

On the explicit formulae of characters for
discrete series representations

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Let us consider a simply connected complex simple Lie group $G_{\mathbb{C}}$ and its connected real simple form G . We denote by $\mathfrak{g}_{\mathbb{C}}$, $\mathfrak{g}$ the Lie algebra of $G_{\mathbb{C}}$, G respectively.

According to a criterion of Harish-Chandra in [3], a square integrable representation ω on G (i.e., one of each matrix coefficients of ω is square integrable with respect to a Haar measure dx on G) exists if and only if G has a compact Cartan subgroup B .

In this paper, we shall study the global characters of square integrable representations which are called the discrete series. Hence we shall assume that G has a compact Cartan subgroup B . For one of each irreducible representation ω of G in the discrete series, we define a distribution H on G by

$$\Theta(f) = \text{Trace} \int_G f(x^{-1}) \omega(x) dx$$

for all C^{∞} -functions f on G with compact support.

Then Θ is a tempered invariant eigendistribution on G , and Θ is real analytic on the set of all regular elements in G (cf. [3]).

We select a maximal compact subgroup K of G containing B . Let $\omega|_K$ be the restriction of ω to K . Therefore we put, for each irreducible representation π of K , by $|\omega|_K : \pi|$ the multiplicity of π occurring in $\omega|_K$. Then the restriction of Θ to B is expressed as

$$\Theta(b) = \sum_{\xi_K} |\omega|_K : \pi| \text{Trace } \pi(b)$$

in the sense of the distributions on B , where ξ_K is the set of all inequivalent classes of irreducible representations on K .

We will summarize Harish-Chandra's parametrization in [1], [3] for the characters of representations in the discrete series.

Let us consider Σ the root system of the pair $(\mathfrak{g}_\mathbb{C}/\mathfrak{b}_\mathbb{C})$ where $\mathfrak{b}_\mathbb{C}$ is the complexification of Lie algebra $\mathfrak{b}$ of B . By P^-, P^+, P^-, L' , we shall denote the set of all positive roots, the set of all noncompact positive roots, the set of all compact positive roots, the set of all regular integral form on $\mathfrak{b}_\mathbb{C}$ respectively.

Let Θ be the character of a fixed irreducible representation ω of G in the discrete series. Then

$$\Theta = \varepsilon(\Lambda)(-1)^{|P^+|} \sum_{s \in W(G/B)} \varepsilon(s) \exp s\Lambda \text{ on } B' \dots (*)$$

for suitably chosen Λ in L'

where $B' = \{b = \exp H; b \in B, \alpha(H) \neq 0 \text{ for all } \alpha \text{ in } P^+\}$,

$W(G/B)$ = the Weyl group of the pair (G/B) ,

$\varepsilon(s)$ = the signature of s in $W(G/B)$,

and $\varepsilon(\Lambda) = \prod_{\alpha \in P} \text{sgn}(\Lambda, \alpha)$.

Conversely, for each form Λ in L' , there exists unique irreducible representation ω of G in the discrete series such that

$\Theta = \text{Trace } \omega$ satisfies the above identity (*). Thus one of each discrete character Θ is parametrized by $\Theta = \Theta_{\Lambda}$ ($\Lambda \in L'$) under the identity (*).

We now state our purpose of this paper. Let us consider a regular dominant integral form Λ on $\mathfrak{g}_C$ and finite dimensional irreducible representation π_{Λ} of G_C with the highest weight $\Lambda - \frac{1}{2} \sum_{\alpha \in P} \alpha$. Therefore we define a distribution $\mathcal{S}_{\Lambda}$ on B as following;

$$\mathcal{S}_{\Lambda}(b) = \sum_{s \in W(G/B) \setminus W} \Theta_{s\Lambda} - (-1)^{|P^+|} \text{Trace } \pi_{\Lambda}(b), \quad b \in B$$

where W is the Weyl group of the pair $(\mathfrak{g}_C/\mathfrak{b}_C)$.

By The identity (*), $\mathcal{S}_{\Lambda} \equiv 0$ on B' . Moreover, since the discrete representations are all infinite dimensional, $\mathcal{S}_{\Lambda} \neq 0$ on B .

