

Certain classes of analytic functions concerned with uniformly starlike and convex functions

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Abstract

Applying the coefficient inequalities of functions $f(z)$ belonging to the subclasses $\mathcal{MD}(\alpha, \beta)$ and $\mathcal{ND}(\alpha, \beta)$ of certain analytic functions in the open unit disk $\mathbb{U}$, two subclasses $\mathcal{MD}^*(\alpha, \beta)$ and $\mathcal{ND}^*(\alpha, \beta)$ are introduced. The object of the present paper is to derive some convolution properties of functions $f(z)$ in the classes $\mathcal{MD}^*(\alpha, \beta)$ and $\mathcal{ND}^*(\alpha, \beta)$.

1 Introduction

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} \mid |z| < 1\}$. Shams, Kulkarni and Jahangiri [4] have studied the subclass $\mathcal{SD}(\alpha, \beta)$ of $\mathcal{A}$ consisting of $f(z)$ which satisfy

$$\operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > \alpha \left| \frac{zf'(z)}{f(z)} - 1 \right| + \beta \quad (z \in \mathbb{U})$$

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for some $\alpha(\alpha \geq 0)$ and for some $\beta(0 \leq \beta < 1)$. The subclass $\mathcal{KD}(\alpha, \beta)$ is defined by $f(z) \in \mathcal{KD}(\alpha, \beta)$ if and only if $zf'(z) \in \mathcal{SD}(\alpha, \beta)$. In view of the classes $\mathcal{SD}(\alpha, \beta)$ and $\mathcal{KD}(\alpha, \beta)$, we introduce the subclass $\mathcal{MD}(\alpha, \beta)$ consisting of all functions $f(z) \in \mathcal{A}$ which satisfy

$$\operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) < \alpha \left| \frac{zf'(z)}{f(z)} - 1 \right| + \beta \quad (z \in \mathbb{U})$$

for some $\alpha(\alpha \leq 0)$ and for some $\beta(\beta > 1)$. The class $\mathcal{ND}(\alpha, \beta)$ is also considered as the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which satisfy $zf'(z) \in \mathcal{MD}(\alpha, \beta)$. We discuss some properties of functions $f(z)$ belonging to the classes $\mathcal{MD}(\alpha, \beta)$ and $\mathcal{ND}(\alpha, \beta)$.

We note if $f(z) \in \mathcal{MD}(\alpha, \beta)$, then, for $\alpha < -1$, $\frac{zf'(z)}{f(z)}$ lies in the region $G \equiv G(\alpha, \beta) \equiv \{w = u + iv : \operatorname{Re} w < \alpha|w - 1| + \beta\}$, that is, part of the complex plane which contains $w = 1$ and is bounded by the ellipse

$$\left(u - \frac{\alpha^2 - \beta}{\alpha^2 - 1} \right)^2 + \frac{\alpha^2}{\alpha^2 - 1} v^2 = \frac{\alpha^2(\beta - 1)^2}{(\alpha^2 - 1)^2}$$

with vertices at the points

$$\left(\frac{\alpha^2 - \beta}{\alpha^2 - 1}, \frac{\beta - 1}{\sqrt{\alpha^2 - 1}} \right), \left(\frac{\alpha^2 - \beta}{\alpha^2 - 1}, -\frac{\beta - 1}{\sqrt{\alpha^2 - 1}} \right), \left(\frac{\alpha + \beta}{\alpha + 1}, 0 \right), \left(\frac{\alpha - \beta}{\alpha - 1}, 0 \right).$$

