

COMPACTIFICATIONS OF SYMMETRIC VARIETIES AND APPLICATIONS TO REPRESENTATION THEORY

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1. INTRODUCTION

In these notes we give applications of equivariant compactifications of group varieties of semisimple groups of adjoint type.

Let k be an algebraically closed field of arbitrary characteristic, and let G be a semisimple linear algebraic group of adjoint type defined over k . Furthermore, let σ be an involutive automorphism of G defined over k . The pair (G, σ) is called a symmetric pair. The compactification in question is an equivariant compactification of G/G^σ uniquely characterized by the fact that it is smooth and that the complement $D = X - G/G^\sigma$ is the union of ℓ irreducible divisors, where ℓ denotes the rank of G/G^σ . A description of this compactification is the subject matter of section 2. This compactification, over the complex numbers, is related to the Oshima compactification, and over fields of characteristic zero, the DeConcini-Procesi compactification. It can also be described as the maximal Satake compactification of G/G^σ , which appears in the work of Goresky-MacPherson.

In section 5, we give an application of this compactification to representation theory of finite groups of Lie type. We give a Langlands type parametrization of character sheaves of such groups. Langlands parametrization takes the following form. Let k be a local or global field. Let W denote the Weil group of k . Then Langlands parametrization associates to an admissible irreducible representation of $G(k)$, a homomorphism of W (or the Weil-Deligne group W') to ${}^L G(\mathbb{C})$. Replacing $\mathbb{F}_q$ for k in this recipe yields little that is interesting: the Weil

group for $\mathbb{F}_q$ is isomorphic to $\mathbb{Z}$, generated by the Frobenius automorphism. The key step here is to replace W by the tame part of the Weil group of F , where F is a local field with residue class field $\mathbb{F}_q$. This replacement has been made earlier by MacDonald[7] for GL_n and for GL_2 by Ilya Piatetski-Shapiro[8]. Both authors use an ad hoc method: by appealing to the classification of irreducible representations in both cases. The classification of irreducible character sheaves is known by Lusztig, hence it is in principle, possible to give such a parametrization for other groups, too. The purpose of these notes is to give a geometric explanation of why the tame part appears.

Let us give a brief review of the arguments here. Character sheaves are analogues over the finite fields of the Harish-Chandra equations for characters of semisimple Lie groups over the reals. The tame Weil group appears as part of the monodromy group around the divisors D_i : since characters correspond to character sheaves which are regular holonomic systems, they detect only the tame part of the fundamental group, thus explaining the appearance of the tame part of the Weil group of F .

In order to complete this sketch, it is then necessary to explain the function-sheaf dictionary for varieties over finite fields. This is the subject matter of section 3. It is then necessary to give the definition of microlocalization of ℓ -adic sheaves. This is done in 4.

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2. THE COMPACTIFICATION

Let k be a field, and let G be a connected semisimple linear algebraic group of adjoint type over k . Let σ denote an involutive automorphism of G defined over k .

The purpose of this section is to give properties of a canonical compactification of G/G^σ .

Let us first recall some basic definitions and properties concerning involutions and tori. The following definition is due to Th. Vust[10].

- Definition 2.0.1.**
1. Let A be a torus of G . Then A is said to be σ -split if and only if σ acts on A by inversion: $\sigma(a) = a^{-1}$ for all $a \in A$.
 2. Let P be a parabolic subgroup of G . Then P is said to be σ -split if and only if $P \cap \sigma(P)$ is a Levi subgroup of P .
 3. A pair of a Borel subgroup B and a maximal torus T contained in B is said to be fundamental if $B = \sigma(B)$ and $T = \sigma(T)$ hold.
 4. A pair of a Borel subgroup B and a maximal torus T is said to be standard if T contains a maximal σ -split torus A .

It is a theorem of Steinberg that a σ -stable Borel subgroup always exists. However, since the automorphism is not semisimple in general,

one cannot conclude the existence of a maximal torus $T \subset B$ such that $\sigma(T) = T$.

