

A uniqueness set for the linear partial differential operators of the second order

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d を 2 以上の自然数とする。 $W \subset \mathbb{C}^{d+1}$ に対して $\mathcal{O}(W)$ で W 上の正則函数のなす空間を表わす。 任意の $\lambda \in \mathbb{C}$ に対して $\mathcal{O}_\lambda(W) = \{ f \in \mathcal{O}(W) ; (\Delta_z + \lambda^2) f = 0 \}$, $\Delta_z = \sum_{j=1}^{d+1} \frac{\partial^2}{\partial z_j^2}$ と定義する。 $M = \{ z \in \mathbb{C}^{d+1} ; \sum_{j=1}^{d+1} z_j^2 = 0 \}$ とおく。 [13] では制限写像 $\alpha_\lambda: f \rightarrow f|_M$ が $\mathcal{O}_\lambda(\mathbb{C}^{d+1})$ から $\mathcal{O}(\mathbb{C}^{d+1})|_M$ への全単射であることを示したが, $\mathcal{O}_\lambda(\mathbb{C}^{d+1})$ の元は M 上の値のみで一意的に決定できるという意味で M を微分作用素 $\Delta_z + \lambda^2$ の uniqueness set と呼ぶことにする。

ここでは $\Delta_z + \lambda^2$ を一般の定数係数 2 階偏微分作用素にかえても類似の現象が起こることを示す (定理 4.1)。 また $\widehat{B}(r)$ を $\mathbb{C}^{d+1}$ 上の半径 r の Lie 球とすると前述の制限写像 α_λ は $\mathcal{O}_\lambda(\widehat{B}(r))$ から $\mathcal{O}(\widehat{B}(r))|_M$ への線型位相同型になり (定理 2.1), さらに $\mathcal{O}_\lambda(\widehat{B}(r))$ は複素球面 $\widehat{S}$ 上の汎函数の空間の Fourier-Borel 変換 $\mathcal{F}_\lambda$ の像として表わされる (定理 3.1)。

Introduction.

Let d be a positive integer and $d \geq 2$. Let us denote by $\mathcal{O}(W)$ the space of holomorphic functions on $W \subset \mathbb{C}^{d+1}$. For any $\lambda \in \mathbb{C}$ we define $\mathcal{Q}_\lambda(W) = \{f \in \mathcal{O}(W); (\Delta_z + \lambda^2)f = 0\}$, where $\Delta_z = \sum_{j=1}^{d+1} (\partial/\partial z_j)^2$. Let $M = \{z \in \mathbb{C}^{d+1}; z_1^2 + z_2^2 + \dots + z_{d+1}^2 = 0\}$. In [13] it was shown that the restriction mapping $\alpha_\lambda: f \rightarrow f|_M$ is a one-to-one linear mapping of $\mathcal{Q}_\lambda(\mathbb{C}^{d+1})$ onto $\mathcal{O}(\mathbb{C}^{d+1})|_M$. In this sense we call M a uniqueness set of the differential operator $\Delta_z + \lambda^2$.

Here we will show that the same kind of phenomenon holds more generally for a linear partial differential operator of the second order with constant coefficients (Theorem 4.1) and that α_λ is a linear topological isomorphism of $\mathcal{Q}_\lambda(\tilde{B}(r))$ onto $\mathcal{O}(\tilde{B}(r))|_M$, where $\tilde{B}(r)$ is the Lie ball of radius r (Theorem 2.1).

The Fourier-Borel transformation P_λ has been studied in [2], [6], [10], [12], [13] etc. We will determine the inverse image of $\mathcal{Q}_\lambda(\tilde{B}(r))$ by the transformation P_λ (Theorem 3.1).

§1. Preliminaries.

Let d be a positive integer and assume $d \geq 2$. $S = \{x \in \mathbb{R}^{d+1}; \|x\| = 1\}$ denotes the unit sphere in $\mathbb{R}^{d+1}$, where $\|x\|^2 = \sum_{j=1}^{d+1} x_j^2$. ds denotes the unique $O(d+1)$ invariant measure on S with $\int_S 1 \, ds$

$= 1$, where $O(k)$ is the orthogonal group of degree k . $\|\cdot\|_\infty$ is the sup norm on S . $H_{n,d}$ is the space of spherical harmonics of degree n in dimension $d + 1$. For spherical harmonics, see Müller [11].

