpha^{-1}\partial_t, \partial_i], \quad (3.105)$$

where $\Gamma_\mu = \frac{1}{8}[\gamma^a, \gamma^b] e_{(a)}^\lambda \nabla_\mu e_{(b)\lambda}$ is the spin connection in the unbarred frame.

Consider now the total action. The term $\bar{\Gamma}^0 D_t$ after the transformation gives rise to $\Gamma^0 \alpha^{-1} D_t$. In particular the factor α^{-1} balances the factor α coming from Γ_t . In this way, every and only time derivatives are accompanied by a factor α^{-1} in front. As we may have expected, we can thus absorb the disformal factor into the time redefinition Eq. (3.54). As a result, we find that the disformal transformation of the Dirac action is given by

$$S = -\frac{i}{2} \int d^3x d\bar{t} N \sqrt{h} \left(\bar{\Psi} \gamma^0 N^{-1} D_{\bar{t}} \Psi + \bar{\Psi} \gamma^i D_i \Psi + m \bar{\Psi} \Psi \right), \quad (3.106)$$

where $D_{\bar{t}} = \partial_{\bar{t}} + ieA_{\bar{t}} + \Gamma_{\bar{t}}$, $\Gamma_{\bar{t}} \equiv \Gamma_t[\partial_{\bar{t}}, \partial_i]$ and we have used the previous result for the electromagnetic field (3.90). A disformal transformation is completely a rescaling of the time coordinate, thus physics is invariant under such time redefinition. Let us end this section by noting that neither the charge e nor the mass m are modified by a disformal transformation.

3.3.3 Frame Independence

What we have learned is that every field is disformal invariant separately. The situation is more interesting if there are two fields coupled to two disformally related metrics. Does our results still apply?

For this reason, let us treat the whole system, gravity and matter. First, we *universally couple* matter to $\bar{g}_{\mu\nu}$. In this way, the weak equivalence principle is satisfied. We could consider different coupling for different matter fields but it would not change the conclusion. For instance, it is known that if radiation and matter couple to different disformal metrics we could see its effects in the CMB [92, 109]. Regarding the gravity sector, let us consider for simplicity the Einstein-Hilbert action plus a scalar field with a metric $g_{\mu\nu}$. Recall that if the total action is written in terms of $g_{\mu\nu}$, we refer to it as gravity frame. On the other hand, if the total action is written in terms of $\bar{g}$ then we call it the matter frame. In this section, we first show that the frame independence holds in such a case and finally we discuss features of having two different metrics.

3.3.3.1 Gravity plus matter

In the presence of two conformally/disformally related metrics between gravity and matter the action is usually cast as [84, 92, 101, 103, 109, 190–192]

$$S = \int d^4x \left\{ \sqrt{-g} \left(\frac{M_{pl}^2}{2} R[g] + \mathcal{L}_\phi(g, \phi) \right) + \sqrt{-f} \mathcal{L}_m(f, \psi_I) \right\}, \quad (3.107)$$

where as we said we chose the Einstein-Hilbert action for simplicity, but in principle one could chose non-minimal couplings as well. The scalar field ϕ is the dominant field in the universe and thus we regard it as part of the gravity action as well. We include matter fields ψ_I minimally coupled to another metric f . The metric of the matter frame is related to that of the gravity frame by

$$f_{\mu\nu} = g_{\mu\nu} + C(\phi, X) \nabla_\mu \phi \nabla_\nu \phi. \quad (3.108)$$

As we did in previous sections, we take the uniform- ϕ slicing as regard the scalar field ϕ as a function of time only. Under this assumption, the two metrics g and f differ only in the lapse function, i.e.

$$N_f^2 = N_g^2 \left(1 - XC(\phi, X) \right) \equiv N_g^2 \beta^2(\phi, X). \quad (3.109)$$

We will now proceed as follows. We first do a disformal transformation of the metric g to a metric $\bar{g}$. We absorb the background change in a redefinition of the time coordinate and the perturbations into a redefinition of the lapse, see Eq. (3.72). Then we study what are the effects on the matter sector. Let us take the relation between g and $\bar{g}$ to be

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + B(\phi, \bar{X}) \nabla_\mu \phi \nabla_\nu \phi. \quad (3.110)$$

Thus, the two lapse function as related by

$$N_g^2 = \bar{N}_g^2 (1 - XB(\phi, \bar{X})) \equiv \bar{N}_g^2 \alpha^2(\phi, \bar{X}), \quad (3.111)$$

where $\bar{X} = (\partial_t \phi)^2 / \bar{N}_g^2$. The background value of α can be absorbed by a time coordinate rescaling given by $d\bar{t} = \alpha_0(t) dt$ and perturbations go into a new lapse, that is

$$N_g dt = \bar{N}_g \alpha dt = e^{\bar{n}_g} \alpha_0(t) dt = e^{\bar{n}_g} d\bar{t}, \quad (3.112)$$

where we used Eq. (3.72) to define $e^{\bar{n}_g} \equiv \bar{N}_g e^{s(t, x^i)}$.

At this point we know that the gravity action is disformal invariant up to a rescaling of time. Does the same rescaling apply to the matter sector? In order to see that we express the matter sector in terms of N_g , i.e.

$$S = \int d^3x dt N_g \sqrt{h} \left\{ \frac{M_{pl}^2}{2} R[g] + \mathcal{L}_\phi(g, \phi) + \beta(\phi, X) \mathcal{L}_m(h, (\beta N_g)^{-1} \partial_t \psi_I, \dots) \right\}. \quad (3.113)$$

We make use of Eq. (3.112) to write the action in terms of the barred time $\bar{t}$, which yields

$$S = \int d^3x d\bar{t} \bar{N}_g \sqrt{\bar{h}} \left\{ \frac{M_{pl}^2}{2} R[\bar{g}] + \mathcal{L}_\phi(\bar{g}, \phi) + \beta(\phi, X_{\bar{g}}) \mathcal{L}_m(h, (\beta \bar{N}_g)^{-1} \partial_{\bar{t}} \psi_I, \dots) \right\}, \quad (3.114)$$

where we defined

$$X_{\bar{g}} \equiv \frac{1}{\bar{N}_g^2} \left(\frac{\partial \phi}{\partial \bar{t}} \right)^2 = X. \quad (3.115)$$

Note that the matter sector in terms of $\bar{t}$ includes a non-minimal coupling with the scalar field. Nevertheless, this non-minimal coupling can be removed if we redefine the metric $\bar{f}$ by

$$\bar{f}_{\mu\nu} = \bar{g}_{\mu\nu} + C(\phi, X_{\bar{g}}) \nabla_\mu \phi \nabla_\nu \phi, \quad (3.116)$$

or alternatively $\bar{N}_f \equiv \bar{N}_g \beta(\phi, X_{\bar{g}})$. With this new metric $\bar{f}$ the total action reads

$$S = \int d^3x d\bar{t} \left\{ \sqrt{-\bar{g}} \left(\frac{M_{pl}^2}{2} R[\bar{g}] + \mathcal{L}_\phi(\bar{g}, \phi) \right) + \sqrt{-\bar{f}} \mathcal{L}_m(\bar{f}, \psi_I) \right\}. \quad (3.117)$$

We readily see that the action looks exactly as the action (3.107) but for the replacement $t \rightarrow \bar{t}$. Furthermore, the relation between the gravity and matter metrics Eq. (3.108) has not been modified according to Eq. (3.116). Therefore, the total system of gravity and matter is unchanged up to a time rescaling, which implies that that physics is disformal invariant.

3.3.3.2 Propagation speed of gravitons and photons

In the previous sections we have seen that the physical observables of a given theory, including gravity and matter, are invariant under a general metric transformation. This fact does not mean that one could not obtain observable features due to, for example, matter being disformally coupled. As we will extensively discuss in Chap. 4, the form of the coupling to matter could be

very relevant. We have advanced in Sec. 3.1.2 that gravitons may propagate at a speed faster or slower than the photons,⁶ as it was pointed out by Bekenstein [81]. This will be the case if matter minimally couples to a metric which is disformally related to the gravity metric.⁷ Then, such relative difference in the propagation speeds exists in any frame.

We can clarify the previous statement with a particular model. Consider an action given by

$$S = \int d^4x \left\{ \sqrt{-g} \left(\frac{M_{pl}^2}{2} R[g] + \mathcal{L}(g, \phi) \right) + \sqrt{-\bar{g}} \mathcal{L}_m(\bar{g}, \psi_I) \right\}, \quad (3.118)$$

where $\bar{g}_{\mu\nu}$ is disformally related to $g_{\mu\nu}$. We used the notation $\bar{g}$ so that it is consistent with Sec. 3.3.2. In this action, photons and gravitons propagate at a different speed. As this point may not be completely clear from Eq.(3.118) let us rewrite it only in terms of the matter metric, that is

$$S = \int d^4x \left\{ \sqrt{-g[\bar{g}]} \left(\frac{M_{pl}^2}{2} R[g[\bar{g}]] + \mathcal{L}(g[\bar{g}], \phi) \right) + \sqrt{-\bar{g}} \mathcal{L}_m(\bar{g}, \psi_I) \right\}. \quad (3.119)$$

Now, we can make use of the transformation rules⁸ for a disformal transformation given by Eq. (3.70) we saw in Sec. 3.1.2. In this way, we are led to a Horndeski action which we can roughly write as

$$S = \int d^4x \sqrt{-\bar{g}} \left\{ \mathcal{L}_{\text{Horndeski}}(\bar{g}, \bar{K}, \bar{R}, \phi, \bar{X}) + \mathcal{L}_m(\bar{g}, \psi_I) \right\}. \quad (3.120)$$

We can see that in the matter frame the interpretation has changed. The gravitons propagate at a speed different than the speed of light because of a *modification of gravity*, not due to the disformal coupling with matter. One can check that, in this Horndeski Lagrangian, the propagation speed of tensor modes will be given by $c_T = \alpha^{-1}$. This result can be alternatively derived from the relative factor α^2 in the disformal transformation. Note that using the results of Sec. 3.3.2.1 we conclude that this effect cannot be observed in the CMB temperature power spectrum [218], even if tensor modes were detected. This can be easily understood by looking at the action Eq. (3.118). Since matter fields are usually irrelevant during inflation, such model is physically equivalent to GR plus a scalar field. This also means that during inflation the propagation speed of tensor modes could be set to unity by going to the Einstein frame (at least perturbatively). This is discussed in detail in Ref. [218]. Nevertheless, we could find the difference in propagation speeds if we test

⁶This depends on the sign of the disformal factor B in Eq. (3.46).

⁷Not to be confused with the fact that a disformal transformation by itself does not change the speed of propagation.

⁸We have to switch the roles of the barred and unbarred metrics.

the theory Eq. (3.118) using gravitational waves and their electromagnetic counterpart.

To end this section, let us say that if we are interested in the causal structure of the whole system, i.e. gravity and matter, we can use a disformal transformation to go to a frame where the propagation speed of the fastest species is set to unity, i.e. equal to the geometrical factor that defines the causal structure often written as c . Thus, in the resulting frame any field has a propagation speed less than one and we can apply the standard notions of causal structure.

3.4 Summary and discussion

We have seen that scalar-tensor theories often involve a non-minimal couplings. In the action, non-minimal couplings can be interpreted in two ways. First, there could be a function of the scalar field in front of the scalar curvature and matter be minimally coupled to the metric. Second, the gravity sector could be GR plus a minimally coupled scalar field but matter interacts non-trivially with the scalar field. This two representations of the same theory are respectively called the *matter (Jordan) frame* and the *gravity (Einstein) frame*.

In section 3.1 we have motivated scalar-tensor theories and non-minimal couplings from higher dimensional theories. We have seen that Jordan and Einstein frames are related by *metric transformations*, which in general are *conformal* and *disformal* transformations. In Secs. 3.1.1 and 3.1.2, we have provided interpretations, transformation rules and applications of metric transformations in cosmology and other areas of physics.

After we showed the mathematical equivalence of frames, we proceeded in Secs. 3.2 and 3.3 to prove that frames related by a metric transformation are actually *physically equivalent*. We started with particular examples in a cosmological background and then we moved to a proof for non-linear cosmological perturbations. We have seen that including first derivatives of the field in the transformation makes it non-trivial and thus we presented an explicit proof of the equivalence. We showed that the same applies for matter fields. Interestingly, if photons couple to a disformally related metric then gravitons and photons, would have different propagation speeds.

We end this section by noting that metric transformations can be also understood within Hamiltonian formulation of GR. In that case a metric transformation is a canonical transformation of variables. Thus, one can show that a (regular) metric transformation does not change the number of degrees of freedom of the theory as it keeps the same constraint algebra (see [230] for

a proof in a general disformal transformation). In Appendix [H](#) we provide a brief introduction to the Hamiltonian formalism and metric transformations.

Signals from the matter frame

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We have devoted Chapter 3 to show that in scalar-tensor theories, physical observables do not depend on the frame one works in. We have seen that a metric transformation give us an equivalent representation of the same model, even if it involves derivatives of the scalar field. Does this mean that all inflationary models which are related by a metric transformation, which in principle is infinitely large, yields exactly the same predictions and thus they are observably indistinguishable? The answer is obviously no. This degeneracy is broken when we include matter fields into our system. For example, as we have seen in Secs. 3.2 and 3.3 the *relation between the gravity and the matter metrics* is relevant. In other words, the important point is what is the form of the *gravity frame* and what is the form of the *matter frame*. In this chapter, we will see that this relation may in fact be *observationally distinguishable*, e.g. has a distinct feature in the power spectrum.

This chapter is organized as follows. First in Sec. 4.2 we discuss *how to define inflation* and review a particular analytic model. In Sec. 4.3 we study two interesting cases of matter frames. We will follow our article in JCAP, Ref. [124].

4.1 Inflation and frames

In Chapter 2 we have defined *inflation* as an *accelerated expansion*. However, if there is a non-minimal coupling as in Chapter 3 two questions appear. First, *which metric do we use to define inflation?* Since, for example, the matter metric need not be inflating (see Sec. 3.2). This has a simple answer and it is that inflation should be defined as an accelerated expansion in the Einstein frame.¹

The second question is: *If there is an Einstein frame, can we say that a non-minimal coupling was present?* During inflation the inflaton field completely dominates the dynamics of the universe and other fields are usually completely negligible. Thus whether there was a non-minimal coupling or not, just becomes a matter of taste.² The answer to this question is more subtle and we will use some examples. We have seen that non-minimal couplings in Scalar-Tensor theories (which includes $f(R)$ theories), without any higher order derivatives in the action, can be transformed into an Einstein-Hilbert action form. The price to pay is a non-minimal coupling between matter fields and the scalar field. Knowing this, *we can turn around the question of what non-minimally coupled theory took place during inflation to what was the coupling to matter?*

Assuming that there exists an Einstein frame, we can write our inflationary Lagrangian in generality as

$$S = \int d^4x \sqrt{-g} \left\{ \frac{M_{pl}^2}{2} R + \mathcal{L}_{\text{inf}}(\phi) \right\} + \int d^4x \sqrt{-\tilde{g}} \mathcal{L}_{\text{matter}}(\chi) , \quad (4.1)$$

where $\mathcal{L}_{\text{inf}}$ and $\mathcal{L}_{\text{matter}}$ respectively are the Lagrangian for the inflaton and matter fields. In the limit where the scalar field ϕ completely dominates the dynamics of the metric, we can split the system into the gravity sector, i.e. metric and ϕ , and the matter sector, here the field χ . Regarding the coupling to matter, we will take a general approach and leave the relation between g and $\tilde{g}$ a free variable. In this way, we will see in which cases the coupling becomes relevant, i.e. observable. What is the form of the Jordan frame? For this purpose, let us be more concrete in the model.

First of all, we will assume that matter metric is related to the gravity metric by

$$g_{\mu\nu} = \Omega^2(\phi) \tilde{g}_{\mu\nu} . \quad (4.2)$$

For the moment Ω is any well-behaved non-zero function of ϕ . Second, we take as a representative of matter fields a canonical scalar field χ with mass

¹In general it does not always exist. However, one can define it perturbatively [116, 180].

²That applies to theories with an existing Einstein Frame.

$\tilde{m}$, i.e.

$$\mathcal{L}_{\text{matter}}(\chi) = -\frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\chi\partial_\nu\chi - \frac{1}{2}\tilde{m}^2\chi^2. \quad (4.3)$$

With this information, we can explicitly write down the form of the Einstein and Jordan frames. The Einstein frame is given by

$$S = \int d^4x \sqrt{-g} \left(\frac{M_{pl}^2}{2} R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) \right) + \Omega^{-2}\sqrt{-g} \left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi - \frac{1}{2}m^2\chi^2 \right), \quad (4.4)$$

where M_{pl} is the Planck mass, ϕ is the inflaton field and $m = \Omega^{-1}\tilde{m}$ (see Sec. 3.1.1). The Jordan frame reads

$$S_g = \int d^4x \sqrt{-\tilde{g}} \left(\frac{M_{pl}^2}{2}\Omega^2(\tilde{\phi})\tilde{R} - \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\tilde{\phi}\partial_\nu\tilde{\phi} - \tilde{V}(\tilde{\phi}) \right) + \sqrt{-\tilde{g}} \left(-\frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\chi\partial_\nu\chi - \frac{1}{2}\tilde{m}^2\chi^2 \right) \quad (4.5)$$

where we have redefined the scalar field so that it takes a canonical form by

$$\left(\frac{d\tilde{\phi}}{d\phi} \right)^2 = \Omega^2(\phi) \left| 1 - 6M_{pl}^2 \left(\frac{\partial \ln \Omega}{\partial \phi} \right)^2 \right| \quad (4.6)$$

and the new potential is

$$\tilde{V}(\tilde{\phi}) = V(\phi(\tilde{\phi}))\Omega^4(\phi(\tilde{\phi})). \quad (4.7)$$

According to what we have discussed in Sec. 3.2 (see also Refs. [204, 205]) both actions lead to the same curvature power spectrum and, a priori, they cannot be distinguished observationally in the temperature power spectrum of the CMB. Note that we will assume that at the end of inflation $\Omega \rightarrow 1$ and therefore the Jordan and Einstein frame coincide. In this way, we do not spoil the predictions of the hot big bang as at late times the form of the coupling is very much constraint [64] (see the discussion in Sec. 1.3).

Now the question we want to address is the following: *Does the functional form of Ω has any effect on the observables in the early universe?* In other words, would there be any observable difference if matter coupled to, say, Ω_1 or Ω_2 and so on? Note that if the answer is negative, that means that plenty of models would be degenerate. To understand that we will make two difference choices of Ω and show that in fact there is an observable difference, theoretically speaking, in section 4.3. Nevertheless, we can advance the main idea as follows. In conformal coordinates and in the uniform- ϕ slicing, the

effect of a conformal transformation is completely encoded in the change of the scale factor, i.e.

$$\tilde{a} = \Omega^{-1} a. \quad (4.8)$$

Furthermore, recall that perturbations of the scalar fields follow the Hubble parameter (see Sec. 2.2) corresponding to $\tilde{a}$, that is

$$\tilde{\mathcal{H}} = \mathcal{H} - \frac{d}{d\eta} \ln \Omega. \quad (4.9)$$

In this way, it is clear that by choosing Ω we are changing how the perturbations evolve in the matter frame.

A note of warning. A theory is specified when both matter and gravity sectors are given. In this chapter, we are considering a functional space of theories, each theory specified by the functional form of Ω , i.e. in notation of distributions $S = S[\Omega]$. Then it is natural that different theories have different predictions. This should not be confused with the frame invariance of observables in Chapter 3. In other words, once Ω is chosen the resulting theory is invariant under metric transformations.

4.2 Inflation in the Einstein frame

In this section, we will study the dynamics of the gravitational sector. Note that if we were to work in the general case, we could not get any useful insight since we have complete freedom in the form of the non-minimal coupling Ω . However, if we just focus on a particular form of Ω , then we would very much narrow our considerations and we would not learn much from it. For these reasons, we will focus on an inflationary model which is simple, exact and analytic and still provides a wide variety of results. That is, we will make use of power-law inflation [125] where the inflaton has an exponential potential and we will assume this is the inflationary model in the Einstein frame. For the interested reader, the exact general solution is shown in Refs. [231, 232].

4.2.1 Power-law inflation

Power-law inflation was first introduced by Lucchin and Matarrese [125] where they considered an exponential potential for the scalar field given by

$$V(\phi) = V_0 e^{-\lambda\phi/M_{pl}}, \quad (4.10)$$

where V_0 is the energy scale of the potential and λ is a given parameter. In a flat FLRW background (see Eq. (2.7) with $K = 0$), recall that is

$$ds^2 = -dt^2 + a^2 d\mathbf{x}^2, \quad (4.11)$$

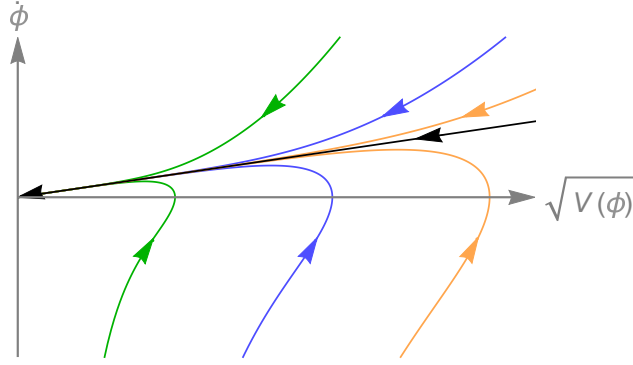


Figure 4.1: Phase space diagram for a given λ . Black line stands for the attractor power-law solution, whereas the colored lines are the two families of general solutions (initially rolling-up solutions below the black line and initially rolling-down solutions above the black line).

the Einstein equations (see Eqs. (2.9), (2.46) and (2.52)) yield

$$3H^2 M_{pl}^2 = \frac{1}{2} \dot{\phi}^2 + V_0 e^{-\lambda\phi/M_{pl}} \quad (4.12)$$

and

$$\ddot{\phi} + 3H\dot{\phi} - \frac{\lambda V_0}{M_{pl}} e^{-\lambda\phi/M_{pl}} = 0. \quad (4.13)$$

We can now plug in a power-law ansatz for the scale factor, e.g.

$$a = a_0 \left(\frac{t}{t_0} \right)^p \quad (0 < t < \infty). \quad (4.14)$$

The solution from this ansatz yields that the scalar field evolves as

$$\phi = \frac{2M_{pl}}{\lambda} \ln \left(\frac{t}{t_0} \right), \quad (4.15)$$

the parameter p is related to λ by

$$p = 2/\lambda^2, \quad (4.16)$$

and the scale t_0 is related to the parameters of the theory by

$$\lambda^2 V_0 t_0^2 = 2M_{pl}^2 (3p - 1). \quad (4.17)$$

The Hubble and slow-roll parameters are given respectively by

$$\begin{aligned} H &\equiv \frac{\dot{a}}{a} = \frac{p}{t}, \\ \varepsilon &\equiv \frac{\dot{H}}{H^2} = \frac{1}{p}, \end{aligned} \quad (4.18)$$

where a dot refers to the derivative with respect to the cosmic time t . We can see that this solution describes an initial big bang followed by an eternal expansion, i.e. the universe never stops expanding. Regarding the graceful exit we will assume the inflaton eventually decays. There is a relation which will be useful later and it is given by

$$\lambda H M_{pl} = \dot{\phi}. \quad (4.19)$$

We have inflation if $\ddot{a} > 0$, which implies $p > 1$. In fact, slow-roll inflation requires $\varepsilon \ll 1$ and thus we will consider $p \gg 1$ (see Eq. (4.18)). The solution we have shown is an attractor solution, i.e. solutions with initial conditions lying outside the attractor solution tend to approach the attractor solution with time (see Fig. 4.1). As one can see in Refs. [231, 232], there are two types of solutions a part from the late time attractor. Either the field starts rolling down faster and then slows down or the field starts rolling up, stops and rolls down again. These two latter class of solutions do not have a well defined initial conditions for the perturbations and thus we decided not to consider them further in this thesis.

Before ending this section, let us rewrite the variables in conformal time, i.e. $dt = a d\eta$, since we will make heavy use of them later.³ First we have that

$$\left(\frac{\eta}{\eta_0}\right) = \left(\frac{t}{t_0}\right)^{-(p-1)} \quad (4.20)$$

where $\eta_0 = -\frac{t_0}{a_0(p-1)}$. Note that η runs from $-\infty < \eta < 0$. Thus,

$$a = a_0 \left(\frac{\eta}{\eta_0}\right)^{-\frac{p}{p-1}}, \quad \mathcal{H} \equiv \frac{a'}{a} = \frac{p}{p-1} \frac{1}{-\eta} \quad (4.21)$$

and

$$\phi = -\lambda M_{pl} \frac{p}{p-1} \ln \left(\frac{\eta}{\eta_0}\right), \quad (4.22)$$

where a prime denotes derivative with respect to conformal time. Let us see next what are the inflationary predictions of this model.

4.2.2 Primordial fluctuations

Now we will compute the power spectrum of quantum fluctuations and find the scale dependence of the power spectrum of temperature fluctuations in the CMB. We will make use of cosmological perturbation theory [18, 233]. We proceed as follows (see Appendix E for a detailed derivation). First we perturb

³Since we do a conformal transformation, conformal time is most appropriate.

the inflaton and the metric around the FLRW background. For convenience we choose to work in the uniform- ϕ or comoving gauge in the conformal time where the metric is given by

$$ds^2 = a^2(\eta) \left(-(1 + 2A) d\eta^2 + 2\partial_i \beta dt dx^i + ((1 + 2\mathcal{R}_c) \delta_{ij} + \gamma_{ij}) dx^i dx^j \right), \quad (4.23)$$

and we only focused on scalar and tensor perturbations as the vector modes are non-dynamical. We expand up to second order the action in the Einstein frame Eq. (4.4) completely neglecting the field β for the moment. Since we know there is only one scalar d.o.f. we have integrate to out the lapse A and shift β , i.e. we take the variation with respect to them and solve the constraints. Once the system is reduced, only one scalar variable survives and yields (see Eq. (E.35)) in Fourier space

$$S_{2,\mathcal{R}_c} = \int d^3x d\eta z^2 \left(\mathcal{R}_c^{2'} - k^2 \mathcal{R}_c^2 \right), \quad (4.24)$$

where $z^2 = a^2 \varepsilon$ and $\mathcal{R}_c$ is the comoving curvature perturbation. The equations of motion are the so-called *Mukhanov-Sasaki equation* [234, 235], namely

$$\mathcal{R}_c'' + 2 \frac{z'}{z} \mathcal{R}_c' + k^2 \mathcal{R}_c = 0, \quad (4.25)$$

where

$$z = a\phi'/\mathcal{H} = \lambda a M_{pl} \quad (4.26)$$

and we used Eq. (4.19) in the last step. Now we have to quantize the system and compute the power spectrum of quantum fluctuations.

