

On the continuity of positive definite functions on conelike semigroups

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Dedicated to the memory of Knud Maack Bisgaard

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Abstract

Let S be a conelike semigroup in $\mathbb{Q}^k$. In [5], P. Ressel showed an integral representation of bounded positive definite functions on S which is continuous at 0. In this paper, we will analyze some integral representations of unbounded positive definite functions on S which is continuous at 0.

1 Introduction

Let S be an abelian semigroup with the identity 0. A function $\varphi : S \rightarrow \mathbb{R}$ is called *positive definite* if

$$\sum_{j,k=1}^n c_j \overline{c_k} \varphi(s_j + s_k) \geq 0$$

for all $n \in \mathbb{N}$, $s_1, \dots, s_n \in S$, $c_1, \dots, c_n \in \mathbb{R}$.

A function $\sigma : S \rightarrow \mathbb{R}$ is called a *character* if it is multiplicative and not identically zero. In particular, if $0 \notin \sigma(S)$, σ is called *zerofree*. The set of characters on S is denoted by S^* . Denote by $\mathcal{A}(S^*)$ the least σ -ring of subsets of S^* rendering the mapping $S^* \ni \sigma \mapsto \sigma(s) \in \mathbb{R}$ measurable for each $s \in S$. A function $\varphi : S \rightarrow \mathbb{R}$ is called a *moment function* if there is a measure μ defined on $\mathcal{A}(S^*)$ such that

$$\varphi(s) = \int_{S^*} \sigma(s) d\mu(\sigma)$$

for all $s \in S$. Note that every moment function is positive definite and every bounded positive definite function on S is a moment function whose representing

measure is unique (see [1], Theorem 4.2.8). But a positive definite function is not necessarily a moment function (see [1], Theorem 6.3.5), and a representing measure is not necessarily unique if any (see [1], Example 6.4.3).

An abelian $*$ -semigroup S is called *determinate* if whenever μ and ν are measures on $\mathcal{A}(S^*)$ such that

$$\int_{S^*} \sigma(s) d\mu(\sigma) = \int_{S^*} \sigma(s) d\nu(\sigma), \quad s \in S$$

then $\mu = \nu$. The semigroup S is called *semiperfect* if every positive definite function $\varphi : S \rightarrow \mathbb{R}$ is a moment function, and *perfect* if S is semiperfect and determinate.

A subset M of a vector space over the scalar field $\mathbb{K}$ ($\mathbb{K} = \mathbb{Q}$ or $\mathbb{R}$) is called *conelike* if for each $s \in M$ there is some $a \in \mathbb{K}$ such that $\alpha s \in M$ for all $\alpha \in \mathbb{K}$ satisfying $\alpha \geq a$.

P. Ressel has proved the following theorem (see [5], Theorem 2):

Ressel's Theorem *Let S be a conelike semigroup in the real vector space $\mathbb{R}^k$, $k \geq 1$, with $\overset{\circ}{S} \neq \emptyset$ and $0 \in \overline{\overset{\circ}{S}}$, where $\tilde{S} := \{s \in S \mid (\mathbb{R}_+ s) \cap \overset{\circ}{S} \neq \emptyset\}$. For a bounded positive definite function $\varphi : S \rightarrow \mathbb{R}$ the following properties are equivalent :*

- (i) φ is uniformly continuous.
- (ii) φ is continuous at 0.
- (iii) $\exists \{s_n\} \subset \tilde{S}$ with $s_n \rightarrow 0$ and $\varphi(s_n) \rightarrow \varphi(0)$.
- (iv) There is a bounded nonnegative measure μ on $S^\square$ such that $\varphi(s) = \int_{S^\square} e^{-\langle v, s \rangle} d\mu(v)$, $s \in S$, where $S^\square := \{v \in \mathbb{R}^k \mid \langle v, s \rangle \geq 0 \text{ for all } s \in S\}$.

