

Oscillatory Property for Second Order Differential Equations

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There are many results on oscillatory property of solutions of differential equations. In this article, we shall discuss oscillatory property of solutions and the existence of a bounded nonoscillatory solution of second order differential equations by applying Liapunov second method.

Consider an equation

$$(1) \quad (r(t)x')' + f(t, x, x') = 0 \quad (' = \frac{d}{dt}),$$

where $r(t) > 0$ is continuous on $I = [0, \infty)$ and $f(t, x, u)$ is continuous on $I \times \mathbb{R} \times \mathbb{R}$, $\mathbb{R} = (-\infty, \infty)$. To discuss oscillatory property of solutions of (1), we consider an equivalent system

$$(2) \quad x' = \frac{y}{r(t)}, \quad y' = -f(t, x, \frac{y}{r(t)}).$$

A solution $x(t)$ of (1) which exists in the future is said to be oscillatory if for every $T > 0$ there is a $t_0 > T$ such that $x(t_0) = 0$. Moreover, the equation (1) is said to be oscillatory if every solution of (1) which exists in the future is oscillatory.

Theorem 1. Assume that there exist two continuous scalar function $V(t, x, y)$ and $W(t, x, y)$ defined on $t \geq T$, $0 < x < K$, $|y| < \infty$ and on $t \geq T$, $-K < x < 0$, $|y| < \infty$, respectively, where T can be large and $K > 0$ or $K = \infty$, and assume that $V(t, x, y)$ and $W(t, x, y)$ satisfy the following conditions;

- (i) $V(t, x, y) \rightarrow \infty$ uniformly for $0 < x < K$ and $-\infty < y < \infty$ as $t \rightarrow \infty$, and
- $W(t, x, y) \rightarrow \infty$ uniformly for $-K < x < 0$ and $-\infty < y < \infty$ as $t \rightarrow \infty$,

(ii) $\dot{V}_{(2)}(t, x(t), y(t)) \leq 0$ for all sufficiently large t , where $\{x(t), y(t)\}$ is a solution of (2) such that $0 < x(t) < K$ for all large t and

$$\dot{V}_{(2)}(t, x(t), y(t)) = \overline{\lim}_{h \rightarrow 0^+} \frac{1}{h} \{ V(t+h, x(t+h), y(t+h)) - V(t, x(t), y(t)) \},$$

(iii) $\dot{W}_{(2)}(t, x(t), y(t)) \leq 0$ for all sufficiently large t , where $\{x(t), y(t)\}$ is a solution of (2) such that $-K < x(t) < 0$ for all large t and

$$\dot{W}_{(2)}(t, x(t), y(t)) = \overline{\lim}_{h \rightarrow 0^+} \frac{1}{h} \{ W(t+h, x(t+h), y(t+h)) - W(t, x(t), y(t)) \}.$$

Then the solution $x(t)$ of (1) such that $|x(t)| < K$ for all large t is oscillatory. Moreover, if $K = \infty$, the equation (1) is oscillatory.

Proof. Let $x(t)$ be a solution of (1) which is defined on $[t_0, \infty)$ and bounded by K for all large t , and suppose that $x(t)$ is not oscillatory. Then $x(t)$ is either positive or negative for all large t . Now assume that $0 < x(t) < K$ for all $t \geq \sigma$ where $\sigma \geq T$. By the condition (i), if t is sufficiently large, say $t \geq t_1$, we have

$$V(\sigma, x(\sigma), y(\sigma)) < V(t, x, y)$$

for all $0 < x < K$, $|y| < \infty$. However, by the condition (ii), we have

$$V(t, x(t), y(t)) \leq V(\sigma, x(\sigma), y(\sigma))$$

for all $t \geq \sigma$, if necessary, choosing a large σ . This contradicts $V(t_1, x(t_1), y(t_1)) > V(\sigma, x(\sigma), y(\sigma))$. When we assume that $-K < x(t) < 0$ for all large t , we have also a contradiction by using $W(t, x(t), y(t))$. Thus we see that $x(t)$ is oscillatory.