Hence $\mathcal{S}_{\Lambda}$ is a singular distribution on B , because B' is open dense in B . We shall, in this paper, give a characterization of

and obtain a global formula of the character $\sum_{s \in W(G/B) \setminus W} \Theta_{s\Lambda}$ under the following assumptions.

(A1): G contains a compact Cartan subgroup B .

(A2): All of noncompact roots in Σ have the same length with each other.

For this purpose, we will state more precisely descriptions.

We define the generating function $\Phi_{s,\Lambda}$ ($s \in W$) on B_c as followings;

$$\Phi_{s,\Lambda} = \prod_{\alpha \in P^-(s)} (1 - \exp(-\alpha)) \exp(s\Lambda + s\vartheta) / \prod_{\alpha \in P^+(s)} (1 - \exp(\alpha))$$

where $\vartheta = \frac{1}{2} \sum_{\alpha \in P} \alpha$, $P^+(s) = \{\alpha \in P^+ \cup -P^+; s^{-1}\alpha > 0\}$, and $P^-(s) = \{\alpha \in P^- \cup -P^-; s^{-1}\alpha > 0\}$.

Then $\Phi_{s,\Lambda}$ is holomorphic on the complex domain

$$D_s = \{b = \exp H \in B_c; |\exp \alpha(H)| < 1 \text{ for each } \alpha \text{ in } P^+(s)\}.$$

The functions $\Phi_{s,\Lambda}$ ($s \in W$) are concerned with Blattner's conjecture.

This phenomenon is stated as followings. Let L be the set of all integral form on $\mathfrak{b}_c$ and $Q_s(\mu)$ ($\mu \in L$) be the partition function on L , which is defined by

$$1 / \prod_{\alpha \in P^+(s)} (1 - \exp \alpha) = \sum_{\mu \in L} Q_s(\mu) \exp \mu.$$

Then $\Phi_{s,\Lambda}$ is expressed as

$$\Phi_{s,\Lambda} = \sum_{\mu \in L} Q_s(\mu) \prod_{\alpha \in P^-(s)} (1 - \exp(-\alpha)) \exp(s\Lambda + s\vartheta + \mu).$$

We now choose an element u in W satisfying $u^{-1}\alpha > 0$ for each α in P^- .

Since $\Delta_K(b^s) = \varepsilon(s)\Delta_K(b)$ ($\Delta_K = \exp \frac{1}{2} \sum_{\alpha \in P^-} \alpha \prod_{\alpha \in P^-} (1 - \exp -\alpha)$)
and $sP^+ \subseteq P^+ \cup -P^+$ for all $b \in B$, $s \in W(G/B)$,

we get

$$\sum_{s \in W(G/B)} \bar{\Phi}_{su, \Lambda} = \sum_{\mu \in L} \sum_{s \in W(G/B)} Q_{su}(\mu) \varepsilon(s) \Delta_K \exp(su\Lambda + s\mathcal{G}^+(u) + \mu)$$

$$\text{where } \mathcal{G}^+(u) = \frac{1}{2} \sum_{\alpha \in P^+(u)} \alpha.$$

Moreover by $P^+(su) = sP^+(u)$, the above equation is rewritten as

$$\begin{aligned} \sum_{s \in W(G/B)} \bar{\Phi}_{su, \Lambda} &= \sum_{\mu \in L} \sum_{s \in W(G/B)} Q_u(\mu) \Delta_K \exp(su\Lambda + s\mathcal{G}^+(u) + s\mu) \\ &= \sum_{\mu \in L} \sum_{s \in W(G/B)} \varepsilon(s) Q_u(s\mu - u\Lambda - \mathcal{G}^+(u)) \exp \mu. \end{aligned}$$

