

Homogeneous Law Invariant Coherent Multiperiod Value Measures and their Limits

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1 Introduction

Let $(\Omega, \mathcal{F}, P)$ be a standard probability space. We denote $L^p(\Omega, \mathcal{F}, P)$ by L^p , $1 \leq p \leq \infty$.

Definition 1 We say that a map $\phi : L^\infty \rightarrow \mathbf{R}$ is a coherent value measure, if the following are satisfied.

- (1) If $X \geq 0$, then $\phi(X) \geq 0$.
- (2) Superadditivity : $\phi(X_1 + X_2) \geq \phi(X_1) + \phi(X_2)$.
- (3) Positive homogeneity : for $\lambda > 0$ we have $\phi(\lambda X) = \lambda \phi(X)$.
- (4) For every constant c we have $\phi(X + c) = \phi(X) + c$.

Then Delbaen [6] essentially proved the following.

Theorem 2 For $\phi : L^\infty \rightarrow \mathbf{R}$, the following conditions are equivalent.

- (1) There is a (closed convex) set of probability measures $\mathcal{Q}$ such that any $Q \in \mathcal{Q}$ is absolutely continuous with respect to P and for $X \in L^\infty$

$$\phi(X) = \inf \{E^Q[X]; Q \in \mathcal{Q}\}.$$

- (2) ϕ is a coherent value measure and satisfies the Fatou property, i.e., if $\{X_n\}_{n=1}^\infty \subset L^\infty$ is uniformly bounded and converging to X in probability, then

$$\phi(X) \geq \limsup \phi(X_n).$$

- (3) ϕ is a coherent value measure and satisfies the following property. If X_n is a uniformly bounded sequence that increases to X , then $\phi(X_n)$ tends to $\phi(X)$.

Now we introduce the following notion.

Definition 3 We say that a map $\phi : L^\infty \rightarrow \mathbf{R}$ is law invariant, if $\phi(X) = \phi(Y)$ whenever $X, Y \in L^\infty$ have the same probability law.

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Let $\mathcal{L}$ denote the set of probability measures on $\mathbf{R}$, $\mathcal{L}_p$, $p \in [1, \infty)$, denote the set of probability measures ν on $\mathbf{R}$ such that $\int_{\mathbf{R}} |x|^p \nu(dx) < \infty$, and $\mathcal{L}_\infty$ denote the set of probability measures ν on $\mathbf{R}$ such that $\nu(\mathbf{R} \setminus [-M, M]) = 0$ for some $M > 0$. Also, $\mathcal{M}_{[0,1]}$ be the set of probability measure on $[0, 1]$.

For $\nu \in \mathcal{L}$, let F_ν be the distribution functions of ν , i.e., $F_\nu(z) = \nu((-\infty, z])$, $z \in \mathbf{R}$. Let us define $Z : [0, 1] \times \mathcal{L} \rightarrow \mathbf{R}$ by

$$Z(x, \nu) = \inf\{z; F_\nu(z) > x\}, \quad x \in [0, 1], \nu \in \mathcal{L}.$$

Then $Z(\cdot, \nu) : [0, 1] \rightarrow \mathbf{R}$ is non-decreasing and right continuous, and the probability law of $Z(\cdot, \nu)$ under Lebesgue measure on $[0, 1]$ is ν (c.f.[9]). For any random variables X , we denote by μ_X the probability law of X .

For each $\alpha \in (0, 1]$, let $\eta_\alpha : \mathcal{L}_1 \rightarrow \mathbf{R}$ be given by

$$\eta_\alpha(\nu) = \alpha^{-1} \int_0^\alpha Z(x, \nu) dx, \quad \nu \in \mathcal{L}_1.$$

Also, we define $\eta_0 : \mathcal{L}_\infty \rightarrow \mathbf{R}$ by

$$\eta_0(\nu) = \inf\{x \in \mathbf{R}; \nu((-\infty, x]) > 0\} \quad X \in \mathcal{L}_\infty.$$

Then we have the following (cf. [8], also see Section).

Theorem 4 Assume that $(\Omega, \mathcal{F}, P)$ is a standard probability space and P is non-atomic. Let $\phi : L^\infty \rightarrow \mathbf{R}$. Then the following conditions are equivalent.