To apply this theorem, the following lemmas play an important role. In the following, a scalar function $v(t, x, y)$ will be called a Liapunov function for (2), if $v(t, x, y)$ is continuous in (t, x, y) in the domain of definition and is locally Lipschitzian in (x, y) . Moreover, we define $\dot{v}_{(2)}(t, x, y)$ by

$$\dot{v}_{(2)}(t, x, y) = \overline{\lim}_{h \rightarrow 0^+} \frac{1}{h} \{ v(t+h, x+h \frac{y}{r(t)}, y-hf(t, x, \frac{y}{r(t)})) - v(t, x, y) \}.$$

Lemma 1. For $t \geq T^*$, $x > 0$, $|y| < \infty$, where T^* can be large, we assume that there exists a Liapunov function $v(t, x, y)$ which satisfies the following conditions;

(i) $yv(t, x, y) > 0$ for $t \geq T^*$, $x > 0$, $y \neq 0$,

(ii) $\dot{v}_{(2)}(t, x, y) \leq -\lambda(t)$, where $\lambda(t)$ is a continuous function defined on $t \geq T^*$ and

$$\lim_{t \rightarrow \infty} \int_T^t \lambda(s) ds \geq 0 \text{ for all large } T.$$

Moreover, we assume that there is a τ and a $w(t, x, y)$ for all large T such that $\tau \geq T$ and $w(t, x, y)$ is a Liapunov function defined on $t \geq \tau$, $x > 0$, $y < 0$, which satisfies the following conditions;

(iii) $y \leq w(t, x, y)$ and $w(\tau, x, y) \leq b(y)$, where $b(y)$ is continuous, $b(0) = 0$ and

$$b(y) < 0 \quad (y \neq 0),$$

(iv) $\dot{w}_{(2)}(t, x, y) \leq -\rho(t)w(t, x, y)$, where $\rho(t) \geq 0$ is continuous and

$$\int_{\tau}^{\infty} \frac{1}{r(t)} \exp \left\{ - \int_{\tau}^t \rho(s) ds \right\} dt = \infty.$$

Then, if $\{x(t), y(t)\}$ is a solution of (2) such that $x(t) > 0$ for all large t , we have $y(t) \geq 0$ for all large t .

We can obtain a similar lemma for a solution $\{x(t), y(t)\}$ of (2) such that $x(t) < 0$ for all large t . For the proof of Lemma 1 and the details, see [5].

Proposition 1. For the equation (1) we assume that

$$(i) \quad \int_0^{\infty} \frac{dt}{r(t)} = \infty,$$

(ii) for $t \geq 0$ and $x \geq 0$, there exist continuous functions $a(t)$ and $a(x)$ such that

$$(3) \quad \lim_{t \rightarrow \infty} \int_T^t a(s) ds \geq 0 \quad \text{for all large } T$$

and that $xa(x) > 0$ ($x \neq 0$), $a'(x) \geq 0$ and for all large t , $x \geq 0$, $|u| < \infty$

$$a(t)a(x) \leq f(t, x, u),$$

(iii) for $t \geq 0$ and $x \leq 0$, there exist continuous functions $b(t)$ and $\beta(x)$ such that

$$\lim_{t \rightarrow \infty} \int_T^t b(s) ds \geq 0 \quad \text{for all large } T$$

and that $x\beta(x) > 0$ ($x \neq 0$), $\beta'(x) \geq 0$ and for all large t , $x \leq 0$, $|u| < \infty$

$$f(t, x, u) \leq b(t)\beta(x).$$

Then, if $\int_0^{\infty} a(t) dt = \infty$ and $\int_0^{\infty} b(t) dt = \infty$, the equation (1) is oscillatory. Moreover, if we have

$$(4) \quad \int_0^{\infty} a(t) dt < \infty, \quad \int_0^{\infty} \left(\frac{1}{r(s)} \int_s^{\infty} a(u) du \right) ds = \infty$$

and

$$(5) \quad \int_0^{\infty} b(t) dt < \infty, \quad \int_0^{\infty} \left(\frac{1}{r(s)} \int_s^{\infty} b(u) du \right) ds = \infty,$$

then all bounded solutions of (1) are oscillatory. In addition to the conditions above, if

$$(6) \quad \int_{\epsilon}^{\infty} \frac{du}{a(u)} < \infty, \quad \int_{-\epsilon}^{-\infty} \frac{du}{\beta(u)} < \infty \quad \text{for some } \epsilon > 0,$$

the equation (1) is oscillatory.