Assuming that the field is in the slow-roll regime, i.e. $p \gg 1$, we can make use of the WKB approximation⁴ to solve Eq.(4.25) in the deep subhorizon

⁴The WKB approximation is a good approximation if the wavelength of the oscillations is much shorter than the length scale of the system. In other words, if the frequency is much faster than the time scale of the system. During inflation this is translated to $k \gg \mathcal{H}$, which also means $k\eta \gg 1$, that is sub-horizon scales. For example, take an equation like

$$\mathcal{R}_c'' + \omega^2(k, \eta) \mathcal{R}_c = 0.$$

The following is the solution at leading order if $\omega^2 \gg \omega'$,

$$\mathcal{R}_c(k, \eta) \sim \frac{1}{\sqrt{\omega(k, \eta)}} e^{-i \int^\eta d\eta' \omega(k, \eta')}.$$

Including a damping term, which is slowly varying in time as well, slightly changes the amplitude of the wave function. For example, if we include $2\mathcal{H}\mathcal{R}_c'$ the amplitude is modified from $\omega^{-1/2}$ to $a^{-1}\omega^{-1/2}$. In this way it is easy to find Eq. (4.27).

limit, that is $k \gg \mathcal{H}$. If we choose as initial conditions the Bunch-Davies vacuum one has, once the canonical commutation relations are imposed,

$$\mathcal{R}_c(k, \eta) = \frac{a^{-1}}{\sqrt{4\varepsilon} k} e^{-ik\eta}. \quad (4.27)$$

After horizon crossing, i.e. $k = \mathcal{H}(\eta_*)$, the mode freezes and becomes constant, that is

$$\mathcal{R}_c(k, \eta) = \frac{H(\eta_*)}{\sqrt{4\varepsilon} k^3}, \quad (4.28)$$

where $aH = \mathcal{H}$ and it is evaluated at horizon crossing. This gives us a relation between k and η at horizon crossing, i.e. $-k\eta_* = p/(p-1)$. Thus we are led to

$$H = \frac{\mathcal{H}}{a} = \frac{1}{-a_0\eta_0} \frac{p}{p-1} \left(\frac{\eta}{\eta_0} \right)^{\frac{1}{p-1}} = H_0 \left(\frac{k}{k_0} \right)^{\frac{-1}{p-1}}, \quad (4.29)$$

where $H_0 = p/t_0$ and k_0 is the scale corresponding to the horizon crossing of η_0 , i.e. $-k_0\eta_0 = p/(p-1)$. The power spectrum of the curvature perturbation is therefore given by

$$\mathcal{P}_{\mathcal{R}_c}(k) = \frac{4\pi k^3}{(2\pi)^3} |\mathcal{R}_c(k)|^2 = \left(\frac{H^2}{2\pi\dot{\phi}} \right)^2 = \frac{p}{8\pi^2} \frac{H^2}{M_{pl}^2} = \frac{p}{8\pi^2} \frac{H_0^2}{M_{pl}^2} \left(\frac{k}{k_0} \right)^{\frac{-2}{p-1}}. \quad (4.30)$$

Similarly, the tensor perturbation spectrum is given by

$$\mathcal{P}_\gamma(k) = \frac{4\pi k^3}{(2\pi)^3} |\gamma_k|^2 = \frac{2}{\pi^2} \frac{H^2}{M_{pl}^2} = \frac{16}{p} \mathcal{P}_{\mathcal{R}_c}(k), \quad (4.31)$$

where in the last step we used Eq. (4.30). These are the standard results of power-law inflation [125] (see Ref. [236] for a more precise calculation).

In passing, we can compare it to observational constraints. The spectral index is given in terms of the parameter p as

$$n_s - 1 = \frac{d \ln \mathcal{P}_{\mathcal{R}_c}}{d \ln k} = \frac{-2}{p-1}, \quad (4.32)$$

which is slightly red, while the tensor to scalar ratio is given by

$$r = \frac{\mathcal{P}_\gamma}{\mathcal{P}_{\mathcal{R}_c}} = \frac{16}{p}. \quad (4.33)$$

The constraints by Planck 2015 [38], give $n_{s,planck} \approx 0.96$ and $r_{planck} < 0.1$ (see Sec. 2.2). From the spectral index we find $p \approx 50$, but at the same time it results in a large r , i.e. $r \approx 0.32$. This is actually a common feature of large

field inflation models. Despite the fact that this model is already excluded as it is, we will consider that the matter scalar field χ is a curvaton [126–128]. As we will see in next section, the discrepancy can be alleviated in the curvaton scenario.⁵ In any case, our purpose is not to resolve the discrepancy but to study the matter’s physics in the Jordan frame. It turns out that we can actually make power-law inflation observationally more attractive, if we regard a curvaton as matter, as will be shown below.

4.3 Matter frame and its signatures

So far we reviewed power-law inflation in the Einstein frame while neglecting the contribution of the matter fields, as it is usually done. To tackle the question of whether the coupling to matter can leave some observable imprints, we need a field which contributes to the curvature power spectrum and, thus, to the temperature fluctuations of the CMB. There is an already existing model capable of doing so and it is called the curvaton scenario [126–128].

Before going into details, let us see how the matter metric variables are related to the power-law Einstein metric after the conformal transformation Eq. (4.2). It is sufficient to consider only the modification of the background. The line element of the matter frame is given by

$$d\tilde{s}^2 = \Omega^{-2}(\phi)ds^2 = \Omega^{-2}(-dt^2 + a^2d\mathbf{x}^2) = -d\tilde{t}^2 + \tilde{a}^2d\mathbf{x}^2, \quad (4.34)$$

and thus the proper time and the scale factor are respectively related by

$$d\tilde{t} = \Omega^{-1/2}dt \quad (4.35)$$

and

$$\tilde{a} = \Omega^{-1/2}a. \quad (4.36)$$

On the other hand, the matter conformal Hubble parameter reads

$$\tilde{\mathcal{H}} = \mathcal{H} - \Omega'/\Omega = \mathcal{H} \left(1 - \lambda \frac{\partial \ln \Omega}{\partial \phi/M_{pl}} \right), \quad (4.37)$$

where (4.19) has been used. With these transformation rules we are ready to study the matter frame.

In section 4.3.1, we will briefly explain the curvaton mechanism and while doing so we will show the equivalence of frames for the curvaton. Next, in Secs. 4.3.2 and 4.3.3, we will consider two examples of the matter coupling, i.e. two different functions of Ω .

⁵Essentially because we will have two contributions to the scalar power spectrum but a single contribution to the tensor power spectrum. Thus r is reduced.

4.3.1 Curvaton

Let us present the basics of the curvaton mechanism, which was proposed by Refs. [126–128]. See Appendix G for details. The curvaton is a scalar field that:

- (i) Has a *non-vanishing* initial background *value* and it is almost *massless*.⁶
- (ii) Its energy density is *sub-dominant during inflation*.
- (iii) After the end of inflation and the subsequent decay of the inflaton, the curvaton dominates the energy density of the universe.⁷
- (iv) The curvaton decays into radiation before primordial nucleosynthesis (see Chapter 2). Its isocurvature (or entropy) perturbation is then converted into *adiabatic curvature perturbation* and contributes to the scalar power spectrum along with the inflaton.

Scalar fields which satisfy the first two conditions are often referred to as *spectator fields*. The curvaton is a spectator field which contributes to the curvature power spectrum. Note that its contribution could be larger than that of the inflation if, for example, the mass of the inflaton is much smaller than 10^{13}GeV .⁸

In order to avoid any confusion let us see that the quantum fluctuations of the curvaton are indeed frame independent. We take the curvaton to be minimally coupled to the matter metric $\tilde{g}_{\mu\nu}$ and therefore the curvaton action is given by

$$S_m = \int d^4x \sqrt{-\tilde{g}} \left(-\frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \chi \partial_\nu \chi - \frac{1}{2} \tilde{m}^2 \chi^2 \right), \quad (4.38)$$

where $\tilde{m}$ is the mass of the curvaton in the matter frame. If we now perturbatively expand the curvaton around its background value, i.e. $\chi(k, \eta) = \chi_0(\eta) + \chi_k(k, \eta)$, up to second order in the action in Fourier space we obtain

$$S_{2,\chi} = \int d^3x d\eta \frac{\tilde{a}^2}{2} (\chi_k'^2 - (k^2 + \tilde{a}^2 \tilde{m}^2) \chi_k^2). \quad (4.39)$$

⁶If it is massive it oscillates and decays.

⁷There might be some amount of radiation if the inflaton decayed into radiation.

⁸If the mass of the inflaton is 10^{13}GeV , the Hubble parameter will be roughly $H \sim 10^{13} \text{GeV}$. This means that the amplitude of the power spectrum is 10^{-12} , i.e. 3 orders of magnitude smaller than what we observe. In that case, the curvaton contribution could dominate as the amplitude of its power spectrum is inversely proportional to its background value.

Note that we assumed that the curvaton is completely sub-dominant and therefore we can neglect metric and inflaton perturbations. The equations of motion for the mode functions χ_k are given by

$$\chi_k'' + 2\tilde{\mathcal{H}}\chi_k' + (k^2 + \tilde{m}^2\tilde{a}^2)\chi_k = 0. \quad (4.40)$$

On the other hand, in the Einstein frame where the action is

$$S_m = \int d^4x \Omega^{-2} \sqrt{-g} \left(-\frac{1}{2} g^{\mu\nu} \partial_\mu \chi \partial_\nu \chi - \frac{1}{2} m^2 \chi^2 \right), \quad (4.41)$$

the action for the perturbation χ_k is given by

$$S_{2,\chi} = \int d^3x d\eta \frac{\Omega^{-2} a^2}{2} (\chi_k'^2 - (k^2 + a^2 \Omega^{-2} \tilde{m}^2) \chi_k^2). \quad (4.42)$$

Thus, the equation for the mode functions reads

$$\chi_k'' + 2(\mathcal{H} - \Omega'/\Omega)\chi_k' + (k^2 + m^2 a^2)\chi_k = 0, \quad (4.43)$$

where we redefined the mass by $m = \Omega^{-1} \tilde{m}$. We proceed as usual with its quantization. Note that if we use the conformal time of the time coordinate, the canonical commutation is trivially equal in both frames. Then the field equation is automatically the same. In any case, for illustrative purposes we presented the mode function equations of motions for both frames. We can compare Eqs. (4.40) and (4.43) and readily see that they are exactly equal but for interpretations. Concretely, Eq. (4.40) is the e.o.m. of a canonical scalar field in an expanding background whereas Eq. (4.43) has explicitly the effect of the coupling between the inflaton and the curvaton. Nevertheless, the important point is that the curvaton perturbation power spectrum is frame independent. Let us now see what is the contribution of the curvaton to the power spectrum.

Assuming the sudden decay approximation [128], the curvaton curvature perturbation spectrum is given by (see Appendix G)

$$\mathcal{P}_\chi(k) = r_\star^2 \frac{4\pi k^3}{(2\pi)^3} \frac{|\chi_k|^2}{\chi_\star^2}, \quad (4.44)$$

where $\chi_\star$ and $r_\star$ respectively are the background value of the curvaton field and the energy density fraction of the curvaton at the time of decay,⁹ that is

$$r_\star = \rho_{\chi_\star} / (\rho_{\text{total}} + p_{\text{total}})_\star. \quad (4.45)$$

⁹The value $\chi_\star$ is related to χ_0 by the background equations of motion, i.e. $\chi'' + 2\tilde{\mathcal{H}}\chi' + \tilde{a}^2 \tilde{m}^2 \chi = 0$. In other words $\chi_\star = \chi_\star(\chi_0)$. See Appendix G.

Note that Eq. (4.44) is valid as long as the curvaton has a non-vanishing background value, which implies $\tilde{H} \gg \tilde{m}$ [128, 237]. We can understand this fact from the e.o.m. of the curvaton, i.e.

$$\ddot{\chi} + 3\tilde{H}\dot{\chi} + \tilde{m}^2\chi^2 = 0. \quad (4.46)$$

If $\tilde{H} \gg \tilde{m}$ then the curvaton cannot settle down to the minimum of its potential due to a large Hubble friction. Then the total scalar power spectrum, i.e. the one which sources the temperature fluctuations in the CMB, has a contribution from inflation and from the curvaton and we express it as

$$\mathcal{P}_{\text{tot}} = \mathcal{P}_{\mathcal{R}_c} + \mathcal{P}_{\chi}, \quad (4.47)$$

It should be noted that as a result the tensor to scalar ratio (4.33) that we observe in the CMB is given by [238]

$$\tilde{r} = \frac{\mathcal{P}_{\gamma}}{\mathcal{P}_{\text{total}}} = \frac{r}{1 + \mathcal{P}_{\chi}/\mathcal{P}_{\mathcal{R}_c}}, \quad (4.48)$$

where we used Eq. (4.33) in the last step. Therefore, we conclude that in general $\tilde{r} < r$. In this way, we can make this scenario more consistent with the current observational data if we assume the curvaton energy density at the time of decay is comparable to or greater than that due to the inflaton and radiation. Then $\tilde{r}$ and the non-gaussianity parameter, f_{NL} , are small enough [239, 240].

In what follows we consider a couple of examples separately. Bear in mind that we assume a slow-roll inflationary Einstein frame, i.e. $p \gg 1$, unless otherwise stated.

4.3.2 Power-law matter frame

We start by considering that the coupling between matter and gravity metrics is a simple conformal transformation given by

$$\Omega_1(\phi) = e^{\alpha\lambda\phi/(2M_{Pl})}, \quad (4.49)$$

where α is, for the moment, a free parameter and the dependence on λ is included for later simplicity. This type of exponential coupling is often present in dilaton models in string theory, for example see Ref. [241]. Note that the parameter α describes the theory space. In other words, different values of α correspond to two different theories. Let us illustrate this point by showing the total action in the matter frame. First, we need to calculate the inflaton

field redefinition Eq. (4.6). For the case of the exponential coupling Eq. (4.49) we have that Eq. (4.6) reads

$$\frac{d\tilde{\phi}}{d\phi} = e^{\alpha\lambda\phi/(2M_{pl})} \sqrt{1 - \frac{3}{2}\alpha^2\lambda^2} = \frac{2\xi^{-1/2}}{\alpha\lambda} e^{\alpha\lambda\phi/(2M_{pl})}, \quad (4.50)$$

where for simplicity we defined

$$\xi \equiv \frac{(\alpha\lambda)^2}{4 - 6(\alpha\lambda)^2} = \frac{\alpha^2}{2(p - 3\alpha^2)}. \quad (4.51)$$

After a simple integration we find that

$$\frac{\tilde{\phi}}{M_{pl}} = \xi^{-1/2} e^{\alpha\lambda\phi/M_{pl}}. \quad (4.52)$$

Now, we can write the Jordan frame Eq. (4.5) in terms of the redefined inflaton field Eq. (4.52), which yields

$$S = \int d^4x \sqrt{-\tilde{g}} \left(\frac{\xi \tilde{\phi}^2 \tilde{R}}{2M_{pl}^4} - \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \tilde{\phi} \partial_\nu \tilde{\phi} - \tilde{V}_0 \left(\frac{\tilde{\phi}}{M_{pl}} \right)^{4 - \frac{2}{\alpha}} \right) \quad (4.53)$$

$$+ \sqrt{-\tilde{g}} \left(-\frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \chi \partial_\nu \chi - \frac{1}{2} \tilde{m}^2 \chi^2 \right), \quad (4.54)$$

where $\tilde{V}_0 = V_0 \xi^{2-1/\alpha}$. Let us note a couple of important points. First, in this form it is clear that each choice of α leads to a different theory, in this case to a different potential for the inflaton in the Jordan frame. For example, if $\alpha \gg 1$ we have a quartic potential and if $\alpha = 1$ then we have a quadratic potential.¹⁰ Second, it should be noted that for $\alpha^2 > 2/(3\lambda^2) = p/3$, the gravitational part of the Jordan frame action becomes ghost-like. Nevertheless since the original Einstein frame action is perfectly normal, the system is stable in spite of its seemingly dangerous form [242].

Let us now explore the corresponding evolution of the inflationary universe in the matter frame. First, the coupling in terms of the cosmic and conformal time respectively reads (see Eqs. (4.15) and (4.22))

$$\Omega_1(\phi) = e^{\alpha\lambda\phi/(2M_{pl})} = \left(\frac{t}{t_0} \right)^\alpha = \left(\frac{\eta}{\eta_0} \right)^{-\frac{\alpha}{p-1}}. \quad (4.55)$$

We thus straightforwardly find that the matter frame scale factor is given by

$$\tilde{a} = \Omega_1^{-1} a = a_0 \left(\frac{\eta}{\eta_0} \right)^{\frac{\alpha-p}{p-1}} \equiv a_0 \left(\frac{\eta}{\eta_0} \right)^{-\frac{\tilde{p}}{\tilde{p}-1}}, \quad (4.56)$$

¹⁰It turns out that the case $\alpha = 1$ is quite particular, leading to an exponential expansion.

where in the last step we defined a new power-law index $\tilde{p}$. Note that we have obtained a new power-law universe with a different power-law index. In fact, it is related to that in the Einstein frame by

$$\tilde{p} - 1 = \frac{p - 1}{1 - \alpha}. \quad (4.57)$$

The conformal hubble parameter is now given by

$$\tilde{\mathcal{H}} = \frac{\tilde{p}}{\tilde{p} - 1} \frac{1}{-\eta}. \quad (4.58)$$

So far, we have worked in conformal time for convenience. However, we know that the cosmic time, the proper time for matter, also changes and it does so according to

$$d\tilde{t} = \Omega_1^{-1} dt. \quad (4.59)$$

Integrating this equation we find

$$\left(\frac{\tilde{t}}{\tilde{t}_0} \right) = \left(\frac{t}{t_0} \right)^{-\alpha+1} \quad (4.60)$$

where

$$\tilde{t}_0 = \frac{t_0}{1 - \alpha}. \quad (4.61)$$

In terms of the matter proper time the scale factor and the Hubble parameter respectively read

$$\tilde{a} = a_0 \left(\frac{\tilde{t}}{\tilde{t}_0} \right)^{\tilde{p}} \quad \text{and} \quad \tilde{H} = \frac{\tilde{p}}{\tilde{t}}. \quad (4.62)$$

This agrees with Ref. [210], where the condition to obtain the scale-invariant tensor spectrum was discussed from the Jordan frame point of view. Depending on the value of α we obtain different power law indexes and, therefore, different evolutions of the universe. Let us list all the possible cases:

1. If $-\infty < \alpha < 1$ we have an *inflationary universe* with $\tilde{p} > 1$ and $0 < \tilde{t} < \infty$.
2. If $\alpha = 1$ we have an *exact exponential expansion*, which corresponds to the limit $\tilde{p} \rightarrow \infty$, and $-\infty < \tilde{t} < \infty$.
3. If $1 < \alpha < p$ we have a *super-inflationary universe* with $\tilde{p} < 0$ and $-\infty < \tilde{t} < 0$. This case is quite interesting and we will study it further below.

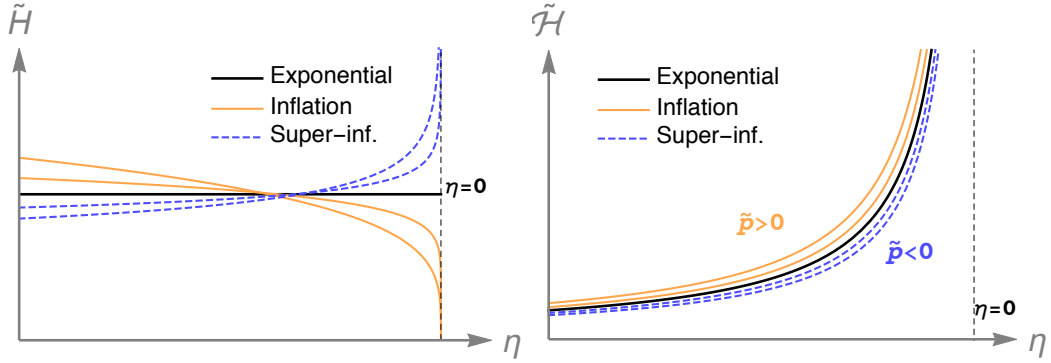


Figure 4.2: Jordan conformal Hubble parameter as a function of conformal time η . The black line is the exponential expansion $\tilde{p} \rightarrow \infty$, orange lines stand for power-law inflation, $\tilde{p} > 1$, whereas green dashed lines stand for super-power-law inflation, $\tilde{p} < 0$. Note how in the super-inflating case $\tilde{H}$ increases and $\tilde{\mathcal{H}}$ increases faster than the exponential case.

4. If $\alpha = p$ we have flat space where $\tilde{p} = 0$ and $-\infty < \tilde{t} < 0$. This case actually corresponds to $\Omega = a$.
5. If $p < \alpha < \infty$, we have a *decelerated contracting universe* with $0 < \tilde{p} < 1$ and $-\infty < \tilde{t} < 0$. Note that $\alpha > p$ implies $\alpha^2 > p/3$ for $p > 1/3$. Hence this last case corresponds to the ghost-like gravity mentioned before.

Interestingly, there are cases where power-law Jordan frame is not restricted to $\tilde{p} > 1$ nor even $\tilde{p} > 0$. Thus, despite the fact that an almost scale invariant spectrum is obtained independent of the frame, the matter frame might not be an inflationary universe. It could even be contracting! In words of Refs. [243], the non-minimal coupling “assists” inflation. That is to say that we have inflation in the Einstein frame although the Jordan frame is not inflationary, due to the non-minimal coupling with the inflaton.

Before computing the power spectrum, let us look in more detail what is the meaning of a super-inflating universe. Take a look at Eqs. (4.58) and (4.62). We see that $\tilde{H}, \tilde{\mathcal{H}} > 0$. The difference with an inflationary universe is that for super-inflation $\tilde{H}$ grows in time, while usually $\tilde{H}$ always decrease. In other words

$$\frac{d\tilde{H}}{d\tilde{t}} > 0 \quad \text{for} \quad \tilde{p} < 0. \quad (4.63)$$

This is the reason why we call it *super-inflation*. It expands faster than an exponential expanding universe. Note that in usual General Relativity one has that $\dot{H} \propto -(p + \rho)$ (see Chapter 2) and thus H always decrease unless the null energy condition (which says $p + \rho > 0$) is violated. This could have catastrophic consequences for the singularity theorems. However, in the

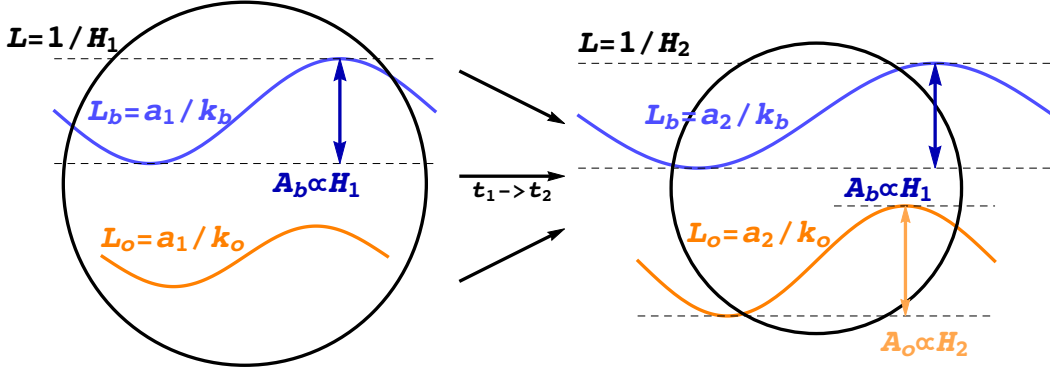


Figure 4.3: Rough idea of the quantum fluctuations in a super-inflationary universe. On the left: the blue mode exits the horizon and freezes. The orange mode keeps oscillating. On the right: the orange mode exits the horizon. The Hubble radius decreased and the amplitude of the orange mode increased with respect to the blue one. Thus we have a blue tilt of the spectrum.

presence of non-minimal couplings this is possible and in fact quite normal in the Jordan frame. Recall that we started from an Einstein frame with canonical scalar fields and therefore our theory is well defined. The behavior of the Hubble and the conformal Hubble parameter for $0 < \alpha < p$ ($\tilde{p} < 0$ and $\tilde{p} > 1$) are respectively shown in Fig. 4.2. Let us turn now to the power spectrum.

To calculate the scalar power spectrum due to the curvaton we assume that we are in the slow-roll regime, i.e. $|\tilde{p}| \gg 1$. Then can make use of the WKB approximation inside the horizon and assume that a mode freezes out instantaneously at horizon crossing, which we call the *instantaneous horizon exit assumption*. The mode function for the curvaton in subhorizon scales is then given by

$$\chi_k(k, \eta) = \frac{\tilde{a}^{-1}}{\sqrt{2}k} e^{-ik\eta} \quad (4.64)$$

and after horizon crossing by

$$\chi_k(k, \eta) = \frac{\tilde{H}(\eta_*)}{\sqrt{2}k^3}. \quad (4.65)$$

The curvaton curvature perturbation spectrum, if we assume the sudden decay approximation [128], is given by

$$\mathcal{P}_\chi(k) = r_*^2 \frac{4\pi k^3}{(2\pi)^3} \frac{|\chi_k|^2}{\chi_*^2} = r_*^2 \frac{\tilde{H}^2}{(2\pi M_{pl} \chi_*)^2} = r_*^2 \frac{\tilde{H}_0^2}{M_{pl}^2 \chi_*^2} \left(\frac{k}{k_0} \right)^{\frac{-2}{\tilde{p}-1}}. \quad (4.66)$$

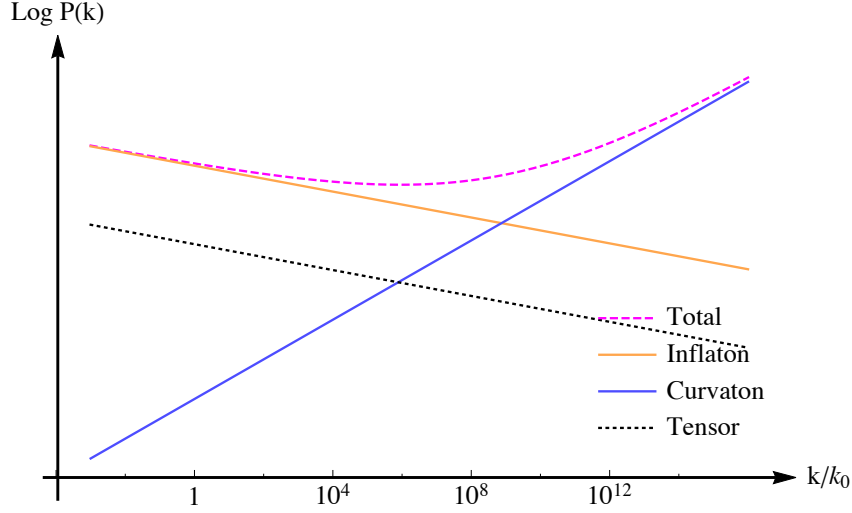


Figure 4.4: Power spectrum in logarithmic scale for the power-law Jordan case. Inflation curvature power spectrum (red line), tensor power spectrum (black line), super-inflationary curvaton power spectrum (blue line) and total scalar power spectrum (green line). We chose $p = 50$ and $\alpha = 4$ ($\tilde{p} \sim -15$).

The curvaton spectral index can be easily extracted from (4.66). We obtain

$$\tilde{n}_\chi - 1 = \frac{-2}{\tilde{p} - 1}, \quad (4.67)$$

which need not be a red index. In particular, a blue tilt is obtained for $\tilde{p} < 0$, which is a common feature of super-inflationary models [244–249]. See Fig. 4.4. This blue spectrum contribution could be important on small scales.¹¹ Interestingly, it could enhance primordial black hole formation which can account for a fair amount of dark matter [251] or explain $60M_\odot$ binary black hole mergers [252]. See Ref. [253] for an interesting application of a similar model to the detected GW and Ref. [250] for values of current constraints. It should be noted that if the amplitude of the power spectrum becomes order unity, one expects a back-reaction from the large curvaton fluctuations. However, well before the back-reaction becomes relevant, we would most likely have an over-abundance of PBHs and it would be ruled out by observations. Thus, there should be a cutoff in this model. In that respect, recall that at the end of inflation we require that $\Omega \rightarrow 1$ and therefore the blue tilt does have a cut-off. A detailed study on such a situation could be interesting but it is beyond the scope of this thesis.

¹¹If it were important at large scales, e.g. on CMB scales, this model would be completely ruled out by observations. Recall that $n_s \sim 0.96$ for CMB scales. Well below that there is no real constraint on n_s . Note that PBH also create spectral distortions on the CMB [250].