Defining the Blattner's number $b_{u, \Lambda}(\mu)$ ($\mu \in L$) and a subset L^+ in L

$$\text{by } b_{u, \Lambda}(\mu) = \sum_{s \in W(G/B)} \varepsilon(s) Q_u(s\mu - u\Lambda - \mathcal{G}^+(u)),$$

and $L_+ = \{ \mu \in L; (\mu, \alpha) > 0 \text{ for each compact positive root } \alpha \}$,

then

$$\begin{aligned} \sum_{s \in W(G/B)} \bar{\Phi}_{su, \Lambda} &= \sum_{\mu \in L_+} b_{u, \Lambda}(\mu) \sum_{s \in W(G/B)} \varepsilon(s) \exp s\mu \\ &= \sum_{\mu \in L_+} b_{u, \Lambda}(\mu) (\Delta_K)^2 \text{Trace } \pi_\mu \end{aligned}$$

where π_μ is the finite dimensional irreducible representation of

K with the highest weight $\mu - \frac{1}{2} \sum_{\alpha \in P^-} \alpha$.

There was conjectured that for the representation $\omega = \omega(u\Lambda)$ with its character $\oplus_{u\Lambda}$,

$$|\omega(u\Lambda)|_{K:\pi} = \begin{cases} b_{u, \Lambda}(\mu) & \text{if } \pi = \pi_\mu, \mu \in L_+ \\ 0 & \text{otherwise} \end{cases}$$

This conjecture implies

$$\sum_{s \in W} \Phi_{s, \Lambda}(b) = (-1)^{|P^-|} |\Delta_K(b)|^2 \sum_{s \in W(G/B) \setminus W} \oplus_{s\Lambda}(b) \dots.. (**).$$

According to these observations, we arrive the following situation; it will be suggested that a calculation of explicit relation between $\sum_{s \in W} \Phi_{s, \Lambda}(b)$ and $\text{Trace } \pi_{\Lambda}(b)$ ($b \in B$) enable us to clear the gap δ_{Λ} .

As far as the autor knows, the several results of multiplicity formulae of characters in the discrete series (Blattner's conjecture) have been obtained in the following papers; [14], [15] (W. Schmid), [10] (R. Hotta and K. R. Parthasarathy).

[4] (H. Hecht and W. Schmid), [17] (N. Wallach).

In the amount of these papers, Blattner's conjecture was completely solved by [4], [15] and by [17] for all real semisimple matrix groups with compact Cartan subgroups. Especially, the explicit multiplicity formulae of characters of discrete series representations were obtained by [4] and [15].

However, we need not apply the multiplicity theorem in [4], [15] to our arguments. Our main results will be stated after the following preparations.

Definition; a subset F in P^+ is strongly orthogonal if and only if F satisfies that for each of two distinct roots α, β in F , $\alpha = -\beta$ and $\alpha \pm \beta \notin \Sigma$.

Definition; two strongly orthogonal system F_1, F_2 in P^+ are

conjugate if and only if there exists t in $W(G/B)$ such that

$$tF_1 \cup -tF_1 = F_2 \cup -F_2.$$

By Γ_0 , we denote a complete representative of all nonconjugate

strongly orthogonal system in P^+ under $W(G/B)$. Choosing a

system F^0 in Γ_0 satisfying $|F^0| = \text{real rank of } \mathfrak{g}$, then we can

assume that $F \subseteq F^0$ for all F in Γ_0 under the condition A1, A2.

Let $B(F)$ ($F \in \Gamma_0$) be the subgroup of B , which is defined by

$B(F) = \{b = \exp H \in B; \alpha(H) = 0 \text{ for each } \alpha \text{ in } F\}$. Let us consider

$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the Cartan decomposition of $\mathfrak{g}$, here $\mathfrak{k}$ is the Lie

algebra of K . Then for each α in F , there exist $X_\alpha, X_{-\alpha}$ in $\mathfrak{g}_\mathbb{C}$

such that $\text{ad}(H)X_{\pm\alpha} = \pm\alpha(H)X_{\pm\alpha}$ ($H \in \mathfrak{b}_\mathbb{C}$), $\sqrt{-1}(X_\alpha + X_{-\alpha}) \in \mathfrak{p}$, $(X_\alpha - X_{-\alpha}) \in \mathfrak{p}$.