The following theorem summarizes results concerning σ -split tori and parabolics.

Theorem 2.0.2. *Let G be a reductive groups scheme defined over an algebraically closed field k . Let σ be a non-trivial involution of G defined over k . Then the following hold true.*

1. *There exists a non-trivial σ -split torus. Any two maximal σ -split tori of G are conjugate under the action of G^σ .*
2. *Let P be a minimal σ -split torus. Set $L = P \cap \sigma(P)$. Then there exists a unique maximal σ -split torus A such that $L = Z_G(A)$.*
3. *The commutator $[L, L]$ of $L = Z_G(A)$ is contained in G^σ .*
4. *Let T be a torus of G which contains a maximal σ -split torus A of G . Then T is σ -stable.*

There is a correspondence between one parameter subgroups (1PS for short) of a σ -split torus A and σ -split parabolics of G . Let $\lambda \in X_*(A)$ be a 1PS of A . The parabolic $P(\lambda)$ associated to λ is given (set theoretically) as follows.

$$P(\lambda) = \{g \in G \mid \lim_{t \rightarrow \infty} \lambda(t)g\lambda(t)^{-1} \text{ exists in } G\}$$

The intersection $P(\lambda) \cap P(\lambda^{-1})$ is the centralizer of λ ; hence it is reductive, and we denote it by $L(\lambda)$.

The compactification in question satisfies the following properties. The rank of a symmetric pair (G, σ) is by definition the dimension of a maximal σ -split torus A .

Theorem 2.0.3. *There exists a unique G -equivariant compactification X of G/G^σ such that the following properties hold.*

1. *The compactification X is smooth, and the complement $D = X - G/G^\sigma$ is a divisor with only normal crossings.*
2. *Let $D = \bigcup_{i=1}^{\ell} D_i$ be the decomposition of D into irreducible components. Then ℓ is equal to the rank of the symmetric pair (G, σ) , and for any subset J of $I = \{1, \dots, \ell\}$, the intersection $\bigcap_{i \in J} D_i$ is the closure of a G -orbit.*
3. *For any subset J of I , there exists a σ -split parabolic subgroup P_J such that there exists a G -equivariant projection:*

$$\pi_J : \bigcap_{i \in J} D_i \rightarrow G/P_J$$

such that the fiber of π_J over P_J/P_J is the canonical compactification of the pair (L_J, σ) , where $L_J = P_J \cap \sigma(P_J)$.

A more intrinsic description can be given as follows. Let A be a maximal σ -split torus of G . Let $W = N_G(A)/Z_G(A)$ denote the little Weyl group of the symmetric pair (G, σ) . Elements of the index set I are in one-to-one correspondence with the set of edges of a Weyl

chamber in $X_*(A) \otimes \mathbb{R}$. A subset J of I defines a wall of the Weyl chamber. Let λ be a generic 1PS in the wall. Then $P(J) = P(\lambda)$, and D_J is equal to the closure of the G -orbit of the limit $\lim_{t \rightarrow \infty} \lambda(t)H/H$.

3. THE DICTIONARY

The purpose of this section is to recall the dictionary of sheaves and functions on varieties over finite fields. This is due to Grothendieck. The main references are [2] and [5].

Let k be a finite field, and let X denote a separated scheme of finite type over k . Let ℓ denote a prime distinct from the characteristic $p > 0$ of k . Let $D_c^b(X, \overline{\mathbb{Q}}_\ell)$ denote the derived category of ℓ -adic sheaves on X . Given K , an element of $D_c^b(X, \overline{\mathbb{Q}}_\ell)$, a function t_K on $X(k)$ is defined as follows. Let $x \in X(k)$. Let F_x denote the geometric Frobenius map relative to k ; it acts on the group $H_c^\bullet(X \otimes_k \overline{k}, K)$. Then $t_K(x)$ is the alternating sum of the trace of F_x on $H_c^i(X \otimes_k \overline{k}, K)$:

$$t_K(x) = \sum_i (-1)^i \text{Tr}(F_x, H_c^i(X \otimes_k \overline{k}, K)).$$

It is now accepted wisdom, instead of considering functions of $X(k)$ per se, but to look for ℓ -adic sheaves whose trace function t_K gives the desired function.