(1) There is a (compact convex) subset $\mathcal{M}_0$ of $\mathcal{M}_{[0,1]}$ such that

$$\phi(X) = \inf\left\{\int_0^1 \eta_\alpha(\mu_X) m(d\alpha); m \in \mathcal{M}_0\right\}, \quad X \in L^\infty.$$

(2) ϕ is a law invariant coherent value measure with the Fatou property.

Definition 5 We say that a map $\eta : \mathcal{L}_\infty \rightarrow \mathbf{R}$ is a mild value measure (MVM), if there is a subset $\mathcal{M}_0$ of $\mathcal{M}_{[0,1]}$ such that

$$\eta(\nu) = \inf\left\{\int_0^1 \eta_\alpha(\nu) m(d\alpha); m \in \mathcal{M}_0\right\}, \quad \nu \in \mathcal{L}_\infty.$$

For any MVM η , we define a subset $\mathcal{M}(\eta)$ of $\mathcal{M}_{[0,1]}$ by

$$\mathcal{M}(\eta) = \{m \in \mathcal{M}; \eta(\nu) \leq \int_0^1 \eta_\alpha(\nu) m(d\alpha) \text{ for all } \nu \in \mathcal{L}_\infty\}.$$

For any $\nu \in \mathcal{L}_1$, we see that $\eta_\alpha(\nu) \leq \eta_1(\nu)$, $\alpha \in [0, 1]$. So any MVM η can be extended to a map from $\mathcal{L}_1$ to $[-\infty, \infty)$ by

$$\eta(\nu) = \inf\left\{\int_0^1 \eta_\alpha(\nu) m(d\alpha); m \in \mathcal{M}(\eta)\right\}, \quad \nu \in \mathcal{L}_1.$$

We denote this map by the same symbol η .

Definition 6 Let η be an MVM and $(\Omega, \mathcal{F}, P)$ be a probability space.

(1) For any integrable random variable X and any sub- σ -algebra $\mathcal{G}$, we define a $\mathcal{G}$ -measurable random variable $\eta(X|\mathcal{G})$ by

$$\eta(X|\mathcal{G}) = \eta(P(X \in dx|\mathcal{G})),$$

where $P(X \in dx|\mathcal{G})$ is a regular conditional probability law of X given a sub- σ -algebra $\mathcal{G}$. We call $\eta(X|\mathcal{G})$ a conditional value measure.

(2) For any integrable random variable X and any filtration $\{\mathcal{F}_k\}_{k=0}^n$, we define an adapted process $\{Z_k\}_{k=0}^n$ inductively by

$$Z_n = \eta(X|\mathcal{F}_n),$$

$$Z_{k-1} = \eta(Z_k|\mathcal{F}_{k-1}), \quad k = n, n-1, \dots, 1.$$

We denote an $\mathcal{F}_0$ -measurable random variable Z_0 by $\eta(X|\{\mathcal{F}_k\}_{k=0}^n)$, and call it a homogeneous filtered value measure.

(3) For any filtration $\{\mathcal{F}_k\}_{k=0}^n$ and any integrable adapted process $\{X_k\}_{k=0}^n$, we define an adapted process $\{Y_k\}_{k=0}^n$ inductively by

$$Y_n = X_n,$$

$$Y_{k-1} = X_{k-1} \wedge \eta(Y_k|\mathcal{F}_{k-1}), \quad k = n, n-1, \dots, 1.$$

We denote an $\mathcal{F}_0$ -measurable random variable Y_0 by $\eta(\{X_k\}_{k=0}^n|\{\mathcal{F}_k\}_{k=0}^n)$, and call it a homogeneous filtered value measure of an adapted process $\{X_k\}_{k=0}^n$.

In this paper, we consider two kinds of limit theorem for homogeneous filtered value measures. Let us introduce the following notion. For any MVM η and $p \in [1, \infty)$, let

$$\Delta_p(\eta) = \sup\left\{\int_0^1 (\alpha^{-1/p} \wedge \frac{(1-\alpha)^{1-1/p}}{\alpha}) m(d\alpha); m \in \mathcal{M}(\eta)\right\}.$$

1.1 Brownian-Poisson Filtration

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $\{B(t); t \in [0, \infty)\}$ be a d -dimensional Brownian motion and $\{N_i(t); t \in [0, \infty)\}$, $i = 1, \dots, \ell$, be Poisson processes with an intensity λ_i . We assume that they are independent. Let $\lambda = \sum_{i=1}^{\ell} \lambda_i$, and let $\mathcal{F}_t = \sigma\{B(s), N_i(s); s \leq t, i = 1, \dots, \ell\}$, $t \geq 0$.

