

A proof of the existence of indiscernible trees without Erdős-Rado theorem

Munhehiro Kobayashi
Institute of Mathematics
The University of Tsukuba

Our interest in this paper is to see the similarity between Erdős-Rado theorem and compactness argument using Ramsey theorem in model theory. Erdős-Rado theorem is a theorem in infinitary combinatorics that generalizes Ramsey theorem to handle uncountable situations. In model theory, compactness arguments are available, so arguments tend to be settled in countable situation.

We give a proof without Erdős-Rado theorem to the next theorem.

Theorem 3.1.16. *Let B be a set of parameters, and $\Gamma(x_{\omega < \omega})$ be a set of $\mathcal{L}_B$ -formulas. If $\Gamma(x_{\omega < \omega})$ has $\mathcal{L}_S$ -subtree property, then Γ is realized by an $\mathcal{L}_S$ -indiscernible tree over B .*

This theorem is proved with Erdős-Rado theorem in [2] and [3], while we use compactness arguments and Ramsey theorem.

Byunghan Kim, Hyeung-Joon Kim, and Lynn Scow recently revised their preprint[4], and it contains essentially the same argument of this paper. We have constructed the content independently.

We work in a complete theory T in a language $\mathcal{L}$ throughout this paper. Let $\mathbb{M}$ be a big model of T . We write $\langle n_1 \dots n_k \rangle$ to refer the element of $\omega^{<\omega}$ of length k whose i -th value is n_i . For $\eta_1, \eta_2 \in \omega^{<\omega}$, we write $\eta_1 \widehat{} \eta_2$ to refer the concatenation of η_1 and η_2 . For a set S and an indexed set $(a_s)_{s \in S}$, we write a_S to denote $(a_s)_{s \in S}$.

1 Theorems in infinitary combinatorics

1.1 Ramsey's theorem and Erdős-Rado theorem

Infinite Ramsey's theorem and Erdős-Rado theorem are theorems in infinitary combinatorics. Erdős-Rado theorem is a generalization of Ramsey's theorem to uncountable situations.

Definition 1.1.1. For cardinals α, β, γ and for $n < \omega$, we write

$$\alpha \rightarrow (\beta)_\gamma^n$$

whenever $|X| = \alpha$ and $f : [X]^n \rightarrow \gamma$, there exists $Y \subset X$ with $|Y| = \beta$ such that $f([Y]^n)$ is a singleton.

Theorem 1.1.2 (Infinite Ramsey's Theorem). *For all $k, n \in \omega$,*

$$\aleph_0 \rightarrow (\aleph_0)_k^n.$$

Theorem 1.1.3 (Erdős-Rado Theorem). *For all $n \in \omega$ and infinite cardinal κ ,*

$$\exp_n(\kappa)^+ \rightarrow (\kappa^+)_\kappa^{n+1},$$

where $\exp_n(\kappa)$ is inductively defined by $\exp_0(\kappa) = \kappa$, $\exp_{n+1}(\kappa) = 2^{\exp_n(\kappa)}$.

2 Indiscernible structures

We introduce indiscernible sequences and $\mathcal{L}_S/\mathcal{L}_1$ -indiscernible trees. We also define subsequence property and $\mathcal{L}_S/\mathcal{L}_1$ -subtree property, which later we prove that they induces the existence of indiscernible structures.

2.1 Indiscernible sequences

Definition 2.1.4 (Indiscernible sequences). Let $\mathcal{L}_o = \{<\}$ and $\mathcal{L}_o$ -structure I be a totally ordered set, and let $B \subset \mathbb{M}$. For $a_I \subset \mathbb{M}$, we say a_I is an indiscernible sequence over B if for all $I_0, I_1 \subset I$ such that $I_0 \cong_{\mathcal{L}_o} I_1$, it holds that $\text{tp}(a_{I_0}/B) = \text{tp}(a_{I_1}/B)$.

Be careful the index set I is not a subset of the big model $\mathbb{M}$ and the I -indexed set a_I is a subset of $\mathbb{M}$.

Subsequence property was introduced by Tsuboi in his lecture note in 1999.

Definition 2.1.5 (Subsequence property). Let $\mathcal{L}_o = \{<\}$ and $\mathcal{L}_o$ -structure I be a totally ordered set. For a set of formulas $\Gamma(x_I)$, we say Γ has subsequence property if

$$\cup \{ \Gamma(x_{\sigma(I)}) \mid \sigma : I \rightarrow I \text{ is an } \mathcal{L}_o\text{-embedding} \}$$

is consistent.