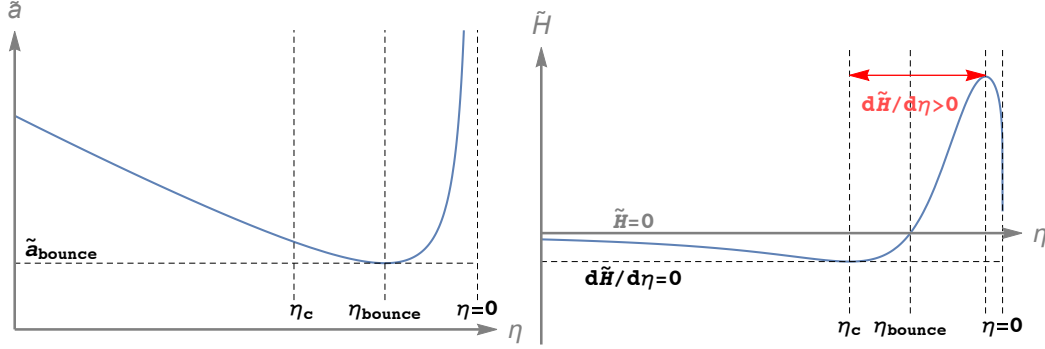


Figure 4.5: On the left: Scale factor of the matter frame $\tilde{a}$ in terms of the conformal time η . On the right: Conformal hubble parameter of the matter frame $\tilde{H}$ in terms of the conformal time η . At the time of bounce $\tilde{a}$ is minimum and $\tilde{H} = 0$. There is another point where $\tilde{H}$ stop decreasing and start increasing, i.e. a saddle point $\tilde{H}'(\eta_c) = 0$.

4.3.3 Bouncing matter frame

Let us now turn into the second and final example. In this case, we consider a slightly more complicated function for the coupling between the Einstein and matter metrics. Concretely, we choose to work with

$$\Omega_2(\phi) = \left(1 + e^{\frac{-\alpha\lambda}{2M_{pl}}\phi}\right)^{-1} = \frac{1}{1 + (t/t_0)^{-\alpha}}, \quad (4.68)$$

where again α is a free parameter which sets the theory space. The reason for such functional form is obvious if we study the two limits in the proper time t . For instance, we have that if $\alpha > 0$ then

$$\Omega_2(t) \approx \begin{cases} (t/t_0)^{-\alpha} & t \ll t_0 \\ 1 & t \gg t_0 \end{cases}, \quad (4.69)$$

We see that at early times ($t \ll t_0$) it reproduces the first example Sec. 4.3.2 and at late times ($t \gg t_0$) it is just identical to the Einstein frame. For $\alpha < 0$ we will have the inverse situation. The case $\alpha > p$ is particularly interesting since at early times the universe is a decelerated contracting universe whereas at late times it is an accelerated expanding universe. Thus, we expect that this model experiences a *bounce*. From now on, we will only study the case $\alpha > p$ and the other cases easily follow.

First of all, we can see that there is a bounce in the matter frame by looking at the conformal Hubble parameter. Using Eq. (4.37) with Eq. (4.68) we are led to

$$\tilde{\mathcal{H}} = \mathcal{H} \left(1 - \frac{\alpha}{p} \frac{e^{\frac{-\alpha\lambda}{2M_{pl}}\phi}}{1 + e^{\frac{-\alpha\lambda}{2M_{pl}}\phi}}\right) = \mathcal{H} \left(1 - \frac{\alpha}{p} \frac{(t/t_0)^{-\alpha}}{1 + (t/t_0)^{-\alpha}}\right). \quad (4.70)$$

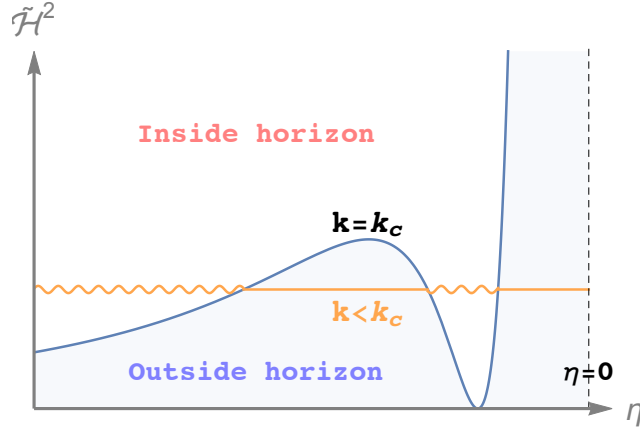


Figure 4.6: Conformal Jordan squared hubble parameter (blue line) for the bouncing universe case and horizontal lines are constant k lines. Shaded region stands for super-horizon regime and the consequent freezing of the modes. In orange we schematically show a mode with $k < k_c$. At η_1 it exits the horizon and freeze. Then, at η_2 it re-enters the horizon and oscillates again.

Then at the time of bounce we have that $\tilde{\mathcal{H}} = 0$. We can see that Eq. (4.70) vanishes at

$$t_{\text{bounce}} = t_0 \left(\frac{\alpha}{p} - 1 \right)^{1/\alpha}. \quad (4.71)$$

Let us now turn into the scale factor and the matter proper time. The scale factor in conformal time is given by

$$\tilde{a} = \Omega_2^{-1} a = a_0 \left(1 + \left(\frac{\eta}{\eta_0} \right)^{\frac{\alpha}{p-1}} \right) \left(\frac{\eta}{\eta_0} \right)^{-\frac{p}{p-1}} \quad (4.72)$$

and the proper time, after integrating, reads

$$\tilde{t} = \frac{t}{t_0} \left(1 - \frac{(t/t_0)^{-\alpha}}{\alpha - 1} \right) \quad (4.73)$$

where the bounce occurs in the matter frame proper time at

$$\tilde{t}_{\text{bounce}} = \frac{\alpha}{p} \frac{\alpha - p - 1}{\alpha - 1} \left(\frac{\alpha}{p} - 1 \right)^{1/\alpha - 1}. \quad (4.74)$$

We can readily see that $-\infty < \tilde{t} < \infty$. The form of the scale factor in terms of the time $\tilde{t}$ is algebraically involved and does not yield any insight. However we can see the approximate behavior using the results of Sec. 4.3.2.

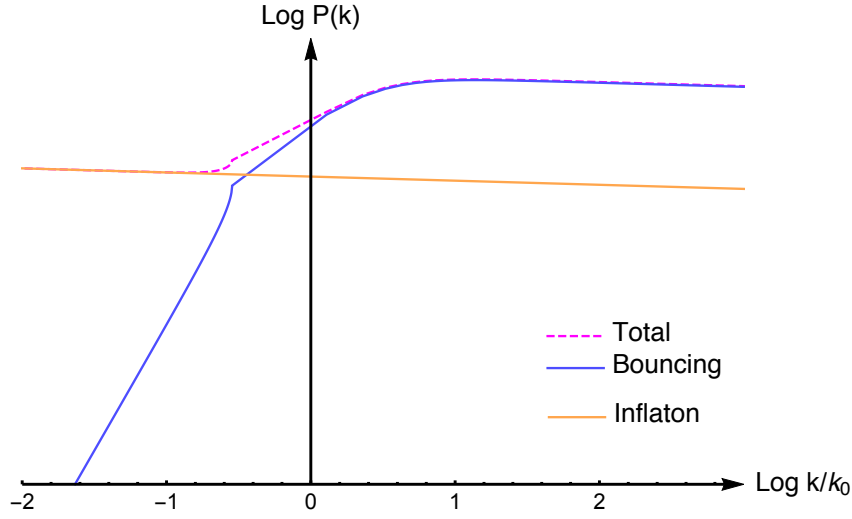


Figure 4.7: Power spectrum for the bouncing universe (blue line), curvature power spectrum (orange line) and total scalar power spectrum for the bouncing case (magenta line). We chose $p = 32$ and $\alpha = 10$ ($\tilde{p} \sim -2.5$).

In that case we find that

$$\tilde{a} \approx \begin{cases} a_0 (-\tilde{t})^{\tilde{p}} & \tilde{t} \ll \tilde{t}_{\text{bounce}} \quad (\tilde{t} < 0) \\ a_0 \tilde{t}^p & \tilde{t} \gg \tilde{t}_{\text{bounce}} \end{cases}, \quad (4.75)$$

Note that when $\alpha > p$ ($0 < \tilde{p} < 1$) the $t = 0$ singularity in the Einstein frame is sent to $\tilde{t} \rightarrow -\infty$ in the matter frame. However, the universe bounces at a finite time in both frames. We show numerically the behavior of $\tilde{a}$ and $\tilde{\mathcal{H}}$ with respect to the conformal time in Fig 4.5. The difference of this model with respect to the usual bouncing cosmologies [254–257] (for a review of bouncing cosmologies see [258]) is that we only “avoided” the initial singularity in the matter frame but it is still present in the Einstein frame. Not to create any confusions, we call this model the Jordan bouncing universe. What is really going on is that the inflaton prevents matter to experience the singularity thanks to its non-canonical coupling. This implies that the initial vacuum state for the curvaton is well defined, as $\tilde{\mathcal{H}}$ vanishes in the limit $\eta \rightarrow -\infty$ (see Eq. (4.70)), and we can compute the curvaton power spectrum in this Jordan bouncing universe.

Before computing the curvaton power spectrum for the Jordan bouncing universe, let us note two important points. First, we have a bounce at $\tilde{\mathcal{H}} = 0$ and, thus, we also have a time when $\tilde{\mathcal{H}}' = 0$, i.e we go from a contracting to an expanding phase (see Fig. 4.5). Therefore, we encounter a critical scale, let us say k_c . We illustrate this point in Fig. 4.6 by showing the behavior of $\tilde{\mathcal{H}}^2$. On one hand, the large scale modes $k < k_c$ exit and re-enter the horizon

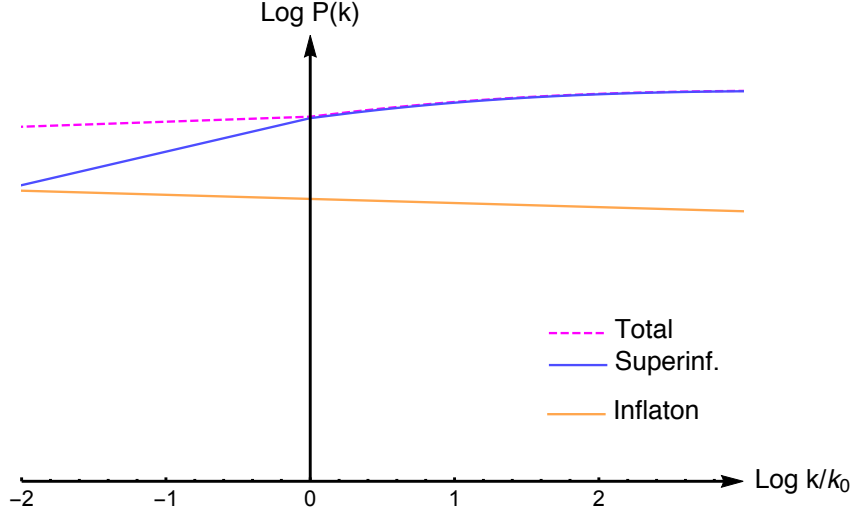


Figure 4.8: Power spectrum for the initially super-inflating universe (blue line), curvature power spectrum (orange line) and total scalar power spectrum for the super-inflating case (magenta line). We chose $p = 32$ and $\alpha = 62$ ($\tilde{p} \sim 0.5$).

once before the bounce, and they finally exit the horizon again during the last inflationary stage. On the other hand, the small scale modes $k > k_c$ remain inside the horizon until the last inflationary stage.

Second, we have seen that we need a non-vanishing background value of the curvaton $\chi = \chi_0(\eta)$ and that it implies that the condition $\tilde{H} \gg \tilde{m}$ has to be fulfilled at all times. We can satisfy the latter if the mass of the curvaton depend on the inflaton, at least as $\tilde{m}(\phi) \propto \Omega(\phi)$ at early times.¹²

We can now proceed to compute the power spectrum. We will assume the instantaneous horizon exit and reentry approximation for $k < k_c$, that is we will consider that the mode function instantaneously freeze after it exits the horizon and we match such value when it re-enters again.¹³ In this way, we find that the mode function of the curvaton before the final horizon crossing can be approximated by

$$\chi_k \approx \begin{cases} \frac{1}{\sqrt{2k}} \frac{\tilde{a}(\eta_2)}{\tilde{a}(\eta_1)} \frac{1}{\tilde{a}(\eta)} e^{-ik\eta + i\alpha_k} & k < k_c \\ \frac{1}{\sqrt{2k}} \frac{1}{\tilde{a}(\eta)} e^{-ik\eta} & k > k_c \end{cases}, \quad (4.76)$$

¹²A dependence on the inflaton in the mass of the curvaton could be present in a general case. However, there would be no frame in which the curvaton does not couple to the inflaton. Nevertheless, we can still define the matter frame where the kinetic term of the curvaton is minimally coupled to the metric.

¹³It is not a good approximation for modes near k_c but it does not affect the general behaviour.

where modes with $k < k_c$ first exited the horizon at η_1 and then reentered at a time η_2 . The factor α_k is an irrelevant phase. It should be noted that although the instantaneous horizon exit and reentry approximation would break down for $\tilde{p} \ll 1$, we can still regard the obtained power spectrum as a rough approximation.

The power spectrum for the curvaton in the bouncing case is given as well by Eq. (4.44). We present a numerical plot of the power spectrum in Fig. 4.7 due to the fact that the scale factor has a non-trivial behavior around the bounce. See how the spectrum becomes blue on large scales. This behavior is also present in bouncing cosmologies. As a last example, we consider an initial super-inflationary stage instead of an initial contracting phase, i.e. $1 < \alpha < p$ and $\tilde{p} < 0$. This yields a blue spectrum at large scales as shown in Fig. 4.8, although the tilt is not as sharp as the Jordan bouncing universe. Note that in this case we do not have to assume any dependence on the inflaton in the curvaton mass. It is interesting to point out that the blue tilt at large scales could potentially explain the *large scale suppression* observed in the CMB [38].

4.4 Summary and discussion

We end this chapter with a short summary and some discussions. We have argued in Sec. 4.2 that inflation, as an accelerated expansion, is best defined in the Einstein frame, even if it is defined perturbatively. Then we reviewed power-law inflation and its predictions in Sec. 4.3. Then, in Sec. 4.3 we considered that matter fields are coupled to a metric related by a conformal transformation with the Einstein frame inflationary metric. We presented the curvaton mechanism and we showed that if the mass of the inflation is smaller than 10^{13} GeV then the curvaton could have the dominant contribution to the CMB temperature power spectrum. Next, we have considered two examples, a super-inflating and a bouncing matter frame. We find that its signals respectively are a blue tilt on small scales or on large scales. The former yields a possible formation of PBHs and the latter could explain the large scale suppression of the CMB power spectrum.

In summary, we have seen that understanding the evolution from the matter frame point of view, considerably simplifies the interpretation and intuition of the physics involved. One could even consider that the matter frame is our interpretation of the evolution of the universe and thus one could provide alternative interpretations of the origins of our universe [107, 108].

To end this chapter let us provide another point of view of the model we built. For instance, it could be understood as a non-minimal coupling between curvaton and inflaton and forget about the notions of matter and gravity frames. We could have started with a given frame and study the physics there. However, there is an important flaw in that method and it is the following. If we are studying a wide class of couplings, i.e. we have a free coupling between the inflaton and the curvaton, it would be harder to find out which model yields interesting result without solving the equations of motion. In this way, our interpretations makes the whole process much simpler and intuitive.

CHAPTER 5

Conclusions

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In the quest of unraveling the universe, we found out that it is made of more components that we do not fully understand than those we do. For example, in our universe's latest stages, dark energy makes up to 70% of the current energy density in our universe. In its earliest epoch, the inflaton completely dominated the evolution before the hot big bang and provided the seeds for the large scale structure of our visible universe. Thus, there is a vast number of open questions than GR alone does not satisfactorily explain. For these reasons, we must explore alternatives to GR, even if it is to find out that GR is still the simplest and best description of our universe.

5.1 Summary of the thesis

We have seen in Chap. 1 that GR is very successful *nowadays*¹ and at *solar system scales*. Thus, alternative theories to GR are very much constraint at these scales. However, as we argued in Sec. 1.1, on cosmological scales where gravity is the dominant force, we find the accelerated expansion due to dark energy and we have robust evidence of inflation in the CMB. We see that we have plenty of room and motivation to study theories of gravity in cosmology, specially during inflation where the universe was in an extremely high energy regime. In Sec. 1.2, we briefly presented alternatives to GR and in Sec 1.3 we focused on a general class of theories, called scalar-tensor theories. In particular, we argued that transformations of the metric are a crucial piece in scalar-tensor theories of gravity. Then to fully grasp the importance of scalar-tensor theories and metric transformation we had to review the standard model of cosmology and the basics of inflation.

¹In a cosmological time scale.

In Chap. 2 we introduced the standard model of cosmology. We saw that our universe is *homogeneous* and *isotropic* on scales larger than 100 Mpc and that it is well described by GR plus radiation (photons and neutrinos), matter (baryons and dark matter) and dark energy (cosmological constant). In Sec. 2.1 we have seen that GR plus a perfect fluid leads in general to an expanding FLRW universe, where the power law of the expansion depends on the equation of state of perfect fluid. For instance, for radiation and matter the expansion is a decelerated expansion. For a fluid violating the strong energy condition, e.g. cosmological constant, the expansion is accelerated. In this way, tracing back in time the evolution of our universe we find a period where the densities and temperatures were extremely high, that is the *hot big bang*.

As successful as the hot big bang is, we saw in Sec. 2.2 that it needs² inflation to explain its initial homogeneity, spatial flatness and the tiny temperature fluctuations. Inflation as a paradigm predicts that the CMB fluctuations are: (i) *gaussian*, (ii) *adiabatic* and (iii) *almost scale invariant*. This is exactly what CMB observations find. Then, we showed in Sec. 2.2.3 that using CMB data one can constrain inflationary models. Interestingly, a simple quadratic potential for the inflaton seems to be excluded at 2σ C.L. and, currently, the favored model is Starobinsky R^2 inflation, which belongs to a sub-class of scalar-tensor theories. Since we had robust evidence of inflation and a hint to consider non-minimal couplings, we moved to the study of scalar-tensor theories in the inflationary epoch.

Scalar-tensor theories are a promising candidate for an alternative to GR. They can be understood as an effective theory, whether they come from quantum corrections (such as $f(R)$ theories) or from higher dimensional theories (like string theory). Their attractive point is that they are general enough so that one can explore a wide range of applications, e.g. inflation and dark energy, and understand symmetries of gravity, e.g. conformal symmetry and metric transformations. In Chap. 3 we motivated non-minimal couplings in scalar-tensor theories from dimensional reduction in higher dimensional theories. In Sec. 3.1 we introduced *conformal* and *disformal transformations* and studied their interpretations and applications. A conformal transformation is a local rescaling of the metric and a disformal transformation is a stretching of the metric in a preferred direction, in our case given by the gradient of a scalar field. We have seen that both transformations preserve causal structure. As two examples, we showed that a conformal transformation maps a static universe to an expanding one and that a disformal transformation in

²At least it make the initial conditions more plausible.

dS space opens or closes the dS hyperboloid.³ We also showed that metric transformations can remove non-minimal couplings and thus one can relate a non-minimally coupled theory to a minimally coupled one. The price to pay is that in the minimally coupled theory, matter interacts non-trivially with the scalar field. This led us to the notion of frames, i.e. theories related by a metric transformation. The *matter (Jordan) frame* where matter minimally couples but there is a non-minimal coupling between the scalar curvature and the scalar field and the *gravity (Einstein) frame* where the scalar field is minimally coupled to gravity but matter directly couples to the scalar field. In the case of a conformal transformation we went to *Brans-Dicke* theories and for a disformal transformation we ended up in *Horndeski* theories.

Then in Secs. 3.2 and 3.3 we proceeded to prove that while two frames related by a metric transformation are indeed mathematically equivalent, they are also physically equivalent. By physical equivalence we mean those quantities that are measurable, which we called *observables*, do not depend on the frame one computes them. This is also known as *frame independence*. Next, we briefly reviewed earlier works on the physical invariance under conformal transformations in Sec. 3.2 and then we explained our work on cosmological disformal invariance in Sec. 3.3. We discussed the gravitational action as well as matter fields and we considered for completeness a scalar, a vector and a fermion field. We also argued that if two massless species couple to two disformally related metrics, they have different relative propagation speed. The main result of this chapter is that even though a disformal transformation contains first derivatives of the scalar field, two disformally related frames are *physically equivalent*. Furthermore, a disformal transformation in cosmology can be understood as a *rescaling of the time coordinate*, as long as perturbation theory holds.

We have extensively discussed inflation and frames in scalar-tensor theories. We have seen that frames related by a metric transformation are physically equivalent in cosmology. Then we asked the question: *are all inflationary theories related by a metric transformation indistinguishable?* We devoted Chap. 4 to show that theoretically these related frames are in principle different theories once the coupling to matter is taken into account. However, often matter is completely negligible during inflation and, therefore, these related frames are usually observationally degenerate. This degeneracy is broken if there is a matter field capable of contributing to observables. Thus we chose to consider a *curvaton* as a representative of matter as it contributes to the curvature perturbations and consequently on the temperature power spectrum in the CMB. We reviewed the curvaton mechanism in Sec. 4.3.1. We have

³See Appendix D for more examples of disformal transformations.

seen that a curvaton is a scalar field that: (i) is *almost massless*, (ii) has a *non-vanishing background value*, (iii) is *subdominant during inflation* and (iv) after the inflaton decays it dominates the universe and *decays into radiation* converting its isocurvature to adiabatic modes.

We started by arguing in Sec. 4.1 that inflation, as an accelerated expansion, is *best defined in the Einstein frame*. The reason being that the Einstein frame is a suitable frame to study gravity as our knowledge of GR straightforwardly applies, for example with the energy theorems. In the Einstein frame inflation is seen an accelerated expansion whereas in the matter frame the universe could look quite different (we showed an example in Sec. 3.2). How different could the universe evolve in the matter frame is discussed in Sec. 4.3. In order to study it analytically, we considered a simple inflationary model in Sec. 4.2.1, so-called power-law inflation, and we reviewed its predictions and compare it with CMB data. Then in Sec. 4.3 we considered that matter is coupled to a metric related to the gravity metric by an arbitrary conformal transformation. We considered two examples where the coupling to matter contains two different functions of the inflaton. First, in Sec. 4.3.2 we considered an exponential (dilatonic) coupling and we showed that the universe in the matter frame need not be inflating at all. In fact, it could be *contracting* or *super-inflating* depending on the parameter of the conformal transformation. In particular, we focused in the super-inflating matter frame. In this case, we obtain a *blue tilt at small scales* in the temperature power spectrum. We argued that this is interesting in the *formation of PBHs* and the possible binary black hole mergers. Lastly, we considered a bouncing matter frame in Sec. 4.3.3 by considering a slight extension of the previous section. We have seen that this leads to a blue tilt on large scales. This could explain the observed *suppression of the power spectrum on large scales* in the CMB. These are the main results of this chapter.

We can briefly summarize the results of this thesis as follows.

- (i) Metric transformations are a crucial element of scalar-tensor theories. In cosmology, they mathematically and physically relate models within scalar-tensor theories.
- (ii) In cosmology a disformal transformation can be understood as a rescaling of the time coordinate.
- (iii) Frames related by a general disformal transformation are physically equivalent in cosmology.

- (iv) The coupling to matter fields might be important during inflation, specially under the curvaton mechanism. In this case the notion of frames gives us intuition on the evolution of matter fields.
- (v) We obtained a blue tilt on small scales if the curvaton (matter) frame is super-inflating. This potentially leads to the formation of PBHs and possible binary black hole mergers.
- (vi) We obtained a blue tilt on large scales if the curvaton (matter) frame bounces. This leads to a large scale suppression of the CMB temperature power spectrum, which although not entirely significant, the CMB data seems to point in this way.

In short, scalar-tensor theories are a general and rich alternative to General Relativity and metric transformations are useful and intuitive tool within this class of theories.

5.2 Implications and future prospects

We have seen that in scalar-tensor theories there is freedom on the definition of metric. Mathematically speaking, one can work in the frame one find most suitable. From a physical point of view, we can map theories which describe the same observable predictions. For example, in the vastness of inflationary models, we can find those with equivalent physics. However, this is not the end of the applications of metric transformations.

First, metric transformations can provide new solutions, for example one can obtain the FLRW or Schwarzschild metrics from flat spacetime with a conformal and disformal transformation respectively. They are also a helpful tool to discover new theories. This is the case of the so-called beyond Horndeski theories [171]. They also helped to realize that a Lagrangian which apparently has Ostrograski ghost contains hidden constraints [169], which are clear in the Hamiltonian formalism and the degeneracy of the kinetic matrix [230]. In this way, Refs. [175, 176] built a general class of scalar-tensor theories with a degenerate kinetic matrix. On the other hand, certain metric transformations could be a symmetry of the universe, for example conformal symmetry in the early universe [54–56] or in mimetic gravity [224, 225]. Thus, the study of scalar-tensor theories and metric transformations is a promising direction.

Second, we have seen that metric transformations allows us to split the matter and the gravity sector, if there is dominant field which drives the evo-

lution of the universe.⁴ This notion of frames provides simple interpretations of the physics in the gravity sector, e.g. inflation, and in the matter sector. In this way, we easily found blue tilt spectrum by coupling a curvaton to a super-inflating or a bouncing universe. Thus, we believe that simple interpretations are necessary to fully grasp a theory. For example, we could have discussed the system in Chap. 4 as a mixing of the curvaton and the inflaton. However, in this approach one loses the meaning of the coupling and the intuition on the curvaton's dynamics as well. We conclude that metric transformations are a powerful tool in the early universe.

It would be certainly interesting to study applications in other areas of cosmology. For example in the early universe, one could look for an efficient reheating model through a general disformal coupling to matter or one could derive an specific model from higher dimensional theories and test it with observations, in similar way to Ref. [166]. In the late time universe, even though we have strong constraints from the solar system scales there are also interesting applications of couplings to matter. For example, chameleon fields [83], screening mechanisms through disformal couplings [101] or explaining dark energy through a disformal coupling of standard model matter with dark matter [89].

We would like to finish by saying that, hopefully, this thesis showed that scalar-tensor theories are an attractive candidate for the theory of gravity. Even though they might not be the fundamental theory, they can be viewed as an effective theory with a wide range of applications. Thus all that was discussed in this thesis could be applied in future theoretical models. We expect that ongoing and future surveys⁵ of CMB polarization and Large Scale Structure will provide us with new data to test cosmological models and shine some light on alternative theories of gravity.

⁴This is certainly the case in cosmology

⁵For example, we have Squared Kilometer Array (SKA), Dark Energy Survey (DES), extended Baryon Oscillation Spectroscopic Survey (eBOSS), Euclid mission among many others.

Transformation rules

In this appendix we derive some useful formulas regarding metric transformations that are used in the main text. Since the order of the (regular) metric transformations commute, we can study the change of variables under a conformal and disformal transformation separately.