Since $\frac{\alpha + \beta}{\alpha + 1} < 1 < \frac{\alpha - \beta}{\alpha - 1} < \beta$, we have $\mathcal{MD}(\alpha, \beta) \subset \mathcal{MD}(0, \beta) \equiv \mathcal{M}(\beta)$. For $\alpha = -1$, if $f(z) \in \mathcal{MD}(\alpha, \beta)$, then $\frac{zf'(z)}{f(z)}$ belongs to the region which contains $w = 0$ and is bounded by parabola

$$u = -\frac{v^2 - \beta^2 + 1}{2(\beta - 1)}.$$

In the case of $f(z) \in \mathcal{ND}(\alpha, \beta)$, $\frac{zf''(z)}{f'(z)}$ lies in the region which contains $w = 0$ and is bounded by the ellipse

$$\left(u + \frac{\beta - 1}{\alpha^2 - 1} \right)^2 + \frac{\alpha^2}{\alpha^2 - 1} v^2 = \frac{\alpha^2(\beta - 1)^2}{(\alpha^2 - 1)^2} \quad (\alpha < -1)$$

with vertices at the points

$$\left(\frac{1 - \beta}{\alpha^2 - 1}, \frac{\beta - 1}{\sqrt{\alpha^2 - 1}} \right), \left(\frac{1 - \beta}{\alpha^2 - 1}, -\frac{\beta - 1}{\sqrt{\alpha^2 - 1}} \right), \left(\frac{1 - \beta}{\alpha - 1}, 0 \right), \left(\frac{\beta - 1}{\alpha + 1}, 0 \right).$$

Since $\frac{\beta - 1}{\alpha + 1} < 0 < \frac{1 - \beta}{\alpha - 1} < \beta$, we have $\mathcal{ND}(\alpha, \beta) \subset \mathcal{ND}(0, \beta) \equiv \mathcal{M}(\beta)$. And for $\alpha = -1$, $\frac{zf''(z)}{f'(z)}$ lies in the domain which contains $w = 0$ and is bounded by parabola

$$u = -\frac{v^2}{2(\beta - 1)} + \frac{\beta - 1}{2}.$$

The classes $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$ were considered by Uralegaddi, Ganigi and Sarangi [3], Nishiwaki and Owa [1], and Owa and Nishiwaki [2].

2 Coefficient inequalities for the classes $\mathcal{MD}(\alpha, \beta)$ and $\mathcal{ND}(\alpha, \beta)$

We try to derive sufficient conditions for $f(z)$ which are given by using coefficient inequalities.

Theorem 2.1. *If $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} \{|n - \beta + 1| + |n - \beta - 1| - 2\alpha(n - 1)\} |a_n| \leq \beta - |2 - \beta|$$

for some $\alpha(\alpha \leq 0)$ and for some $\beta(\beta > 1)$, then $f(z) \in \mathcal{MD}(\alpha, \beta)$.

Proof. Let us suppose that

$$\sum_{n=2}^{\infty} \{|n - \beta + 1| + |n - \beta - 1| - 2\alpha(n - 1)\} |a_n| \leq \beta - |2 - \beta|$$

for $f(z) \in \mathcal{A}$. It suffices to show that

$$\left| \frac{\left(\frac{zf'(z)}{f(z)} - \alpha \left| \frac{zf'(z)}{f(z)} - 1 \right| - \beta \right) + 1}{\left(\frac{zf'(z)}{f(z)} - \alpha \left| \frac{zf'(z)}{f(z)} - 1 \right| - \beta \right) - 1} \right| < 1 \quad (z \in \mathbb{U}).$$

We have

$$\begin{aligned} & \left| \frac{\left(\frac{zf'(z)}{f(z)} - \alpha \left| \frac{zf'(z)}{f(z)} - 1 \right| - \beta \right) + 1}{\left(\frac{zf'(z)}{f(z)} - \alpha \left| \frac{zf'(z)}{f(z)} - 1 \right| - \beta \right) - 1} \right| = \left| \frac{zf'(z) - \alpha e^{i\theta} |zf'(z) - f(z)| - \beta f(z) + f(z)}{zf'(z) - \alpha e^{i\theta} |zf'(z) - f(z)| - \beta f(z) - f(z)} \right| \\ &= \left| \frac{z + \sum_{n=2}^{\infty} na_n z^n - \alpha e^{i\theta} \left| \sum_{n=2}^{\infty} (n-1)a_n z^n \right| - \beta z - \beta \sum_{n=2}^{\infty} a_n z^n + z + \sum_{n=2}^{\infty} a_n z^n}{z + \sum_{n=2}^{\infty} na_n z^n - \alpha e^{i\theta} \left| \sum_{n=2}^{\infty} (n-1)a_n z^n \right| - \beta z - \beta \sum_{n=2}^{\infty} a_n z^n - z - \sum_{n=2}^{\infty} a_n z^n} \right| \\ &= \left| \frac{(2 - \beta) + \sum_{n=2}^{\infty} (n - \beta + 1)a_n z^{n-1} - \alpha e^{i\theta} \left| \sum_{n=2}^{\infty} (n-1)a_n z^{n-1} \right|}{-(\beta - \sum_{n=2}^{\infty} (n - \beta - 1)a_n z^{n-1} + \alpha e^{i\theta} \left| \sum_{n=2}^{\infty} (n-1)a_n z^{n-1} \right|)} \right| \end{aligned}$$