This turns out to be a fruitful approach, since functions occurring in representation theory usually satisfy a system of differential equations. This is the case for characters, to which we return in a moment. In practice, the system of differential equations that arise are regular holonomic systems [4], [3]. The sheaf of solutions become a particular type of bounded complex of constructible sheaves called perverse sheaves. The definition of perverse sheaves have been carried over to $D_c^b(X, \overline{\mathbb{Q}}_\ell)$, it is denoted by $\text{Perv}(X, \overline{\mathbb{Q}}_\ell)$ [1].

Characters of semisimple Lie groups over the reals satisfy a system of differential equations first studied by Harish-Chandra. This is known to be a regular holonomic system by the work of R. Hotta and M. Kashiwara. The right analogue over finite fields have been defined and studied extensively by G. Lusztig in a series of papers; [6] is a good introduction to the subject.

4. MICROLOCALIZATION OF ℓ -ADIC SHEAVES

Throughout this section, let k be a field of characteristic $p > 0$. Let X be a scheme of finite type over k , Y be a smooth subscheme of X and let $\mathcal{F}$ be an ℓ -adic sheaf defined over k .

By the smoothness assumption, the normal cone $C_{Y \setminus X}$ of Y in X is a vector bundle over Y ; the dual vector bundle is the conormal bundle of Y in X denoted by $C_{Y \setminus X}^*$. Let us fix an additive character $\varphi : \mathbb{F}_q \rightarrow \mathbb{Q}_\ell^*$. The Fourier-Deligne transformation F gives a functor from $D_c^b(C_{Y \setminus X})$ to $D_c^b(C_{Y \setminus X}^*)$.

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Assume that $\mathcal{F}$ is tamely ramified along Y . The specialization $\nu_Y(\mathcal{F})$ has been defined by J.L. Verdier[9]. The microlocalization of $\mathcal{F}$ along Y is defined as follows.

Definition 4.0.4. Let $\nu_Y(\mathcal{F})$ denote the specialization of $\mathcal{F}$ along Y ; this is an ℓ -adic sheaf on the normal cone $C_{Y \setminus X}$ of Y in X .

The microlocalization of $\mathcal{F}$ along Y is the Fourier-Deligne transformation of $\nu_Y(\mathcal{F})$; it is a monodromic ℓ -adic sheaf on the conormal cone $C_{Y \setminus X}^*$ of Y in X .

The basic properties of the microlocalization are as follows.

- Proposition 4.0.5.**
1. The microlocalization functor ν is a functor from $D_c^b(X, A)$ to $\text{Mon}(C_{Y \setminus X}^*, A)$, the derived category of bounded complexes of constructible monodromic sheaves.
 2. The microlocalization functor commutes with the Verdier duality functor.
 3. Microlocalization respects proper and smooth base changes.

In contrast to the specialization functor, the microlocalization functor is not local with respect to the étale topology of X .

5. LANGLANDS PARAMETRIZATION

Let k be the finite field of q elements. Let G be a semisimple group of adjoint type over k . We denote by ${}^L G$ the Langlands dual group of G . Let F be a local field with residue class field k . Let $W^{\text{tame}}(k)$ denote the tame part of the Weil group of F . Alternately, one can define the group as the semi-direct product of $\mathbb{Z}$ and $\text{proj lim}_n \mathbb{F}_{q^n}^\times$, where the action of $\mathbb{Z}$ is via the Frobenius map.

Theorem 5.0.6. Let $\mathcal{F}$ be a character sheaf on G . Then there exists a homomorphism $W^{\text{tame}} \times \text{sl}_2 \rightarrow {}^L G$.

This map is given by applying the microlocalization functor defined in the previous section to character sheaves.

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