Proof. Under our assumptions, if we consider a function $v(t, x, y) = \frac{y}{a(x)}$ for large t , this function satisfies the conditions in Lemma 1 with $\lambda(t) = a(t)$. Since the condition (3) implies that for all large T , there is a τ such that $\tau \geq T$ and $\int_{\tau}^t a(s)ds \geq 0$ for all $t \geq \tau$, a function $w(t, x, y) = y + a(x) \int_{\tau}^t a(s)ds$ defined on $t \geq \tau$, $x > 0$, $y < 0$ satisfies the conditions in Lemma 1 with $\rho(t) \equiv 0$. Thus we can see that if $\{x(t), y(t)\}$ is a solution of (2) such that $x(t) > 0$ for all large t , then $y(t) \geq 0$ for all large t . We can also see that for a solution such that $x(t) < 0$ for all large t , $y(t) \leq 0$ for all large t .

In the case where we assume that $\int_0^{\infty} a(t)dt = \infty$ and $\int_0^{\infty} b(t)dt = \infty$, for large t , if we define $V(t, x, y)$ and $W(t, x, y)$ by

$$V(t, x, y) = \begin{cases} \frac{y}{a(x)} + \int_0^t a(s)ds & (y \geq 0) \\ \int_0^t a(s)ds & (y < 0) \end{cases}$$

and

$$W(t, x, y) = \begin{cases} \int_0^t b(s)ds & (y > 0) \\ \frac{y}{\beta(x)} + \int_0^t b(s)ds & (y \leq 0), \end{cases}$$

we can see that these functions satisfy the conditions in Theorem 1 for $K = \infty$, and hence the equation (1) is oscillatory.

In the case where we assume (4) and (5), letting $K > 0$ be a constant, se.

$$V(t, x, y) = \int_x^K \frac{du}{a(u)} + \int_0^t \left(\frac{1}{r(s)} \int_s^\infty a(u) du \right) ds$$

for $t \geq 0$, $0 < x < K$ and $|y| < \infty$. For a solution $x(t)$ of (2) which satisfies $0 < x(t) < K$ for all large t , there is a $\sigma > 0$ such that $0 < x(t) < K$ and $y(t) \geq 0$ for $t \geq \sigma$, and hence

$$\dot{V}_{(2)}(t, x(t), y(t)) = \frac{1}{r(t)} \left\{ -\frac{y(t)}{a(x(t))} + \int_t^\infty a(u) du \right\}.$$

If we set $V^*(t, x, y) = -\frac{y}{a(x)} + \int_t^\infty a(u) du$, we have $\lim_{t \rightarrow \infty} V^*(t, x(t), y(t)) \leq 0$. On the other hand, we have $\dot{V}_{(2)}^*(t, x, y) \geq 0$, and hence $V^*(t, x(t), y(t)) \leq 0$, which implies that $\dot{V}_{(2)}(t, x(t), y(t)) \leq 0$ for $t \geq \sigma$. For $t \geq 0$, $-K < x < 0$ and $|y| < \infty$, define $W(t, x, y)$ by

$$W(t, x, y) = \int_x^{-K} \frac{du}{\beta(u)} + \int_0^t \left(\frac{1}{r(s)} \int_s^\infty b(u) du \right) ds.$$

Then the conclusion follows from Theorem 1, because K is arbitrary.

In addition, when we assume (6), we can set $K = \infty$ in $V(t, x, y)$ and $W(t, x, y)$ above, and hence the equation (1) is oscillatory.

The result above contains Coles' result [2] and Macki and Wong's result [3].

Remark. It is clear that we can combine the conditions on $a(t)$ and $b(t)$. The Liapunov's method is also applicable to obtain Bobisud's [1] and Opial's [4] results, see [5].

Now we shall discuss the existence of a bounded nonoscillatory solution of (1). The following theorems will be applied. Consider an equation of the second order

$$(7) \quad x'' = F(t, x, x'),$$

where $F(t, x, y)$ is continuous on $I \times \mathbb{R} \times \mathbb{R}$. Let $\underline{\omega}(t)$ and $\overline{\omega}(t)$ be two functions defined

on I , twice differentiable and bounded on I with their derivatives. We assume that $\underline{\omega}(t) \leq \bar{\omega}(t)$,

$$(8) \quad \bar{\omega}''(t) \leq F(t, \bar{\omega}(t), \bar{\omega}'(t))$$

and

$$(9) \quad \underline{\omega}''(t) \geq F(t, \underline{\omega}(t), \underline{\omega}'(t))$$

for all $t \geq 0$.