$$< \frac{|2 - \beta| + \sum_{n=2}^{\infty} |n - \beta + 1| |a_n| - \alpha \sum_{n=2}^{\infty} (n - 1) |a_n|}{\beta - \sum_{n=2}^{\infty} |n - \beta - 1| |a_n| + \alpha \sum_{n=2}^{\infty} (n - 1) |a_n|}.$$

The last expression is bounded above by 1 if

$$|2 - \beta| + \sum_{n=2}^{\infty} |n - \beta + 1| |a_n| - \alpha \sum_{n=2}^{\infty} (n - 1) |a_n| \leq \beta - \sum_{n=2}^{\infty} |n - \beta - 1| |a_n| - \alpha \sum_{n=2}^{\infty} (n - 1) |a_n|$$

which is equivalent to our condition

$$\sum_{n=2}^{\infty} \{|n - \beta + 1| + |n - \beta - 1| - 2\alpha(n - 1)\} |a_n| \leq \beta - |2 - \beta|$$

of the theorem. This completes the proof of the theorem. $\square$

By using Theorem2.1, we have

Corollary 2.1. *If $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} n \{|n - \beta + 1| + |n - \beta - 1| - 2\alpha(n - 1)\} |a_n| \leq \beta - |2 - \beta|$$

for some $\alpha(\alpha \leq 0)$ and for some $\beta(\beta > 1)$, then $f(z) \in \mathcal{ND}(\alpha, \beta)$

Proof. From $f(z) \in \mathcal{ND}(\alpha, \beta)$ if and only if $zf'(z) \in \mathcal{MD}(\alpha, \beta)$, replacing a_n by na_n in Theorem2.1, we have the corollary. $\square$

3 Relation for $\mathcal{MD}^*(\alpha, \beta)$ and $\mathcal{ND}^*(\alpha, \beta)$

By Theorem2.1, the class $\mathcal{MD}^*(\alpha, \beta)$ is considered as the subclass of $\mathcal{MD}(\alpha, \beta)$ consisting of $f(z)$ satisfying

$$(3.1) \quad \sum_{n=2}^{\infty} \{|n - \beta + 1| + |n - \beta - 1| - 2\alpha(n - 1)\} |a_n| \leq \beta - |2 - \beta|$$

for some $\alpha(\alpha \leq 0)$ and for some $\beta(\beta > 1)$. The class $\mathcal{ND}^*(\alpha, \beta)$ is also considered as the subclass of $\mathcal{ND}(\alpha, \beta)$ consisting of $f(z)$ which satisfy

$$(3.2) \quad \sum_{n=2}^{\infty} n \{|n - \beta + 1| + |n - \beta - 1| - 2\alpha(n - 1)\} |a_n| \leq \beta - |2 - \beta|$$

for some $\alpha(\alpha \leq 0)$ and for some $\beta(\beta > 1)$. By the coefficient inequalities for the classes $\mathcal{MD}^*(\alpha, \beta)$ and $\mathcal{ND}^*(\alpha, \beta)$, we see

Theorem 3.1. *If $f(z) \in \mathcal{A}$, then*

$$\mathcal{MD}^*(\alpha_1, \beta) \subset \mathcal{MD}^*(\alpha_2, \beta)$$

for some α_1 and α_2 ($\alpha_1 \leq \alpha_2 \leq 0$).