Example 2.1.6. Let $\Gamma(x_\omega)$ be the set of formulas expressing “ x_ω is an indiscernible sequence.” Then, Γ has subsequence property. Γ can be concretely written as

$$\left\{ \varphi(x_I) \leftrightarrow \varphi(x_J) \mid \varphi \in \mathcal{L}, I, J \subset \omega, I \cong_{\mathcal{L}_o} J \right\}.$$

Example 2.1.7. Let $\Gamma(x_\omega, y_\omega)$ be the set of formulas expressing “ $(x_i, y_i)_{i \in \omega}$ witnesses the order property of $\varphi(x, y)$.” Then, Γ has the subsequence property.

Γ can be concretely written as

$$\{ \varphi(x_i, y_j) \mid i < j < \omega \} \cup \{ \neg \varphi(x_j, y_i) \mid j \leq i < \omega \}.$$

The following lemma guarantees the existence of indiscernible sequences.

Lemma 2.1.8 (Tsuboi 1999). *Let B be a set of parameters, and $\Gamma(x_\omega)$ be a set of $\mathcal{L}_B$ -formulas. If $\Gamma(x_\omega)$ has subsequence property, then Γ is realized by an indiscernible sequence over B .*

Proof. We show $\Gamma(x_\omega) \cup$ “ x_ω is an indiscernible sequence over B ” is consistent, where

“ x_ω is an indiscernible sequence over B ” =

$$\left\{ \varphi(x_I) \leftrightarrow \varphi(x_J) \mid \varphi \in \mathcal{L}_B, I, J \subset_{\text{fin}} \omega, I \cong_{\mathcal{L}_o} J \right\}.$$

We use compactness argument. We fix $\mathcal{L}_B$ -formulas $\varphi_1, \dots, \varphi_m$ each of which has n free variables from x_I . It is sufficient to show

$$\tilde{\Gamma} = \Gamma \cup \left\{ \varphi_k(x_{I_0}) \leftrightarrow \varphi_k(x_{I_1}) \mid k = 1, \dots, m, I_0, I_1 \underset{n \text{ elem}}{\subset} \omega, I_0 \underset{\mathcal{L}_0}{\simeq} I_1 \right\}$$

is consistent. We fix a realization $A \models \Gamma$, and we define $F : A^n \rightarrow 2^n$ by

$$F(\bar{a}) = \sum_{k=1}^n i_k 2^k, \text{ where } \begin{cases} i_k = 0 & \text{if } \neg \varphi_k(\bar{a}) \text{ holds} \\ i_k = 1 & \text{if } \varphi_k(\bar{a}) \text{ holds.} \end{cases} \quad \text{for } \bar{a} \in A^n$$

By Ramsey's theorem, there is an infinite $A' \subset A$ such that $F|_{A'^n}$ is constant. This A' is a witness of $\tilde{\Gamma}$, for φ_k have the same truth value on A'^n , and $A' \models \Gamma$ by subsequence property. $\square$

2.2 Indiscernible trees

Definition 2.2.9. Let $\mathcal{L}_1 = \{\cap, <_{\text{len}}, <_{\text{lex}}, <_{\text{ini}}\}$, and let $\mathcal{L}_S = \mathcal{L}_1 \cup \{P_n \mid n \in \omega\}$.

Here, we use the notation $\mathcal{L}_S$ instead of the original notation $\mathcal{L}_0$ in [2].

Definition 2.2.10. Let the interpretation of $\mathcal{L}_1$ and $\mathcal{L}_S$ in $\omega^{<\omega}$ as follows:

- $\eta \cap \nu$ = the longest common initial segment of η and ν .
- $\eta <_{\text{len}} \nu \Leftrightarrow \eta$ has the less length than ν .
- $\eta <_{\text{lex}} \nu \Leftrightarrow \eta$ is less than ν in the lexicographic order.
- $\eta <_{\text{ini}} \nu \Leftrightarrow \eta$ is a proper initial segment of ν .
- $P_n(\eta) \Leftrightarrow \eta$ has the length of n .

We refer $\mathcal{L}_S$ or $\mathcal{L}_1$ -substructures of $\omega^{<\omega}$ by the word ‘trees’.

Definition 2.2.11 (Indiscernible trees). Let $B \subset \mathbb{M}$.