A.1 Conformal transformation

We start first with a conformal transformation given by

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}. \quad (\text{A.1})$$

The connection, i.e. the Christoffel symbols, are related by [154]

$$\mathcal{C}_{\mu\nu}^\sigma = \tilde{\Gamma}_{\mu\nu}^\sigma - \Gamma_{\mu\nu}^\sigma, \quad (\text{A.2})$$

where one can explicitly compute that

$$\mathcal{C}_{\mu\nu}^\alpha = 2\delta_{(\mu}^\alpha \nabla_{\nu)} \ln \Omega - g_{\mu\nu} \nabla^\alpha \ln \Omega. \quad (\text{A.3})$$

With that, the Ricci scalar transforms as

$$\tilde{R} = \Omega^{-2} \left[R - (D-1) \left(2 \frac{\square \Omega}{\Omega} - (D-4) \frac{g^{\mu\nu} \nabla_\mu \Omega \nabla_\nu \Omega}{\Omega^2} \right) \right]. \quad (\text{A.4})$$

A.2 Disformal transformation

We present now equivalent formulas for a disformal transformation given by

$$\bar{g}_{\mu\nu} = g_{\mu\nu} + B \nabla_\mu \phi \nabla_\nu \phi. \quad (\text{A.5})$$

One can find that the inverse metric is given by

$$\bar{g}^{\mu\nu} = g^{\mu\nu} - \frac{B}{1 - BX} \nabla^\mu \phi \nabla^\nu \phi, \quad (\text{A.6})$$

and the determinants are related by

$$\sqrt{-\bar{g}} = \sqrt{-g} \sqrt{1 - BX}, \quad (\text{A.7})$$

where $X \equiv -\nabla_\mu \phi \nabla^\mu \phi$. Let us call the change in the connection as

$$\mathcal{D}_{\mu\nu}^\sigma = \bar{\Gamma}_{\mu\nu}^\sigma - \Gamma_{\mu\nu}^\sigma. \quad (\text{A.8})$$

Then, one can find that the explicit expression in terms of ϕ is

$$\begin{aligned} \mathcal{D}_{\mu\nu}^\sigma &= \frac{B}{1-BX} \nabla^\sigma \phi \left(\nabla_\mu \nabla_\nu \phi + \nabla_{(\mu} \phi \nabla_{\nu)} \ln B + \frac{1}{2} \nabla^\gamma \phi \nabla_\gamma B \nabla_\mu \phi \nabla_\nu \phi \right) \\ &\quad - \frac{1}{2} \nabla^\sigma B \nabla_\mu \phi \nabla_\nu \phi. \end{aligned} \quad (\text{A.9})$$

The Ricci scalar is longer than in the conformal case and we find

$$\begin{aligned} \bar{R} &= R - \frac{B}{1-BX} \left((\Box \phi)^2 - \nabla_\mu \nabla_\nu \phi \nabla^\mu \nabla^\nu \phi + \nabla^\mu \ln B \left(\frac{1}{2} \nabla_\mu X + \nabla_\mu \phi \Box \phi \right) \right) \\ &\quad + \frac{1}{\sqrt{1-BX}} \nabla^\mu \left(\frac{B}{\sqrt{1-BX}} (\nabla_\mu \phi \nabla^\nu \phi \nabla_\nu \ln B + X \nabla_\mu \ln B + \nabla_\mu X + 2\Box \phi \nabla_\mu \phi) \right), \end{aligned} \quad (\text{A.10})$$

where note that the last line is a total derivative with respect to the metric $\bar{g}_{\mu\nu}$. We also give here the expression for the Ricci tensor as we make use of it to compute the transformation rules of $\mathcal{L}_5$, that is

$$\begin{aligned} \bar{\mathcal{R}}_{\mu\nu} &= \mathcal{R}_{\mu\nu} - \frac{B}{1-BX} (\Box \phi \nabla_\mu \nabla_\nu \phi - \nabla_\mu \nabla^\alpha \phi \nabla_\nu \nabla_\alpha \phi - \mathcal{R}_{\alpha\mu\lambda\nu} \nabla^\alpha \phi \nabla^\lambda \phi) \\ &\quad + \frac{1}{4} \frac{B^2}{(1-BX)^2} (\nabla_\mu X \nabla_\nu X - 2\nabla^\alpha \phi \nabla_\alpha X \nabla_\mu \nabla_\nu \phi) \\ &\quad - \frac{1}{2} \Box B \nabla_\mu \phi \nabla_\nu \phi + \frac{1}{1-BX} \left(\Box \phi \nabla_{(\mu} \phi \nabla_{\nu)} B \right. \\ &\quad + (-X \nabla_a B \nabla^a B + 2B \Box \phi \nabla_a \phi \nabla^a B + 2B \nabla^a \phi \nabla^b \phi \nabla_a \nabla_b B) \frac{1}{4} \nabla_\mu \phi \nabla_\nu \phi \\ &\quad + \nabla^a \phi \nabla_{(\mu} \phi \nabla_{\nu)} \nabla_a B - \nabla^a B \nabla_{(\mu} \phi \nabla_{\nu)} \nabla_a \phi + \frac{1}{2} X \nabla_\mu \nabla_\nu B + \nabla_a \phi \nabla^a B \nabla_\mu \nabla_\nu \phi \Big) \\ &\quad + \frac{1}{(1-BX)^2} \left(\frac{1}{2} \nabla_{(\mu} B \nabla_{\nu)} X + \frac{1}{4} X^2 \nabla_\mu B \nabla_\nu B \right. \\ &\quad \left. + \frac{1}{4} \nabla^a \phi \nabla_a (BX) (2\nabla_{(\mu} \phi \nabla_{\nu)} B + B \nabla_b \phi \nabla^b B \nabla_\mu \phi \nabla_\nu \phi) \right). \end{aligned} \quad (\text{A.11})$$

It is also useful to show

$$\begin{aligned} \bar{G}_{\mu\nu} \bar{\nabla}^\mu \phi \bar{\nabla}^\nu \bar{X} &= \frac{1}{(1-BX)^3} \left(G_{\mu\nu} \nabla^\mu \phi \nabla^\nu X \right. \\ &\quad \left. + \frac{1}{2} \frac{B}{1-BX} \left[\nabla^\mu X \nabla^\nu X (\Box \phi g_{\mu\nu} + \nabla_\mu \nabla_\nu \phi) + \nabla_\alpha \phi \nabla^\alpha X ((\Box \phi)^2 - \nabla_\mu \nabla_\nu \phi \nabla^\mu \nabla^\nu \phi) \right] \right) \end{aligned} \quad (\text{A.12})$$

APPENDIX B

Horndeski Lagrangian

The Generalized Galileon or Horndeski Lagrangian [79] is a general Lagrangian with second order equations of motion. Its form follows from particular anti-symmetric combinations which eliminate second time derivative squared terms in the action. It is given by

$$S = \int d^4x \sqrt{-g} \sum_i \mathcal{L}_i, \quad (\text{B.1})$$

where

$$\mathcal{L}_2 = G_2(\phi, X), \quad (\text{B.2})$$

$$\mathcal{L}_3 = G_3(\phi, X) \square \phi, \quad (\text{B.3})$$

$$\mathcal{L}_4 = G_4(\phi, X) R + G_{4,X} ((\square \phi)^2 - \nabla_\mu \nabla_\nu \phi \nabla^\mu \nabla^\nu \phi), \quad (\text{B.4})$$

$$\begin{aligned} \mathcal{L}_5 = & G_5(\phi, X) G^{\mu\nu} \nabla_\mu \nabla_\nu \phi \\ & + G_{5,X} ((\square \phi)^3 - 3 \square \phi \nabla_\mu \nabla_\nu \phi \nabla^\mu \nabla^\nu \phi + 2 \nabla_\mu \nabla_\nu \phi \nabla^\mu \nabla^\alpha \phi \nabla_\alpha \nabla_\mu \phi). \end{aligned} \quad (\text{B.5})$$

We have defined $\square \phi \equiv \nabla^\mu \nabla_\mu \phi$ and $G^{\mu\nu}$ is the Einstein tensor. Note the particular relation between the coefficients in front of R and $(\square \phi)^2$ as well as in $\mathcal{L}_5$. In fact, we now know that this relation can be relaxed in several extensions of the Horndeski Lagrangian. In this extensions they either use the fact that the kinetic matrix might be degenerate, i.e. lead to extra constraints, or they add higher derivatives spatially covariant terms [174–179].

We can see how the coefficients of the Horndeski Lagrangian change under a disformal transformation given by

$$\bar{g}_{\mu\nu} = g_{\mu\nu} + B \nabla_\mu \phi \nabla_\nu \phi. \quad (\text{B.6})$$

We take B to be constant for simplicity. Then we see that

$$G_2[X] \rightarrow G_2[\bar{X}] \sqrt{1 + B\bar{X}}, \quad (\text{B.7})$$

$$G_{3,X}[X] \rightarrow G_{3,\bar{X}}[\bar{X}] (1 + B\bar{X})^{-5/2}, \quad (\text{B.8})$$

$$G_4[X] \rightarrow G_4[\bar{X}] \sqrt{1 + B\bar{X}} \quad (\text{B.9})$$

and

$$G_{5,X}[X] \rightarrow G_{5,\bar{X}}[\bar{X}] (1 + B\bar{X})^{-5/2}. \quad (\text{B.10})$$

Singular metric transformations: Mimetic gravity

In the main body of the thesis we discussed what a metric transformation is and how this notion is relevant for gravity. In particular, we found that a general form of metric transformations is given by the so-called disformal transformation (see Appendix A), that is

$$\bar{g}_{\mu\nu} = A(\phi, X) (g_{\mu\nu} + B(\phi, X) \nabla_\mu \phi \nabla_\nu \phi) , \quad (\text{C.1})$$

where $g_{\mu\nu}$ is the metric, ∇_μ is the covariant derivative of $g_{\mu\nu}$, ϕ is a scalar field, X is its kinetic term $X \equiv -\nabla_\mu \phi \nabla^\mu \phi$ and A and B are two free functions. This transformation has a singular case though [115, 169]. It is easy to see if we consider the inverse metric, i.e.

$$\bar{g}^{\mu\nu} = A^{-1}(\phi, X) \left(g^{\mu\nu} - \frac{B(\phi, X)}{1 - X B(\phi, X)} \nabla^\mu \phi \nabla^\nu \phi \right) , \quad (\text{C.2})$$

and contract it with the scalar field derivatives $\nabla_\mu \phi \nabla_\nu \phi$. This yields

$$\bar{g}^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi = \frac{A^{-1}(\phi, X)}{1 - X B(\phi, X)} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi . \quad (\text{C.3})$$

We can guess that there is a singular behavior if $\bar{g}^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi = b(\phi)$, which gives

$$B(\phi, X) = \frac{1}{X} + \frac{1}{A(\phi, X) b(\phi)} . \quad (\text{C.4})$$

This result can also be obtained from looking at the Jacobian of the transformation. For simplicity let us take $b = -1$. This actually gives rise to a constraint in the original action given by

$$\bar{g}^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + 1 = 0 \quad (\text{C.5})$$

We can introduce such constraint directly into the action and study its behavior.

For simplicity let us consider a simple example to understand why it is called mimetic gravity. First take the action to be

$$S = \int d^4x \sqrt{-g} \left\{ \frac{M_{pl}^2}{2} R + \mathcal{L}(\phi, X) \right\} \quad (\text{C.6})$$

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and we do a singular conformal transformation which is given by

$$g_{\mu\nu} = X_f f_{\mu\nu}. \quad (\text{C.7})$$

where $X = -g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi$ and $X_f = -f^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi$, and we have mimetic gravity from a singular transformation. In that case, the action can be rewritten as

$$S = \int d^4x \sqrt{-g} \left\{ \frac{M_{pl}^2}{2} R + \mathcal{L}(\phi) + \lambda(X - 1) \right\} \quad (\text{C.8})$$

The equations of motion are given by

$$M_{pl}^2 G_{\mu\nu} = g_{\mu\nu} \mathcal{L} + \lambda \nabla_\mu \phi \nabla_\nu \phi, \quad (\text{C.9})$$

$$\nabla_\mu (\lambda \nabla^\mu \phi) + \mathcal{L}_\phi = 0, \quad (\text{C.10})$$

$$X = 1. \quad (\text{C.11})$$

Now consider a FLRW space-time (2.7). The constraint $X = 1$ in facts sets a preferred reference frame where $\phi = t$. Plugging this into the field equations of motion (C.11) we find

$$\frac{1}{a^3} \frac{d}{dt} (a^3 \lambda) = \mathcal{L}_\phi. \quad (\text{C.12})$$

Note that if $\mathcal{L} = 0$ then we have that $\lambda \propto a^{-3}$. This tells us that actually this constraint obtained from a singular metric transformation behaves just like dust. Even the energy-momentum tensor of ϕ , i.e. $T_{\mu\nu} = \lambda \nabla_\mu \phi \nabla_\nu \phi$ is identical to a pressureless perfect fluid with 4-velocity equals to the gradient of the scalar field $\nabla_\mu \phi$. The name mimetic gravity comes from the fact that ϕ mimics pressureless matter for $\mathcal{L} = 0$. There are several generalizations of this model by including higher derivatives. For example, see Ref. [259] and references within. It is interesting to note that mimetic gravity is a particular limit of Hořava-Lifshitz gravity and Einstein-Aether theories. It has been shown recently that generalized mimetic gravity contains a ghost [259] which can be cured by including higher spatial derivatives [260].

Disformal transformation: Examples

In this appendix we will see some examples of the disformal invariance in cosmology and some interesting cases where disformal transformation connects solutions of the Einstein Equations.

D.1 General gauge

Take a disformal transformation given by

$$\bar{g}_{\mu\nu} = g_{\mu\nu} + B\nabla_\mu\phi\nabla_\nu\phi. \quad (\text{D.1})$$

Under the 3 + 1-decomposition along the n_μ direction we have

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu. \quad (\text{D.2})$$

The line element is

$$ds^2 = -N^2 dt^2 + h_{ij} (N^i dt + dx^i) (N^j dt + dx^j), \quad (\text{D.3})$$

where $n_\mu dx^\mu = -Ndt$, N is the lapse and N^i is the shift vector. If we do a disformal transformation the hypersurface orthogonal vector transforms as

$$\bar{n}_\mu = n_\mu \sqrt{\frac{1 - BX}{1 + BD_i\phi D^i\phi}}, \quad (\text{D.4})$$

and the spatial metric as

$$\bar{h}_{ij} = h_{ij} + BD_i\phi D_j\phi, \quad (\text{D.5})$$

where D_i is the covariant derivative with respect to h_{ij} . Consequently, one can check that the lapse and the shift respectively transform as

$$\bar{N} = N \sqrt{\frac{1 - BX}{1 + BD_i\phi D^i\phi}} \quad \text{and} \quad \bar{N}^i = N^i + \frac{BD^i\phi}{1 + BD_i\phi D^i\phi} \mathcal{L}_n\phi, \quad (\text{D.6})$$

where $\mathcal{L}_n\phi = n^\mu \nabla_\mu \phi$ is the Lie derivative along the orthonormal direction.

D.2 Cosmology

Let us see two applications to cosmology. In the first example I will show that although a disformal transformation between two metric is interpreted redefinition of the time coordinate, the two metric are indeed different and have different evolutions. In the second example, I will show you that even so the redshift is frame independent.

Example 1: power-law frames. Consider an expanding universe where the scale factor is a power-law of time with power-law index $p > 1$. We have seen in Chapter 4 that the line element is given by

$$ds^2 = -dt^2 + a^2 d\mathbf{x}^2 = a^2(\eta) (-d\eta^2 + d\mathbf{x}^2), \quad (\text{D.7})$$

where η is the conformal time and the scale factor a is

$$a \propto t^p \quad \Rightarrow \quad a \propto \eta^{-\frac{p}{p-1}}. \quad (\text{D.8})$$

The time coordinates respectively run from

$$0 < t < \infty \quad \Rightarrow \quad -\infty < \eta < 0. \quad (\text{D.9})$$

For this universe the comoving hubble radius is given by $\mathcal{H}^{-1}$, where

$$\mathcal{H} = \frac{1}{a} \frac{da}{d\eta} = \frac{p}{p-1} \frac{1}{-\eta}. \quad (\text{D.10})$$

See how the comoving hubble radius decrease, as it is usual in an accelerated expanding universe. Now, do a disformal transformation given by

$$\bar{g}_{\mu\nu} = g_{\mu\nu} - (a^2(t) - 1)\delta_\mu^t \delta_\nu^t. \quad (\text{D.11})$$

This transformation leads us to a line element given by

$$d\bar{s}^2 = -d\eta^2 + A(\eta)^2 d\mathbf{x}^2 = A^2(\tau) (-d\tau^2 + d\mathbf{x}^2). \quad (\text{D.12})$$

In this barred frame the scale factor evolves as

$$A(\eta) \propto \eta^{-\frac{p}{p-1}} \quad \Rightarrow \quad A(\tau) \propto \tau^p, \quad (\text{D.13})$$

where the time coordinates respectively runs from

$$-\infty < \eta < 0 \quad \Rightarrow \quad 0 < \tau < \infty. \quad (\text{D.14})$$

The barred comoving hubble radius is given by $\bar{\mathcal{H}}^{-1}$, where

$$\bar{\mathcal{H}} = \frac{p}{\tau}. \quad (\text{D.15})$$

This time the comoving hubble radius increase! It is like the super-inflationary universe in Chapter 4.

Example 2: Redshift. Let us briefly check the frame independence of the redshift as we did in the conformal case. First, we start with matter universally coupled to a flat FLRW background in the matter frame,

$$d\bar{s}^2 = \bar{a}^2(\bar{\eta}) (-d\bar{\eta}^2 + d\mathbf{x}^2) . \quad (\text{D.16})$$

We can go now to another frame, e.g. the gravity frame, related by a disformal transformation Eq. (3.48), which yields

$$ds^2 = -\bar{a}^2(\bar{\eta})\alpha^{-2}(\bar{\eta})d\bar{\eta}^2 + \bar{a}^2(\bar{\eta})d\mathbf{x}^2 = a^2(\eta) (-d\eta^2 + d\mathbf{x}^2) . \quad (\text{D.17})$$

Note that $\bar{a}(\bar{\eta}) = a(\eta(\bar{\eta}))$ and in the last step we used the time redefinition $d\bar{\eta} = \alpha d\eta$.

Let us focus now on photons since we are the source of the observation. Photons follow null geodesics of the metric $\bar{g}_{\mu\nu}$, that is $d\bar{s}^2 = 0$. In the matter frame, the four-momentum of the photon is given by $\bar{k}^\mu = (\bar{k}^\eta, \bar{\mathbf{k}})$. The energy measure by a comoving matter observer, i.e. its four-momentum reads $\bar{u}_\mu = (-\bar{a}(\bar{\eta}), 0)$, is given by

$$\bar{\mathcal{E}} = -\bar{k}^\mu \bar{u}_\mu = \bar{a}(\bar{\eta}) \bar{k}^\eta. \quad (\text{D.18})$$

It is important to remind the reader that scalar quantities are invariant under a disformal transformation and so it is the above energy measured by a comoving observer. We could also obtain the same results if we use the transformation rules for a vector (3.61), that is $\bar{k}^\eta = \alpha^{-1} k^\eta$. For instance, the energy in the unbarred frame reads

$$\mathcal{E} = -k^\mu u_\mu = a(\eta) k^\eta. \quad (\text{D.19})$$

Now using that $\bar{k}^\eta = \alpha \bar{k}^\eta = k^\eta$ we are led to conclude the measured energy by the comoving matter observer is frame independent, $\bar{\mathcal{E}} = \mathcal{E}$. As a result the redshift defined as the ratio between the measured photon energy at emission and observation,

$$1 + z \equiv \mathcal{E}_{emit}/\mathcal{E}_{obs} = \frac{a(\eta_{obs})}{a(\eta_{emit})} = \frac{\bar{a}(\bar{\eta}_{obs})}{\bar{a}(\bar{\eta}_{emit})}, \quad (\text{D.20})$$

is also frame independent. Note that again this result is consistent with cosmological disformal transformations being equivalent to a rescaling of the time.

As a final remark, the (physical) speed of light measured by a *matter observer* is frame independent as well. To see this, take the null geodesics of the photon, i.e. $d\bar{s}^2 = 0$, which also means that $\bar{k}^\eta = |\bar{\mathbf{k}}| = k^\eta$. Extensions of this result include the frame independence of the luminosity distance. These results agree Refs. [92, 109] if matter is universally coupled.

D.3 Kerr-Schild ansatz

There is one interesting application of disformal transformations. So far they are only a mathematical trick, but if it were realized dynamically then we would obtain an interesting model. In any case, it is known that we can use the Kerr-Schild ansatz to obtain new solutions of Einstein Equations from known solutions. For example, we can obtain Schwarzschild geometry by

$$d\bar{s}_{\text{BH}}^2 = ds_0^2 + \frac{2GM}{r} (k_\mu dx^\mu)^2, \quad (\text{D.21})$$

where ds_{BH}^2 is the Schwarzschild geometry, ds_0^2 is flat minkowsky spacetime and k_μ is a null vector with respect to both metrics, i.e. $g_0^{\mu\nu} k_\mu k_\nu = \bar{g}_{\text{BH}}^{\mu\nu} k_\mu k_\nu = 0$. Note that it is exactly a disformal transformation! For example, take $k_\mu dx^\mu = -dt + dr$ in the minkowsky metric and see that you obtain the Schwarzschild geometry in Eddington-Finkelstein coordinates, i.e.

$$d\bar{s}_{\text{BH}}^2 = - \left(1 - \frac{2GM}{r}\right) dt^2 - \frac{4GM}{r} dt dr + \left(1 + \frac{2GM}{r}\right) dr^2 + r^2 d\Omega. \quad (\text{D.22})$$

We can recover the Schwarzschild coordinates by changing the time coordinate to

$$d\bar{t}_{\text{BH}} = dt + \frac{2GM/r}{1 - 2GM/r} dr. \quad (\text{D.23})$$

One can also obtain Kerr and Vaidya metrics in this way [200].

Another straightforward application is that we can obtain dS space from flat Minkowsky as well. This time take

$$ds_{\text{dS}}^2 = ds_0^2 + \frac{r^2}{\ell^2} k_\mu k_\nu, \quad (\text{D.24})$$

with again $k_\mu dx^\mu = -dt + dr$. We obtain dS in the analogy of Eddington-Finkelstein coordinates, i.e.

$$d\bar{s}_{\text{BH}}^2 = - \left(1 - \frac{r^2}{\ell^2}\right) dt^2 - 2\frac{r^2}{\ell^2} dt dr + \left(1 + \frac{r^2}{\ell^2}\right) dr^2 + r^2 d\Omega. \quad (\text{D.25})$$

We can recover the static dS metric by transforming time to

$$d\bar{t}_{\text{dS}} = dt + \frac{r^2/\ell^2}{1 - r^2/\ell^2} dr. \quad (\text{D.26})$$

It is particularly interesting that a null disformal transformation creates a horizon in the original spacetime.

Cosmological perturbations during inflation

In this appendix we give details on calculations of cosmological perturbations at linear order in perturbation, second order in the action. I will consider the following action:

$$S = \int d^4x \sqrt{-g} \left\{ \frac{M_{pl}^2}{2} R + P(\phi, X) \right\} , \quad (\text{E.1})$$

where R is the Ricci scalar and $P(\phi, X)$ is a general function of ϕ is the inflaton field and X its kinetic term, i.e. $X \equiv -\nabla_\mu \phi \nabla^\mu \phi$. Note that $P_X > 0$ so that the system is stable. This model is known as K-inflation [261]. We work in the ADM decomposition [262], that is

$$ds^2 = -N^2 dt^2 + h_{ij} (N^i dt + dx^i) (N^j dt + dx^j) , \quad (\text{E.2})$$

where N is the lapse function, N_i is the shift vector and h_{ij} is the spatial metric, where $h_{ij} = a^2 \delta_{ij}$ for an homogeneous, isotropic and spatially flat universe. In this decomposition we have

$$S = \int d^4x \sqrt{-g} \frac{M_{pl}^2}{2} R = \int d^3x dt N \sqrt{h} \frac{M_{pl}^2}{2} \{ {}^{(3)}R + K_{ij} K^{ij} - K^2 \} , \quad (\text{E.3})$$

where ${}^{(3)}R$ is the Ricci scalar of h_{ij} and K_{ij} is the extrinsic curvature, which is given by

$$K_{ij} = \frac{1}{2N} \left(\dot{h}_{ij} - 2D_{(i} N_{j)} \right) , \quad (\text{E.4})$$

where D_i is the covariant derivative of h_{ij} . For the inflaton field we have that

$$X = \frac{1}{N^2} \left(\dot{\phi} - N^i D_i \phi \right)^2 - D_i \phi D^i \phi . \quad (\text{E.5})$$

The background equations of motion are given by

$$3H^2 M_{pl}^2 = 2XP_X - P, \quad (\text{E.6})$$

$$\dot{H} M_{pl}^2 = -XP_X \quad (\text{E.7})$$

and

$$\frac{1}{a^3} \frac{d}{dt} (a^3 P_X \dot{\phi}) = P_\phi. \quad (\text{E.8})$$

Note that from the equations of motion we can identify the scalar field as a perfect fluid with energy density and pressure respectively given by

$$\rho = 2XP_X - P \quad \text{and} \quad p = P. \quad (\text{E.9})$$

With this identification we find that the equation of state is

$$\omega = \frac{p}{\rho} = -1 + \frac{2}{3}\varepsilon. \quad (\text{E.10})$$

The slow-roll parameters for this model are given by

$$\varepsilon \equiv \frac{-\dot{H}}{H^2} \quad ; \quad \eta \equiv \frac{\dot{\varepsilon}}{H\varepsilon} \quad ; \quad s \equiv \frac{\dot{c}_s}{Hc_s} \quad ; \quad \delta \equiv \frac{\ddot{\phi}}{H\dot{\phi}} \quad \text{and} \quad q \equiv \frac{\dot{P}_X}{HP_X}. \quad (\text{E.11})$$

where c_s is the scalar sound speed, i.e. $c_s^{-2} \equiv 1 + 2XP_{XX}/P_X$. There is a relation among them, i.e. $\eta = 2(\varepsilon + \delta) + q$ and only one has to require $\varepsilon, \eta, s \ll 1$ for inflation to happen.

E.1 Perturbations

We can perturb the metric and the inflaton in a general way, that is

$$\begin{aligned} \phi &= \phi_0 + \delta\phi \quad ; \quad N = 1 + A \quad ; \quad N_i = \partial_i \beta + B_i; \\ h_{ij} &= a^2(t) \left((1 + 2\psi) \delta_{ij} + 2\partial_i \partial_j E + 2\partial_{(i} F_{j)} + \gamma_{ij} \right), \end{aligned} \quad (\text{E.12})$$

where $\partial_i B^i = \partial_i F^i = \partial_i \gamma^{ij} = 0$ and $\delta^{ij} \gamma_{ij} = 0$. The trace of the perturbed spacial metric is called the curvature perturbation and it is given by

$$\mathcal{R} = \psi - \frac{1}{3} \Delta E, \quad (\text{E.13})$$

where $\Delta = \partial_i \partial^i$. However not all variables are independent since we have coordinate transformation degrees of freedom, i.e. diffeomorphism invariance. For that, we need to study how do the variables change under a coordinate transformation.