For $S_n \in H_{n,d}$ $\tilde{S}_n$ denotes the unique homogeneous harmonic polynomial of degree n on C^{d+1} such that $\tilde{S}_n|_S = S_n$.

The Lie norm $L(z)$ and dual Lie norm $L^*(z)$ on C^{d+1} are defined as follows:

$$L(z) = L(x + iy) = [\|x\|^2 + \|y\|^2 + 2\{\|x\|^2\|y\|^2 - (x \cdot y)^2\}^{\frac{1}{2}}]^{\frac{1}{2}},$$

$$L^*(z) = \sup\{|z \cdot \zeta|; L(\zeta) \leq 1\},$$

where $z, \zeta \in C^{d+1}$, and $z \cdot \zeta = z_1 \zeta_1 + z_2 \zeta_2 + \dots + z_{d+1} \zeta_{d+1}$,

$x, y \in R^{d+1}$ (see Drużkowski [1]). We put

$$\tilde{B}(r) = \{z \in C^{d+1}; L(z) < r\} \quad \text{for } 0 < r \leq \infty$$

and

$$\tilde{B}[r] = \{z \in C^{d+1}; L(z) \leq r\} \quad \text{for } 0 \leq r < \infty.$$

Let us denote by $\mathcal{O}(\tilde{B}(r))$ the space of holomorphic functions on $\tilde{B}(r)$. Then $\mathcal{O}(\tilde{B}(r))$ is an FS space. $\mathcal{O}(\tilde{B}(\infty)) = \mathcal{O}(C^{d+1})$ is the space of entire functions on C^{d+1} . Let us define

$$\mathcal{O}(\tilde{B}[r]) = \text{ind} \lim_{r' > r} \mathcal{O}(\tilde{B}(r')).$$

Then $\mathcal{O}(\tilde{B}[r])$ is a DFS space. For $\lambda \in \mathbb{C}$, we put $\mathcal{O}_\lambda(\tilde{B}(r)) = \{f \in \mathcal{O}(\tilde{B}(r)); (\Delta_z + \lambda^2)f = 0\}$ and $\mathcal{O}_\lambda(\tilde{B}[r]) = \{f \in \mathcal{O}(\tilde{B}[r]); (\Delta_z + \lambda^2)f = 0\}$, where $\Delta_z = (\partial/\partial z_1)^2 + (\partial/\partial z_2)^2 + \dots + (\partial/\partial z_{d+1})^2$. $P_n(\mathbb{C}^{d+1})$ denotes the space of homogeneous polynomials of degree n on $\mathbb{C}^{d+1}$.

For $r > 0$ we put

$$X_{r,L} = \{f \in \mathcal{O}(\mathbb{C}^{d+1}); \sup_{z \in \mathbb{C}^{d+1}} |f(z)| \exp(-rL(z)) < \infty\}.$$

Then $X_{r,L}$ is a Banach space with respect to the norm

$$\|f\|_{r,L} = \sup_{z \in \mathbb{C}^{d+1}} |f(z)| \exp(-rL(z)).$$

Define

$$\text{Exp}(\mathbb{C}^{d+1}; (r:L)) = \text{proj} \lim_{r' > r} X_{r',L} \quad \text{for } 0 \leq r < \infty,$$

$$\text{Exp}(\mathbb{C}^{d+1}; [r:L]) = \text{ind} \lim_{r' < r} X_{r',L} \quad \text{for } 0 < r \leq \infty.$$

$\text{Exp}(\mathbb{C}^{d+1}; (r:L))$ is an FS space and $\text{Exp}(\mathbb{C}^{d+1}; [r:L])$ is a DFS space.

$\text{Exp}(\mathbb{C}^{d+1}) = \text{Exp}(\mathbb{C}^{d+1}; [\infty:L])$ is called the space of entire

functions of exponential type. $\text{Exp}'(\mathbb{C}^{d+1}; (r:L))$ and

$\text{Exp}'(\mathbb{C}^{d+1}; [r:L])$ denote the spaces dual to $\text{Exp}(\mathbb{C}^{d+1}; (r:L))$ and

$\text{Exp}(\mathbb{C}^{d+1}; [r:L])$ respectively.