Proof. For $\alpha_1 \leq \alpha_2 \leq 0$, we obtain

$$\begin{aligned} & \sum_{n=2}^{\infty} \{|n - \beta + 1| + |n - \beta - 1| - 2\alpha_2(n - 1)\} |a_n| \\ & \leq \sum_{n=2}^{\infty} \{|n - \beta + 1| + |n - \beta - 1| - 2\alpha_1(n - 1)\} |a_n|. \end{aligned}$$

Therefore, if $f(z) \in \mathcal{MD}^*(\alpha_1, \beta)$, then $f(z) \in \mathcal{MD}^*(\alpha_2, \beta)$. Hence we get the required result. $\square$

By using Theorem 3.1, we also have

Corollary 3.1. *If $f(z) \in \mathcal{A}$, then*

$$\mathcal{ND}^*(\alpha_1, \beta) \subset \mathcal{ND}^*(\alpha_2, \beta)$$

for some α_1 and α_2 ($\alpha_1 \leq \alpha_2 \leq 0$).

4 Convolution of the classes $\mathcal{MD}^*(\alpha, \beta)$ and $\mathcal{ND}^*(\alpha, \beta)$

For analytic functions $f_j(z)$ given by

$$f_j(z) = z + \sum_{n=2}^{\infty} a_{n,j} z^n \quad (j = 1, 2, \dots, p),$$

the Hadamard product (or convolution) of $f_1(z), f_2(z), \dots, f_p(z)$ is defined by

$$(f_1 * f_2 * \dots * f_p)(z) = z + \sum_{n=2}^{\infty} \left(\prod_{j=1}^p a_{n,j} \right) z^n.$$

Thus we have

Theorem 4.1. *If $f_1(z) \in \mathcal{MD}^*(\alpha, \beta_1)$ and $f_2(z) \in \mathcal{MD}^*(\alpha, \beta_2)$ for some α ($\alpha \leq 2 - \sqrt{5}$) and β_1, β_2 ($1 < \beta_1, \beta_2 \leq 2$), then $(f_1 * f_2) \in \mathcal{MD}^*(\alpha, \beta)$, where*

$$\beta = 1 + \frac{(\beta_1 - 1)(\beta_2 - 1)(2 - \alpha)}{(\beta_1 - 1)(\beta_2 - 1) + (2 - \alpha - \beta_1)(2 - \alpha - \beta_2)}.$$

Proof. From (3.1), for $f(z) \in \mathcal{MD}^*(\alpha, \beta)$ with $1 < \beta \leq 2$, we have

$$\begin{aligned} \sum_{n=2}^{\infty} \{(n+1-\beta) + (n-1-\beta) - 2\alpha(n-1)\} |a_n| &\leq \sum_{n=2}^{\infty} \{(n+1-\beta) + |n-1-\beta| - 2\alpha(n-1)\} |a_n| \\ &\leq 2(\beta-1), \end{aligned}$$

that is, if $f(z) \in \mathcal{MD}^*(\alpha, \beta)$, then

$$(4.1) \quad \sum_{n=2}^{\infty} \frac{n(1-\alpha) - \beta + \alpha}{\beta-1} |a_n| \leq 1.$$

Conversely, if $f(z)$ satisfies

$$(4.2) \quad \sum_{n=2}^{\infty} \frac{n(1-\alpha) + 1 - \beta + \alpha}{\beta-1} |a_n| \leq 1,$$

then $f(z) \in \mathcal{MD}^*(\alpha, \beta)$. From (4.1), if $f_1(z) \in \mathcal{MD}^*(\alpha, \beta_1)$, then

$$(4.3) \quad \sum_{n=2}^{\infty} \frac{n(1-\alpha) - \beta_1 + \alpha}{\beta_1-1} |a_{n,1}| \leq 1,$$

and also if $f_2(z) \in \mathcal{MD}^*(\alpha, \beta_2)$, then

$$(4.4) \quad \sum_{n=2}^{\infty} \frac{n(1-\alpha) - \beta_2 + \alpha}{\beta_2-1} |a_{n,2}| \leq 1.$$