E.1.1 Gauge transformations

Let us study how the perturbations variables we have defined in Eq. (E.12) transforms under a general coordinate transformation given by

$$x^\alpha \rightarrow \tilde{x}^\alpha = x^\alpha + \xi^\alpha, \quad (\text{E.14})$$

where

$$\xi^\alpha = (T, \partial^i L + S^i) \quad (\text{E.15})$$

and $\partial_i S^i = 0$. We know that under an infinitesimal coordinate transformation tensors transforms according to its lie derivative along ξ [154], that is

$$\tilde{Q} = Q - \mathcal{L}_\xi Q, \quad (\text{E.16})$$

where Q is any tensor with any indexes. In particular for the metric we have

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} - \mathcal{L}_\xi g_{\mu\nu}, \quad (\text{E.17})$$

where

$$\mathcal{L}_\xi g_{\mu\nu} = 2g_{\sigma(\mu} \partial_{\nu)} \xi^\sigma + \xi^\sigma \partial_\sigma g_{\mu\nu}. \quad (\text{E.18})$$

Applying this formula to Eq. (E.12) we find the following transformation rules:

$$\delta\phi \rightarrow \delta\phi - \dot{\phi} T, \quad (\text{E.19})$$

$$A \rightarrow A - \dot{T}, \quad (\text{E.20})$$

$$\beta \rightarrow \beta - \dot{L} + T, \quad (\text{E.21})$$

$$\psi \rightarrow \psi - HT - \frac{1}{3}\Delta L, \quad (\text{E.22})$$

$$E \rightarrow E - L, \quad (\text{E.23})$$

$$B_i \rightarrow B_i - \dot{S}_i, \quad (\text{E.24})$$

$$F_i \rightarrow F_i - S_i, \quad (\text{E.25})$$

$$\gamma_{ij} \rightarrow \gamma_{ij}. \quad (\text{E.26})$$

In particular note that the tensor modes, i.e. the traceless and transverse part of the spatial metric γ_{ij} , are gauge invariant at first order. At second order in perturbation tensor modes are not gauge invariant and scalar, vector and tensor mode mix. One way to deal with this gauge freedom is to work with gauge invariant variables.

E.1.1.1 Gauge invariant variables

One can build many gauge invariant variables as combinations of the previous variables. Here we only focus on the gauge invariant curvature perturbation, that is given by

$$\zeta = \psi - \frac{H}{\dot{\phi}} \delta\phi - \frac{1}{3} \Delta E = \mathcal{R} - \frac{H}{\dot{\phi}} \delta\phi. \quad (\text{E.27})$$

It is easy to check that it is gauge invariant. For other variables see Refs. [18, 135]. This can be generalized to the case of a perfect fluid. Then one has that

$$\zeta = \psi - \frac{1}{3}\Delta E - \frac{1}{3}\frac{\delta\rho}{\rho + p}, \quad (\text{E.28})$$

where ρ and p respectively are the energy density and pressure of the fluid and $\delta\rho$ is the perturbation of the energy density. When one studies inflationary cosmological perturbations there two popular gauge choices. We call slicing when we fix the time coordinate degree of freedom and gauge when we fix time and spatial gauges d.o.f.

Uniform- ϕ or comoving slicing: In this slicing we choose $\delta\phi = 0$ and then ζ becomes the comoving curvature perturbation, i.e.

$$\mathcal{R}_c = \zeta \Big|_{\delta\phi=0} = \psi_c - \frac{1}{3}\Delta E_c, \quad (\text{E.29})$$

where the subindex c means that quantities are evaluated in the comoving slicing. The comoving gauge is when $\delta\phi = E = F_i = 0$.

Flat slicing: In the flat slicing one chooses a gauge where the spatial curvature is zero, i.e. $\mathcal{R} = 0$, which yields

$$\zeta \Big|_{\mathcal{R}=0} = -\frac{H}{\dot{\phi}}\delta\phi_f, \quad (\text{E.30})$$

where the subindex f means that quantities are evaluated in the flat slicing. Then one usually works with the inflaton perturbation $\delta\phi_f$. The flat gauge is when $\psi = E = F_i = 0$. Thus, we readily see that the relation among these previous two gauges is

$$\mathcal{R}_c = -\frac{H}{\dot{\phi}}\delta\phi_f. \quad (\text{E.31})$$

Newtonian gauge: In the Newtonian gauge one chooses $\alpha = \Psi$, $\psi = \Phi$ and $\beta = S^i = E = 0$. Then

$$\zeta_N = \Phi - \frac{H}{\dot{\phi}}\delta\phi_N, \quad (\text{E.32})$$

where the subindex N stands for Newtonian gauge. The $i - j$ Einstein equations yield $\Psi = -\Phi$. With this result, after solving the $0 - i$ Einstein Equations [18] one has

$$\dot{\Phi} + H\Phi = -XP_X \frac{\delta\phi_N}{\dot{\phi}} \quad (\text{E.33})$$

and thus

$$\zeta_N = \Phi + \frac{\Phi + \dot{\Phi}/H}{\varepsilon}. \quad (\text{E.34})$$

This is what we used in Chapter 2 Eq. (2.65), where $1 + \omega = \frac{2}{3}\varepsilon$.

E.1.2 Second order action

We are now ready to find the equations of motion for the perturbations and later quantize it. The way to proceed is as follows. One expands the action up to second order (the first order vanish as a consequence of the background equations of motion) and find out that A , β and S^i are non-dynamical. Taking the variation with respect to them yield the so-called Hamiltonian and Momentum constraint. We have to solve them and use the solution in the action. This is called the Faddeev-Jackiw method [263]. To simplify the calculations we will choose a particular gauge a priori and reduce the system. Furthermore, scalar, vector and tensor perturbations decouple at first order so we can treat them separately [18].

E.1.2.1 Scalar

The second order action in the uniform- ϕ slicing with $E = 0$ and in conformal time, that is $dt = a d\eta$, is given by

$$S_{2,S} = \int d\eta d^3x \frac{a^2 \varepsilon M_{pl}^2}{c_s^2} (\mathcal{R}_c'^2 - c_s^2 (\partial \mathcal{R}_c)^2), \quad (\text{E.35})$$

where the constraints give [264]

$$A_c = \frac{\dot{\mathcal{R}}_c}{H} \quad \text{and} \quad \Delta\beta_c = -\frac{\Delta\mathcal{R}_c}{H} + \frac{a^2 \varepsilon}{c_s^2} \dot{\mathcal{R}}_c. \quad (\text{E.36})$$

In this form, the scalar mode is indeed massless. On the other hand, the second order action in the flat gauge is given by

$$S_{2,S} = \int d\eta d^3x \frac{a^2 P_X}{2c_s^2} (\delta\phi_f'^2 - c_s^2 (\partial\delta\phi_f)^2 - m_{eff}^2 \delta\phi_f^2), \quad (\text{E.37})$$

where

$$m_{eff}^2 \equiv -(\varepsilon + \delta)H^2 \left(2 + p + \delta - 2s + \frac{\dot{\varepsilon} + \dot{\delta}}{H(\varepsilon + \delta)} \right). \quad (\text{E.38})$$

It can be shown that the effective mass is very small, i.e. $m_{eff}/H \ll 1$, even though we did not require neither p nor δ to be small. This time the constraints yield

$$A_f = \frac{\dot{\phi} P_X}{2H} \delta\phi_f \quad \text{and} \quad \Delta\beta_f = -\frac{a^2 \varepsilon}{c_s^2} \frac{d}{dt} \left(\frac{H}{\dot{\phi}} \delta\phi_f \right). \quad (\text{E.39})$$

E.1.2.2 Vector

For the vector perturbations we have that in both slicings the constraints yield $S^i = 0$ and we obtain

$$S_{2,V} = \int d\eta d^3x a^2 M_{pl}^2 \partial_i F_j' \partial^i F'^j. \quad (\text{E.40})$$

This means that the vector modes are non-dynamical and decay as a^{-3} . We knew from the beginning that vector modes are non-dynamical for two reason. The first one is that the inflaton is a scalar field and thus it cannot source vector perturbations. The second is that from the Hamiltonian analysis we know that GR has only 2 tensor degrees of freedom (see Appendix H).

E.1.2.3 Tensor

Lastly, for the tensor mode one has

$$S_{2,T} = \int d\eta d^3x a^2 M_{pl}^2 \left(\gamma'_{ij} \gamma'^{ij} - \partial_k \gamma_{ij} \partial^k \gamma^{ij} \right). \quad (\text{E.41})$$

They essentially behave like a massless scalar field in an expanding background.

E.2 Quantization of perturbations**E.2.1 Scalar modes**

Proceeding as usual in canonical quantizations, one first goes to Fourier space, that is

$$\mathcal{R}_c(\eta, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \tilde{\mathcal{R}}_c(\eta, k) e^{i\vec{k} \cdot \vec{x}}. \quad (\text{E.42})$$

Then we quantize the system by introducing creation and annihilation operators, i.e.

$$\tilde{\mathcal{R}}_c(\eta, k) = u_k(\eta) a_{\mathbf{k}} + u_{-k}^*(\eta) a_{-\mathbf{k}}^\dagger, \quad (\text{E.43})$$

with the commutation relations

$$[a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger] = (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}') \quad (\text{E.44})$$

which are equivalent to

$$\{u_k, \pi_{u,k}\} = u_k^*(\eta) u_k'(\eta) - u_k(\eta) u_k^{*\prime}(\eta) = -i \quad (\text{E.45})$$

where $\pi_{u,k} = \delta\mathcal{L}/\delta u_k$. Recall that u_k satisfies the same equations of motion as $\tilde{\mathcal{R}}_c(\eta, k)$ which are

$$\tilde{\mathcal{R}}_c'' + 2\frac{z'}{z}\tilde{\mathcal{R}}_c' + c_s^2 k^2 \tilde{\mathcal{R}}_c = 0, \quad (\text{E.46})$$

where $z^2 \equiv a^2 \varepsilon / c_s^2$. This is the so-called *Mukhanov-Sasaki* equation [234, 235]. First note that in the limit $c_s k \ll aH$ the e.o.m. for $\tilde{\mathcal{R}}_c$ have a constant solution. This means that the mode does not evolve on super-horizon scales, i.e. the mode is “frozen”. Now, if the initial vacuum state on subhorizon scales, i.e. $c_s k \gg aH$, is the lowest energy state, so-called Bunch-Davies, one has that under WKB approximation and assuming $\varepsilon, \eta, s \ll 1$ the mode function is given by, after imposing Eq. (E.45),

$$\tilde{\mathcal{R}}_c(k, \eta) = \frac{a^{-1}}{M_{pl} \sqrt{4c_s \varepsilon} k} e^{-ikc_s \eta}, \quad (\text{E.47})$$

in the comoving gauge. It should be noted in curved space time the vacuum is degenerate to arbitrary Bogoliubov transformations [43]. In inflation we assume that the vacuum deep inside the horizon is given by the de Sitter invariant vacuum, a.k.a. Euclidean vacuum, which is de Sitter invariant. The vacuum is defined by

$$a_{\mathbf{k}}|0\rangle = 0. \quad (\text{E.48})$$

Then we can compute the two point function of the curvature perturbation, i.e. the power spectrum, to be after horizon crossing at η_* , i.e. at $aH = c_s k$,

$$\langle 0 | \tilde{\mathcal{R}}_c(k, \eta_*) \tilde{\mathcal{R}}_c(k', \eta_*) | 0 \rangle = (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}') \left| \tilde{\mathcal{R}}_c(k, \eta_*) \right|^2 \quad (\text{E.49})$$

where

$$\left| \tilde{\mathcal{R}}_c(k, \eta) \right|^2 = \frac{H^2}{4c_s \varepsilon k^3 M_{pl}^2}. \quad (\text{E.50})$$

Note that we solved the curvature perturbation in the sub-horizon regime and match it with the super-horizon constant solution. This is a good approximation as long as $\varepsilon, \eta, s \ll 1$. Thus the power spectrum, i.e.

$$\langle 0 | \mathcal{R}_c(\mathbf{x}) \mathcal{R}_c(\mathbf{x}') | 0 \rangle = \int d \ln k \mathcal{P}_{\mathcal{R}_c}(k), \quad (\text{E.51})$$

is given by

$$\mathcal{P}_{\mathcal{R}_c}(k) = \frac{4\pi k^3}{(2\pi)^3} |\tilde{\mathcal{R}}_{c,k}|^2 = \frac{1}{(2\pi)^2} \frac{H^2}{2c_s \varepsilon M_{pl}^2} \Big|_{H=k/a}. \quad (\text{E.52})$$

E.2.2 Tensor modes

We do a similar procedure for the tensor modes taking into account their two polarizations. Thus we decompose them as

$$\gamma_{ij}(\eta, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \sum_{s=+, \times} \varepsilon_{ij}^s(k) \tilde{\gamma}_{\mathbf{k}}^s(\eta) e^{i\vec{k} \cdot \vec{x}}, \quad (\text{E.53})$$

where ε_{ij}^s are the polarization tensors. They satisfy $\varepsilon_{ii} = k^i \varepsilon_{ij} = 0$ and

$$\varepsilon_{ij}^s(k) \varepsilon_{ij}^{s'}(k) = 2\delta_{ss'}. \quad (\text{E.54})$$

We now decompose $\tilde{\gamma}_{\mathbf{k}}^s(\eta)$ into creation and annihilation operators and proceed as we did before. Then, the equations of motion for the tensor modes are given by

$$\tilde{\gamma}_{\mathbf{k}}'' + 2\mathcal{H}\tilde{\gamma}_{\mathbf{k}}' + k^2\tilde{\gamma}_{\mathbf{k}} = 0, \quad (\text{E.55})$$

for both polarizations. Again, on superhorizon scales, that is $k \ll aH$, we have a constant solution. On the other hand, deep inside the horizon, i.e. $k \gg aH$ we have

$$\tilde{\gamma}_{\mathbf{k}}^s(\eta) = \frac{a^{-1}}{M_{pl}\sqrt{2k}} e^{-ik\eta}. \quad (\text{E.56})$$

Thus power spectrum for the tensor modes after horizon crossing is found to be

$$\mathcal{P}_T(k) = \sum_s \frac{4\pi k^3}{(2\pi)^3} |\tilde{\gamma}_k^s|^2 \approx 2 \left(\frac{H}{M_{pl}\pi} \right)_{H=k/a}^2. \quad (\text{E.57})$$

It is important to note that if tensor modes were detected then we would know the value of H during inflation, i.e. the energy scale during inflation.

APPENDIX F

δN formalism

The idea behind δN formalism [265, 266] is quite simple. First, note that, by definition, the total number of e-folds N contains the information about all the expansion of the universe at both linear and *non-linear* level. Formally speaking, N is defined as the integral over time of the expansion parameter, i.e. $\theta \equiv \nabla_\mu n^\mu$ where n^μ is the orthonormal vector of our space-like hypersurface [154, 267], namely

$$N(s_i, s_f, \mathbf{x}) = \int_{\tau_i(s_i)}^{\tau_f(s_f)} d\tau \frac{\theta}{3}. \quad (\text{F.1})$$

Notice that the total value of $N(s_i, s_f, \mathbf{x})$, in particular the perturbations, depends on how one define the initial and final time slices, we called them respectively s_i and s_f . That being said, the δN formalism makes use of a smart choice of slicing to subtract the fluctuations in the total number of e-folds from the background value. Then, it relates them to the final comoving curvature perturbation.¹

To be concrete, on super-horizon scales, where the spatial variation is negligible compared to the time variation, i.e. $c_s k \ll H a$, one has that the spatial metric at first order in gradient expansion [272, 273] is given by

$$h_{ij} = a^2(t) e^{2\mathcal{R}(t, \mathbf{x})} \delta_{ij}. \quad (\text{F.2})$$

The total number of e-folds, eq. (F.1), is then given by

$$N(s_i, s_f, \mathbf{x}) \approx \int_{\tau_i(s_i)}^{\tau_f(s_f)} d\tau (H + \mathcal{R}') = \ln a(\tau_f)/a(\tau_i) + \mathcal{R}(\tau_f, \mathbf{x}; s_f) - \mathcal{R}(\tau_i, \mathbf{x}; s_i), \quad (\text{F.3})$$

where the semicolon in the last two term indicates that they must be evaluated at the slice s_f and s_i respectively. From this expression it is straightforward to identify the perturbations in the total number of e-folds, i.e.

$$\delta N(s_f, s_i, \mathbf{x}) \approx \mathcal{R}(\tau_f, \mathbf{x}; s_f) - \mathcal{R}(\tau_i, \mathbf{x}; s_i). \quad (\text{F.4})$$

¹This is related to the separate universe approach, where each point evolve like the background as if it were independent [268–271].

At this point, it is clear that if one further chooses the final slice s_f to be a comoving slice and the initial slice s_i to be a flat slice, δN is formally related the final comoving curvature perturbation, i.e.

$$\delta N(s_f = \text{comoving}, s_i = \text{flat}, \mathbf{x}) = \mathcal{R}_c(\tau_f, \mathbf{x}) + O\left(\frac{\varepsilon}{aH}\right). \quad (\text{F.5})$$

Once that relation is known, one would like to estimate δN in a more practical way. For that purpose, one integrates the total number of e-folds in terms of the inflaton field. With the same prescriptions one finds

$$\begin{aligned} N(s_f = \text{comoving}, s_i = \text{flat}, \mathbf{x}) &= \int_{\phi(\tau_i) + \delta\phi(\tau_i, \mathbf{x})}^{\phi(\tau_f)} H(\phi) \frac{d\phi}{\phi} \\ &= \ln a(\tau_f)/a(\tau_i) + N_\phi \delta\phi(\tau_i, \mathbf{x}) + \frac{1}{2} N_{\phi\phi} \delta\phi^2(\tau_i, \mathbf{x}) + \dots, \end{aligned} \quad (\text{F.6})$$

where we rewrote the integral in terms of ϕ so that the fact that we chose a comoving slice is showed in the integration limit. In the last equality, we used a Taylor series expansion and since we count the number of e-folds N from the end of inflation we have

$$N_\phi \equiv \frac{\partial N}{\partial \phi} = -\frac{H}{\dot{\phi}}. \quad (\text{F.7})$$

Interestingly, the implications of equations (F.5) and (F.6) is that one can estimate the power spectrum and non-gaussianities on superhorizon scales by knowing the background evolution and the initial fluctuations of the inflaton field. A straightforward comparison with the definition of local non-gaussianity [274], i.e.

$$\mathcal{R}_c = \mathcal{R}_{c,g} + \frac{3}{5} f_{NL} \mathcal{R}_{c,g}^2, \quad (\text{F.8})$$

where the subindex g indicates it is assumed to be gaussian, yields

$$f_{NL}^{\text{naive}} = \frac{5}{6} \frac{N_{\phi\phi}}{N_\phi} = \frac{5}{6} (\varepsilon + \delta) = \frac{5}{12} (\eta - p), \quad (\text{F.9})$$

where in the last step we used the slow-roll parameters of K-inflation [275], i.e. $\eta = 2(\varepsilon + \delta) + p$ (see Appendix E). We called it naive due to the fact that there is another contribution one should take into account. The intrinsic non-gaussianity of the field. For a discussion on that topic see our work Ref. [275].

APPENDIX G

Curvaton model

In this appendix we will see in more detail the curvaton model. We make use of the δN formalism explained in Appendix F. I will follow reference [276]. The curvaton model goes as follows [126–128]. Assume that a part from the inflaton field, which drives inflation, there is another scalar field which we call curvaton. The energy density of curvaton is much smaller than that of the inflaton and therefore it has a negligible contribution onto the dynamics of the universe. This is called a spectator field. Nevertheless, the curvaton field fluctuates as any quantum scalar field in an expanding universe. The curvaton contributes as an isocurvature component and the curvature perturbation is not conserved. However, if the curvaton decays into radiation before nucleosynthesis, once the inflaton decayed, the isocurvature component is converted into adiabatic one. If the spectator field has a non-vanishing energy density then it contributes and may dominate the curvature perturbation. At the time of decay the curvaton oscillates around the minimum of its potential and therefore it behaves like pressureless matter. Let us compute the amplitude of the curvature perturbation as well as the non-gaussianity in this case.

Since we deal with super-horizon scales, we make use of the separate universe approach. The δN formalism in the case of multi-components can be written as [139, 277]

$$\zeta = \delta N(t, \mathbf{x}) + \frac{1}{3} \int_{\bar{\rho}(t)}^{\rho(t, \mathbf{x})} \frac{d\tilde{\rho}}{\tilde{\rho} + \tilde{p}}, \quad (\text{G.1})$$

where δN is the perturbed expansion, ρ and p are the energy density and pressure respectively and an upper bar refers to the background value. In the epoch of interest we consider that there is radiation and the curvaton. We may expand ζ perturbatively to find

$$\zeta = \zeta_1 + \frac{3}{5} f_{NL} \zeta_1^2 + \dots, \quad (\text{G.2})$$

where ζ_1 is the first order perturbation and f_{NL} is the local non-gaussianity parameter.¹ The curvature perturbation for the curvaton, which we call χ , in

¹The factor 3/5 comes from the definition of non-gaussianity in Ref. [274] which uses the Newtonian potential instead of the curvature perturbation. See Sec. 2.2.3.

the uniform-curvaton density slicing is given by

$$\zeta_\chi = \delta N(t, \mathbf{x}) + \frac{1}{3} \int_{\bar{\rho}_\chi(t)}^{\rho_\chi(t, \mathbf{x})} \frac{d\tilde{\rho}_\chi}{\tilde{\rho}_\chi}. \quad (\text{G.3})$$

On the other hand for radiation we have

$$\zeta_{\text{rad}} = \delta N(t, \mathbf{x}) + \frac{1}{4} \int_{\bar{\rho}_{\text{rad}}(t)}^{\rho_{\text{rad}}(t, \mathbf{x})} \frac{d\tilde{\rho}_{\text{rad}}}{\tilde{\rho}_{\text{rad}}}. \quad (\text{G.4})$$

We can use (G.3) to find that the curvaton density on spatially flat hypersurface is

$$\rho_\chi|_{\delta N=0} = e^{3\zeta_\chi} \bar{\rho}_\chi. \quad (\text{G.5})$$

When the curvaton oscillates around the minimum of the potential we can approximated it with a simple squared potential. Thus, the energy density of the curvaton around its minimum when it oscillates is given by

$$\rho_\chi = \frac{1}{2} m^2 \chi^2, \quad (\text{G.6})$$

where m is the mass of the curvaton. Recall that the curvaton was present during inflation and it developed quantum fluctuations that exit the horizon and froze. Let us split the value of the curvaton at horizon crossing as

$$\chi_* = \bar{\chi}_* + \delta\chi_*, \quad (\text{G.7})$$

where a subindex $*$ refers to horizon crossing and $\delta\chi$ is the perturbations. We can then related the value of the curvaton at horizon crossing χ_* to the amplitude of the oscillation χ by a given function $\chi = g(\chi_*)$. Thus during the curvaton oscillation we have

$$\bar{\rho}_\chi = \frac{1}{2} m^2 \bar{g}^2, \quad (\text{G.8})$$

for the background and

$$\rho_\chi = \frac{1}{2} m^2 \left[\bar{g} + g' \left(\frac{\bar{g}}{g'} \frac{\delta\chi}{\bar{\chi}} \right) + \frac{1}{2} g'' \left(\frac{\bar{g}}{g'} \frac{\delta\chi}{\bar{\chi}} \right)^2 + \dots \right]^2, \quad (\text{G.9})$$

for the perturbation expansion. The same applies to the curvature perturbation on uniform-curvaton density slicing, that is

$$e^{3\zeta_\chi} = \frac{1}{\bar{g}^2} \left[\bar{g} + g' \left(\frac{\bar{g}}{g'} \frac{\delta\chi}{\bar{\chi}} \right) + \frac{1}{2} g'' \left(\frac{\bar{g}}{g'} \frac{\delta\chi}{\bar{\chi}} \right)^2 + \dots \right]^2. \quad (\text{G.10})$$

We are now left to expand and identify term by term. We find from Eq. (G.9) that

$$\delta^{(1)}\rho_\chi = m^2 g \delta\chi \quad (\text{G.11})$$

and

$$\delta^{(2)}\rho_\chi = m^2 \left(1 + \frac{gg''}{g'^2}\right) (\delta\chi)^2. \quad (\text{G.12})$$

From the curvature perturbation Eq. (G.10) we have

$$\zeta_\chi^{(1)} = \frac{2}{3} \frac{\delta\chi}{\chi} \quad (\text{G.13})$$

and

$$\zeta_\chi^{(2)} = -\frac{3}{2} \left(1 - \frac{gg''}{g'^2}\right) (\zeta_\chi^{(1)})^2. \quad (\text{G.14})$$

We can then identify that the local non-gaussianity is given by

$$f_{NL}^\chi = -\frac{5}{4} \left(1 - \frac{gg''}{g'^2}\right). \quad (\text{G.15})$$

Note that depending on the form of the curvaton potential, i.e. the form of the function g , we can have large or small non-gaussianity, i.e. bigger or smaller than 1.

Now the question is, what is the contribution to the power spectrum? For that we need to consider the radiation field as well. For simplicity, we assume the sudden decay approximation. That is the curvaton decays instantaneously into radiation. In the uniform-total energy density slicing we have that the total energy density is given by the contribution of radiation and the curvaton, i.e.

$$\bar{\rho}(t_{\text{dec}}) = \rho_{\text{rad}}(t_{\text{dec}}, \mathbf{x}) + \rho_\chi(t_{\text{dec}}, \mathbf{x}), \quad (\text{G.16})$$

note that this include perturbations. On the other hand, if the total energy density is homogeneous we have from Eq. (G.1)

$$\zeta = \delta N. \quad (\text{G.17})$$

From Eqs. (G.3) and (G.4) we can find the local energy density of each field, i.e.

$$\zeta_{\text{rad}} = \zeta + \frac{1}{4} \ln \left(\frac{\rho_{\text{rad}}}{\bar{\rho}_{\text{rad}}} \right) \quad (\text{G.18})$$

and

$$\zeta_\chi = \zeta + \frac{1}{3} \ln \left(\frac{\rho_\chi}{\bar{\rho}_\chi} \right). \quad (\text{G.19})$$

Thus we can rewrite Eq. (G.16) dividing by the total energy density, that is

$$(1 - \Omega_{\chi, \text{dec}}) e^{4(\zeta_{\text{rad}} - \zeta)} + \Omega_{\chi, \text{dec}} e^{3(\zeta_\chi - \zeta)} = 1, \quad (\text{G.20})$$

where $\Omega_{\chi,\text{dec}} = \frac{\bar{\rho}_\chi}{\bar{\rho}_\chi + \bar{\rho}_{\text{rad}}}$. We have the relation between the curvature perturbation and the energy densities. For simplicity, let us assume that $\zeta_{\text{rad}} = 0$. Then expanding at first order we have

$$\zeta_1 = r_* \zeta_\chi^{(1)}, \quad (\text{G.21})$$

where

$$r_* = \left. \frac{3\bar{\rho}_\chi}{3\bar{\rho}_\chi + 4\bar{\rho}_{\text{rad}}} \right|_{\text{dec}}. \quad (\text{G.22})$$

This gives the well known result that the power spectrum from the curvaton is

$$\mathcal{P}_\chi(k) = \frac{4\pi k^3}{(2\pi)^3} \frac{r_*^2 |\delta\chi_k|^2}{\chi_*^2}. \quad (\text{G.23})$$

If we expand up to second order we have

$$\zeta_2 = \left(\frac{3}{2r_*} \left(1 + \frac{gg''}{g'^2} \right) - 2 - r \right) \zeta_1^2. \quad (\text{G.24})$$

Thus the non-gaussianity after the decay of the curvaton is given by

$$f_{NL} = \frac{5}{4r_*} \left(1 + \frac{gg''}{g'^2} \right) - \frac{5}{3} - \frac{5r_*}{6}. \quad (\text{G.25})$$

Again, by choosing g we can make non-gaussianity of $O(1)$ or less.

APPENDIX H

Hamiltonian Formalism

In this appendix we review the Hamiltonian formalism for General Relativity [278, 279]. We study the number of degrees of freedom and show that there are only two tensor modes, i.e. gravitational waves. Then we show that a conformal transformation is a type of canonical transformation, which does not change the number of degrees of freedom as long as the transformation is regular.