Theorem 2. Suppose that there exist two Liapunov functions $V(t, x, y)$ and $W(t, x, y)$ defined on $0 \leq t < \infty$, $\underline{\omega}(t) \leq x \leq \bar{\omega}(t)$, $y \geq K$ and on $0 \leq t < \infty$, $\underline{\omega}(t) \leq x \leq \bar{\omega}(t)$, $y \leq -K$, respectively, where $K > 0$ can be large, and assume that $V(t, x, y)$ and $W(t, x, y)$ satisfy the following conditions;

- (i) $V(t, x, y) \leq b(y)$ and $W(t, x, y) \leq b(|y|)$, where $b(r) > 0$ is continuous,
- (ii) $V(t, x, y) \rightarrow \infty$ as $y \rightarrow \infty$, $W(t, x, y) \rightarrow \infty$ as $y \rightarrow -\infty$, uniformly for t, x ,
- (iii) in the interior of their domains of definition

$$\dot{V}(t, x, y) = \overline{\lim}_{h \rightarrow 0^+} \frac{1}{h} \{ V(t+h, x+hy, y+hF(t, x, y)) - V(t, x, y) \} \geq 0$$

and

$$\dot{W}(t, x, y) = \overline{\lim}_{h \rightarrow 0^+} \frac{1}{h} \{ W(t+h, x+hy, y+hF(t, x, y)) - W(t, x, y) \} \leq 0$$

or

- (iii)' in the interior of their domains of definition

$$\dot{V}(t, x, y) \leq 0 \quad \text{and} \quad \dot{W}(t, x, y) \geq 0.$$

Then the equation (7) has a solution $x(t)$ such that $\underline{\omega}(t) \leq x(t) \leq \bar{\omega}(t)$ and $x'(t)$ is bounded for all $t \geq 0$.

Theorem 3. Under the assumptions in Theorem 2, if $\underline{\omega}(0) = \overline{\omega}(0)$ and

$$\dot{V}(t, x, y) \leq 0 \quad \text{and} \quad \dot{W}(t, x, y) \leq 0$$

in the interiors of their domains of definition, then the equation (7) has a solution $x(t)$ such that $\underline{\omega}(t) \leq x \leq \overline{\omega}(t)$ and $x'(t)$ is bounded for all $t \geq 0$.

For the proofs, see [6].

In discussing the existence of a bounded nonoscillatory solution of (1), we assume that the derivative of $r(t)$ is continuous, and consequently the equation (1) can be written as

$$(10) \quad x'' + \frac{r'(t)}{r(t)} x' + \frac{1}{r(t)} f(t, x, x') = 0.$$

Proposition 2. Suppose that there exist functions $b(t)$ and $\beta(x)$ which satisfy the following conditions;

- (i) $b(t)$ is continuous on I and $b(t) \geq 0$ for $t \geq T$, where T can be large,
- (ii) $\beta(x)$ is continuous on $x \geq 0$,
- (iii) for $t \geq T$, $x > 0$ and all y ,

$$(11) \quad f(t, x, y) \leq b(t)\beta(x).$$

Moreover, we assume that there is a $c > 0$ such that $f(t, c, 0) \geq 0$ for $t \geq T$. Then, if

$$(12) \quad 0 < \epsilon \leq r(t) \leq \rho \quad \text{for some } \epsilon, \rho \text{ and all } t \geq 0$$

and

$$(13) \quad \int_0^\infty tb(t)dt < \infty$$

or if, there is an $A > 0$ such that

$$(14) \quad \left| \frac{r'(t)}{r(t)} \right| < A \quad \text{for } t \geq 0$$

and we have

$$(15) \quad \int_0^\infty \left(\frac{1}{r(s)} \int_s^\infty b(u) du \right) ds < \infty,$$

the equation (10) has a bounded nonoscillatory solution.