Therefore we put $\mathfrak{a}_R(F)$, $\mathfrak{z}_\mathbb{C}(F)$, $\mathfrak{g}_\mathbb{C}(F)$, $\mathfrak{b}(F)$ ($\equiv \mathfrak{a}_I(F)$), $\mathfrak{a}(F)$, and

$\mathfrak{b}_\mathbb{C}(F)$ by the followings;

$\mathfrak{a}_R(F) = \sum_{\alpha \in F} \sqrt{-1} R(X_\alpha + X_{-\alpha})$, $\mathfrak{z}_\mathbb{C}(F)$ = the centralizer of $\mathfrak{a}_R(F)$ in $\mathfrak{g}_\mathbb{C}$,

$\mathfrak{g}_\mathbb{C}(F)$ = the orthogonal complement of $\mathfrak{a}_R(F)$ in $\mathfrak{z}_\mathbb{C}(F)$ with

respect to the Killing form on $\mathfrak{g}_\mathbb{C}$,

$\mathfrak{b}(F)$ = the Lie algebra of $B(F)$, $\mathfrak{a}(F) = \mathfrak{a}_I(F) + \mathfrak{a}_R(F)$, and

$\mathfrak{b}_\mathbb{C}(F)$ = the complexification of $\mathfrak{b}(F)$ in $\mathfrak{g}_\mathbb{C}$.

Then $\mathfrak{g}_\mathbb{C}(F)$ is a reductive subalgebra of $\mathfrak{g}_\mathbb{C}$ with Cartan

subalgebra $\mathfrak{b}(F)$. Moreover, the positive (noncompact positive)

root system of $(\mathfrak{g}_\mathbb{C}(F)/\mathfrak{b}_\mathbb{C}(F))$ can be identified with the set

$P(F) = \{\alpha \in P; (\alpha, \beta) = 0, \beta \in F\}$ (resp. $P^+(F) = \{\alpha \in P^+; \alpha \pm \beta \notin \Sigma \text{ for all } \beta \in F\}$). We now consider, for a fixed system F in Γ_0 , a singular distribution $X_{s\Lambda}(F; b)$ on B , which is given by

$$\langle X_{s\Lambda}(F; b), f(b) \rangle = \frac{1}{|W(G/B)|} \sum_{u \in W(G/B)} \int_{B(F)} \sum_{t \in W(\mathfrak{g}_c(F)/\mathfrak{b}_c(F))} \xi(ut) \times$$

$$|\Delta_{K(F)}(b)|^2 f(b) \frac{\exp \sum_{\alpha \in P(F)} \alpha(\log b)}{\prod_{\alpha \in P(F)} (1 - \exp - \alpha(\log b))} db$$

for all C^∞ -functions f on B , where db is the Haar measure on $B(F)$ normalized as $\int_{B(F)} db = 1$,

$$\Delta_{K(F)} = \exp\left(\frac{1}{2} \sum_{\alpha \in P^-(F)} \alpha\right) \prod_{\alpha \in P^-(F)} (1 - \exp - \alpha), \quad P^-(F) = P(F) - P^+(F).$$

and $W(\mathfrak{g}_c(F)/\mathfrak{b}_c(F))$ = the Weyl group of $(\mathfrak{g}_c(F)/\mathfrak{b}_c(F))$.

Our first main result is stated below.

Theorem I. Suppose G fulfills the conditions A1, A2. Then, for the functionals $\Phi_{s, \Lambda}$ ($s \in W$), $X_{s\Lambda}(F; b)$ on B , we have

$$\sum_{s \in W} (-1)^{|P^+|} \Phi_{s, \Lambda}(b) = \sum_{\substack{s \in W(F) \setminus W \\ s^{-1}F > 0}} \xi(s) (-1)^{|P^+| + |F|} X_{s\Lambda}(F; b)$$

for all b in B , where $W(F) = \{s \in W; sF \subseteq F \cup -F\}$.

Let us consider, for a fixed system F in Γ_0 , the Cartan subgroup $A(F)$ of G , which is corresponded to $\mathfrak{a}(F)$. Let $Z(F)$ be the centralizer of $\mathfrak{a}_R(F)$ in G . Then $\mathfrak{g} \cap \mathfrak{z}_c(F)$ is the Lie algebra of $Z(F)$.