- (1) Let S be an $\mathcal{L}_S$ -substructure of $\omega^{<\omega}$. For $a_S \subset \mathbb{M}$, we say a_S is an $\mathcal{L}_S$ -indiscernible tree over B if for all $S_0, S_1 \subset S$ such that $S_0 \underset{\mathcal{L}_S}{\simeq} S_1$, it holds that $\text{tp}(a_{S_0}/B) = \text{tp}(a_{S_1}/B)$.
- (2) Let S be an $\mathcal{L}_1$ -substructure of $\omega^{<\omega}$. For $a_S \subset \mathbb{M}$, we say a_S is an $\mathcal{L}_1$ -indiscernible tree over B if for all $S_0, S_1 \subset S$ such that $S_0 \underset{\mathcal{L}_1}{\simeq} S_1$, it holds that $\text{tp}(a_{S_0}/B) = \text{tp}(a_{S_1}/B)$.

Be careful the index set S is not a subset of the big model $\mathbb{M}$ and the S -indexed set a_S is a subset of $\mathbb{M}$.

Example 2.2.12. For $\eta, \nu \in \omega^{<\omega}$, we say η is an ancestor or a descendant of ν if either of the nodes is a proper initial segment of the other, and we say η and ν are siblings if η and ν has the same length n and the length of $\eta \cap \nu$ is $n - 1$.

Let T be the theory of random graph in the language $\{R(*, *)\}$. For distinct vertices a_ω, a_ν in the big model that satisfies for all $\eta, \nu \in \omega^{<\omega}$

$$\models R(a_\eta, a_\nu) \Leftrightarrow \text{“}\eta \text{ is an ancestor or a descendant of } \nu\text{”}$$

form an $\mathcal{L}_S$ and $\mathcal{L}_1$ -indiscernible tree.

Let $b_{\omega < \omega}$ be the tree-indexed subset such that for all $\eta, \nu \in \omega^{<\omega}$,

$$\models R(a_\eta, a_\nu) \Leftrightarrow \text{"}\eta \text{ is an ancestor or a descendant of } \nu\text{" or "}\eta \text{ and } \nu \text{ are siblings."}$$

Then, b_ω is an $\mathcal{L}_S$ -indiscernible tree but not an $\mathcal{L}_1$ -indiscernible tree. In fact,

$$\{\emptyset, \langle 0 \rangle, \langle 1 \rangle\} \simeq_{\mathcal{L}_1} \{\emptyset, \langle 00 \rangle, \langle 10 \rangle\} \text{ but } \models R(b_{\langle 0 \rangle}, b_{\langle 1 \rangle}) \wedge \neg R(b_{\langle 00 \rangle}, b_{\langle 10 \rangle}).$$

Definition 2.2.13 (Subtree property [2], [3]). Let $B \subset \mathbb{M}$.

- (1) Let S be an $\mathcal{L}_S$ -substructure of $\omega^{<\omega}$. For a set of $\mathcal{L}_B$ -formulas $\Gamma(x_S)$, we say Γ has $\mathcal{L}_S$ -subtree property if

$$\cup \{ \Gamma(x_{\sigma(S)}) \mid \sigma : I \rightarrow I \text{ is an } \mathcal{L}_S\text{-embedding} \}$$

is consistent.

- (2) Let S be an $\mathcal{L}_1$ -substructure of $\omega^{<\omega}$. For a set of $\mathcal{L}_B$ -formulas $\Gamma(x_S)$, we say Γ has $\mathcal{L}_S$ -subtree property if

$$\cup \{ \Gamma(x_{\sigma(S)}) \mid \sigma : I \rightarrow I \text{ is an } \mathcal{L}_1\text{-embedding} \}$$

is consistent.

Example 2.2.14. $\Gamma(x_{\omega < \omega}) = \text{"}x_{\omega < \omega} \text{ witnesses the } k\text{-tree property of } \varphi(x, y)\text{"}$ has the $\mathcal{L}_S$ -subtree property (if Γ is consistent).

Γ can be concretely written as

$$\Gamma(y_\omega \times \omega) = \bigcup_{i \in \omega} \left\{ \neg \exists x \left(\bigwedge_{i < k} \varphi(x, y_{\hat{i} j_i}) \right) \mid j_0, \dots, j_{k-1} \in \omega \right\} \cup \bigcup_{\nu \in \omega^\omega} \left\{ \exists x \left(\bigwedge_{i < n} \varphi(x, y_{\nu|_i}) \right) \mid n \in \omega \right\}.$$

3 Existence of indiscernible trees

In this section, we prove that subtree property implies the existence of an indiscernible tree without Erdős-Rado theorem.

The existence of $\mathcal{L}_S$ -indiscernible trees is proved with the following theorem in [2], [3].