$\tilde{S} = \{z \in \mathbb{C}^{d+1}; z_1^2 + z_2^2 + \dots + z_{d+1}^2 = 1\}$ is the complex

sphere. We define $\text{Exp}(\tilde{S}: [r:L]) = \text{Exp}(C^{d+1}: [r:L])|_{\tilde{S}}$ and $\text{Exp}(\tilde{S}: (r:L)) = \text{Exp}(C^{d+1}: (r:L))|_{\tilde{S}}$. The topology of $\text{Exp}(\tilde{S}: [r:L])$ is defined to be the quotient topology $\text{Exp}(C^{d+1}: [r:L])/I_{\text{exp}[r:L]}(C^{d+1})$, where we put $I_{\text{exp}[r:L]}(C^{d+1}) = \{f \in \text{Exp}(C^{d+1}: [r:L]); f = 0 \text{ on } \tilde{S}\}$. We also define the topology of $\text{Exp}(\tilde{S}: (r:L))$ similarly. $\text{Exp}'(\tilde{S}: [r:L])$ and $\text{Exp}'(\tilde{S}: (r:L))$ denote the spaces dual to $\text{Exp}(\tilde{S}: [r:L])$ and $\text{Exp}(\tilde{S}: (r:L))$ respectively.

If f is a function or a functional on S , we denote by $S_n(f; \cdot)$ the n -th spherical harmonic component of f :

$$(1.1) \quad S_n(f; s) = N(n, d) \langle f, P_{n,d}(\cdot, s) \rangle \quad \text{for } s \in S,$$

where

$$(1.2) \quad N(n, d) = \dim H_{n,d} = \frac{(2n + d - 1)(n + d - 2)!}{n! (d - 1)!}$$

and $P_{n,d}$ is the Legendre polynomial of degree n and of dimension $d + 1$. We put $L_n(x) = \|x\|^n P_{n,d}(\alpha \cdot x / \|x\|)$ for fixed $\alpha \in S$. Then L_n is the unique homogeneous harmonic polynomial of degree n with the following properties:

$$L_n(Ax) = L_n(x) \quad \text{for all } A \in O(d+1) \text{ such that } A\alpha = \alpha.$$

$$L_n(\alpha) = 1.$$

We see that $S_n(f; \cdot)$ belongs to $H_{n,d}$ for $n = 0, 1, \dots$.

Put $\Lambda_+ = \{(n, k) \in \mathbb{Z}_+^2; n \equiv k \pmod{2} \text{ and } n \geq k\}$, where $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$. For any $F \in \mathcal{O}(\widetilde{B}(r))$ we can determine uniquely $S_{n,k}(F; \cdot) \in H_{k,d}$ for every $(n, k) \in \Lambda_+$ in such a way that

$$(1.3) \quad F(z) = \sum_{(n,k) \in \Lambda_+} (\sqrt{z^2})^{n-k} \widetilde{S}_{n,k}(F; z),$$

where $z^2 = z_1^2 + z_2^2 + \dots + z_{d+1}^2$, and the right hand side of (1.3) converges uniformly on every compact set of $\widetilde{B}(r)$. The $S_{n,k}(F; \cdot)$ is called the (n, k) -component of F (see [8][9]).

Next we consider a complex cone M as follows:

$$M = \{z \in \mathbb{C}^{d+1}, z^2 = 0\}.$$

M is identified with the cotangent bundle on S minus its zero section. $P_n(M)$ denotes the restriction to M of $P_n(\mathbb{C}^{d+1})$. We put the subset N of M as follows:

$$N = \{z = x + iy \in M; \|x\| = \|y\| = 1\},$$

where $x, y \in \mathbb{R}^{d+1}$. The unit cotangent bundle to S is identified with N and we have $N \simeq O(d+1)/O(d-1)$. dN denotes the unique $O(d+1)$ -invariant measure on N with $\int_N 1 \, dN(z) = 1$.

It is known that for any $f_n \in H_{n,d}$ and any $g_m \in H_{m,d}$

$$(1.4) \quad \int_S f_n(s) \overline{g_m(s)} \, ds \\ = \frac{n! \, N(n,d) \, \Gamma((d+1)/2)}{2^{2n} \, \Gamma(n + \frac{d+1}{2})} \int_N \widehat{f}_n(z) \overline{\widehat{g}_m(z)} \, dN(z),$$

(see, for example [4][5][13]).

§ 2. A uniqueness set for the differential operator $\Delta_z + \lambda^2$.