Applying the Shwarz's inequality, we have the following inequality

$$(4.5) \quad \sum_{n=2}^{\infty} \sqrt{\frac{\{n(1-\alpha) - \beta_1 + \alpha\} \{n(1-\alpha) - \beta_2 + \alpha\}}{(\beta_1-1)(\beta_2-1)}} \sqrt{|a_{n,1}| |a_{n,2}|} \leq 1$$

by (4.3) and (4.4). From (4.2) and (4.5), if the following inequality

$$\begin{aligned} (4.6) \quad &\sum_{n=2}^{\infty} \frac{n(1-\alpha) + 1 - \beta + \alpha}{\beta-1} |a_{n,1}| |a_{n,2}| \\ &\leq \sum_{n=2}^{\infty} \sqrt{\frac{\{n(1-\alpha) - \beta_1 + \alpha\} \{n(1-\alpha) - \beta_2 + \alpha\}}{(\beta_1-1)(\beta_2-1)}} \sqrt{|a_{n,1}| |a_{n,2}|} \end{aligned}$$

is satisfied, then we say that $f(z) \in \mathcal{MD}^*(\alpha, \beta)$. This inequality holds true if

$$(4.7) \quad \frac{n(1-\alpha) + 1 - \beta + \alpha}{\beta-1} \sqrt{|a_{n,1}| |a_{n,2}|} \leq \sqrt{\frac{\{n(1-\alpha) - \beta_1 + \alpha\} \{n(1-\alpha) - \beta_2 + \alpha\}}{(\beta_1-1)(\beta_2-1)}}$$

for all $n \geq 2$. Therefore, we have

$$(4.8) \quad \frac{n(1-\alpha) + 1 - \beta + \alpha}{\beta-1} \leq \frac{\{n(1-\alpha) - \beta_1 + \alpha\} \{n(1-\alpha) - \beta_2 + \alpha\}}{(\beta_1-1)(\beta_2-1)}$$

which is equivalent to

$$(4.9) \quad \beta \geq 1 + \frac{(\beta_1 - 1)(\beta_2 - 1)\{n(1 - \alpha) + \alpha\}}{(\beta_1 - 1)(\beta_2 - 1) + \{n(1 - \alpha) - \beta_1 + \alpha\}\{n(1 - \alpha) - \beta_2 + \alpha\}}$$

for all $n \geq 2$.

Let $G(n)$ be the right hand side of the last inequality. Then $G(n)$ is decreasing for $n \geq 2$ for $\alpha \leq 2 - \sqrt{5}$. Thus $G(2)$ is the maximum of $G(n)$ for $\alpha \leq 2 - \sqrt{5}$. This completes the proof of the theorem. $\square$

For the functions $f(z)$ belonging to the class $\mathcal{ND}^*(\alpha, \beta)$, we also have

Corollary 4.1. *If $f_1(z) \in \mathcal{ND}^*(\alpha, \beta_1)$ and $f_2(z) \in \mathcal{ND}^*(\alpha, \beta_2)$ for some α and β_1, β_2 , ($1 < \beta_1, \beta_2 \leq 2$) then $(f_1 * f_2)(z) \in \mathcal{ND}^*(\alpha, \beta)$, where*

$$\beta = 1 + \frac{(\beta_1 - 1)(\beta_2 - 1)(2 - \alpha)}{(\beta_1 - 1)(\beta_2 - 1) + 2(2 - \alpha - \beta_1)(2 - \alpha - \beta_2)}.$$

By virtue of Theorem 4.1, we have the following theorem.