It is natural to consider this theorem for unbounded positive definite functions. In general, every unbounded positive definite function is not a moment function. But every conelike semigroup in the rational vector space $\mathbb{Q}^k$, $k \geq 1$, is perfect (see [4], Theorem 3.3, [2], Theorem 6). In section 3, we will prove a Ressel-type theorem for unbounded positive definite functions on conelike semigroups in $\mathbb{Q}$. In section 4, we will show that such a Ressel-type theorem in $(\mathbb{Q}_+ \setminus \{0\})^2 \cup \{(0, 0)\}$ does not hold. In section 5, for some conelike semigroups in $\mathbb{Q}^k$, we will prove that the implication (ii) $\Rightarrow$ (iv) holds.

Throughout this paper, an abelian semigroup S in $\mathbb{Q}^k$ (or $\mathbb{R}^k$) is conelike, and the composition on S is the ordinary addition. See [1] for other details on positive definite and moment functions, and see [3] on positive definite functions on conelike semigroups.

2 Preliminaries

In this section, we will determine explicitly the zerofree characters on S with $\overset{\circ}{S}_\mathbb{Q} \neq \emptyset$, where $\overset{\circ}{S}_\mathbb{Q}$ is the interior of S in the rational vector space $\mathbb{Q}^k$ with the relative topology. This argument is similar to P. Ressel's (cf. [5]).

Proposition 1 *Let S be a conelike subsemigroup of $\mathbb{Q}^k$ with $\overset{\circ}{S}_{\mathbb{Q}} \neq \emptyset$. Then every zerofree character $\sigma \in S^*$ is of the form*

$$\sigma(s) = \exp\langle v, s \rangle$$

for some $v \in \mathbb{R}^k$.

Put $\widetilde{S}_{\mathbb{Q}} := \{s \in S \mid (\mathbb{Q}_+ s) \cap \overset{\circ}{S}_{\mathbb{Q}} \neq \emptyset\}$. The set $\widetilde{S}_{\mathbb{Q}}$ contains $\overset{\circ}{S}_{\mathbb{Q}}$. By the similar proof of [5], Lemma 3, we have the following.

Lemma 2 *Let S be a conelike subsemigroup of $\mathbb{Q}^k$ with $\overset{\circ}{S}_{\mathbb{Q}} \neq \emptyset$, and $\sigma \in S^*$ is not zerofree. Then $\sigma \equiv 0$ on $\widetilde{S}_{\mathbb{Q}}$, in particular on $\overset{\circ}{S}_{\mathbb{Q}}$.*

Define the sets

$$\begin{aligned} W &:= \{\sigma \in S^* \mid \sigma : \text{zerofree}\}, \\ N &:= \{\sigma \in S^* \mid \sigma : \text{not zerofree}\}. \end{aligned}$$

If $\overset{\circ}{S}_{\mathbb{Q}} \neq \emptyset$, by Proposition 1, W is topological semigroup isomorphic to $\mathbb{R}^k$ by the correspondence

$$f : (s \mapsto \exp\langle v, s \rangle) \mapsto v.$$

Since S is perfect, every positive definite function φ on S has the following integral representation with the unique measure μ on S^* :

$$\varphi(s) = \int_{S^*} \sigma(s) d\mu(\sigma), \quad s \in S.$$

Since every character $\sigma \in N$ is identically zero on $\overset{\circ}{S}_{\mathbb{Q}}$ by Lemma 2, then

$$\varphi(s) = \int_{\mathbb{R}^k} \exp\langle v, s \rangle d\nu(v), \quad s \in \overset{\circ}{S}_{\mathbb{Q}},$$

where ν is the image measure defined by $\nu := \mu^f$.

3 In the Case of S in $\mathbb{Q}$

In the case of $S \subseteq \mathbb{Q}$ with $\overset{\circ}{S}_{\mathbb{Q}} \neq \emptyset$, it is easily obtained that $S^* = W \cup N = W \cup \{1_{\{0\}}\}$, where $1_{\{0\}}$ is the indicator function of $\{0\}$. We have the following:

Theorem 3 *Let S be a conelike semigroup in the rational vector space $\mathbb{Q}$ with $\overset{\circ}{S}_{\mathbb{Q}} \neq \emptyset$ and $0 \in \widetilde{S}_{\mathbb{Q}}$. For a positive definite function $\varphi : S \rightarrow \mathbb{R}$ the following properties are equivalent :*

- (i) φ is continuous.
- (ii) φ is continuous at 0.
- (iii) $\exists \{s_n\} \subset \widetilde{S}_{\mathbb{Q}}$ with $s_n \rightarrow 0$ and $\varphi(s_n) \rightarrow \varphi(0)$.