Theorem (Shelah, Theorem 2.6 of [5, p.662]). For all $k, n \in \omega$ and ordinal μ , there exists an ordinal λ such that for any $f : (\lambda^{<n})^k \rightarrow \mu$, there is an $\mathcal{L}_S$ -substructure $S \subset \lambda^{<n}$ with $S \simeq_{\mathcal{L}_S} \omega^{<\omega}$ satisfying $f(X) = f(Y)$ for all $X, Y \in S^k$ with $X \simeq_{\mathcal{L}_S} Y$.

This is a variation of Erdős-Rado theorem regarding trees. We want to show the existence of indiscernible trees without this theorem.

3.1 $\mathcal{L}_S$ -indiscernible trees

Proposition 3.1.15 ([3]). Let B be a set of parameters, and $\Gamma(x_\omega^{<n})$ be a set of $\mathcal{L}_B$ -formulas for $n \in \omega$. If $\Gamma(x_{\omega < n})$ has the $\mathcal{L}_S$ -subtree property, then Γ is realized by an $\mathcal{L}_S$ -indiscernible tree over B .

Proof. We show $\Gamma(x_{\omega < n}) \cup "x_{\omega < n} \text{ is an } \mathcal{L}_S\text{-indiscernible tree over } B"$ is consistent, where

$$\begin{aligned} & "x_{\omega < n} \text{ is an } \mathcal{L}_S\text{-indiscernible tree over } B" = \\ & \left\{ \varphi(x_S) \leftrightarrow \varphi(x_T) \mid \varphi \in \mathcal{L}_B, S, T \subseteq \omega^{<n}, S \underset{\mathcal{L}_S}{\simeq} T \right\}. \end{aligned}$$

We show this by induction on n . The case $n = 1$ is clear because $\omega^{<1} = \{\emptyset\}$.

Suppose the n case holds. We write $k \widehat{\omega}^{<n}$ to denote the set $\{\sigma \in \omega^{<n+1} \mid \sigma(0) = k\}$ and X_k to denote the set of variables $x_{k \widehat{\omega}^{<n}}$.

Claim A. $\Gamma(x_{\omega < n+1}) \cup \left(\bigcup_{k \in \omega} \Sigma_k(x_{\omega < n+1}) \right)$ is consistent, where

$$\begin{aligned} \Sigma_k &= "X_k \text{ is an } \mathcal{L}_S\text{-indiscernible tree over } Bx_\emptyset X_0 X_1 \dots X_{k-1} X_{k+1} \dots" \\ &= \left\{ \varphi(x_S) \leftrightarrow \varphi(x_T) \mid \begin{array}{l} \varphi \in \mathcal{L}(Bx_\emptyset X_0 X_1 \dots X_{k-1} X_{k+1} \dots), \\ S, T \subseteq k \widehat{\omega}^{<n}, S \underset{\mathcal{L}_S}{\simeq} T \end{array} \right\}. \end{aligned}$$

Proof of Claim A. Let $a_{\omega < n+1} = a_\emptyset A_0 A_1 \dots \models \Gamma$, where $A_k = a_{k \widehat{\omega}^{<n}}$. First, observe that for any tree S with $S \underset{\mathcal{L}_S}{\simeq} \omega^{<n}$, the tree $\emptyset \ 0 \widehat{S} \ 1 \widehat{\omega}^{<n} \ 2 \widehat{\omega}^{<n} \dots$ becomes an $\mathcal{L}_S$ -substructure that is isomorphic to whole $\omega^{<n+1}$. Therefore $\Gamma(a_\emptyset X_0 A_1 A_2 \dots)$ has $\mathcal{L}_S$ -subtree property over $a_\emptyset A_1 A_2 \dots$ by the $\mathcal{L}_S$ -subtree property of $\Gamma(x_{\omega < n})$. By induction hypothesis, $\Gamma(a_\emptyset X_0 A_1 \dots)$ is realized by A'_0 which is an $\mathcal{L}_S$ -indiscernible tree over $a_\emptyset A_1 A_2 \dots$, i.e. $\Gamma \cup \Sigma_0$ is consistent.

Similarly, $(\Gamma \cup \Sigma_0)(a_\emptyset A'_0 X_1 A_2 \dots)$ has subtree property over $a_\emptyset A'_0 A_2 \dots$. Again by induction hypothesis $(\Gamma \cup \Sigma_0)(a_\emptyset A'_0 X_1 A_2 \dots)$ is realized by A'_1 , an $\mathcal{L}_S$ -indiscernible tree over $a_\emptyset A'_0 A_2 \dots$. Notice A'_0 is still an $\mathcal{L}_S$ -indiscernible tree over $a_\emptyset A'_1 A_2 \dots$, since especially $\Sigma_0(a_\emptyset A'_0 A'_1 A_2 \dots)$ holds. Hence, $\Gamma \cup \Sigma_0 \cup \Sigma_1$ is consistent.