Let η_n , $n = 1, 2, \dots$, be MVM's. We assume the following.

(A-1) There is a constant $C > 0$ such that $\Delta_2(\eta_n) \leq C2^{-n/2}$, $n = 1, 2, \dots$.

Let $F_0(y; \alpha, \beta)$, $y \in \mathbf{R}^{\ell}$, $0 \leq \alpha \leq \beta \leq 1$, be given by

$$F_0(y; \alpha, \beta) = \inf\left\{\int_0^{\gamma} Z(x, \lambda^{-1} \sum_{i=1}^{\ell} \lambda_i \delta_{y_i}) dx; \alpha \leq \gamma \leq \beta\right\}$$

$$= \inf\left\{\gamma \eta_{\gamma}(\lambda^{-1} \sum_{i=1}^{\ell} \lambda_i \delta_{y_i}); \alpha \leq \gamma \leq \beta\right\},$$

and let $b_n : \mathbf{R}^d \times \mathbf{R}^\ell \rightarrow \mathbf{R}$, $n = 1, 2, \dots$, be given by

$$b_n(x, y) = \inf \left\{ |x| 2^{n/2} \left(\int_0^1 \eta_\alpha(\mu_0) m(d\alpha) \right) \right. \\ \left. + \lambda \left(\int_0^1 m(d\alpha) \alpha^{-1} F_0(y; 0 \vee (1 - (2^n \lambda^{-1} (1 - \alpha))), 1 \wedge 2^n \lambda^{-1} \alpha) \right); m \in \mathcal{M}(\eta_n) \right\}.$$

Here μ_0 is a standard normal distribution.

Then $b_n : \mathbf{R}^d \times \mathbf{R}^\ell \rightarrow \mathbf{R}$ is concave,

$$b_n(sx, sy) = s b_n(x, y), \quad x \in \mathbf{R}^d, \quad y \in \mathbf{R}^\ell, \quad s \geq 0,$$

and

$$b_n(x, y_1, \dots, y_\ell) \leq b_n(x', y'_1, \dots, y'_\ell),$$

if $|x| \geq |x'|$, $y_1 \leq y'_1, \dots, y_\ell \leq y'_\ell$.

Let us assume the following furthermore..

(A-2) There is a continuous function $b : \mathbf{R}^d \times \mathbf{R}^\ell \rightarrow \mathbf{R}$ such that $b_n \rightarrow b$, $n \rightarrow \infty$, uniformly on compacts in $\mathbf{R}^d \times \mathbf{R}^\ell$.

Let K be a compact convex set in $\mathbf{R}^d \times \mathbf{R}^\ell$ given by

$$K = \{(z, w) \in \mathbf{R}^d \times [0, \infty)^\ell; b(x, y) \leq x \cdot z + \sum_{i=1}^{\ell} \lambda_i y_i w_i \text{ for all } (x, y) \in \mathbf{R}^d \times \mathbf{R}^\ell\}.$$

Also, let $\mathcal{K}$ be a set of martingales $\rho(t)$ such that there are predictable processes $\varphi : [0, \infty) \times \Omega \rightarrow \mathbf{R}^d$, $\psi_i : [0, \infty) \times \Omega \rightarrow [0, \infty)$, $i = 1, \dots, \ell$, for which

$$P((\varphi(t), \psi_1(t), \dots, \psi_\ell(t)) \in K \text{ for any } t \in [0, T]) = 1$$

and

$$\rho(t) = \prod_{i=1}^{\ell} \left(\prod_{s \in (0, t], \Delta N_i(s) \neq 0} \psi_i(s) \right) \exp \left(\int_0^t \varphi(s) dB(s) - \frac{1}{2} \int_0^t |\phi(s)|^2 ds - \sum_{i=1}^{\ell} \lambda_i \int_0^t (\psi_i(s) - 1) ds \right),$$

$t \geq 0$.