Our main theorem in this section is the following

Theorem 2.1 ([14]). (i) The restriction mapping $F \rightarrow F|_M$ defines the following bijections:

$$(2.1) \quad \alpha_\lambda: \mathcal{O}_\lambda(\widetilde{B}(r)) \longrightarrow \mathcal{O}(\widetilde{B}(r))|_M \quad \text{for any } \lambda \in \mathbb{C}.$$

(ii) If $f \in \mathcal{O}(\widetilde{B}(r))|_M$ then $\alpha_\lambda^{-1}f$ can be expressed as

follows:

$$(2.2) \quad \alpha_\lambda^{-1}f(z) = \int_N f(\rho z'/2) K_\lambda(z, \frac{\bar{z}'}{\rho}) \, dN(z') \quad \text{for } z \in \widetilde{B}(r),$$

where $L(z) < \rho < r$ and

$$(2.3) \quad K_\lambda(z, \zeta) = \sum_{n=0}^{\infty} \left\{ N(n,d) \, \Gamma(n + \frac{d+1}{2}) \, (\lambda \sqrt{z^2}/2)^{-n - \frac{d-1}{2}} \right. \\ \left. J_{n+\frac{d-1}{2}}(\lambda \sqrt{z^2}) \, (z \cdot \zeta)^n \right\}.$$

In particular, if $\lambda = 0$ we have a "Poisson" formula:

$$(2.4) \quad \mathcal{A}_0^{-1} f(z) = \int_N f(\rho z'/2) \frac{(1 + \bar{z}' \cdot \frac{z}{\rho})}{(1 - \bar{z}' \cdot \frac{z}{\rho})^d} dN(z').$$

(iii) $\mathcal{A}_\lambda$ is a linear topological isomorphism of $\mathcal{O}_\lambda(\tilde{B}(r))$ onto $\mathcal{O}(\tilde{B}(r))|_M$ if we equip $\mathcal{O}(\tilde{B}(r))|_M$ with the topology of uniform convergence on every compact set of $\tilde{B}(r) \cap M$.

We need the following lemmas in order to prove the theorem.

Lemma 2.2. Let $F \in \mathcal{O}_\lambda(\tilde{B}(r))$ and $S_{n,k}$ be the (n,k) -component of F . Then we have

$$(2.5) \quad S_{n,k} = \frac{(i\lambda/2)^{n-k} \Gamma(k + \frac{d+1}{2})}{\Gamma(\frac{n-k}{2} + 1) \Gamma(\frac{n+k+d+1}{2})} S_{k,k}$$

for $(n,k) \in \Lambda_+$ and

$$(2.6) \quad \limsup_{n \rightarrow \infty} \|S_{n,n}\|_\infty^{1/n} \leq 1/r.$$

Conversely if we are given a sequence of spherical harmonics $\{S_{n,k}\}_{(n,k) \in \Lambda_+}$ satisfying (2.5) and (2.6) and if we put for $z \in \tilde{B}(r)$

$$(2.7) \quad F(z) = \sum_{(n,k) \in \Lambda_+} (\sqrt{z^2})^{n-k} \tilde{S}_{n,k}(z)$$

then the right hand side of (2.7) converges uniformly and absolutely on every compact set of $\tilde{B}(r)$ and F belongs to $\mathcal{O}_\lambda(\tilde{B}(r))$. Furthermore we have

$$\widetilde{S}_{n,k}(z) = \widetilde{S}_{n,k}(F; z) \quad \text{for } (n,k) \in \Lambda_+.$$

Remark the case $\lambda = 0$ is known (see [9]).

Sketch of the proof. By [8] Theorem 3.2 we have

$$\begin{aligned} (2.8) \quad \Delta_z F(z) &= \sum_{(n,k) \in \Lambda_+} \Delta_z ((\sqrt{z^2})^{n-k} \widetilde{S}_{n,k}(z)) \\ &= \sum_{\substack{(n,k) \in \Lambda_+ \\ n > k}} (n-k)(n+k+d-1) (\sqrt{z^2})^{n-k-2} \widetilde{S}_{n,k}(z). \end{aligned}$$

(2.8) gives us, for $0 \leq k \leq n-2$ with $n \equiv k \pmod{2}$,

$$(2.9) \quad (n-k)(n+k+d-1) S_{n,k} = -\lambda^2 S_{n-2,k},$$

because $F \in \mathcal{O}_\lambda(\tilde{B}(r))$ and $H_{n,d} \perp H_{m,d}$ if $n \neq m$. (2.5) follows from (2.9). (2.6) follows from [8] Theorem 3.2 (3.33).