Iterating this procedure m times, $\Gamma(x_{\omega < n+1}) \cup \left(\bigcup_{k=0}^{m-1} \Sigma_k(x_{\omega < n+1}) \right)$ is consistent. By compactness, we have shown the claim. end of the proof of Claim A

$$\text{Let } \Gamma'(x_{\omega < n+1}) = \Gamma(x_{\omega < n+1}) \cup \left(\bigcup_{k \in \omega} \Sigma_k(x_{\omega < n+1}) \right).$$

Claim B. $\Gamma'(x_{\omega < n+1}) \cup "X_0 X_1 \dots \text{ is an indiscernible sequence over } Bx_\emptyset"$ is consistent, where

$$\begin{aligned} & "X_0 X_1 \dots \text{ indiscernible sequence over } Bx_\emptyset" \\ &= \left\{ \varphi(X_{i_0}, \dots, X_{i_m}) \leftrightarrow \varphi(X_{j_0}, \dots, X_{j_m}) \mid \varphi \in \mathcal{L}_{Bx_\emptyset}, i_0 < \dots < i_m, j_0 < \dots < j_m \right\}. \end{aligned}$$

Proof of Claim B. First, observe that for any subsequence $(i_k \widehat{\omega}^{<n})_{k \in \omega}$ of $(i \widehat{\omega}^{<n})_{i \in \omega}$, the tree $x_\emptyset \ i_0 \widehat{\omega}^{<n} \ i_1 \widehat{\omega}^{<n} \ i_2 \widehat{\omega}^{<n} \dots$ is $\mathcal{L}_S$ -isomorphic to the whole $x_{\omega < n+1}$. Since $\Gamma'(x_{\omega < n+1})$ has subtree property over B , $\Gamma'(x_\emptyset X_0 X_1 \dots)$ has subsequence property over $Bx_\emptyset$. Therefore, there is a realization $a_{\omega < n+1} = a_\emptyset A_0 A_1 \dots$ of Γ' , where $A_k = a_{k \widehat{\omega}^{<n}}$, such that $A_0 A_1 \dots$ is an indiscernible sequence over $Ba_\emptyset$. This can be shown by an argument similar to the proof of Lemma 2.1.8. end of the proof of Claim B

$$\text{Let } \Gamma''(x_{\omega < n+1}) = \Gamma'(x_{\omega < n+1}) \cup "X_0 X_1 \dots \text{ is an indiscernible sequence over } Bx_\emptyset".$$

Claim C. A realization of $\Gamma''(x_{\omega < n+1})$ is an $\mathcal{L}_S$ -indiscernible tree realizing Γ .

Proof of Claim C. Let $\varphi \in \mathcal{L}_B$, $S, T \subseteq \omega^{<n+1}$ such that $S \cong_{\mathcal{L}_S} T$, and $\theta \equiv \varphi(x_S) \leftrightarrow \varphi(x_T)$. We show $\Gamma'' \vdash \theta$. S, T have the form of

$$S = \bigcup_{k=1}^m S_{i_k}, \quad S_{i_k} = \{ \nu \in S \mid \nu(0) = i_k \}, \quad i_0 < \dots < i_m,$$

$$T = \bigcup_{k=1}^m T_{j_k}, \quad T_{j_k} = \{ \nu \in T \mid \nu(0) = j_k \}, \quad j_0 < \dots < j_m.$$

Let $\sigma : \bigcup_{k=1}^m i_k \hat{\omega}^{<n} \rightarrow \bigcup_{k=1}^m j_k \hat{\omega}^{<n}$ be the natural isomorphism. Since $\Gamma''(x_{\omega^{<n+1}}) \supset "X_0 X_1 \dots$ is an indiscernible sequence over $Bx_\emptyset$ ",

$$\Gamma''(x_{\omega^{<n+1}}) \vdash \varphi(x_\emptyset x_{S_{i_0}} \dots x_{S_{i_m}}) \leftrightarrow \varphi(x_\emptyset x_{\sigma(S_{i_0})} \dots x_{\sigma(S_{i_m})}).$$