Start with the Einstein-Hilbert action, that is

$$S = \int d^4x \sqrt{-g} \frac{M_{pl}^2}{2} R. \quad (\text{H.1})$$

We now perform a 3+1 decomposition with the line element given by

$$ds^2 = -N^2 dt^2 + h_{ij} (dx^i + N^i dt) (dx^j + N^j dt), \quad (\text{H.2})$$

where N is the lapse and N^i is the shift vector. N and N^i describe the foliation of our spacetime. In this decomposition the action is given by

$$S = \int d^3x dt N \sqrt{h} \frac{M_{pl}^2}{2} (R^{(3)}[h] + K_{ij} K^{ij} - K^2) \quad (\text{H.3})$$

where

$$K_{ij} = \frac{1}{2N} (\dot{h}_{ij} - 2D_{(i} N_{j)}) , \quad (\text{H.4})$$

$K = h^{ij} K_{ij}$ and D_i is the covariant derivative of h_{ij} . We proceed to compute the Hamiltonian of this system, that is

$$\mathcal{H} = \Pi^{ij} \dot{h}_{ij} - \mathcal{L} = N \mathcal{H}_N + N^i \mathcal{H}_i, \quad (\text{H.5})$$

where $\Pi^{ij} = \delta \mathcal{L} / \delta \dot{h}_{ij} = \frac{1}{2} M_{pl}^2 \sqrt{h} (K^{ij} - h^{ij} K)$,

$$\mathcal{H}_N = \frac{1}{M_{pl}^2 \sqrt{h}} (2\Pi^{ij} \Pi_{ij} - \Pi^2) - \frac{M_{pl}^2 \sqrt{h}}{2} R^{(3)} \quad (\text{H.6})$$

and

$$\mathcal{H}_i = -2h_{ij} D_k (\Pi^{kj} / h) . \quad (\text{H.7})$$

Note that $\pi_N = \delta\mathcal{L}/\delta\dot{N} = 0$ and $\pi_N^i = \delta\mathcal{L}/\delta\dot{N}^i = 0$. These are called primary constraints. Then, the total Hamiltonian is given by

$$H = \int d^3x N \mathcal{H}_N + N^i \mathcal{H}_i. \quad (\text{H.8})$$

See that the lapse N and shift N^i play the role of Lagrange multipliers. Variation with respect to the lapse and shift are known as *Hamiltonian* and *Momentum constraints*:

$$\frac{\delta}{\delta N} \Rightarrow \mathcal{H}_N = 0 \quad \text{and} \quad \frac{\delta}{\delta N^i} \Rightarrow \mathcal{H}_i = 0. \quad (\text{H.9})$$

The evolution of the system is completely described by the Hamiltonian of the system by taking the Poisson bracket, i.e.

$$\dot{Q} = \{Q, \mathcal{H}\} = \frac{\delta Q}{\delta h_{ij}} \frac{\delta H}{\delta \Pi^{ij}} - \frac{\delta Q}{\delta \Pi^{ij}} \frac{\delta H}{\delta h_{ij}}. \quad (\text{H.10})$$

In this way, the Hamiltonian and Momentum constraints are classified as secondary constraints, i.e. they come from the Poisson brackets of the primary constraints with the Hamiltonian. In other words, we require that the time derivative of primary constraints vanishes as well.

An important property of the Hamiltonian formalism is that a coordinate transformation is also given in terms of the Hamiltonian. Diffeomorphism invariance thus tells us that the Poisson algebra of the constraints is closed. In particular, for the smeared form of the constraints, i.e. $H[N] = \int d^3x N \mathcal{H}$ and $H_i[N^i] = \int d^3x N^i \mathcal{H}_i$, we have

$$\begin{aligned} \{H[N], H[M]\} &= \int d^3x (N D^i M - M D^i N) \mathcal{H}_i \\ \{H_i[N^i], H[M]\} &= \int d^3x N^k D_k M \mathcal{H} \\ \{H_i[N^i], H_j[M^j]\} &= \int d^3x (N^k D_k M^l - M^k D_k N^l) \mathcal{H}_l \end{aligned} \quad (\text{H.11})$$

Note that if we assume that the primary and secondary constraints are satisfied then the latter Poisson brackets do not yield any further constraint. The presence of constraints means that we are not dealing with the true degrees of freedom. For example, a priori metric has 10 independent components:¹ the lapse (1), the shift vector (3) and the spatial metric (6). That means that our phase has 20 dimensions. Once the constraints are solved the phase space is reduced.

¹It is a symmetric matrix. In d dimensions a symmetric matrix has $d \times d - \frac{d(d-1)}{2}$ independent components.

The constraints can be classified as first class if their Poisson bracket with all other constraints vanishes² and second class otherwise. First class constraints reduce 2 variables in the phase space and second class 1 only.³ In the Hamiltonian formulation of GR without gauge fixing we have that primary and secondary constraints are first class and thus we are left with 4 variables in the phase space, i.e.

$$20 - 1 \times 2 \times 2 (\text{Hamiltonian}) - 3 \times 2 \times 2 (\text{Momentum}) = 4. \quad (\text{H.12})$$

This means that in GR we have 2 d.o.f. which actually corresponds to the 2 polarizations of *gravitational waves*. This is the reason why we know that during inflation the vector modes are non-dynamical and thus are not generated.

H.1 Application to conformal transformations

Let us briefly see how conformal transformations are understood in the Hamiltonian formalism. To do so it is better to work in the conformal decomposition of the spatial metric, that is

$$h_{ij} = e^{2\Psi} \Upsilon_{ij}, \quad (\text{H.13})$$

where $\sqrt{h} = e^{3\Psi}$ and $\det \Upsilon = 1$. In this case we have that to preserve the Poisson brackets the transformation must be a *canonical transformation*. This condition leads us to

$$h_{ij} \Pi^{ij} = \frac{\Pi_\Psi}{2} \quad \text{and} \quad \Pi^{ij} = e^{-2\Psi} \left(P^{ij} + \frac{\Pi_\Psi}{6} \Upsilon^{ij} \right). \quad (\text{H.14})$$

In this new variables the Hamiltonian is given by

$$H = \int d^3x N \mathcal{H}_N + N^i \mathcal{H}_i, \quad (\text{H.15})$$

where

$$\mathcal{H}_N = M_{pl}^{-2} e^{-3\Psi} \left(2P^{ij} P_{ij} - \frac{\Pi_\Psi^2}{12} \right) + M_{pl}^2 e^\Psi \left(2\Upsilon^{ij} D_i D_j \Psi + \Upsilon^{ij} D_i \Psi D_j \Psi - \frac{1}{2} R^{(3)} \right) \quad (\text{H.16})$$

and

$$\mathcal{H}_i = \Pi_\Psi D_i \Psi - \frac{1}{3} D_i \Pi_\Psi - 2\Upsilon_{ij} D_k P^{kj}. \quad (\text{H.17})$$

²They are associated to gauge transformations.

³Note however that by definition they come in pairs.

H.1.1 Conformal transformation

Let us now do a conformal transformation given by

$$g_{\mu\nu} = e^{2\omega} \tilde{g}_{\mu\nu} \quad (\text{H.18})$$

This is equivalent to do

$$\Psi \rightarrow \Psi + \omega \quad (\text{H.19})$$

and

$$\Pi_\omega \rightarrow \Pi_\omega - \Pi_\Psi, \quad (\text{H.20})$$

so that it is a canonical transformation and we ignored the transformation of the lapse since it is a Lagrange multiplier and we are free to redefine it. Note that in doing the conformal transformation we introduced a new variable ω , which could be a general function of the scalar field. However, ω was not present in the original theory which means that initially we have a trivial constraint given by

$$\Pi_\omega = 0. \quad (\text{H.21})$$

This constraint is obviously first class and has no secondary constraint. Thus it removes the degrees of freedom of ω . After the conformal transformation we mixed the conformal d.o.f. of the metric Ψ with ω . Thus, we have a primary constraint in the resulting theory given by

$$\Pi_\omega - \Pi_\Psi = 0. \quad (\text{H.22})$$

Due to the symmetry $\Psi + \omega$ in the Hamiltonian, the secondary constraint trivially vanishes, i.e.

$$C_\omega = \{\Pi_\omega - \Pi_\Psi, H\} = 0, \quad (\text{H.23})$$

and does not yield any new constraint. Thus we see that (H.22) is a first class constraint which removes one degree of freedom. In fact, we know it removes the degrees of freedom of ω .

What if ω is a function of a scalar field. First if $\omega = \omega(\phi)$ and ϕ is already present in the theory, e.g. the inflation, this transformation does not yield any new constraint and we have 3 d.o.f. in the theory, i.e. 2 tensor and 1 scalar mode. If $\omega = \omega(X)$ we are led to beyond Horndeski theories which potentially has an Ostrogradski ghost [169]. However, in this case we can introduce a new field replacing X with a Lagrange multiplier, that is

$$\lambda(\chi - X). \quad (\text{H.24})$$

Then we can do the conformal transformation $\Psi \rightarrow \Psi + \omega(X)$ and find that we have a primary constraint

$$\Pi_\chi = 0 \quad \rightarrow \quad \Pi_\chi - \omega_\chi \Pi_\psi = 0. \quad (\text{H.25})$$

This time, due to the fact that we introduced higher order derivatives the secondary constraint does not trivially vanish and leads to another constraint, i.e.

$$C_\chi = \{\Pi_\chi - \omega_\chi \Pi_\psi, H\} = 0. \quad (\text{H.26})$$

This two constraints are first class if the scalar field was not present from the beginning thus removing the d.o.f. from ϕ and χ . If the scalar field was present in the theory the constraints are second class and only remove χ . There is a singular case to that and it is the mimetic gravity case. In this case, the χ field transforms as

$$\tilde{\chi} = e^{2\omega(\chi)} \chi, \quad (\text{H.27})$$

since recall it is by definition X . Then the secondary constraint which comes from the variation of χ , i.e.

$$\frac{\delta}{\delta\chi} = \frac{\delta\tilde{\chi}}{\delta\chi} \frac{\delta}{\delta\chi} = \partial_\chi (\chi e^{-2\omega(\chi)}) \frac{\delta}{\delta\chi}, \quad (\text{H.28})$$

will trivially vanish if

$$e^{2\omega(\chi)} = \chi. \quad (\text{H.29})$$

Then we lose the secondary constraint, which would have been first class, and the primary constraint is then first class and remove the degrees of freedom of χ but not ϕ . Thus we have a new degree of freedom, which is the mimetic dark matter. See Ref. [280] for a detailed calculation of the mimetic case and [230] for a study in a general case.

APPENDIX I

de Sitter space

The de Sitter (dS) spacetime is a universe with a positive cosmological constant. The geometry of dS can be easily visualized by embedding it in an extra dimensional Minkowsky spacetime (for a review see [30, 281]). For example, dS in 4D is an hyperboloid of radius ℓ embedded in a 5D Minkowsky spacetime, i.e.

$$ds_{5D}^2 = -dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2, \quad (\text{I.1})$$

where the hyperboloid is given by

$$-x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = \ell^2. \quad (\text{I.2})$$

Depending on how we choose the time coordinate, i.e. on how we slice this hyperboloid, we obtain different coordinate charts of dS. This is illustrated in Figs. I.1 and I.2. Let me illustrate three interesting cases.

First, consider the radius of the S_3 , i.e.

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = r^2, \quad (\text{I.3})$$

and the hyperboloid between x_0 and r , that is

$$x_0 = \ell \sinh \chi / \ell \quad \text{and} \quad r = \ell \cosh \chi / \ell. \quad (\text{I.4})$$

Then metric in the 4D slice is

$$ds_{4D}^2 = -d\chi^2 + \ell^2 \cosh^2(\chi / \ell) d\Omega_3, \quad (\text{I.5})$$

and describes a closed universe. We have that

$$d\Omega_3 = d\theta^2 + \sin \theta^2 d\Omega_2. \quad (\text{I.6})$$

This is called *closed slicing* and it corresponds to the right hand side of Fig. I.1. Note that the slices correspond to slices proportional to constant x_0 and thus they are horizontal slices.

This time start considering the radius of the S_2 , i.e.

$$x_1^2 + x_2^2 + x_3^2 = R^2, \quad (\text{I.7})$$

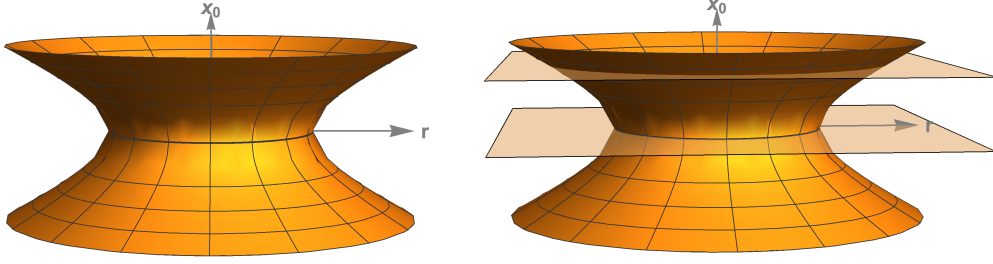


Figure I.1: On the left: dS hyperboloid in 5D. On the right: Closed slicing of dS. Note how every space-like slice contains a spatially finite universe.

and the hyperboloid between x_0 and R , that is

$$x_0 = \rho \cosh \xi \quad \text{and} \quad R = \rho \sinh \xi . \quad (\text{I.8})$$

The 5D metric in this coordinates is given by

$$ds_{5D}^2 = -d\rho^2 + \rho^2 d\xi^2 + \rho^2 \sinh^2 \xi d\Omega_2 + dx_4^2 , \quad (\text{I.9})$$

Now we slice the hyperboloid with ρ and x_4 , i.e.

$$\rho = \ell \sinh \tau / \ell \quad \text{and} \quad x_4 = \ell \cosh \tau / \ell . \quad (\text{I.10})$$

Then metric in the 4D slice is

$$ds_{4D}^2 = -d\tau^2 + \ell^2 \sinh^2 (\tau / \ell) (d\xi^2 + \sinh^2 \xi d\Omega_2) . \quad (\text{I.11})$$

Note that the last term in brackets correspond to an Hyperbolic space. Thus, this is called open slicing, that is

$$ds_{4D}^2 = -d\tau^2 + \ell^2 \sinh^2 (\tau / \ell) dH_3 , \quad (\text{I.12})$$

which corresponds to the left hand side of Fig. I.2. Note that the slicing corresponds to slices of constant x_4 , with $x_4 > \ell$.

There is also the static slicing which is given by

$$ds_{4D}^2 = - \left(1 - \frac{\tilde{r}^2}{\ell^2} \right) dt^2 + \left(1 - \frac{\tilde{r}^2}{\ell^2} \right)^{-1} d\tilde{r}^2 + \tilde{r}^2 d\Omega_2 . \quad (\text{I.13})$$

We can find it by considering the radius of the S_2 , i.e.

$$x_1^2 + x_2^2 + x_3^2 = R^2 , \quad (\text{I.14})$$

and the hyperboloid between x_0 and x_4 , that is

$$x_0 = \rho \sinh \xi \quad \text{and} \quad x_4 = \rho \cosh \xi . \quad (\text{I.15})$$

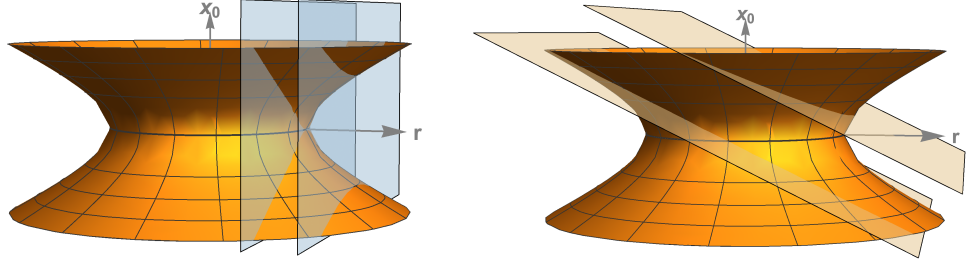


Figure I.2: On the left: Open slicing of dS. See how the slice cuts two disconnected regions. On the right: Flat slicing of dS. Note how both slicings contain an spatially infinite universe.

The 5D metric in this coordinates is given by

$$ds_{5D}^2 = -\rho^2 d\xi^2 + d\rho^2 + dR^2 + R^2 d\Omega_2, \quad (\text{I.16})$$

Now we slice it with the circle given by ρ and R , i.e.

$$\rho = \ell \sin \tau / \ell \quad \text{and} \quad R = \ell \cos \tau / \ell. \quad (\text{I.17})$$

Then metric in the 4D slice is

$$ds_{4D}^2 = -\ell^2 \sin^2(\tau/\ell) d\xi^2 + d\tau^2 + \ell^2 \cos^2(\tau/\ell) d\Omega_2. \quad (\text{I.18})$$

We can define $\tilde{r} = \ell \cos(\tau/\ell)$ and $\xi = t/\ell$ to get the static metric (I.13). In this form, we can see the *cosmological horizon* at $\tilde{r} = \ell = H^{-1}$ where H is the Hubble parameter.

Bibliography

- [1] J. Bicak and T. Ledvinka, *Proceedings, Relativity and Gravitation: Perspectives 100 years after Einstein's stay in Prague, Fundam. Theor. Phys.* **177** (2014) pp.1–535. (Cited on page 1.)
- [2] G. T. Gillies, *The Newtonian gravitational constant: recent measurements and related studies, Rept. Prog. Phys.* **60** (1997) 151–225. (Cited on page 2.)
- [3] E. G. Adelberger, B. R. Heckel, and A. E. Nelson, *Tests of the gravitational inverse square law, Ann. Rev. Nucl. Part. Sci.* **53** (2003) 77–121, [[hep-ph/0307284](#)]. (Cited on page 2.)
- [4] D. Psaltis, *Probes and Tests of Strong-Field Gravity with Observations in the Electromagnetic Spectrum, Living Rev. Rel.* **11** (2008) 9, [[arXiv:0806.1531](#)]. (Cited on pages 2 and 4.)
- [5] G. Bertone, D. Hooper, and J. Silk, *Particle dark matter: Evidence, candidates and constraints, Phys. Rept.* **405** (2005) 279–390, [[hep-ph/0404175](#)]. (Cited on page 2.)
- [6] J. Frieman, M. Turner, and D. Huterer, *Dark Energy and the Accelerating Universe, Ann. Rev. Astron. Astrophys.* **46** (2008) 385–432, [[arXiv:0803.0982](#)]. (Cited on page 2.)
- [7] D. H. Weinberg, M. J. Mortonson, D. J. Eisenstein, C. Hirata, A. G. Riess, and E. Rozo, *Observational Probes of Cosmic Acceleration, Phys. Rept.* **530** (2013) 87–255, [[arXiv:1201.2434](#)]. (Cited on pages 2 and 31.)
- [8] J. Martin, *Everything You Always Wanted To Know About The Cosmological Constant Problem (But Were Afraid To Ask), Comptes Rendus Physique* **13** (2012) 566–665, [[arXiv:1205.3365](#)]. (Cited on page 2.)
- [9] A. Padilla, *Lectures on the Cosmological Constant Problem, arXiv:1502.05296*. (Cited on page 2.)
- [10] R. H. Brandenberger, *Alternatives to the inflationary paradigm of structure formation, Int. J. Mod. Phys. Conf. Ser.* **01** (2011) 67–79, [[arXiv:0902.4731](#)]. (Cited on page 2.)

- [11] R. H. Brandenberger, *The Matter Bounce Alternative to Inflationary Cosmology*, [arXiv:1206.4196](#). (Cited on page 2.)
- [12] D. Battefeld and P. Peter, *A Critical Review of Classical Bouncing Cosmologies*, *Phys. Rept.* **571** (2015) 1–66, [[arXiv:1406.2790](#)]. (Cited on page 2.)
- [13] R. Brout, F. Englert, and E. Gunzig, *The Creation of the Universe as a Quantum Phenomenon*, *Annals Phys.* **115** (1978) 78. (Cited on page 2.)
- [14] A. A. Starobinsky, *Spectrum of relict gravitational radiation and the early state of the universe*, *JETP Lett.* **30** (1979) 682–685. [*Pisma Zh. Eksp. Teor. Fiz.* 30,719(1979)]. (Cited on pages 2 and 5.)
- [15] A. H. Guth, *The Inflationary Universe: A Possible Solution to the Horizon and Flatness Problems*, *Phys. Rev.* **D23** (1981) 347–356. (Cited on page 2.)
- [16] K. Sato, *First Order Phase Transition of a Vacuum and Expansion of the Universe*, *Mon. Not. Roy. Astron. Soc.* **195** (1981) 467–479. (Cited on page 2.)
- [17] V. F. Mukhanov and G. V. Chibisov, *The Vacuum energy and large scale structure of the universe*, *Sov. Phys. JETP* **56** (1982) 258–265. [*Zh. Eksp. Teor. Fiz.* 83,475(1982)]. (Cited on page 2.)
- [18] H. Kodama and M. Sasaki, *Cosmological Perturbation Theory*, *Prog. Theor. Phys. Suppl.* **78** (1984) 1–166. (Cited on pages 2, 76, 114 and 115.)
- [19] C. M. Will, *The Confrontation between General Relativity and Experiment*, *Living Rev. Rel.* **17** (2014) 4, [[arXiv:1403.7377](#)]. (Cited on pages 3, 4 and 5.)
- [20] E. Berti et al., *Testing General Relativity with Present and Future Astrophysical Observations*, *Class. Quant. Grav.* **32** (2015) 243001, [[arXiv:1501.07274](#)]. (Cited on pages 3 and 5.)
- [21] T. Damour and G. Esposito-Farese, *Tensor multiscalar theories of gravitation*, *Class. Quant. Grav.* **9** (1992) 2093–2176. (Cited on page 4.)
- [22] I. H. Stairs, *Testing general relativity with pulsar timing*, *Living Reviews in Relativity* **6** (2003), no. 1 5. (Cited on page 4.)

- [23] B. Jain, V. Vikram, and J. Sakstein, *Astrophysical Tests of Modified Gravity: Constraints from Distance Indicators in the Nearby Universe*, *Astrophys. J.* **779** (2013) 39, [[arXiv:1204.6044](#)]. (Cited on page 4.)
- [24] **Supernova Search Team** Collaboration, A. G. Riess et al., *Observational evidence from supernovae for an accelerating universe and a cosmological constant*, *Astron. J.* **116** (1998) 1009–1038, [[astro-ph/9805201](#)]. (Cited on pages 4 and 12.)
- [25] **Supernova Cosmology Project** Collaboration, S. Perlmutter et al., *Measurements of the cosmological parameters Omega and Lambda from the first 7 supernovae at $z \geq 0.35$* , *Astrophys. J.* **483** (1997) 565, [[astro-ph/9608192](#)]. (Cited on pages 4 and 12.)
- [26] F. Bernardeau, S. Colombi, E. Gaztanaga, and R. Scoccimarro, *Large scale structure of the universe and cosmological perturbation theory*, *Phys. Rept.* **367** (2002) 1–248, [[astro-ph/0112551](#)]. (Cited on page 4.)
- [27] Y.-S. Song and K. Koyama, *Consistency test of general relativity from large scale structure of the Universe*, *JCAP* **0901** (2009) 048, [[arXiv:0802.3897](#)]. (Cited on page 4.)
- [28] R. Reyes, R. Mandelbaum, U. Seljak, T. Baldauf, J. E. Gunn, L. Lombriser, and R. E. Smith, *Confirmation of general relativity on large scales from weak lensing and galaxy velocities*, *Nature* **464** (2010) 256–258, [[arXiv:1003.2185](#)]. (Cited on page 4.)
- [29] **Planck** Collaboration, P. A. R. Ade et al., *Planck 2015 results. XIII. Cosmological parameters*, *Astron. Astrophys.* **594** (2016) A13, [[arXiv:1502.01589](#)]. (Cited on pages 4 and 34.)
- [30] V. Mukhanov, *Physical Foundations of Cosmology*. Cambridge University Press, Oxford, 2005. (Cited on pages 4, 9, 12, 18, 26 and 131.)
- [31] J. Martin, C. Ringeval, and V. Vennin, *Encyclopædia Inflationaris*, *Phys. Dark Univ.* **5-6** (2014) 75–235, [[arXiv:1303.3787](#)]. (Cited on pages 4 and 6.)
- [32] A. Ijjas, P. J. Steinhardt, and A. Loeb, *Cosmic inflation theory faces challenges*, 2017. <https://www.scientificamerican.com/article/cosmic-inflation-theory-faces-challenges/>. (Cited on page 4.)

- [33] A. H. Guth, D. I. Kaiser, A. D. Linde, and Y. Nomura, *A cosmic controversy*, 2017. <https://blogs.scientificamerican.com/observations/a-cosmic-controversy/>. (Cited on page 4.)
- [34] **LIGO Scientific Collaboration and Virgo Collaboration**, B. P. Abbott et al., *Observation of gravitational waves from a binary black hole merger*, *Phys. Rev. Lett.* **116** (Feb, 2016) 061102. (Cited on page 4.)
- [35] **Virgo, LIGO Scientific Collaboration**, B. P. Abbott et al., *Tests of general relativity with GW150914*, *Phys. Rev. Lett.* **116** (2016), no. 22 221101, [[arXiv:1602.03841](#)]. (Cited on pages 4 and 5.)
- [36] S. Dodelson, *Modern Cosmology*. Academic Press, Amsterdam, 2003. (Cited on pages 4, 9 and 12.)
- [37] K. Koyama, *Cosmological Tests of Modified Gravity*, *Rept. Prog. Phys.* **79** (2016), no. 4 046902, [[arXiv:1504.04623](#)]. (Cited on pages 5 and 8.)
- [38] **Planck Collaboration**, P. A. R. Ade et al., *Planck 2015 results. XX. Constraints on inflation*, [arXiv:1502.02114](#). (Cited on pages 5, 33, 78 and 92.)
- [39] T. Clifton, P. G. Ferreira, A. Padilla, and C. Skordis, *Modified gravity and cosmology*, *Physics Reports* **513** (2012), no. 1 1–189, [[arXiv:1106.2476](#)]. (Cited on pages 5 and 8.)
- [40] R. Utiyama and B. S. DeWitt, *Renormalization of a classical gravitational field interacting with quantized matter fields*, *J. Math. Phys.* **3** (1962) 608–618. (Cited on page 5.)
- [41] D. M. Capper and M. J. Duff, *Conformal Anomalies and the Renormalizability Problem in Quantum Gravity*, *Phys. Lett.* **A53** (1975) 361. (Cited on page 5.)
- [42] P. C. W. Davies, S. A. Fulling, S. M. Christensen, and T. S. Bunch, *Energy Momentum Tensor of a Massless Scalar Quantum Field in a Robertson-Walker Universe*, *Annals Phys.* **109** (1977) 108–142. (Cited on page 5.)
- [43] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space*. Cambridge Monographs on Mathematical Physics. Cambridge Univ. Press, Cambridge, UK, 1984. (Cited on pages 5, 43, 63 and 117.)