(iv) There is a nonnegative measure ν on $\mathbb{R}$ such that $\varphi(s) = \int_{\mathbb{R}} e^{vs} d\nu(v)$, $s \in S$.

Corollary 4 Let S be a conelike semigroup in the real vector space $\mathbb{R}$ and define $S_Q := S \cap \mathbb{Q}$. Suppose that $\overset{\circ}{S} \neq \emptyset$, $0 \in \overline{S_Q}$ and $S = \overline{S_Q}$. Then a function $\varphi : S \rightarrow \mathbb{R}$ is continuous and positive definite if and only if there exists a nonnegative measure ν on $\mathbb{R}$ such that

$$\varphi(s) = \int_{\mathbb{R}} e^{vs} d\nu(v), \quad s \in S.$$

4 In the Case of S in $\mathbb{Q}^2$

In the case of S in $\mathbb{Q}$, we proved a Ressel-type theorem for unbounded positive definite functions. But, in the case of S in $\mathbb{Q}^2$, a Ressel-type theorem such as Theorem 3 does not hold. In this section, we will show some counterexamples. Throughout this section, let S be the abelian semigroup $(\mathbb{Q}_+ \setminus \{0\})^2 \cup \{(0,0)\}$.

Example 1 (Counterexample of (iv) $\Rightarrow$ (ii)) For each $k \in \mathbb{N}$, define $v_k \in \mathbb{R}^2$ by $v_k = (k, -k^2)$. Let m be the measure $\sum_{k=1}^{\infty} \frac{1}{k^2} \varepsilon_{v_k}$ on $\mathbb{R}^2$, where ε_{v_k} is the Dirac measure supported by $\{v_k\}$. Define

$$\varphi(x, y) := \int_{\mathbb{R}^2} e^{(v, (x, y))} dm(v) = \sum_{k=1}^{\infty} k^{-2} e^{kx - k^2 y} < \infty, \quad (x, y) \in S.$$

Now φ is not continuous at $(0,0)$. In fact, let $\{x_n\}$ be any sequence of positive numbers tending to 0. For each n , since $\varphi(x_n, y) \rightarrow \infty$ as $y \rightarrow 0$, we can choose y_n such that $0 < y_n < \frac{1}{n}$ and $\varphi(x_n, y_n) > n$. Then $(x_n, y_n) \rightarrow (0,0)$ but $\varphi(x_n, y_n) \rightarrow \infty$.

Example 2 (Counterexample of (iii) + (iv) $\Rightarrow$ (ii)) Let φ be the function as above. We only have to show that there is a sequence $\{s_n\}$ in $\overset{\circ}{S_Q}$ such that $s_n \rightarrow 0$ and $\varphi(s_n) \rightarrow \varphi(0)$ as $n \rightarrow \infty$. For each $n \in \mathbb{N}$, define a continuous mapping γ_n on $(-1,1)$ by $\gamma_n(-t) = \left(\frac{1-t}{n}, \frac{1}{n}\right)$ and $\gamma_n(t) = \left(\frac{1}{n}, \frac{1-t}{n}\right)$ for $0 \leq t < 1$. We can easily prove that $\varphi(\gamma_n(-t)) \downarrow \sum_{k=1}^{\infty} \frac{1}{k^2} e^{\frac{k-k^2}{n}} < \varphi(0)$ and $\varphi(\gamma_n(t)) \rightarrow \infty$ as $0 \leq t \uparrow 1$. By continuity, we can choose $t_n \in (-1,1) \cap \mathbb{Q}$ such that $\varphi(\gamma_n(t_n)) = \varphi(0)$. Putting $s_n = \gamma_n(t_n) \in \overset{\circ}{S_Q}$, we have that $s_n = \gamma_n(t_n) \rightarrow 0$ and $\varphi(s_n) = \varphi(\gamma_n(t_n)) = \varphi(0)$. Then we can obtain the result.