We have $S \cong_{\mathcal{L}_S} T$ and so $\sigma(S_{i_k}) \cong_{\mathcal{L}_S} T_{j_k}$ for each $k = 1, \dots, m$. Since $\Gamma''(x_{\omega^{<n+1}}) \supset "X_k$ is an $\mathcal{L}_S$ -indiscernible tree over $Bx_\emptyset X_0 X_1 \dots X_{k-1} X_{k+1} \dots$ " for all $k \in \omega$, it holds that

$$\Gamma''(x_{\omega^{<n+1}}) \vdash \varphi(x_\emptyset x_{\sigma(S_{i_0})} \dots x_{\sigma(S_{i_m})}) \leftrightarrow \varphi(x_\emptyset x_{T_{j_0}} \dots x_{T_{j_m}}).$$

Thus we have shown $\Gamma''(x_{\omega^{<n+1}}) \vdash \theta$.

end of the proof of Claim C

From the above argument, we have shown the $n + 1$ case of proposition. $\square$

Theorem 3.1.16 ([3]). *Let B be a set of parameters, and $\Gamma(x_{\omega^{<\omega}})$ be a set of $\mathcal{L}_B$ -formulas. If $\Gamma(x_{\omega^{<\omega}})$ has the $\mathcal{L}_S$ -subtree property, then Γ is realized by an $\mathcal{L}_S$ -indiscernible tree over B .*

Proof. This is an immediate consequence from Proposition 3.1.15 and Compactness. $\square$

Example 3.1.17. $\Gamma(x_{\omega^{<\omega}}) = "x_{\omega^{<\omega}}$ witnesses the k -tree property of $\varphi(x, y)"$ is realized by an $\mathcal{L}_S$ -indiscernible tree (if Γ is consistent).

3.2 $\mathcal{L}_1$ -indiscernible trees

Definition 3.2.18 ([3]). Let X be a substructure of $\omega^{<\omega}$, i.e. X is closed under the binary function $\cap$. We define $\text{level}(X)$ by $\text{level}(X) = \{ \text{dom}(\eta) \mid \eta \in X \}$.

Lemma 3.2.19 ([3]). *Let $n \in \omega$ and X, Y be n -element substructures of $\omega^{<\omega}$. $X \cong_{\mathcal{L}_S} Y$ if and only if $X \cong_{\mathcal{L}_1} Y$ and $\text{level}(X) = \text{level}(Y)$.*

Proof. If we have $X \cong_{\mathcal{L}_S} Y$, then $X \cong_{\mathcal{L}_1} Y$ and $\text{level}(X) = \text{level}(Y)$ clearly holds.

Suppose $X \cong_{\mathcal{L}_1} Y$ and $\text{level}(X) = \text{level}(Y)$ holds. We put $l = |\text{level}(X)| = |\text{level}(Y)|$ and fix the $\mathcal{L}_1$ -isomorphism $\sigma : X \rightarrow Y$. Let $(\eta_i)_{i < n}$ enumerates X and $\nu_i = \sigma(\eta_i)$ for $i < n$. There are $i_1, \dots, i_l$ such that $\eta_{i_1} <_{\text{ini}} \dots <_{\text{ini}} \eta_{i_l}$ and so $\nu_{i_1} <_{\text{ini}} \dots <_{\text{ini}} \nu_{i_l}$. By the condition $\text{level}(X) = \text{level}(Y)$, we have $\text{dom}(\eta_{i_k}) = \text{dom}(\nu_{i_k})$ for each $1 \leq k \leq l$. Since $\mathcal{L}_1$ -isomorphisms do not change the relation of having the same length, we have $\text{dom}(\eta) = \text{dom}(\sigma(\eta))$ thus $P_m(\eta) \leftrightarrow P_m(\sigma(\eta))$ for all $\eta \in X$ and $m \in \omega$. Hence σ is the $\mathcal{L}_S$ -isomorphism between X and Y . $\square$

Theorem 3.2.20 ([3]). *Let B be a set of parameters, and $\Gamma(x_{\omega^{<\omega}})$ be a set of $\mathcal{L}_B$ -formulas. If $\Gamma(x_{\omega^{<\omega}})$ has the $\mathcal{L}_1$ -subtree property, then Γ is realized by an $\mathcal{L}_1$ -indiscernible tree over B .*

Proof. We show the set of $\mathcal{L}_B$ -formulas

$$\bar{\Gamma}(x_{\omega^{<\omega}}) = \Gamma \cup \left\{ \varphi(x_{X_1}) \leftrightarrow \varphi(x_{X_2}) \mid \begin{array}{l} \varphi \text{ is an } \mathcal{L}_B\text{-formula,} \\ X_1, X_2 \text{ are finite subsets of } \omega^{<\omega} \text{ with } X_1 \underset{\mathcal{L}_1}{\simeq} X_2 \end{array} \right\}$$

is consistent.