- [44] T. P. Sotiriou and V. Faraoni, *f(R) Theories Of Gravity*, *Rev. Mod. Phys.* **82** (2010) 451–497, [[arXiv:0805.1726](#)]. (Cited on pages 5 and 6.)
- [45] A. De Felice and S. Tsujikawa, *f(R) theories*, *Living Rev. Rel.* **13** (2010) 3, [[arXiv:1002.4928](#)]. (Cited on pages 5 and 6.)
- [46] A. A. Starobinsky, *A New Type of Isotropic Cosmological Models Without Singularity*, *Phys. Lett.* **B91** (1980) 99–102. (Cited on page 5.)
- [47] P. Horava, *Membranes at Quantum Criticality*, *JHEP* **03** (2009) 020, [[arXiv:0812.4287](#)]. (Cited on page 5.)
- [48] S. Mukohyama, *Hořava-lifshitz cosmology: a review*, *Classical and Quantum Gravity* **27** (2010), no. 22 223101. (Cited on page 5.)
- [49] C. de Rham and G. Gabadadze, *Generalization of the Fierz-Pauli Action*, *Phys. Rev.* **D82** (2010) 044020, [[arXiv:1007.0443](#)]. (Cited on pages 5 and 46.)
- [50] S. F. Hassan and R. A. Rosen, *Bimetric Gravity from Ghost-free Massive Gravity*, *JHEP* **02** (2012) 126, [[arXiv:1109.3515](#)]. (Cited on page 5.)
- [51] A. Padilla, *Ghost free brane world bigravity*, *Class. Quant. Grav.* **21** (2004) 2899–2918, [[hep-th/0402079](#)]. (Cited on page 5.)
- [52] Y. Yamashita and T. Tanaka, *Mapping the ghost free bigravity into braneworld setup*, *JCAP* **1406** (2014) 004, [[arXiv:1401.4336](#)]. (Cited on page 5.)
- [53] A. De Felice, T. Nakamura, and T. Tanaka, *Possible existence of viable models of bi-gravity with detectable graviton oscillations by gravitational wave detectors*, *PTEP* **2014** (2014) 043E01, [[arXiv:1304.3920](#)]. (Cited on page 5.)
- [54] R. Kallosh, L. Kofman, A. D. Linde, and A. Van Proeyen, *Superconformal symmetry, supergravity and cosmology*, *Class. Quant. Grav.* **17** (2000) 4269–4338, [[hep-th/0006179](#)]. [Erratum: *Class. Quant. Grav.* **21**, 5017(2004)]. (Cited on pages 5 and 99.)
- [55] G. 't Hooft, *A class of elementary particle models without any adjustable real parameters*, *Found. Phys.* **41** (2011) 1829–1856, [[arXiv:1104.4543](#)]. (Cited on pages 5 and 99.)

- [56] R. Kallosh and A. Linde, *Universality Class in Conformal Inflation*, *JCAP* **1307** (2013) 002, [[arXiv:1306.5220](#)]. (Cited on pages 5 and 99.)
- [57] P. A. M. Dirac, *Cosmological models and the large numbers hypothesis*, *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences* **338** (1974), no. 1615 439–446, [<http://rspa.royalsocietypublishing.org/content/338/1615/439.full.pdf>]. (Cited on page 6.)
- [58] P. Jordan, *Zum gegenwärtigen stand der diracschen kosmologischen hypothesen*, *Zeitschrift für Physik* **157** (1959), no. 1 112–121. (Cited on page 6.)
- [59] C. Brans and R. H. Dicke, *Mach’s principle and a relativistic theory of gravitation*, *Physical Review* **124** (1961), no. 3 925. (Cited on pages 6, 8, 35 and 38.)
- [60] N. Deruelle and M. Sasaki, *Conformal transformations and Nordstrom’s scalar theory of gravity*, *Prog. Theor. Phys. Suppl.* **190** (2011) 143–154, [[arXiv:1012.5386](#)]. (Cited on page 6.)
- [61] F. Englert and R. Brout, *Broken Symmetry and the Mass of Gauge Vector Mesons*, *Phys. Rev. Lett.* **13** (1964) 321–323. (Cited on page 6.)
- [62] P. W. Higgs, *Broken symmetries, massless particles and gauge fields*, *Phys. Lett.* **12** (1964) 132–133. (Cited on page 6.)
- [63] CMS Collaboration, S. Chatrchyan et al., *Observation of a new boson at a mass of 125 GeV with the CMS experiment at the LHC*, *Phys. Lett.* **B716** (2012) 30–61, [[arXiv:1207.7235](#)]. (Cited on page 6.)
- [64] Y. Fujii, K.-i. Maeda, et al., *The scalar-tensor theory of gravitation*. Cambridge University Press, 2003. (Cited on pages 6, 7, 8, 35, 39 and 73.)
- [65] P. M. Lavrov and I. L. Shapiro, *Renormalization of gauge theories in curved space-time*, *Physical Review D* **81** (2010), no. 4 044026, [[arXiv:0911.4579](#)]. (Cited on page 6.)
- [66] S. Mukohyama, *Emergence of time in power-counting renormalizable Riemannian theory of gravity*, *Phys. Rev.* **D87** (2013), no. 8 085030, [[arXiv:1303.1409](#)]. (Cited on page 6.)

- [67] S. Tsujikawa, *Dark energy: investigation and modeling*, [arXiv:1004.1493](#). (Cited on page 6.)
- [68] S. Tsujikawa, *Modified gravity models of dark energy*, *Lect. Notes Phys.* **800** (2010) 99–145, [[arXiv:1101.0191](#)]. (Cited on page 6.)
- [69] R. H. Dicke, *Mach’s principle and invariance under transformation of units*, *Physical Review* **125** (1962), no. 6 2163. (Cited on page 6.)
- [70] V. Faraoni and S. Nadeau, *(pseudo) issue of the conformal frame revisited*, *Physical Review D* **75** (2007), no. 2 023501, [[gr-qc/0612075](#)]. (Cited on pages 6 and 48.)
- [71] E. Abbott, M. Roberts Brothers (Boston, J. Wilson, Son, and M. University Press (Cambridge, *Flatland: Romance of Many Dimensions*. Roberts Brothers, 1885. (Cited on page 7.)
- [72] O. Klein, *Quantum Theory and Five-Dimensional Theory of Relativity. (In German and English)*, *Z. Phys.* **37** (1926) 895–906. [Surveys High Energ. Phys.5,241(1986)]. (Cited on page 7.)
- [73] L. Randall and R. Sundrum, *A Large mass hierarchy from a small extra dimension*, *Phys. Rev. Lett.* **83** (1999) 3370–3373, [[hep-ph/9905221](#)]. (Cited on page 7.)
- [74] G. R. Dvali, G. Gabadadze, and M. Porrati, *4-D gravity on a brane in 5-D Minkowski space*, *Phys. Lett.* **B485** (2000) 208–214, [[hep-th/0005016](#)]. (Cited on page 7.)
- [75] P. Brax, C. van de Bruck, and A.-C. Davis, *Brane world cosmology*, *Rept. Prog. Phys.* **67** (2004) 2183–2232, [[hep-th/0404011](#)]. (Cited on page 7.)
- [76] R. Maartens and K. Koyama, *Brane-World Gravity*, *Living Rev. Rel.* **13** (2010) 5, [[arXiv:1004.3962](#)]. (Cited on page 7.)
- [77] J.-P. Uzan, *Varying Constants, Gravitation and Cosmology*, *Living Rev. Rel.* **14** (2011) 2, [[arXiv:1009.5514](#)]. (Cited on page 7.)
- [78] V. Faraoni, *Cosmology in scalar-tensor gravity*, vol. 139. Springer, 2004. (Cited on pages 7 and 35.)
- [79] G. W. Horndeski, *Second-order scalar-tensor field equations in a four-dimensional space*, *International Journal of Theoretical Physics* **10** (1974), no. 6 363–384. (Cited on pages 7, 45 and 103.)

- [80] C. Cheung, P. Creminelli, A. L. Fitzpatrick, J. Kaplan, and L. Senatore, *The Effective Field Theory of Inflation*, *JHEP* **03** (2008) 014, [[arXiv:0709.0293](#)]. (Cited on page 7.)
- [81] J. D. Bekenstein, *Relation between physical and gravitational geometry*, *Physical Review D* **48** (1993), no. 8 3641, [[gr-qc/9211017](#)]. (Cited on pages 8, 45, 53 and 68.)
- [82] J. Khoury and A. Weltman, *Chameleon cosmology*, *Phys. Rev.* **D69** (2004) 044026, [[astro-ph/0309411](#)]. (Cited on page 8.)
- [83] J. Khoury and A. Weltman, *Chameleon fields: Awaiting surprises for tests of gravity in space*, *Phys. Rev. Lett.* **93** (2004) 171104, [[astro-ph/0309300](#)]. (Cited on pages 8 and 100.)
- [84] J. Sakstein, *Disformal theories of gravity: from the solar system to cosmology*, *Journal of Cosmology and Astroparticle Physics* **2014** (2014), no. 12 012, [[arXiv:1409.1734](#)]. (Cited on pages 8 and 66.)
- [85] J. Sakstein and S. Verner, *Disformal Gravity Theories: A Jordan Frame Analysis*, *Phys. Rev.* **D92** (2015), no. 12 123005, [[arXiv:1509.05679](#)]. (Cited on page 8.)
- [86] R. Saito, D. Yamauchi, S. Mizuno, J. Gleyzes, and D. Langlois, *Modified gravity inside astrophysical bodies*, *JCAP* **1506** (2015) 008, [[arXiv:1503.01448](#)]. (Cited on page 8.)
- [87] J. Sakstein, E. Babichev, K. Koyama, D. Langlois, and R. Saito, *Towards Strong Field Tests of Beyond Horndeski Gravity Theories*, *Phys. Rev.* **D95** (2017), no. 6 064013, [[arXiv:1612.04263](#)]. (Cited on page 8.)
- [88] E. Babichev, K. Koyama, D. Langlois, R. Saito, and J. Sakstein, *Relativistic Stars in Beyond Horndeski Theories*, *Class. Quant. Grav.* **33** (2016), no. 23 235014, [[arXiv:1606.06627](#)]. (Cited on page 8.)
- [89] L. Berezhiani, J. Khoury, and J. Wang, *A Universe Without Dark Energy: Cosmic Acceleration from Dark Matter-Baryon Interactions*, [[arXiv:1612.00453](#)]. (Cited on pages 8 and 100.)
- [90] C. Burrage and J. Sakstein, *A Compendium of Chameleon Constraints*, *JCAP* **1611** (2016), no. 11 045, [[arXiv:1609.01192](#)]. (Cited on page 8.)

- [91] S. Tsujikawa, K. Uddin, S. Mizuno, R. Tavakol, and J. Yokoyama, *Constraints on scalar-tensor models of dark energy from observational and local gravity tests*, *Phys. Rev.* **D77** (2008) 103009, [[arXiv:0803.1106](#)]. (Cited on page 8.)
- [92] P. Brax, C. Burrage, A.-C. Davis, and G. Gubitosi, *Cosmological tests of the disformal coupling to radiation*, *Journal of Cosmology and Astroparticle Physics* **2013** (2013), no. 11 001, [[arXiv:1306.4168](#)]. (Cited on pages 8, 65, 66 and 109.)
- [93] P. Brax and C. Burrage, *Constraining disformally coupled scalar fields*, *Physical Review D* **90** (2014), no. 10 104009, [[arXiv:1407.1861](#)]. (Cited on page 8.)
- [94] H. Y. Ip, J. Sakstein, and F. Schmidt, *Solar System Constraints on Disformal Gravity Theories*, *JCAP* **1510** (2015) 051, [[arXiv:1507.00568](#)]. (Cited on page 8.)
- [95] S. G. Turyshev and J. G. Williams, *Space-based tests of gravity with laser ranging*, *Int. J. Mod. Phys.* **D16** (2007) 2165–2179, [[gr-qc/0611095](#)]. (Cited on page 8.)
- [96] A. I. Vainshtein, *To the problem of nonvanishing gravitation mass*, *Phys. Lett.* **B39** (1972) 393–394. (Cited on page 8.)
- [97] E. Babichev, C. Deffayet, and R. Ziour, *k-Mouflage gravity*, *Int. J. Mod. Phys.* **D18** (2009) 2147–2154, [[arXiv:0905.2943](#)]. (Cited on page 8.)
- [98] K. Hinterbichler and J. Khoury, *Symmetron Fields: Screening Long-Range Forces Through Local Symmetry Restoration*, *Phys. Rev. Lett.* **104** (2010) 231301, [[arXiv:1001.4525](#)]. (Cited on page 8.)
- [99] M. Zumalacarregui, T. Koivisto, D. Mota, and P. Ruiz-Lapiente, *Disformal scalar fields and the dark sector of the universe*, *Journal of Cosmology and Astroparticle Physics* **2010** (2010), no. 05 038, [[arXiv:1004.2684](#)]. (Cited on pages 8 and 46.)
- [100] A. De Felice, R. Kase, and S. Tsujikawa, *Vainshtein mechanism in second-order scalar-tensor theories*, *Phys. Rev.* **D85** (2012) 044059, [[arXiv:1111.5090](#)]. (Cited on page 8.)
- [101] T. S. Koivisto, D. F. Mota, and M. Zumalacárregui, *Screening modifications of gravity through disformally coupled fields*, *Physical*

- review letters* **109** (2012), no. 24 241102, [[arXiv:1205.3167](#)]. (Cited on pages 8, 66 and 100.)
- [102] E. Babichev and C. Deffayet, *An introduction to the Vainshtein mechanism*, *Class. Quant. Grav.* **30** (2013) 184001, [[arXiv:1304.7240](#)]. (Cited on page 8.)
- [103] J. Sakstein, *Towards viable cosmological models of disformal theories of gravity*, *Physical Review D* **91** (2015), no. 2 024036, [[arXiv:1409.7296](#)]. (Cited on pages 8 and 66.)
- [104] P. Gratia, W. Hu, A. Joyce, and R. H. Ribeiro, *Double screening*, *JCAP* **1606** (2016), no. 06 033, [[arXiv:1604.00395](#)]. (Cited on page 8.)
- [105] C. Wetterich, *Universe without expansion*, *Physics of the Dark Universe* **2** (2013), no. 4 184–187, [[arXiv:1303.6878](#)]. (Cited on page 8.)
- [106] C. Wetterich, *Eternal universe*, *arXiv preprint arXiv:1404.0535* (2014). (Cited on page 8.)
- [107] C. Wetterich, *Hot big bang or slow freeze?*, *arXiv preprint arXiv:1401.5313* (2014). (Cited on pages 8 and 92.)
- [108] A. Ijjas and P. J. Steinhardt, *The anamorphic universe*, *JCAP* **1510** (2015), no. 10 001, [[arXiv:1507.03875](#)]. (Cited on pages 8 and 92.)
- [109] C. van de Bruck, J. Morrice, and S. Vu, *Constraints on nonconformal couplings from the properties of the cosmic microwave background radiation*, *Physical review letters* **111** (2013), no. 16 161302, [[arXiv:1303.1773](#)]. (Cited on pages 8, 65, 66 and 109.)
- [110] C. van de Bruck and J. Morrice, *Disformal couplings and the dark sector of the universe*, *Journal of Cosmology and Astroparticle Physics* **1504** (2015), no. 04 036, [[arXiv:1501.03073](#)]. (Cited on pages 8 and 46.)
- [111] C. Burrage, S. Céspedes, and A.-C. Davis, *Disformal transformations on the CMB*, *JCAP* **1608** (2016), no. 08 024, [[arXiv:1604.08038](#)]. (Cited on page 8.)
- [112] N. Makino and M. Sasaki, *The density perturbation in the chaotic inflation with non-minimal coupling*, *Progress of Theoretical Physics* **86** (1991), no. 1 103–118. (Cited on pages 8 and 48.)

- [113] N. Deruelle and M. Sasaki, *Conformal equivalence in classical gravity: the example of "veiled" general relativity*, in *Cosmology, Quantum Vacuum and Zeta Functions*, pp. 247–260. Springer, 2011. [arXiv:1007.3563](#). (Cited on pages 8, 38, 41, 48 and 50.)
- [114] T. Chiba and M. Yamaguchi, *Conformal-frame (in) dependence of cosmological observations in scalar-tensor theory*, *Journal of Cosmology and Astroparticle Physics* **2013** (2013), no. 10 040, [[arXiv:1308.1142](#)]. (Cited on pages 8, 48 and 51.)
- [115] N. Deruelle and J. Rua, *Disformal transformations, veiled general relativity and mimetic gravity*, *Journal of Cosmology and Astroparticle Physics* **2014** (2014), no. 09 002, [[arXiv:1407.0825](#)]. (Cited on pages 8, 46, 53 and 105.)
- [116] D. Bettoni and S. Liberati, *Disformal invariance of second order scalar-tensor theories: Framing the horndeski action*, *Physical Review D* **88** (2013), no. 8 084020, [[arXiv:1306.6724](#)]. (Cited on pages 8, 45, 46 and 72.)
- [117] M. Sasaki, “Lecture notes on cosmology.” <http://www2.yukawa.kyoto-u.ac.jp/~misao.sasaki/>. (Cited on page 9.)
- [118] M. Sasaki, *Inflation and Birth of Cosmological Perturbations*, *Fundam. Theor. Phys.* **177** (2014) 305–321, [[arXiv:1210.7880](#)]. (Cited on page 9.)
- [119] D. Baumann, *Inflation*, in *Physics of the large and the small, TASI 09, proceedings of the Theoretical Advanced Study Institute in Elementary Particle Physics, Boulder, Colorado, USA, 1-26 June 2009*, pp. 523–686, 2011. [arXiv:0907.5424](#). (Cited on page 9.)
- [120] D. Baumann and L. McAllister, *Inflation and String Theory*. Cambridge University Press, 2015. (Cited on page 9.)
- [121] D. Baumann, “Lecture notes on inflation and string theory.” <http://www.damtp.cam.ac.uk/user/db275/>. (Cited on page 9.)
- [122] D. Tong, “Lecture notes on theoretical physics.” <http://www.damtp.cam.ac.uk/user/tong/>. (Cited on page 9.)
- [123] G. Domènech, A. Naruko, and M. Sasaki, *Cosmological disformal invariance*, *JCAP* **1510** (2015), no. 10 067, [[arXiv:1505.00174](#)]. (Cited on pages 9, 46 and 51.)

- [124] G. Domènech and M. Sasaki, *Conformal frame dependence of inflation*, *Journal of Cosmology and Astroparticle Physics* **2015** (2015), no. 04 022, [[arXiv:1501.07699](#)]. (Cited on pages 9 and 71.)
- [125] F. Lucchin and S. Matarrese, *Power-law inflation*, *Physical Review D* **32** (1985), no. 6 1316. (Cited on pages 10, 74 and 78.)
- [126] T. Moroi and T. Takahashi, *Cosmic density perturbations from late-decaying scalar condensations*, *Physical Review D* **66** (2002), no. 6 063501, [[hep-ph/0206026](#)]. (Cited on pages 10, 79, 80 and 121.)
- [127] K. Enqvist and M. S. Sloth, *Adiabatic cmb perturbations in pre-big-bang string cosmology*, *Nuclear Physics B* **626** (2002), no. 1 395–409, [[hep-ph/0109214](#)]. (Cited on pages 10, 79, 80 and 121.)
- [128] D. H. Lyth and D. Wands, *Generating the curvature perturbation without an inflaton*, *Physics Letters B* **524** (2002), no. 1 5–14, [[hep-ph/0110002](#)]. (Cited on pages 10, 79, 80, 81, 82, 86 and 121.)
- [129] E. Hubble, *A relation between distance and radial velocity among extra-galactic nebulae*, *Proc. Nat. Acad. Sci.* **15** (1929) 168–173. (Cited on page 12.)
- [130] S. Davidson, E. Nardi, and Y. Nir, *Leptogenesis*, *Phys. Rept.* **466** (2008) 105–177, [[arXiv:0802.2962](#)]. (Cited on page 19.)
- [131] M. Trodden, *Electroweak baryogenesis*, *Rev. Mod. Phys.* **71** (1999) 1463–1500, [[hep-ph/9803479](#)]. (Cited on page 19.)
- [132] D. E. Morrissey and M. J. Ramsey-Musolf, *Electroweak baryogenesis*, *New J. Phys.* **14** (2012) 125003, [[arXiv:1206.2942](#)]. (Cited on page 19.)
- [133] A. Vilenkin and E. P. S. Shellard, *Cosmic Strings and Other Topological Defects*. Cambridge University Press, 2000. (Cited on page 22.)
- [134] L. Kofman, A. D. Linde, and A. A. Starobinsky, *Reheating after inflation*, *Phys. Rev. Lett.* **73** (1994) 3195–3198, [[hep-th/9405187](#)]. (Cited on page 26.)
- [135] V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger, *Theory of cosmological perturbations. Part 1. Classical perturbations. Part 2. Quantum theory of perturbations. Part 3. Extensions*, *Phys. Rept.* **215** (1992) 203–333. (Cited on pages 27 and 114.)

- [136] A. Albrecht, P. Ferreira, M. Joyce, and T. Prokopec, *Inflation and squeezed quantum states*, *Phys. Rev.* **D50** (1994) 4807–4820, [[astro-ph/9303001](#)]. (Cited on page 27.)
- [137] D. Polarski and A. A. Starobinsky, *Semiclassicality and decoherence of cosmological perturbations*, *Class. Quant. Grav.* **13** (1996) 377–392, [[gr-qc/9504030](#)]. (Cited on page 27.)
- [138] J. Martin, V. Vennin, and P. Peter, *Cosmological Inflation and the Quantum Measurement Problem*, *Phys. Rev.* **D86** (2012) 103524, [[arXiv:1207.2086](#)]. (Cited on page 27.)
- [139] D. H. Lyth, K. A. Malik, and M. Sasaki, *A General proof of the conservation of the curvature perturbation*, *JCAP* **0505** (2005) 004, [[astro-ph/0411220](#)]. (Cited on pages 28 and 121.)
- [140] D. J. Eisenstein, H.-J. Seo, and M. White, *On the robustness of the acoustic scale in the low-redshift clustering of matter*, *The Astrophysical Journal* **664** (August 1, 2007, 2007) 660–674. n/a. (Cited on page 31.)
- [141] P. Ade, N. Aghanim, M. Arnaud, M. Ashdown, J. Aumont, C. Baccigalupi, A. Banday, R. Barreiro, et al., *Planck 2015 results. xiii. cosmological parameters*, [arXiv:1502.01589](#). (Cited on page 34.)
- [142] P. Ade, N. Aghanim, M. Arnaud, M. Ashdown, J. Aumont, C. Baccigalupi, A. Banday, R. Barreiro, N. Bartolo, E. Battaner, et al., *Planck 2015 results. xiv. dark energy and modified gravity*, [arXiv:1502.01590](#). (Cited on page 34.)
- [143] V. Sahni, A. Shafieloo, and A. A. Starobinsky, *Model-independent evidence for dark energy evolution from baryon acoustic oscillations*, *The Astrophysical Journal Letters* **793** (2014), no. 2 L40, [[arXiv:1406.2209](#)]. (Cited on page 34.)
- [144] P. Ade, N. Aghanim, M. Arnaud, F. Arroja, M. Ashdown, J. Aumont, C. Baccigalupi, M. Ballardini, A. Banday, R. Barreiro, et al., *Planck 2015. xx. constraints on inflation*, [arXiv:1502.02114](#). (Cited on page 34.)
- [145] B. Whitt, *Fourth-order gravity as general relativity plus matter*, *Physics Letters B* **145** (1984), no. 3 176–178. (Cited on page 35.)

-
- [146] A. Jakubiec and J. Kijowski, *On theories of gravitation with nonlinear lagrangians*, *Physical Review D* **37** (1988), no. 6 1406. (Cited on page 35.)
- [147] A. D. Linde, M. Sasaki, and T. Tanaka, *CMB in open inflation*, *Phys. Rev.* **D59** (1999) 123522, [[astro-ph/9901135](#)]. (Cited on page 35.)
- [148] S. R. Coleman and F. De Luccia, *Gravitational Effects on and of Vacuum Decay*, *Phys. Rev.* **D21** (1980) 3305. (Cited on page 35.)
- [149] X. Chen, *Primordial Features as Evidence for Inflation*, *JCAP* **1201** (2012) 038, [[arXiv:1104.1323](#)]. (Cited on page 35.)
- [150] X. Chen and M. H. Namjoo, *Standard Clock in Primordial Density Perturbations and Cosmic Microwave Background*, *Phys. Lett.* **B739** (2014) 285–292, [[arXiv:1404.1536](#)]. (Cited on page 35.)
- [151] K. Horiguchi, K. Ichiki, and J. Yokoyama, *Revisiting the oscillations in the CMB angular power spectra at $\ell \sim 120$ in the Planck2015 data*, [arXiv:1705.02770](#). (Cited on page 35.)
- [152] C. T. Byrnes and K.-Y. Choi, *Review of local non-Gaussianity from multi-field inflation*, *Adv. Astron.* **2010** (2010) 724525, [[arXiv:1002.3110](#)]. (Cited on page 35.)
- [153] T. Takahashi, *Primordial non-Gaussianity and the inflationary Universe*, *PTEP* **2014** (2014), no. 6 6B105–0. (Cited on page 35.)
- [154] R. M. Wald, *General Relativity*. 1984. (Cited on pages 41, 101, 113 and 119.)
- [155] G. Kang, *On black hole area in Brans-Dicke theory*, *Phys. Rev.* **D54** (1996) 7483–7489, [[gr-qc/9606020](#)]. (Cited on page 41.)
- [156] S. W. Hawking, *Black holes in the brans-dicke theory of gravitation*, *Comm. Math. Phys.* **25** (1972), no. 2 167–171. (Cited on page 41.)
- [157] M. A. Scheel, S. L. Shapiro, and S. A. Teukolsky, *Collapse to black holes in Brans-Dicke theory. 2. Comparison with general relativity*, *Phys. Rev.* **D51** (1995) 4236–4249, [[gr-qc/9411026](#)]. (Cited on page 41.)
- [158] G. Domènech and M. Sasaki, *Conformal frames in cosmology*, *Int. J. Mod. Phys.* **D25** (2016), no. 13 1645006, [[arXiv:1602.06332](#)]. [[56\(2017\)](#)]. (Cited on pages 41 and 48.)

- [159] V. Faraoni, E. Gunzig, and P. Nardone, *Conformal transformations in classical gravitational theories and in cosmology*, *Fund. Cosmic Phys.* **20** (1999) 121, [[gr-qc/9811047](#)]. (Cited on page 43.)
- [160] E. Pajer, G. L. Pimentel, and J. V. S. Van Wijck, *The Conformal Limit of Inflation in the Era of CMB Polarimetry*, [arXiv:1609.06993](#). (Cited on page 43.)
- [161] M. K. Benna and I. R. Klebanov, *Gauge-String Dualities and Some Applications*, in *String theory and the real world: From particle physics to astrophysics. Proceedings, Summer School in Theoretical Physics, 87th Session, Les Houches, France, July 2-27, 2007*, pp. 611–662, 2008. [arXiv:0803.1315](#). (Cited on page 43.)
- [162] N. Beisert et al., *Review of AdS/CFT Integrability: An Overview*, *Lett. Math. Phys.* **99** (2012) 3–32, [[arXiv:1012.3982](#)]. (Cited on page 43.)
- [163] U. Leonhardt, *Optical conformal mapping*, *Science* **312** (2006), no. 5781 1777–1780, [<http://science.sciencemag.org/content/312/5781/1777.full.pdf>]. (Cited on page 43.)
- [164] T. S. Koivisto and D. E. Wills, *Matters on a moving brane*, *Int. J. Mod. Phys. D* **22** (2013) 1342024, [[arXiv:1312.3462](#)]. (Cited on page 44.)
- [165] T. Koivisto, D. Wills, and I. Zavala, *Dark D-brane Cosmology*, *JCAP* **1406** (2014) 036, [[arXiv:1312.2597](#)]. (Cited on page 44.)
- [166] F. Larrouturou, S. Mukohyama, R. Namba, and Y. Watanabe, *Where does curvaton reside? Differences between bulk and brane frames*, *Phys. Rev. D* **95** (2017), no. 6 063509, [[arXiv:1609.06191](#)]. (Cited on pages 44 and 100.)
- [167] C. Deffayet, X. Gao, D. Steer, and G. Zahariade, *From k-essence to generalised Galileons*, *Phys.Rev. D* **84** (2011) 064039, [[arXiv:1103.3260](#)]. (Cited on page 45.)
- [168] T. Kobayashi, M. Yamaguchi, and J. Yokoyama, *Generalized g-inflation with the most general second-order field equations*, *Progress of Theoretical Physics* **126** (2011), no. 3 511–529, [[arXiv:1105.5723](#)]. (Cited on page 45.)