Then we have the following.

Theorem 7 Under the assumption (A-1) and (A-2), we have the following.

For any $X \in L^2(\Omega, \mathcal{F}_T, P)$, $T > 0$,

$$\lim_{n \rightarrow \infty} \eta_n(X | \{\mathcal{F}_{2^{-n}k}\}_{k=0}^{2^{2n}}) = \inf \{E[\rho(T)X]; \rho \in \mathcal{K}\}.$$

We prove this theorem in Section 5 via a nonlinear partial differential equation.

1.2 Collective Risk

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $K \geq 1$, $p \in (1, \infty)$, $p_k \in \mathbf{R}$, $\lambda_k > 0$, and $\nu_k \in \mathcal{L}_p$, $k = 1, \dots, K$. Let $Z_i^{(k)}$, $\tau_i^{(k)}$, $k = 1, \dots, K$, $i = 1, 2, \dots$, be independent random variables such that the distribution of $Z_i^{(k)}$ is ν_k , and $P(\tau_i^{(k)} > t) = \exp(-\lambda_k t)$, $t \geq 0$, for $k = 1, \dots, K$, $i = 1, 2, \dots$. Let $N_i^{(k)}(t) = 1_{\{\tau_i^{(k)} \leq t\}}$, and $X_i^{(k)}(t) = Z_i^{(k)} N_i^{(k)}(t) + p_k(\tau_i^{(k)} \wedge t)$ for $t \geq 0$, $k = 1, \dots, K$, $i = 1, 2, \dots$.

Let $\mathcal{F}_t = \sigma\{X_i^{(k)}(s); s \in [0, t], k = 1, \dots, K, i = 1, 2, \dots\}$, $t \geq 0$. Also, let

$$X(t; m_1, \dots, m_K) = \sum_{k=1}^K \sum_{i=1}^{m_k} X_i^{(k)}(t)$$

for any $t \geq 0$, and any $m_1, \dots, m_K \in \mathbf{Z}_{\geq 0}$. Here $\mathbf{Z}_{\geq 0}$ denotes the set of non-negative integers.

Theorem 8 *Let η be MVM. Assume that $\Delta_p(\eta) < \infty$. Let $\Phi : [0, \infty)^K \times \mathbf{R}^K \rightarrow \mathbf{R}$ be given by*

$$\Phi(x, \xi) = \eta\left(\sum_{\ell_1, \dots, \ell_K=0}^{\infty} \left(\prod_{k=1}^K (\exp(-\lambda_k x_k) \frac{(\lambda_k x_k)^{\ell_k}}{\ell_k!})\right) (\nu_1 - \xi_1)^{* \ell_1} \cdots (\nu_K - \xi_K)^{* \ell_K}\right) + \sum_{k=1}^K p_k x_k,$$

for $x \in [0, \infty)^K$, $\xi \in \mathbf{R}^K$. Here $*$ stands for the convolution and $\nu + a$ denotes a probability measure on $\mathbf{R}$ given by the following for any probability measure ν on $\mathbf{R}$ and $a \in \mathbf{R}$.

$$(\nu + a)(A) = \nu(\{x \in \mathbf{R}; x - a \in A\}) \text{ for any Borel set } A \text{ in } \mathbf{R}.$$

Assume that there is a C^1 function $u : [0, \infty) \times [0, \infty)^K \rightarrow \mathbf{R}$ such that $u(0, x) = 0$, $x \in [0, \infty)^K$, and satisfies the following Hamilton-Jacobi equation

$$\frac{\partial}{\partial t} u(t, x) = \Phi(x, \frac{\partial}{\partial x^1} u(t, x), \dots, \frac{\partial}{\partial x^K} u(t, x)), \quad (t, x) \in [0, \infty) \times [0, \infty)^K.$$

Then we have the following.

$$\sup\{|h\eta(X(t; m_1, \dots, m_K) | \{\mathcal{F}_{jh}\}^{[h-2]}) - u(t, m_1 h, \dots, m_K h)|;$$

$$t, m_1 h, \dots, m_K h \in [0, R], m_1, \dots, m_K \in \mathbf{Z}_{\geq 0}\} \rightarrow 0,$$

as $h \downarrow 0$, for any $R > 0$.

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