Claim. For a finite substructure X of $\omega^{<\omega}$ and an $\mathcal{L}_B$ -formula $\varphi(x_X)$,

$$\Gamma_\varphi(x_{\omega^{<\omega}}) = \Gamma \cup \left\{ \varphi(x_{X_1}) \leftrightarrow \varphi(x_{X_2}) \mid X_1, X_2 \text{ are subsets of } \omega^{<\omega} \text{ with } X_1 \underset{\mathcal{L}_1}{\simeq} X_2 \underset{\mathcal{L}_1}{\simeq} X \right\}$$

is consistent.

Proof of Claim. We put $k = |\text{level}(X)|$. Γ has $\mathcal{L}_1$ -subtree property so $\mathcal{L}_S$ -subtree property. By Proposition 3.1.16, Γ has a realization $a_{\omega^{<\omega}}$ that is an $\mathcal{L}_S$ -indiscernible tree over B . We define the function $f : [\omega]^k \rightarrow \{0, 1\}$ by

$$f(\{n_1, \dots, n_k\}) = \begin{cases} 1 & \text{if } \varphi(a_Y) \text{ holds for all } Y \underset{\mathcal{L}_1}{\simeq} X \text{ with } \text{level}(Y) = \{n_1, \dots, n_k\} \\ 0 & \text{if } \neg\varphi(a_Y) \text{ holds for all } Y \underset{\mathcal{L}_1}{\simeq} X \text{ with } \text{level}(Y) = \{n_1, \dots, n_k\}. \end{cases}$$

This is well defined because $X \underset{\mathcal{L}_1}{\simeq} Y$ and $\text{level}(X) = \text{level}(Y)$ imply $X \underset{\mathcal{L}_S}{\simeq} Y$ and $a_{\omega^{<\omega}}$ is an $\mathcal{L}_S$ -indiscernible tree over B . By Ramsey's theorem, there is an infinite $H \subset \omega$ such that f is constant on $[H]^k$. Let h_ω enumerate the elements of H in increasing order. For $\eta \in \omega^{<\omega}$ we define $\sigma_H : \omega^{<\omega} \rightarrow \omega^{<\omega}$ by $\text{dom}(\sigma_H(\eta)) = h_{\text{dom}(\eta)}$ and

$$\sigma_H(\eta)(n) = \begin{cases} 0 & \text{if } n \notin H \\ \eta(i) & \text{if } n = h_i \end{cases}$$

$$\text{i.e. } \sigma_H(\eta) = \left\langle \underbrace{0 \dots 0}_{h_0} \eta(0) \underbrace{0 \dots 0}_{h_1 - h_0 - 1} \eta(1) \underbrace{0 \dots 0}_{h_2 - h_1 - 1} \dots \eta(d-1) \underbrace{0 \dots 0}_{h_{d-1} - h_{d-2} - 1} \right\rangle, \text{ where } d = \text{dom}(\eta).$$

Observe that for $\eta, \mu, \nu \in \omega^{<\omega}$ if $\eta <_{\text{len}} \nu$, $\eta <_{\text{ini}} \nu$, $\eta <_{\text{lex}} \nu$, $\eta \cap \nu = \mu$ holds, then we have $\sigma_H(\eta) <_{\text{len}} \sigma_H(\nu)$, $\sigma_H(\eta) <_{\text{ini}} \sigma_H(\nu)$, $\sigma_H(\eta) <_{\text{lex}} \sigma_H(\nu)$, $\sigma_H(\eta) \cap \sigma_H(\nu) = \sigma_H(\mu)$ respectively. Thus σ_H is an $\mathcal{L}_1$ -embedding.

By the $\mathcal{L}_1$ -indiscernibility of Γ , $(a_{\sigma_H(\eta)})_{\eta \in \omega^{<\omega}}$ is also a realization of Γ , and by the choice of H , $(a_{\sigma_H(\eta)})_{\eta \in \omega^{<\omega}}$ satisfies Γ_φ . Hence Γ_φ is consistent. end of the proof of Claim

Since for any $\mathcal{L}_B$ -formula φ and $X \subset \omega^{<\omega}$, Γ_φ in the above claim also has the $\mathcal{L}_1$ -subtree property, we can show the finite satisfiability of $\bar{\Gamma}$ using the claim iteratively. $\square$

Example 3.2.21. Let T be NTP_2 theory. If $\varphi(x, y)$ has the k -tree property, then there exists $k' \in \omega$ such that the set of formulas $\Gamma_{k'}(x_{\omega^{<\omega}}) = \{ \varphi(x, y) \mid y_{\omega^{<\omega}} \text{ witnesses the } k'\text{-tree property of } \varphi(x, y) \}$ has the $\mathcal{L}_1$ -subtree property, hence $\Gamma_{k'}$ is realized by an $\mathcal{L}_1$ -indiscernible tree.