Theorem 4.2. *If $f_j \in \mathcal{MD}^*(\alpha, \beta_j)$ ($j = 1, 2, \dots, p$) for some α ($\alpha \leq 2 - \sqrt{5}$) and β_j ($1 < \beta_j \leq 2$), then $(f_1 * f_2 * \dots * f_p) \in \mathcal{MD}^*(\alpha, \beta)$, where*

$$\beta = 1 + \frac{A_p}{B_p - C_p D_p + E_p} \quad (p \geq 2),$$

$$A_p = \prod_{j=1}^p (\beta_j - 1)(2 - \alpha)^{p-1}, \quad B_p = (2 - \alpha)^{p-2} \prod_{j=1}^p (\beta_j - 1),$$

$$C_p = \sum_{m=1}^{p-2} (2 - \alpha)^{p-m-2} (1 - \alpha)^{m-1}, \quad D_p = \prod_{j=1}^{p-m} (\beta_j - 1) \prod_{l=p-m+1}^p (2 - \alpha - \beta_l),$$

and

$$E_p = (1 - \alpha)^{p-2} \prod_{j=1}^p (2 - \alpha - \beta_j).$$

Proof. When $p = 2$, we have

$$\beta = 1 + \frac{(\beta_1 - 1)(\beta_2 - 1)(2 - \alpha)}{(\beta_1 - 1)(\beta_2 - 1) + (2 - \alpha - \beta_1)(2 - \alpha - \beta_2)}.$$

Let us suppose that $(f_1 * \dots * f_k) \in \mathcal{MD}^*(\alpha, \beta_0)$ and $f_{k+1} \in \mathcal{MD}^*(\alpha, \beta_{k+1})$, where

$$\beta_0 = 1 + \frac{A_k}{B_k - C_k D_k + E_k} \quad (k \geq 2).$$

Using Theorem 4.1 and replacing β_1 by β_0 and β_2 by β_{k+1} , we see that

$$\begin{aligned}\beta &= 1 + \frac{(\beta_0 - 1)(\beta_{k+1} - 1)(2 - \alpha)}{(\beta_0 - 1)(\beta_{k+1} - 1) + (2 - \alpha - \beta_0)(2 - \alpha - \beta_{k+1})} \\ &= 1 + \frac{A_{k+1}}{B_{k+1} - \{B_k(2 - \alpha - \beta_{k+1}) + (1 - \alpha)C_k D_k(2 - \alpha - \beta_{k+1})\} + E_{k+1}} \\ &= 1 + \frac{A_{k+1}}{B_{k+1} - \{B_k(2 - \alpha - \beta_{k+1}) + C_k^+ D_{k+1}\} + E_{k+1}} \\ &= 1 + \frac{A_{k+1}}{B_{k+1} - C_{k+1} D_{k+1} + E_{k+1}},\end{aligned}$$

where

$$C_k^+ = \sum_{m=2}^{k-1} (2 - \alpha)^{k-m-1} (1 - \alpha)^{m-1}.$$

This completes the proof of the Theorem. $\square$

Finally we have

Corollary 4.2. *If $f_j \in \mathcal{ND}^*(\alpha, \beta_j)$ ($j = 1, 2, \dots, p$) for some α and $\beta_j (1 < \beta_j \leq 2)$, then $(f_1 * f_2 * \dots * f_p) \in \mathcal{ND}^*(\alpha, \beta)$, where*

$$\begin{aligned}\beta &= 1 + \frac{A_p}{B_p - C_p^* D_p + 2^{p-1} E_p} \quad (p \geq 2), \\ A_p &= \prod_{j=1}^p (\beta_j - 1)(2 - \alpha)^{p-1}, \quad B_p = (2 - \alpha)^{p-2} \prod_{j=1}^p (\beta_j - 1), \\ C_p^* &= \sum_{m=1}^{p-2} 2^m (2 - \alpha)^{p-m-2} (1 - \alpha)^{m-1}, \quad D_p = \prod_{j=1}^{p-m} (\beta_j - 1) \prod_{l=p-m+1}^p (2 - \alpha - \beta_l),\end{aligned}$$

and

$$E_p = (1 - \alpha)^{p-2} \prod_{j=1}^p (2 - \alpha - \beta_j).$$

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