Proof. Under the conditions (12) and (13), the condition (13) implies that $\int_0^\infty b(t) dt < \infty$, and consequently $\int_s^\infty b(u) du$ exists and is small if s is sufficiently large, because $b(t) \geq 0$ eventually. Since $\epsilon \leq r(t) \leq \rho$, we have

$$\int_0^\infty \left(\frac{1}{r(s)} \int_s^\infty b(u) du \right) ds < \infty \quad \text{and} \quad \frac{1}{r(t)} \int_t^\infty b(u) du < \infty.$$

There is an $L > 0$ such that $\beta(c) \leq \frac{L}{2}$ and there is a $\delta > 0$ such that $\beta(x) \leq L$ if $|x - c| \leq \delta$.

Choose $t_0 \geq T$ so large that

$$0 \leq L \int_{t_0}^t \left(\frac{1}{r(s)} \int_s^\infty b(u) du \right) ds \leq \delta \quad \text{for all } t \geq t_0.$$

For $t_0 \leq t < \infty$, define $\underline{\omega}(t)$ and $\overline{\omega}(t)$ by

$$\underline{\omega}(t) = c \quad \text{and} \quad \overline{\omega}(t) = c + L \int_{t_0}^t \left(\frac{1}{r(s)} \int_s^\infty b(u) du \right) ds.$$

Then $0 < \underline{\omega}(t) \leq \overline{\omega}(t) \leq c + \delta$ for all $t \geq t_0$, and $\underline{\omega}(t)$, $\overline{\omega}(t)$ are bounded with their derivatives.

Clearly we have $\underline{\omega}''(t) \geq -\frac{r'(t)}{r(t)} \underline{\omega}'(t) - \frac{1}{r(t)} f(t, \underline{\omega}(t), \underline{\omega}'(t))$. On the other hand, $\overline{\omega}'(t) = \frac{L}{r(t)} \int_t^\infty b(u) du$ and $\overline{\omega}''(t) = -\frac{r'(t)}{r^2(t)} L \int_t^\infty b(u) du - \frac{L}{r(t)} b(t)$. Thus, using (11),

we have

$$\begin{aligned}
 -\frac{r'(t)}{r(t)} \bar{\omega}'(t) - \frac{1}{r(t)} f(t, \bar{\omega}(t), \bar{\omega}'(t)) &\geq -\frac{r'(t)}{r^2(t)} L \int_t^\infty b(u) du - \frac{1}{r(t)} b(t) \beta(\bar{\omega}(t)) \\
 &\geq -\frac{r'(t)}{r^2(t)} L \int_t^\infty b(u) du - \frac{L}{r(t)} b(t) \\
 &\geq \bar{\omega}''(t),
 \end{aligned}$$

since $c \leq \bar{\omega}(t) \leq c + \delta$ for all $t \geq t_0$ and hence $\beta(\bar{\omega}(t)) \leq L$ for $t \geq t_0$.

For $t \geq t_0$, $\underline{\omega}(t) \leq x \leq \bar{\omega}(t)$ and $y \geq K$, define $V(t, x, y)$ by

$$V(t, x, y) = L \int_{t_0}^t b(s) ds + r(t)y$$

and for $t \geq t_0$, $\underline{\omega}(t) \leq x \leq \bar{\omega}(t)$ and $y \leq -K$, define $W(t, x, y)$ by

$$W(t, x, y) = -L \int_{t_0}^t b(s) ds - r(t)y.$$

Then it is clear that $V(t, x, y)$ and $W(t, x, y)$ satisfy the conditions (i) and (ii) in Theorem 2.

Since $\beta(x) \leq L$ for $\underline{\omega}(t) \leq x \leq \bar{\omega}(t)$, we have

$$\begin{aligned}
 \dot{V}(t, x, y) &= Lb(t) + r'(t)y + r(t) \left\{ -\frac{r'(t)}{r(t)} y - \frac{1}{r(t)} f(t, x, y) \right\} \\
 &= Lb(t) - f(t, x, y) \\
 &\geq Lb(t) - b(t)\beta(x) \geq Lb(t) - Lb(t) = 0
 \end{aligned}$$

and we have also $\dot{W}(t, x, y) \leq 0$. Therefore, it follows from Theorem 2 that the equation (10) has a solution $x(t)$ such that

$$0 < c \leq x(t) \leq c + \delta \quad \text{for all } t \geq t_0$$

and that $x'(t)$ is bounded for all $t \geq t_0$.