Conversely, suppose we are given a sequence $\{S_{n,k}\}$ satisfying (2.5) and (2.6). Then we can show that the right hand side of (2.7) converges uniformly and absolutely on every compact set of $\tilde{B}(r)$ since the Šilov boundary of $\tilde{B}(f)$ is $\{f e^{i\theta} s; 0 \leq \theta < 2\pi, s \in S\}$ (see Hua [3]). Therefore F belongs to $\mathcal{O}(\tilde{B}(r))$ and $S_{n,k} = S_{n,k}(F; \cdot)$.

It is easy to show that $\Delta_z F = -\lambda^2 F$.

q.e.d.

Lemma 2.3. For $F \in \mathcal{O}(\tilde{B}(r))$ we have for any $z \in \mathbb{C}^{d+1}$

$$(2.10) \quad \tilde{S}_{n,n}(F; z) = N(n, d) \int_N F(\rho z'/2) (\bar{z}' \cdot \frac{z}{\rho})^n dN(z'),$$

where ρ is any real number such that $0 < \rho < r$ and the right hand side of (2.10) is independent of ρ .

Sketch of the proof. Since $S_{n,n}(F;) \in H_{n,d}$ it is valid for any $s \in S$

$$(2.11) \quad S_{n,n}(F; s) = N(n, d) \int_S S_{n,n}(F; s') P_{n,d}(s \cdot s') ds'.$$

By (2.11) and (1.4) we have (2.10) because $P_n(M) \perp P_m(M)$ ($n \neq m$)

and $\tilde{P}_{n,d}(z \cdot s) = \{2^n \Gamma(n + \frac{d+1}{2}) (z \cdot s)^n / (n! N(n, d) \Gamma(\frac{d+1}{2}))\}$ on N .

Sketch of the proof of Theorem 2.1. (i) For any $\lambda \in \mathbb{C}$ it is clear that $\mathcal{O}_\lambda(\tilde{B}(r))|_M \subset \tilde{\mathcal{U}}(\tilde{B}(r))|_M$. Let $F \in \mathcal{U}_\lambda(\tilde{B}(r))$. Then for any $z \in M \cap \tilde{B}(r)$ we have

$$(2.12) \quad F(z) = \sum_{n=0}^{\infty} \tilde{S}_{n,n}(F; z).$$

By (2.12) we have for any $z' \in N$

$$(2.13) \quad F(rz'/4) = \sum_{n=0}^{\infty} (r/4)^n \widetilde{S}_{n,n}(F; z'),$$

because $(r/4)N \subset \widetilde{B}(r) \cap M$. If $\alpha_\lambda(F) = 0$, $\widetilde{S}_{n,n} = 0$ on N by (2.13) and the orthogonality of homogeneous polynomials on N . So the spherical harmonic function $S_{n,n} = 0$ by (1.4) and $F = 0$ by (2.5). Therefore α_λ is injective.

Next for $f \in \mathcal{O}(\widetilde{B}(r))$ we define the function F as follows:

$$F(z) = \sum_{(n,k) \in \Lambda_+} (\sqrt{z^2})^{n-k} \widetilde{S}_{n,k}(z),$$

where

$$\widetilde{S}_{n,k}(z) = \frac{\Gamma(k + \frac{d+1}{2}) (i\lambda/2)^{n-k}}{\Gamma(\frac{n-k}{2} + 1) \Gamma(\frac{n+k+d+1}{2})} \widetilde{S}_{k,k}(f; z).$$

As $f \in \mathcal{O}(\widetilde{B}(r))$, $\limsup_{n \rightarrow \infty} \|S_{n,n}\|_\infty^{1/n} = \limsup_{n \rightarrow \infty} \|S_{n,n}(f; \cdot)\|_\infty^{1/n} \leq 1/r$

by [8] Theorem 3.2. Hence by Lemma 2.2 $F \in \mathcal{O}_\lambda(\widetilde{B}(r))$ and $F|_M = f|_M$.

Therefore α_λ is surjective.

(ii) By the proof of surjectivity of α_λ in (i) and (2.10) we get (2.2). In particular, we obtain (2.4) since $K_0(z, \frac{\bar{z}'}{\rho}) = \sum_{k=0}^{\infty} N(k,d) (\bar{z}' \cdot \frac{z}{\rho})^k$ and $\sum_{k=0}^{\infty} N(k,d) x^k = (1+x)(1-x)^{-d}$ for $x \in \mathbb{C}$, $|x| < 1$ (see, for example Müller [11] Lemma 3).