Here, we give a proof for this example.

Proof. Since the theory is NTP_2 , there is $l \in \omega$ that satisfies the following condition: for all array of parameters $c_{l \times \omega}$, if $\{ \varphi(x, c_{i,j}) \mid j \in \omega \}$ is k -inconsistent for all $i < l$, then there exists $\nu \in \omega^l$ such that $\{ \varphi(x, c_{i,\nu(i)}) \mid i < l \}$ is inconsistent. Let k' be $k \times l$, and for $N \in \omega$, let $\Gamma_N(y_{\omega^{<\omega}})$ be the set of formulas " $y_{\omega^{<\omega}}$ witnesses the N -tree property of $\varphi(x, y)$ "

Claim. $\Gamma_{k'}$ has the $\mathcal{L}_1$ -subtree property.

Proof of Claim. We confirm the consistency of $\cup \{ \Gamma_{k'}(x_{\sigma(I)}) \mid \sigma : I \rightarrow I \text{ is an } \mathcal{L}_1\text{-embedding} \}$. Since Γ_k has the $\mathcal{L}_S$ -subtree property, we can apply Theorem 3.1.16 to obtain an $\mathcal{L}_S$ -indiscernible tree $b_{\omega < \omega}$ which realizes Γ_k . Clearly, $b_{\omega < \omega}$ also realizes $\Gamma_{k'}$. We show $b_{\omega < \omega}$ is a realization of $\Gamma_{k'}(y_{\sigma(\omega < \omega)})$ for all $\mathcal{L}_1$ -embedding σ . The condition “ $\{ \varphi(x, b_{\sigma(\nu|_n)}) \mid n \in \omega \}$ is consistent for all $\nu \in \omega^\omega$ ” clearly holds because an $\mathcal{L}_1$ -embedding sends a path into a path and $b_{\omega < \omega}$ is a witness of the k -tree property of φ .

For the condition “ $\{ \varphi(x, b_{\sigma(\eta|_n)}) \mid n \in \omega \}$ is k' -inconsistent for all $\eta \in \omega^{<\omega}$,” since an $\mathcal{L}_1$ -embedding preserves the relation of having the same length, it suffices to show any subset $A \subset \omega^{<\omega}$ of k' elements that have the same length, $\{ \varphi(x, b_\eta) \mid \eta \in A \}$ is inconsistent. Let A be a subset of k' elements in $\omega^{<\omega}$ each of which element has the same length, then either the case happens:

- (1) There is k -element subset $A_1 \subset A$ that belongs to the same sequence of siblings.
- (2) There is l -element subset $A_2 \subset A$ whose parents are pairwise distinct.

In the case (1), $\{ \varphi(x, b_\eta) \mid \eta \in A_1 \}$ is inconsistent, since all elements in A_1 are contained in a particular sequence of siblings and $b_{\omega < \omega}$ is a witness of the k -tree property of φ .

In the case (2), we put $A_2 = \{ \eta_1, \dots, \eta_l \}$ and let $\theta^i \subset \omega^{<\omega}$ be the sequence of siblings that contains η_i for $i = 1, \dots, l$. Observe $\{ \varphi(x, b_\mu) \mid \mu \in \theta^i \}$ is k -inconsistent for each $i = 1, \dots, l$. Because of the way we chose l , there is a path ν in the array $(b_{\theta^1} \dots b_{\theta^l})$ such that $\{ \varphi(x, b_{\nu(i)}) \mid i = 1, \dots, l \}$ is inconsistent. By $\mathcal{L}_S$ -indiscernibility of $b_{\omega < \omega}$, it holds that $b_{\nu(1)}, \dots, b_{\nu(l)} \equiv b_{\eta_1}, \dots, b_{\eta_l}$, thus $\{ \varphi(x, b_{\eta_i}) \mid i = 1, \dots, l \}$ is inconsistent. end of the proof of Claim

By the Theorem 3.2.20, we have $\Gamma_{k'}$ is realized by an $\mathcal{L}_1$ -indiscernible tree. □

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