- [169] M. Zumalacárregui and J. García-Bellido, *Transforming gravity: from derivative couplings to matter to second-order scalar-tensor theories beyond the horndeski lagrangian*, *Physical Review D* **89** (2014), no. 6 064046, [[arXiv:1308.4685](#)]. (Cited on pages 45, 99, 105 and 128.)
- [170] J. Gleyzes, D. Langlois, F. Piazza, and F. Vernizzi, *Essential building blocks of dark energy*, *Journal of Cosmology and Astroparticle Physics* **2013** (2013), no. 08 025, [[arXiv:1304.4840](#)]. (Cited on page 45.)
- [171] J. Gleyzes, D. Langlois, F. Piazza, and F. Vernizzi, *Healthy theories beyond Horndeski*, *Phys. Rev. Lett.* **114** (2015), no. 21 211101, [[arXiv:1404.6495](#)]. (Cited on pages 45 and 99.)
- [172] J. Gleyzes, D. Langlois, F. Piazza, , and F. Vernizzi, *Exploring gravitational theories beyond horndeski*, *Journal of Cosmology and Astroparticle Physics* **2015** (2015), no. 02 018, [[arXiv:1408.1952](#)]. (Cited on page 45.)
- [173] J. Gleyzes, D. Langlois, and F. Vernizzi, *A unifying description of dark energy*, *International Journal of Modern Physics D* **23** (2014), no. 13 1443010, [[arXiv:1411.3712](#)]. (Cited on pages 45 and 51.)
- [174] X. Gao, *Unifying framework for scalar-tensor theories of gravity*, *Phys.Rev.* **D90** (2014), no. 8 081501, [[arXiv:1406.0822](#)]. (Cited on pages 45 and 103.)
- [175] M. Crisostomi, K. Koyama, and G. Tasinato, *Extended Scalar-Tensor Theories of Gravity*, *JCAP* **1604** (2016), no. 04 044, [[arXiv:1602.03119](#)]. (Cited on pages 45, 99 and 103.)
- [176] J. Ben Achour, D. Langlois, and K. Noui, *Degenerate higher order scalar-tensor theories beyond Horndeski and disformal transformations*, *Phys. Rev.* **D93** (2016), no. 12 124005, [[arXiv:1602.08398](#)]. (Cited on pages 45, 99 and 103.)
- [177] J. Ben Achour, M. Crisostomi, K. Koyama, D. Langlois, K. Noui, and G. Tasinato, *Degenerate higher order scalar-tensor theories beyond Horndeski up to cubic order*, *JHEP* **12** (2016) 100, [[arXiv:1608.08135](#)]. (Cited on pages 45 and 103.)
- [178] M. Crisostomi, M. Hull, K. Koyama, and G. Tasinato, *Horndeski: beyond, or not beyond?*, *JCAP* **1603** (2016), no. 03 038, [[arXiv:1601.04658](#)]. (Cited on pages 45 and 103.)

- [179] D. Langlois, M. Mancarella, K. Noui, and F. Vernizzi, *Effective Description of Higher-Order Scalar-Tensor Theories*, [arXiv:1703.03797](#). (Cited on pages 45 and 103.)
- [180] T. Fujita, X. Gao, and J. Yokoyama, *Spatially covariant theories of gravity: disformal transformation, cosmological perturbations and the Einstein frame*, *JCAP* **1602** (2016), no. 02 014, [[arXiv:1511.04324](#)]. (Cited on pages 45, 46 and 72.)
- [181] K. Takahashi, H. Motohashi, T. Suyama, and T. Kobayashi, *General invertible transformation and physical degrees of freedom*, *Phys. Rev. D* **95** (2017), no. 8 084053, [[arXiv:1702.01849](#)]. (Cited on page 45.)
- [182] U. Gen and M. Sasaki, *Radion on the de Sitter brane*, *Prog. Theor. Phys.* **105** (2001) 591–606, [[gr-qc/0011078](#)]. (Cited on page 46.)
- [183] C. de Rham and A. J. Tolley, *DBI and the Galileon reunited*, *JCAP* **1005** (2010) 015, [[arXiv:1003.5917](#)]. (Cited on page 46.)
- [184] P. Brax, C. Burrage, and A.-C. Davis, *Shining light on modifications of gravity*, *Journal of Cosmology and Astroparticle Physics* **2012** (2012), no. 10 016, [[arXiv:1206.1809](#)]. (Cited on page 46.)
- [185] M. Zumalacárregui, T. S. Koivisto, and D. F. Mota, *Dbi galileons in the einstein frame: Local gravity and cosmology*, *Physical Review D* **87** (2013), no. 8 083010, [[arXiv:1210.8016](#)]. (Cited on page 46.)
- [186] N. Kaloper, *Disformal inflation*, *Physics Letters B* **583** (2004), no. 1 1–13, [[hep-ph/0312002](#)]. (Cited on page 46.)
- [187] C. van de Bruck, T. Koivisto, and C. Longden, *Disformally coupled inflation*, *JCAP* **1603** (2016), no. 03 006, [[arXiv:1510.01650](#)]. (Cited on page 46.)
- [188] T. S. Koivisto, *Disformal quintessence*, [arXiv:0811.1957](#). (Cited on page 46.)
- [189] M. Clayton and J. Moffat, *A scalar-tensor cosmological model with dynamical light velocity*, *Physics Letters B* **506** (2001), no. 1 177–186, [[gr-qc/0101126](#)]. (Cited on page 46.)
- [190] J. Moffat, *Bimetric gravity theory, varying speed of light and the dimming of supernovae*, *International Journal of Modern Physics D* **12** (2003), no. 02 281–298, [[gr-qc/0202012](#)]. (Cited on pages 46 and 66.)

-
- [191] J. Magueijo, *New varying speed of light theories*, *Reports on Progress in Physics* **66** (2003), no. 11 2025, [[astro-ph/0305457](#)]. (Cited on pages 46 and 66.)
- [192] J. Magueijo, *Bimetric varying speed of light theories and primordial fluctuations*, *Physical Review D* **79** (2009), no. 4 043525, [[arXiv:0807.1689](#)]. (Cited on pages 46 and 66.)
- [193] P. Brax and C. Burrage, *Explaining the Proton Radius Puzzle with Disformal Scalars*, *Phys.Rev.* **D91** (2015), no. 4 043515, [[arXiv:1407.2376](#)]. (Cited on page 46.)
- [194] J. M. Ezquiaga, J. García-Bellido, and M. Zumalacárregui, *Towards the most general scalar-tensor theories of gravity: a unified approach in the language of differential forms*, *Phys. Rev.* **D94** (2016), no. 2 024005, [[arXiv:1603.01269](#)]. (Cited on page 46.)
- [195] J. M. Ezquiaga, J. García-Bellido, and M. Zumalacárregui, *Field redefinitions in theories beyond Einstein gravity using the language of differential forms*, *Phys. Rev.* **D95** (2017), no. 8 084039, [[arXiv:1701.05476](#)]. (Cited on page 46.)
- [196] L. Heisenberg, *Generalization of the Proca Action*, *JCAP* **1405** (2014) 015, [[arXiv:1402.7026](#)]. (Cited on page 46.)
- [197] L. Heisenberg, R. Kase, and S. Tsujikawa, *Beyond generalized Proca theories*, *Phys. Lett.* **B760** (2016) 617–626, [[arXiv:1605.05565](#)]. (Cited on page 46.)
- [198] R. Kimura, A. Naruko, and D. Yoshida, *Extended vector-tensor theories*, *JCAP* **1701** (2017), no. 01 002, [[arXiv:1608.07066](#)]. (Cited on page 46.)
- [199] R. P. Kerr, *Discovering the Kerr and Kerr-Schild metrics*, in *Kerr Fest: Black Holes in Astrophysics, General Relativity and Quantum Gravity Christchurch, New Zealand, August 26-28, 2004*, 2007. [arXiv:0706.1109](#). (Cited on page 46.)
- [200] M. Visser, *The Kerr spacetime: A Brief introduction*, in *Kerr Fest: Black Holes in Astrophysics, General Relativity and Quantum Gravity Christchurch, New Zealand, August 26-28, 2004*, 2007. [arXiv:0706.0622](#). (Cited on pages 46 and 110.)

- [201] E. Babichev, V. Mukhanov, and A. Vikman, *k-Essence, superluminal propagation, causality and emergent geometry*, *JHEP* **02** (2008) 101, [[arXiv:0708.0561](#)]. (Cited on page 47.)
- [202] A. Vikman, “K-essence: cosmology.” November, 2007. (Cited on page 47.)
- [203] H. A. Kastrup, *On the advancements of conformal transformations and their associated symmetries in geometry and theoretical physics*, *Annalen der Physik* **17** (2008), no. 9-10 631–690, [[arXiv:0808.2730](#)]. (Cited on page 48.)
- [204] J.-O. Gong, J.-c. Hwang, W. I. Park, M. Sasaki, and Y.-S. Song, *Conformal invariance of curvature perturbation*, *Journal of Cosmology and Astroparticle Physics* **2011** (2011), no. 09 023, [[arXiv:1107.1840](#)]. (Cited on pages 48, 51 and 73.)
- [205] J. White, M. Minamitsuji, and M. Sasaki, *Curvature perturbation in multi-field inflation with non-minimal coupling*, *Journal of Cosmology and Astroparticle Physics* **2012** (2012), no. 07 039, [[arXiv:1306.6186](#)]. (Cited on pages 48, 51 and 73.)
- [206] L. Järv, P. Kuusk, M. Saal, and O. Vilson, *Invariant quantities in the scalar-tensor theories of gravitation*, *Phys. Rev. D* **91** (Jan, 2015) 024041, [[arXiv:1411.1947](#)]. (Cited on page 48.)
- [207] R. Catena, M. Pietroni, and L. Scarabello, *Einstein and jordan frames reconciled: A frame-invariant approach to scalar-tensor cosmology*, *Physical Review D* **76** (2007), no. 8 084039, [[astro-ph/0604492](#)]. (Cited on page 48.)
- [208] T. Chiba and M. Yamaguchi, *Extended slow-roll conditions and rapid-roll conditions*, *Journal of Cosmology and Astroparticle Physics* **2008** (2008), no. 10 021, [[arXiv:0807.4965](#)]. (Cited on page 48.)
- [209] T. Qiu, *Reconstruction of a nonminimal coupling theory with scale-invariant power spectrum*, *Journal of Cosmology and Astroparticle Physics* **2012** (2012), no. 06 041, [[arXiv:1204.0189](#)]. (Cited on page 48.)
- [210] M. Li, *Generating scale-invariant tensor perturbations in the non-inflationary universe*, *Physics letters B* **736** (2014) 488–493, [[arXiv:1405.0211](#)]. (Cited on pages 48 and 84.)

- [211] S. Capozziello, S. Nojiri, S. Odintsov, and A. Troisi, *Cosmological viability of $f(r)$ -gravity as an ideal fluid and its compatibility with a matter dominated phase*, *Physics Letters B* **639** (2006), no. 3 135–143, [[astro-ph/0604431](#)]. (Cited on page 48.)
- [212] S. Nojiri and S. D. Odintsov, *Modified $f(r)$ gravity consistent with realistic cosmology: From a matter dominated epoch to a dark energy universe*, *Physical Review D* **74** (2006), no. 8 086005, [[hep-th/0608008](#)]. (Cited on page 48.)
- [213] F. Briscese, E. Elizalde, S. Nojiri, and S. Odintsov, *Phantom scalar dark energy as modified gravity: Understanding the origin of the big rip singularity*, *Physics Letters B* **646** (2007), no. 2 105–111, [[hep-th/0612220](#)]. (Cited on page 48.)
- [214] S. Nojiri and S. D. Odintsov, *Unified cosmic history in modified gravity: from $f(r)$ theory to lorentz non-invariant models*, *Physics Reports* **505** (2011), no. 2 59–144, [[arXiv:1011.0544](#)]. (Cited on page 48.)
- [215] D. P. George, S. Mooij, and M. Postma, *Quantum corrections in higgs inflation: the real scalar case*, *Journal of Cosmology and Astroparticle Physics* **2014** (2014), no. 02 024, [[arXiv:1310.2157](#)]. (Cited on page 48.)
- [216] A. Y. Kamenshchik and C. F. Steinwachs, *Frame dependence of quantum corrections in cosmology*, *arXiv* (2014) [[arXiv:1408.5769](#)]. (Cited on page 48.)
- [217] H. Motohashi and J. White, *Disformal invariance of curvature perturbation*, [arXiv:1504.00846](#). (Cited on page 51.)
- [218] P. Creminelli, J. Gleyzes, J. Noreña, and F. Vernizzi, *Resilience of the standard predictions for primordial tensor modes*, *Physical review letters* **113** (2014), no. 23 231301, [[arXiv:1407.8439](#)]. (Cited on pages 51 and 68.)
- [219] M. Minamitsuji, *Disformal transformation of cosmological perturbations*, *Physics letters B* **737** (2014) 139–150, [[arXiv:1409.1566](#)]. (Cited on page 51.)
- [220] S. Tsujikawa, *Disformal invariance of cosmological perturbations in a generalized class of Horndeski theories*, *Journal of Cosmology and Astroparticle Physics* **1504** (2015) 043, [[arXiv:1412.6210](#)]. (Cited on page 51.)

- [221] Y. Watanabe, A. Naruko, and M. Sasaki, *Multi-disformal invariance of non-linear primordial perturbations*, *Europhys. Lett.* **111** (2015) 39002, [[arXiv:1504.00672](#)]. (Cited on page 51.)
- [222] J. Gleyzes, D. Langlois, M. Mancarella, and F. Vernizzi, *Effective Theory of Interacting Dark Energy*, *Journal of Cosmology and Astroparticle Physics* **1508** (2015), no. 08 054, [[arXiv:1504.05481](#)]. (Cited on page 51.)
- [223] Y. Watanabe, A. Naruko, and M. Sasaki, *Multi-disformal invariance of non-linear primordial perturbations*, *Europhys. Lett.* **111** (2015) 39002, [[arXiv:1504.00672](#)]. (Cited on page 52.)
- [224] A. H. Chamseddine and V. Mukhanov, *Mimetic dark matter*, *Journal of High Energy Physics* **2013** (2013), no. 11 1–5, [[arXiv:1308.5410](#)]. (Cited on pages 53 and 99.)
- [225] A. H. Chamseddine, V. Mukhanov, and A. Vikman, *Cosmology with mimetic matter*, *Journal of Cosmology and Astroparticle Physics* **2014** (2014), no. 06 017, [[arXiv:1403.3961](#)]. (Cited on pages 53 and 99.)
- [226] G. F. Ellis and J.-P. Uzan, *c is the speed of light, isn't it?*, *American journal of physics* **73** (2005), no. 3 240–247, [[gr-qc/0305099](#)]. (Cited on page 55.)
- [227] G. F. Ellis, *Note on varying speed of light cosmologies*, *General Relativity and Gravitation* **39** (2007), no. 4 511–520, [[astro-ph/0703751](#)]. (Cited on page 55.)
- [228] J. Maldacena, *Non-gaussian features of primordial fluctuations in single field inflationary models*, *Journal of High Energy Physics* **2003** (2003), no. 05 013, [[astro-ph/0210603](#)]. (Cited on page 59.)
- [229] R. M. Wald, *General relativity*. University of Chicago press, 2010. (Cited on page 62.)
- [230] G. Domènech, S. Mukohyama, R. Namba, A. Naruko, R. Saitou, and Y. Watanabe, *Derivative-dependent metric transformation and physical degrees of freedom*, *Phys. Rev.* **D92** (2015), no. 8 084027, [[arXiv:1507.05390](#)]. (Cited on pages 69, 99 and 129.)
- [231] J. G. Russo, *Exact solution of scalar field cosmology with exponential potentials and transient acceleration*, *Physics Letters B* **600** (2004), no. 3 185–190, [[hep-th/0403010](#)]. (Cited on pages 74 and 76.)

- [232] A. A. Andrianov, F. Cannata, and A. Y. Kamenshchik, *General solution of scalar field cosmology with a (piecewise) exponential potential*, *Journal of Cosmology and Astroparticle Physics* **2011** (2011), no. 10 004, [[arXiv:1105.4515](#)]. (Cited on pages 74 and 76.)
- [233] V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger, *Theory of cosmological perturbations*, *Physics Reports* **215** (1992), no. 5 203–333. (Cited on page 76.)
- [234] V. F. Mukhanov, *Gravitational Instability of the Universe Filled with a Scalar Field*, *JETP Lett.* **41** (1985) 493–496. (Cited on pages 77 and 117.)
- [235] M. Sasaki, *Large scale quantum fluctuations in the inflationary universe*, *Progress of Theoretical Physics* **76** (1986), no. 5 1036–1046. (Cited on pages 77 and 117.)
- [236] D. H. Lyth and E. D. Stewart, *The curvature perturbation in power law (eg extended) inflation*, *Physics Letters B* **274** (1992), no. 2 168–172. (Cited on page 78.)
- [237] N. Bartolo and A. R. Liddle, *Simplest curvaton model*, *Physical Review D* **65** (2002), no. 12 121301, [[astro-ph/0203076](#)]. (Cited on page 82.)
- [238] T. Fujita, M. Kawasaki, and S. Yokoyama, *Curvaton in large field inflation*, *Journal of Cosmology and Astroparticle Physics* **2014** (2014), no. 09 015, [[arXiv:1404.0951](#)]. (Cited on page 82.)
- [239] K. Enqvist and T. Takahashi, *Mixed inflaton and spectator field models after planck*, *Journal of Cosmology and Astroparticle Physics* **2013** (2013), no. 10 034, [[arXiv:1306.5958](#)]. (Cited on page 82.)
- [240] C. T. Byrnes, M. Cortês, and A. R. Liddle, *Comprehensive analysis of the simplest curvaton model*, *Physical Review D* **90** (2014), no. 2 023523, [[arXiv:1403.4591](#)]. (Cited on page 82.)
- [241] R. Blumenhagen, B. Körs, D. Lust, and S. Stieberger, *Four-dimensional string compactifications with d-branes, orientifolds and fluxes*, *Physics reports* **445** (2007), no. 1 1–193, [[hep-th/0610327](#)]. (Cited on page 82.)
- [242] K.-i. Maeda, *Towards the Einstein-Hilbert Action via Conformal Transformation*, *Phys.Rev.* **D39** (1989) 3159. (Cited on page 83.)

- [243] S. Tsujikawa, *Power-law inflation with a nonminimally coupled scalar field*, *Physical Review D* **62** (2000), no. 4 043512, [[hep-ph/0004088](#)]. (Cited on page 85.)
- [244] Y.-S. Piao, S. Tsujikawa, and Z. Xinmin, *Inflation in string-inspired cosmology and suppression of cmb low multipoles*, *Classical and Quantum Gravity* **21** (2004), no. 18 4455, [[hep-th/0312139](#)]. (Cited on page 87.)
- [245] Y.-S. Piao and Y.-Z. Zhang, *Phantom inflation and primordial perturbation spectrum*, *Physical Review D* **70** (2004), no. 6 063513, [[astro-ph/0401231](#)]. (Cited on page 87.)
- [246] M. Cicoli, S. Downes, B. Dutta, F. G. Pedro, and A. Westphal, *Just enough inflation: power spectrum modifications at large scales*, *Journal of Cosmology and Astroparticle Physics* **2014** (2014), no. 12 030, [[arXiv:1407.1048](#)]. (Cited on page 87.)
- [247] T. Biswas and A. Mazumdar, *Super-inflation, non-singular bounce, and low multipoles*, *Classical and Quantum Gravity* **31** (2014), no. 2 025019, [[arXiv:1304.3648](#)]. (Cited on page 87.)
- [248] Z.-G. Liu, Z.-K. Guo, and Y.-S. Piao, *Cmb anomalies from an inflationary model in string theory*, *The European Physical Journal C* **74** (2014), no. 8 1–7, [[arXiv:1311.1599](#)]. (Cited on page 87.)
- [249] Y.-S. Piao and E. Zhou, *Nearly scale-invariant spectrum of adiabatic fluctuations may be from a very slowly expanding phase of the universe*, *Physical Review D* **68** (2003), no. 8 083515, [[hep-th/0308080](#)]. (Cited on page 87.)
- [250] Y. Ali-Haïmoud and M. Kamionkowski, *Cosmic microwave background limits on accreting primordial black holes*, *Phys. Rev.* **D95** (2017), no. 4 043534, [[arXiv:1612.05644](#)]. (Cited on page 87.)
- [251] A. M. Green, *Primordial Black Holes: sirens of the early Universe*, *Fundam. Theor. Phys.* **178** (2015) 129–149, [[arXiv:1403.1198](#)]. (Cited on page 87.)
- [252] M. Sasaki, T. Suyama, T. Tanaka, and S. Yokoyama, *Primordial Black Hole Scenario for the Gravitational-Wave Event GW150914*, *Phys. Rev. Lett.* **117** (2016), no. 6 061101, [[arXiv:1603.08338](#)]. (Cited on page 87.)

- [253] B. Carr, F. Kuhnel, and M. Sandstad, *Primordial Black Holes as Dark Matter*, *Phys. Rev. D* **D94** (2016), no. 8 083504, [[arXiv:1607.06077](#)]. (Cited on page 87.)
- [254] L. E. Allen and D. Wands, *Cosmological perturbations through a simple bounce*, *Physical Review D* **70** (2004), no. 6 063515, [[astro-ph/0404441](#)]. (Cited on page 90.)
- [255] D. H. Lyth, *The primordial curvature perturbation in the ekpyrotic universe*, *Physics Letters B* **524** (2002), no. 1 1–4, [[hep-ph/0106153](#)]. (Cited on page 90.)
- [256] Y.-F. Cai, T. Qiu, X. Zhang, Y.-S. Piao, and M. Li, *Bouncing universe with quintom matter*, *Journal of High Energy Physics* **2007** (2007), no. 10 071, [[arXiv:0704.1090](#)]. (Cited on page 90.)
- [257] T. Qiu, J. Evslin, Y.-F. Cai, M. Li, and X. Zhang, *Bouncing galileon cosmologies*, *Journal of Cosmology and Astroparticle Physics* **2011** (2011), no. 10 036, [[arXiv:1108.0593](#)]. (Cited on page 90.)
- [258] D. Battfeld and P. Peter, *A critical review of classical bouncing cosmologies*, *Physics Reports* (2015) [[arXiv:1406.2790](#)]. (Cited on page 90.)
- [259] F. Arroja, N. Bartolo, P. Karmakar, and S. Matarrese, *The two faces of mimetic Horndeski gravity: disformal transformations and Lagrange multiplier*, *JCAP* **1509** (2015) 051, [[arXiv:1506.08575](#)]. (Cited on page 106.)
- [260] S. Hirano, S. Nishi, and T. Kobayashi, *Healthy imperfect dark matter from effective theory of mimetic cosmological perturbations*, [arXiv:1704.06031](#). (Cited on page 106.)
- [261] J. Garriga and V. F. Mukhanov, *Perturbations in k-inflation*, *Phys. Lett. B* **458** (1999) 219–225, [[hep-th/9904176](#)]. (Cited on page 111.)
- [262] R. L. Arnowitt, S. Deser, and C. W. Misner, *The Dynamics of general relativity*, *Gen. Rel. Grav.* **40** (2008) 1997–2027, [[gr-qc/0405109](#)]. (Cited on page 111.)
- [263] L. D. Faddeev and R. Jackiw, *Hamiltonian Reduction of Unconstrained and Constrained Systems*, *Phys. Rev. Lett.* **60** (1988) 1692–1694. (Cited on page 115.)

- [264] K. Koyama, *Non-Gaussianity of quantum fields during inflation*, *Class. Quant. Grav.* **27** (2010) 124001, [[arXiv:1002.0600](#)]. (Cited on page 115.)
- [265] A. A. Starobinsky, *Multicomponent de Sitter (Inflationary) Stages and the Generation of Perturbations*, *JETP Lett.* **42** (1985) 152–155. [*Pisma Zh. Eksp. Teor. Fiz.* 42,124(1985)]. (Cited on page 119.)
- [266] M. Sasaki and E. D. Stewart, *A General analytic formula for the spectral index of the density perturbations produced during inflation*, *Prog. Theor. Phys.* **95** (1996) 71–78, [[astro-ph/9507001](#)]. (Cited on page 119.)
- [267] E. Poisson, *A Relativist’s Toolkit*. Cambridge University Press, 2004. Cambridge Books Online. (Cited on page 119.)
- [268] H. Kodama and T. Hamazaki, *Evolution of cosmological perturbations in the long wavelength limit*, *Phys. Rev.* **D57** (1998) 7177–7185, [[gr-qc/9712045](#)]. (Cited on page 119.)
- [269] Y. Nambu and A. Taruya, *Solutions of gauge invariant cosmological perturbations in long wavelength limit*, *Class. Quant. Grav.* **15** (1998) 2761–2772, [[gr-qc/9801021](#)]. (Cited on page 119.)
- [270] G. I. Rigopoulos and E. P. S. Shellard, *The separate universe approach and the evolution of nonlinear superhorizon cosmological perturbations*, *Phys. Rev.* **D68** (2003) 123518, [[astro-ph/0306620](#)]. (Cited on page 119.)
- [271] T. Hamazaki, *Long wavelength limit of evolution of nonlinear cosmological perturbations*, *Phys. Rev.* **D78** (2008) 103513, [[arXiv:0811.2366](#)]. (Cited on page 119.)
- [272] Y.-i. Takamizu, S. Mukohyama, M. Sasaki, and Y. Tanaka, *Non-Gaussianity of superhorizon curvature perturbations beyond δN formalism*, *JCAP* **1006** (2010) 019, [[arXiv:1004.1870](#)]. (Cited on page 119.)
- [273] A. Naruko, Y.-i. Takamizu, and M. Sasaki, *Beyond δN formalism*, *PTEP* **2013** (2013) 043E01, [[arXiv:1210.6525](#)]. (Cited on page 119.)
- [274] E. Komatsu and D. N. Spergel, *Acoustic signatures in the primary microwave background bispectrum*, *Phys. Rev.* **D63** (2001) 063002, [[astro-ph/0005036](#)]. (Cited on pages 120 and 121.)

-
- [275] G. Doménech, J.-O. Gong, and M. Sasaki, *Consistency relation and inflaton field redefinition in the δN formalism*, *Phys. Lett.* **B769** (2017) 413–417, [[arXiv:1606.03343](#)]. (Cited on page 120.)
- [276] M. Sasaki, J. Valiviita, and D. Wands, *Non-Gaussianity of the primordial perturbation in the curvaton model*, *Phys. Rev.* **D74** (2006) 103003, [[astro-ph/0607627](#)]. (Cited on page 121.)
- [277] M. Sasaki and T. Tanaka, *Superhorizon scale dynamics of multiscalar inflation*, *Prog. Theor. Phys.* **99** (1998) 763–782, [[gr-qc/9801017](#)]. (Cited on page 121.)
- [278] E. Poisson, *A Relativist’s Toolkit: The Mathematics of Black-Hole Mechanics*. Cambridge University Press, 2004. (Cited on page 125.)
- [279] M. Bojowald, *Canonical Gravity and Applications: Cosmology, Black Holes, and Quantum Gravity*. Cambridge University Press, 2010. (Cited on page 125.)
- [280] M. Chaichian, J. Kluson, M. Oksanen, and A. Tureanu, *Mimetic dark matter, ghost instability and a mimetic tensor-vector-scalar gravity*, *JHEP* **12** (2014) 102, [[arXiv:1404.4008](#)]. (Cited on page 129.)
- [281] M. Spradlin, A. Strominger, and A. Volovich, *Les Houches lectures on de Sitter space*, in *Unity from duality: Gravity, gauge theory and strings. Proceedings, NATO Advanced Study Institute, Euro Summer School, 76th session, Les Houches, France, July 30-August 31, 2001*, pp. 423–453, 2001. [hep-th/0110007](#). (Cited on page 131.)