(iii) It is clear that α_λ is continuous. (2.2) gives that for any ρ and ρ' with $0 < \rho' < \rho < r$

$$(2.14) \quad \sup_{L(z) \leq \rho'} |\alpha_\lambda^{-1} f(z)| \leq \exp \frac{1}{4} (|\lambda|^2 \rho'^2) \cdot (1 + \frac{\rho'}{\rho}) (1 - \frac{\rho'}{\rho})^{-d} \sup_{z \in \frac{\rho}{2} N} |f(z)|.$$

As $(\rho/2)N$ is the compact set of $\widetilde{B}(r) \cap M$ (2.14) shows that α_λ^{-1} is continuous. q.e.d.

Corollary 2.4. If F belongs to $\mathcal{O}_\lambda(\widetilde{B}(\rho))$ for $0 < \rho < r$ and $F|_M$ belongs to $\mathcal{O}(\widetilde{B}(r))|_M$ then F belongs to $\mathcal{O}_\lambda(\widetilde{B}(r))$.

§3. The Fourier-Borel transformation.

The Fourier-Borel transformation P_λ for a functional $T \in \text{Exp}'(C^{d+1}; [r:L])$ is defined by

$$(3.1) \quad P_\lambda T(z) = \langle T_\lambda, \exp(i\lambda \zeta \cdot z) \rangle \quad \text{for } z \in C^{d+1},$$

where $\lambda \in C \setminus \{0\}$ is a fixed constant. In this section we will determine the functional space on $\widetilde{S}$ whose image by P_λ coincides with $\mathcal{O}_\lambda(\widetilde{B}(r))$. Our main theorem in this section is the following

Theorem 3.1([14]). The transformation P_λ establishes linear topological isomorphisms

$$(3.2) \quad P_\lambda : \text{Exp}'(\widetilde{S}; [|\lambda|r/2 : L]) \xrightarrow{\sim} \mathcal{O}_\lambda(\widetilde{B}(r)) \quad (0 < r \leq \infty),$$

$$(3.3) \quad P_\lambda : \text{Exp}'(\widetilde{S}; (|\lambda|r/2 : L)) \xrightarrow{\sim} \mathcal{O}_\lambda(\widetilde{B}[r]) \quad (0 \leq r < \infty).$$

Remark. The above theorem was proved by Morimoto [10] in the case of $r = \infty$.

We need the following lemma in order to prove the theorem.

Lemma 3.2. If S_n is the spherical harmonic component of f' then

$$(3.4) \quad f' \in \text{Exp}'(\tilde{S}: [r:L]) \iff \limsup_{n \rightarrow \infty} (\|S_n\|_\infty/n!)^{1/n} \leq 1/r,$$

$$(3.5) \quad f' \in \text{Exp}'(\tilde{S}: (r:L)) \iff \limsup_{n \rightarrow \infty} (\|S_n\|_\infty/n!)^{1/n} < 1/r.$$

Lemma 3.2 can be proved in the same way as in the proof of [10] Theorem 6.1.

Sketch of the proof of Theorem 3.1. By using the results of [6] and Theorem 2.1, Corollary 2.4 and Lemma 3.2, Theorem 3.1 can be proved. q.e.d.

§4. A uniqueness set for linear partial differential operators of the second order.

Consider the differential operator

$$(4.1) \quad P(D) = \sum_{i=1}^{d+1} \sum_{j=1}^{d+1} a_{i,j} \frac{\partial^2}{\partial z_i \partial z_j} + \sum_{j=1}^{d+1} b_j \frac{\partial}{\partial z_j} + c,$$

where $a_{i,j}$, b_j , $c \in \mathbb{C}$ and $a_{i,j} = a_{j,i}$ for $i, j = 1, 2, \dots, d+1$,

and we put

$$(4.2) \quad A = \begin{pmatrix} a_{i,j} \end{pmatrix}.$$

As A is a symmetric matrix there is some $T \in U(d+1)$ and $\lambda_1, \lambda_2, \dots, \lambda_{d+1} \in \mathbb{R}$ such that

$$(4.3) \quad TA^tT = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_{d+1} \end{pmatrix},$$

where $U(K)$ is the unitary group of degree k . We define

$$(4.4) \quad A_0 = {}^tT \begin{pmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_{d+1}) \end{pmatrix} T,$$

where we put

$$f(\lambda) = \begin{cases} 1/\lambda & (\lambda \neq 0) \\ 0 & (\lambda = 0). \end{cases}$$

If $TA^tT = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_{d+1} \end{pmatrix}$ and $SA^tS = \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_{d+1} \end{pmatrix}$ for some $T, S \in U(d+1)$ and $\lambda_j, \mu_j \in \mathbb{C}$ ($j = 1, 2, \dots, d+1$) then

$${}^tT \begin{pmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_{d+1}) \end{pmatrix} T = {}^tS \begin{pmatrix} f(\mu_1) & & \\ & \ddots & \\ & & f(\mu_{d+1}) \end{pmatrix} S.$$

When A is regular we have $A_0 = A^{-1}$.