Under the conditions (14) and (15), we can use the same $\underline{\omega}(t)$ and $\overline{\omega}(t)$, since (14) and (15) imply that $\frac{1}{r(t)} \int_t^\infty b(u) du < \infty$. Moreover, (14) and (15) imply that $\int_0^\infty \frac{b(s)}{r(s)} ds < \infty$, and hence it is sufficient to consider

$$V(t, x, y) = y + Ax + L \int_{t_0}^t \frac{b(s)}{r(s)} ds$$

and

$$W(t, x, y) = -y + Ax - L \int_{t_0}^t \frac{b(s)}{r(s)} ds.$$

Remark. Assuming the existence of functions $a(t)$ and $a(x)$ such that $a(t)a(x) \leq f(t, x, y)$ for $t \geq T$, $x < 0$ and all y , we can obtain a result similar to Proposition 2.

By applying Theorem 3, we shall now prove the following proposition.

Proposition 3. Suppose that there exist two functions $b(t)$ and $\beta(x, y)$ which satisfy the following conditions;

- (i) $b(t)$ is continuous on I and $b(t) \geq 0$ for $t \geq T$, where T can be large,
- (ii) $\beta(x, y)$ is continuous on $x \geq 0$ and $y \geq 0$,
- (iii) for $t \geq T$, $x > 0$ and $y \geq 0$,

$$(16) \quad f(t, x, y) \leq b(t)\beta(x, y).$$

Moreover, we assume that there is a $c > 0$ such that

$$(17) \quad f(t, c, 0) \geq 0 \quad \text{for } t \geq T$$

and that for some $M > 0$ such that $c < M$

$$(18) \quad f(t, x, y) \geq 0 \quad \text{for } t \geq T, \quad M \geq x \geq c \quad \text{and } y > K,$$

$$(19) \quad f(t, x, y) \leq 0 \quad \text{for } t \geq T, \quad M \geq x \geq c \quad \text{and } y < -K,$$

where K can be large. If $0 < \epsilon \leq r(t) \leq \rho$ for some ϵ, ρ and all $t \geq 0$ and

$$(20) \quad \int_0^\infty t b(t) dt < \infty$$

or if there is an $A > 0$ such that

$$(21) \quad -A < \frac{r'(t)}{r(t)} \quad \text{for all } t \geq 0$$

and we have

$$(22) \quad \int_0^\infty \left(\frac{1}{r(s)} \int_s^\infty b(u) du \right) ds < \infty,$$

then the equation (10) has a bounded nonoscillatory solution.

Proof. In both cases, we have (22) and the fact that $\frac{1}{r(t)} \int_t^\infty b(u) du \rightarrow 0$ as $t \rightarrow \infty$. For the c , $\beta(c, 0) \leq \frac{L}{2}$ for some L . Since $\beta(x, y)$ is continuous, there is a $\delta > 0$ such that if $|x - c| \leq \delta$ and $0 \leq y \leq \delta$, we have $\beta(x, y) \leq L$. Choose $t_0 \geq T$ so large that

$$L \int_{t_0}^t \left(\frac{1}{r(s)} \int_s^\infty b(u) du \right) ds \leq \min(M - c, \delta) \quad \text{for } t \geq t_0$$

and

$$L \frac{1}{r(t)} \int_t^\infty b(u) du \leq \delta \quad \text{for } t \geq t_0.$$

Then, in the same way as in the proof of Proposition 2, we can see that

$$\underline{\omega}(t) \equiv c \quad \text{and} \quad \overline{\omega}(t) = c + L \int_{t_0}^t \left(\frac{1}{r(s)} \int_s^\infty b(u) du \right) ds$$

satisfy the conditions (8) and (9) for $t \geq t_0$.

In the case where we have $\epsilon \leq r(t) \leq \rho$, for $t \geq t_0$, $\underline{\omega}(t) \leq x \leq \bar{\omega}(t)$ and $y > K$, define $V(t, x, y)$ by $V(t, x, y) = r(t)y$. Then we have

$$\begin{aligned} \dot{V}(t, x, y) &= r'(t)y + r(t) \left\{ -\frac{r'(t)}{r(t)}y - \frac{1}{r(t)}f(t, x, y) \right\} \\ &= -f(t, x, y) \\ &\leq 0. \end{aligned}$$

For $t \geq t_0$, $\underline{\omega}(t) \leq x \leq \bar{\omega}(t)$ and $y < -K$, $W(t, x, y) = -r(t)y$ satisfies $\dot{W}(t, x, y) \leq 0$.