Moreover, there exists a unique parabolic subgroup $Q(F)$ of G such that the reductive part coincides with $Z(F)$. Therefore we define a representation $\pi_{s\Lambda}^F$ of G , which is induced from a finite dimensional irreducible representation of $Q(F)$.

For the simplicity of our notations, we will identify Σ , W with the root system of $(\mathfrak{g}_c/\mathfrak{a}_c(F))$, the Weyl group of $(\mathfrak{g}_c/\mathfrak{a}_c(F))$ by using the Cayley transform. Putting

$$T_\Lambda = \sum_{F \in \Gamma_0} \sum_{\substack{s \in W(F) \setminus W \\ s^{-1}F > 0, s^{-1}P(F) > 0}} \varepsilon(s) \text{Trace } \pi_{s\Lambda}^F, \text{ then the dist-}$$

tribution T_Λ on G is an extension of $\sum_{s \in W} \Phi_{s, \Lambda}$ to G . Speaking more precisely,

$$\sum_{s \in W} \Phi_{s, \Lambda} = (-1)^{|P|} |\Delta_K|^2 T_\Lambda \text{ on } B \quad \dots\dots(***) .$$

We notice that the equation $(***)$ is corresponded to $(**)$.

Theorem II. Under the same assumptions as in Theorem I,

$$\begin{aligned} & \xi_R(F; h) \Delta(F; h) T_\Lambda(h) \\ &= \sum_{s \in W} \sum_{u \in V_0(F) \setminus W(G/A(F))} \prod_F \varepsilon(s^{-1}\alpha) \exp s\Lambda(\log h_I) \times \end{aligned}$$

$$\prod_{\alpha \in F} \exp -|\alpha(\log(h_R)^u)| |(s\Lambda, \alpha)|/|\alpha|^2$$

for all regular elements $h = h_I h_R$ ($h_I \in \overline{A(F) \cap K}$, $h_R \in A(F) \cap \exp p$)

where $W(G/A(F)) =$ the Weyl group of $(G/A(F))$,

$$V_0(F) = \{s \in W(G/A(F)); sF \subseteq F \cup -F\},$$

$$\Delta(F;h) = \exp\left(\frac{1}{2} \sum_{\alpha \in P} \alpha\right)(\log h) \prod_{\alpha \in P} (1 - \exp(-\alpha(\log h))),$$

$$\xi_R(F;h) = \prod_{\substack{\alpha \in P \\ \alpha \neq 0 \text{ on } \mathfrak{g}_R(F)}} (1 - \exp(-\alpha(\log h))),$$

and $\xi(\alpha)$ = the signature of root α .

Theorem III. We keep the same conditions as in Theorem I.

Then $T_\Lambda = \sum_{s \in W(G/B) \setminus W} \oplus_{s\Lambda}$. Consequently, the right hand side in the equation of Theorem I obtains a global formula for the character $\sum_{s \in W(G/B) \setminus W} \oplus_{s\Lambda}$.

The formulae in Theorem I, II will be calculated out by using the certain properties for the groups $W(F)$, $V_0(F)$ ($F \in \overline{\mathbb{Q}}$).

The relation in Theorem III can be proved by using Theorem II, and by using the uniqueness of tempered invariant eigen-distributions in [1], the explicit formulae of $\oplus_{s\Lambda}$ ($s \in W$) on B' (cf. [3]).

For the global character formulae of discrete series representations, there were known the following cases; real rank one groups in [2], pp.120-122 (Harish-Chandra), indefinite unitary groups, $S_p(n, R)$ in [8], [9] (T. Hirai) respectively. On the other hand, [14] (W. Schmid) has given the explicit formulae of characters for discrete series representations on split real Cartan subgroup

of $S_p(n, \mathbb{R})$. This calculation is based on the relation of c) in Theorem (4.15), [15] (W. Schmid).

The same relation as in Theorem III have been observed by [12] for real rank one case, and by [18] (G. Zuckerman) for real rank one case, indefinite unitary groups. In essence, our direction of this paper is similar to the one in [12], [18]: Harish-Chandra has given general principle for these relations. However, the explicit relations are not completely known in general.

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