Example 3 (Counterexample of (iii) $\Rightarrow$ (iv)) Let φ and $\{s_n\}$ be as above, and let μ be the representing measure of φ on S^* . Choose a number α such that $\sum_{k=1}^{\infty} \frac{1}{k^2} e^{\frac{k-k^2}{n}} < \alpha < \varphi(0)$. Define the function ψ as follows:

$$\psi(x, y) = \begin{cases} \varphi(x, y) & ((x, y) \in S \setminus \{(0,0)\}) \\ \alpha & ((x, y) = (0,0)) \end{cases}$$

Then ψ is positive definite on S . By the similar argument to take $\{t_n\}$, we can choose $\tilde{t}_n \in (-1, 1) \cap \mathbb{Q}$ such that $\psi(\gamma_n(\tilde{t}_n)) = \alpha$. Putting $\tilde{s}_n = \gamma_n(\tilde{t}_n) \in \overset{\circ}{S}_Q$, we have that $\tilde{s}_n = \gamma_n(\tilde{t}_n) \rightarrow 0$ and $\psi(\tilde{s}_n) = \psi(\gamma_n(\tilde{t}_n)) = \psi(0)$. But the support of the representing measure of ψ contains $\{1_{\{0\}}\}$. In fact, Since ψ is a moment function on S , there exists the measure μ_0 on S^* such that

$$\psi(s) = \int_{S^*} \sigma(s) d\mu_0(\sigma), \quad s \in S.$$

Put $H := S \setminus \{(0, 0)\}$. By [6], Lemma 2.2, the mapping $f : \sigma \mapsto \sigma|_H$ is a one-to-one correspondence between $S^* \setminus \{1_{\{0\}}\}$ and H^* . Let $\tilde{\mu}$ and $\tilde{\mu}_0$ be the images of μ and μ_0 , respectively, i.e., $\tilde{\mu} = \mu^f$ and $\tilde{\mu}_0 = \mu_0^f$. For $s \in H$,

$$\begin{aligned} \int_{H^*} \sigma(s) d\tilde{\mu}_0(\sigma) &= \int_{S^*} \sigma(s) d\mu_0(\sigma) = \psi(s) \\ &= \varphi(s) = \int_{S^*} \sigma(s) d\mu(\sigma) = \int_{H^*} \sigma(s) d\tilde{\mu}(\sigma). \end{aligned}$$

By [6], Theorem 3.2, H is perfect (see [6] for the definition of perfectness of H). By [6], Proposition 3.1, $\tilde{\mu} = \tilde{\mu}_0$ on H^* . Suppose $\mu_0(\{1_{\{0\}}\}) = 0$, then $\mu = \mu_0$ on S^* , hence $\varphi = \psi$ on S . This contradicts to $\varphi \neq \psi$. Therefore $\mu_0(\{1_{\{0\}}\}) \neq 0$.

5 In the case of S in $\mathbb{Q}^k$

In the case of S in $\mathbb{Q}^2$, a Ressel-type theorem such as Theorem 3 does not hold. But, under an assumption of S , we will show the implication (ii) $\Rightarrow$ (iv).

Proposition 5 *Let S be a conelike semigroup in the rational vector space $\mathbb{Q}^k$, $k \geq 2$, such that $\overset{\circ}{S}_Q \neq \emptyset$ and there exists a sequence $\{s_n\}$ of $\widetilde{S}_Q$ satisfying $\lim_{n \rightarrow \infty} s_n = 0$ and $\dim(\text{linspan}\{s_n\}) = 1$. For a continuous and positive definite function φ on S there exists the nonnegative measure ν on $\mathbb{R}^k$ such that*

$$\varphi(s) = \int_{\mathbb{R}^k} e^{\langle v, s \rangle} d\nu(v), \quad s \in S.$$

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