We define $\mathcal{O}_P(\mathbb{C}^{d+1}) = \{f \in \mathcal{O}(\mathbb{C}^{d+1}); P(D)f = 0\}$ and $M_P = \{z \in \mathbb{C}^{d+1}; A_0 z \cdot z = 0\}$. $\text{Holo}(M_P) = \mathcal{O}(\mathbb{C}^{d+1})|_{M_P}$ denotes the space

of holomorphic functions on the complex cone M_P . We define spaces of germs of holomorphic functions as follows:

$$\mathcal{O}_P(\{0\}) = \text{ind} \lim_{0 \in W} \mathcal{O}_P(W),$$

$$\text{Holo}_{M_P}(\{0\}) = \text{ind} \lim_{0 \in W} \mathcal{O}(W)|_{M_P},$$

where W runs through the neighborhoods of 0 . We equip $\mathcal{O}_P(W)$ and $\mathcal{O}(W)|_{M_P}$ with the topologies of uniform convergence on every compact set of W and every compact set of $W \cap M_P$ respectively. It is known $\mathcal{O}_P(W)$ is an FS space. The topologies of $\mathcal{O}_P(\{0\})$ and $\text{Holo}_{M_P}(\{0\})$ are the locally convex inductive limit of the topologies of $\mathcal{O}_P(W)$ and $\mathcal{O}(W)|_{M_P}$ respectively. It is known $\mathcal{O}_P(\{0\})$ is a DFS space.

Our main theorem in this section is the following:

Theorem 4.1 ([15]). Let $\text{rank } A \geq 2$. The restriction mapping

$\alpha_P : F \rightarrow F|_{M_P}$ defines the following linear topological isomorphisms:

$$(4.5) \quad \alpha_P : \mathcal{O}_P(C^{d+1}) \xrightarrow{\sim} \text{Holo}(M_P),$$

$$(4.6) \quad \alpha_P : \mathcal{O}_P(\{0\}) \xrightarrow{\sim} \text{Holo}_{M_P}(\{0\}).$$

Let d' be a positive integer and $1 \leq d' \leq d$. $B_L(r)$ denotes the Lie ball on $C^{d'+1}$. Let $d'' = d - d'$, $\xi \in C^{d'+1}$ and $t \in C^{d''}$. Let

W be an open set in $C^{d''}$. For $f \in \mathcal{O}(B_L(r) \times W)$ ($r > 0$) $S_{n,k}(f; \zeta, t)$ denotes the (n, k) -component of f with respect to ζ ($(n, k) \in \Lambda_+$).

Then we can write

$$(4.7) \quad f(\zeta, t) = \sum_{(n,k) \in \Lambda_+} (\sqrt{\zeta^2})^{n-k} \widetilde{S}_{n,k}(f; \zeta, t).$$

When t is fixed, the right hand side of (4.7) converges to $f(\zeta, t)$ uniformly and absolutely on every compact set of $B_L(r)$ and $S_{n,k}(f; \zeta, t)$ is a homogeneous harmonic polynomial of degree k . For fixed $\zeta \in C^{d'+1}$ $\widetilde{S}_{n,k}(f; \zeta, t)$ belongs to $\mathcal{O}(W)$.

In order to prove the theorem we need following lemmas.

Lemma 4.2. Suppose f belongs to $\mathcal{O}(C^{d+1})$ and satisfies the differential equation

$$(4.8) \quad \Delta_{\zeta} f = \left(\sum_{j=1}^{d''} b_j \frac{\partial}{\partial t_j} + c \right) f,$$

where $\Delta_{\zeta} = (\partial/\partial \zeta_1)^2 + \dots + (\partial/\partial \zeta_{d'+1})^2$ and $b_j, c \in C$. Then we have

$$(4.9) \quad \widetilde{S}_{n,k}(f; \zeta, t) = \frac{\Gamma(k + \frac{d'+1}{2}) \left(\sum_{j=1}^{d''} b_j \frac{\partial}{\partial t_j} + c \right)^{\frac{n-k}{2}}}{2^{n-k} ((n-k)/2)! \Gamma((n+k+d'+1)/2)} \widetilde{S}_{k,k}(f; \zeta, t)$$

for $(n, k) \in \Lambda_+$.