In the case where we have the condition (21), $V(t, x, y) = y - Ax$ and $W(t, x, y) = -y + Ax$ are Liapunov functions that we desire. Therefore, it follows from Theorem 3 that the equation (10) has a bounded nonoscillatory solution.

Remark. Assuming the existence of functions $a(t)$ and $a(x, y)$ such that $a(t)a(x, y) \leq f(t, x, y)$ for $t \geq T$, $x < 0$ and $y \leq 0$, we can obtain a result similar to Proposition 3.

Now consider the equation (1). We assume that there exist continuous functions $a(t)$, $b(t)$, $a(x)$ and $\beta(x)$ which satisfy the following conditions;

- (i) $a(t)$ and $b(t)$ are nonnegative for $t \geq T$, where T can be large,
- (ii) $xa(x) > 0$ and $x\beta(x) > 0$ for $x \neq 0$, and $a'(x) \geq 0$, $\beta'(x) \geq 0$,
- (iii) $a(t)a(x) \leq f(t, x, u) \leq b(t)\beta(x)$ for $t \geq T$, $|x| < \infty$ and $|u| < \infty$.

Moreover, we assume that the derivative of $r(t)$ is continuous and that $0 < \epsilon \leq r(t) \leq \rho$ for some ϵ, ρ and all $t \geq 0$.

Under the assumptions above, the following results follow immediately from Propositions 1 and 2 with the remark.

Proposition 4. A necessary and sufficient condition in order that all bounded solutions of (1) are oscillatory is that

$$\int_0^{\infty} ta(t)dt = \infty \quad \text{and} \quad \int_0^{\infty} tb(t)dt = \infty.$$

Proposition 5. A necessary and sufficient condition in order that the equation (1) has a bounded nonoscillatory solution is that

$$\int_0^{\infty} ta(t)dt < \infty \quad \text{or} \quad \int_0^{\infty} tb(t)dt < \infty.$$

Proposition 6. In addition to the assumptions above, if

$$\int_{\epsilon}^{\infty} \frac{du}{a(u)} < \infty \quad \text{and} \quad \int_{-\epsilon}^{-\infty} \frac{du}{\beta(u)} < \infty \quad \text{for some } \epsilon > 0,$$

a necessary and sufficient condition in order that the equation (1) is oscillatory is that

$$\int_0^{\infty} ta(t)dt = \infty \quad \text{and} \quad \int_0^{\infty} tb(t)dt = \infty.$$

For the equation (1), we assume that there exist continuous functions $a(t)$, $b(t)$, $a(x)$ and $\beta(x)$ mentioned above. Moreover, we assume that the derivative of $r(t)$ is continuous, $\int_0^{\infty} \frac{dt}{r(t)} = \infty$ and there is an $A > 0$ such that $\left| \frac{r'(t)}{r(t)} \right| < A$ for $t \geq 0$. Then, by Proposition 1 if we have the condition A that

$$(23) \quad \begin{cases} \int_0^{\infty} a(s)ds = \infty \\ \text{or} \\ \int_0^{\infty} a(s)ds < \infty, \quad \int_0^{\infty} \left(\frac{1}{r(s)} \int_s^{\infty} a(u)du \right) ds = \infty \end{cases}$$

and

$$(24) \quad \begin{cases} \int_0^\infty b(s)ds = \infty \\ \text{or} \\ \int_0^\infty b(s)ds < \infty, \quad \int_0^\infty \left(\frac{1}{r(s)} \int_s^\infty b(u)du \right) ds = \infty, \end{cases}$$

all bounded solutions of (1) are oscillatory. On the other hand, if we have the condition B that

$$(25) \quad \int_0^\infty a(s)ds < \infty \quad \text{and} \quad \int_0^\infty \left(\frac{1}{r(s)} \int_s^\infty a(u)du \right) ds < \infty$$

or

$$(26) \quad \int_0^\infty b(s)ds < \infty \quad \text{and} \quad \int_0^\infty \left(\frac{1}{r(s)} \int_s^\infty b(u)du \right) ds < \infty,$$

the equation (1) has a bounded nonoscillatory solution. Therefore the condition A is a necessary and sufficient condition in order that all bounded solutions of (1) are oscillatory, and the condition B is a necessary and sufficient condition in order that the equation (1) has a bounded nonoscillatory solution.

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