Lemma 4.3. Let $P(D) = \Delta_{\zeta} - \sum_{j=1}^{d''} b_j \frac{\partial}{\partial t_j} - c$, $M_{d'} = \{(\zeta, t) \in$

$C^{d+1}; \zeta^2 = 0\}$, $\text{Holo}(M_d) = \mathcal{O}(C^{d+1})|_{M_d}$, and $\text{Holo}_{M_d}(\{0\}) =$

$\text{ind} \lim_{0 \in W} \mathcal{O}(W)|_{M_d}$. The restriction mapping $\alpha: f \rightarrow f|_{M_d}$ defines

the following bijections:

$$(4.10) \quad \alpha: \mathcal{O}_P(C^{d+1}) \longrightarrow \text{Holo}(M_d),$$

$$(4.11) \quad \alpha: \mathcal{O}_P(\{0\}) \longrightarrow \text{Holo}_{M_d}(\{0\}).$$

We can prove Lemmas 4.2 and 4.3 in the same way as in the proofs Theorem 2.1 and Lemma 2.2.

Sketch of the proof of Theorem 4.1. There exists some $T \in$

$U(d+1)$ such that

$$T A^t T = \begin{pmatrix} \lambda_1 & & & & \\ & \ddots & & & \\ & & \lambda_{d'+1} & & \\ & & & 0 & \ddots \\ & & & & 0 \end{pmatrix},$$

where d' is a positive integer with $0 < d' \leq d$ and $\lambda_j \neq 0$ ($j = 1,$

$2, \dots, d'+1$). Consider

$$\Lambda' = \begin{pmatrix} 1/\sqrt{\lambda_1} & & & & \\ & \ddots & & & \\ & & 1/\sqrt{\lambda_{d'+1}} & & \\ & & & 1 & \ddots \\ & & & & 1 \end{pmatrix}$$

and $(B_1, B_2, \dots, B_{d'+1}, C_{d'+2}, \dots, C_{d+1}) \in C^{d+1}$ which satisfies

$$(2B_1\sqrt{\lambda_1} \dots 2B_{d'+1}\sqrt{\lambda_{d'+1}} C_{d'+2} \dots C_{d+1}) = (b_1 \ b_2 \ \dots \ b_{d+1})^t T.$$

It is clear that α_P is a continuous linear mapping of $\mathcal{O}_P(\mathbb{C}^{d+1})$ into $\text{Holo}(M_P)$. For any $g \in \mathcal{O}(\mathbb{C}^{d+1})$ we put

$$h(z) = g((\wedge' T)^{-1} z) \exp\left(\sum_{j=1}^{d'+1} B_j z_j\right).$$

Then it is easy to show that

$$(4.12) \quad P(D)g(z)$$

$$= \exp\left(-\sum_{j=1}^{d'+1} B_j w_j\right) \left(\sum_{j=1}^{d'+1} (\partial/\partial w_j)^2 + \sum_{j=d'+2}^{d+1} C_j (\partial/\partial w_j) + c - \sum_{j=1}^{d'+1} B_j^2 \right) h,$$

where we put $w = \wedge' Tz$. Since $M_{d'} = (\wedge' T)M_P$, we can prove that the bijectivity of α_P from (4.12) and Lemmas 4.2 and 4.3.

Furthermore, we have for any $f \in \text{Holo}(M_P)$ and for any ρ and ρ' with $\rho > r$ and $\rho' > r'$

$$(4.13) \quad \sup \left\{ |\alpha_P^{-1} f(z)| ; z \in (\wedge' T)^{-1}(\overline{B_L(r)} \times B[r']) \right\} \\ \leq C \sup \left\{ |f(z)| ; z \in (\wedge' T)^{-1}((\rho/2)N_{d'} \times B[\rho']) \right\},$$

where $N_{d'} = \{\zeta \in \mathbb{C}^{d'+1}; \zeta^2 = 0\} \cap B_L(2)$ and $B[r'] = \{t \in \mathbb{C}^{d''}; \|t\| \leq r'\}$, and C is a constant which depends on r, r', ρ and ρ' . We can see that α_P^{-1} is continuous by (4.13).

We can prove (4.6) similarly.

q.e.d.

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