

SOME PROPERTIES OF CERTAIN ANALYTIC FUNCTIONS

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1. Introduction.

Let $A(p, n)$ denote the class of functions of the form

$$(1.1) \quad f(z) = z^p + \sum_{k=n}^{\infty} a_{p+k} z^{p+k} \quad (p, n \in \mathbb{N} = \{1, 2, 3, \dots\})$$

which are analytic in the unit disk $U = \{z: |z| < 1\}$.

Further, we define a function $F_{\lambda}(z)$ by

$$(1.2) \quad F_{\lambda}(z) = (1-\lambda)f(z) + \lambda z f'(z)$$

for $\lambda \geq 0$, and $f(z) \in A(p, n)$. A function $f(z)$ belonging to $A(p, 1) = A(p)$ is said to be in the class $P(p, \alpha)$ if and only if it satisfies

$$(1.3) \quad \operatorname{Re} \{ f^{(p)}(z) \} > \alpha$$

for some α ($0 \leq \alpha < p!$) and for all $z \in U$.

The Hadamard product or convolution of two power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is defined as the power series

$$(1.4) \quad (f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

Let the function $f(z)$ and $g(z)$ be analytic in U . Then the function $f(z)$ is said to be subordinate to $g(z)$ if there exists a function $w(z)$ analytic in U , with $w(0) = 0$ and $|w(z)| < 1$ ($z \in U$), such that

$$f(z) = g(w(z))$$

for $z \in U$. We denote the subordination by

$$(1.5) \quad f(z) \prec g(z).$$

2. Inequalities for functions in the class $A(p, n)$.

We begin with the statement of the following lemma due to Miller [1].

Lemma 1. Let $\phi(u, v)$ be a complex valued function such that

$$\phi : D \rightarrow \mathbb{C}, \quad D \subset \mathbb{C} \times \mathbb{C} \quad (\mathbb{C} \text{ is the complex plane}),$$

and let $u = u_1 + iu_2$, $v = v_1 + iv_2$. Suppose that the function $\phi(u, v)$ satisfies

- (i) $\phi(u, v)$ is continuous in D ,
- (ii) $(1, 0) \in D$ and $\operatorname{Re}\{\phi(1, 0)\} > 0$,
- (iii) for all $(iu_2, v_1) \in D$ such that $v_1 \leq -n(1+u_2^2)/2$, $\operatorname{Re}\{\phi(iu_2, v_1)\} \leq 0$.

Let $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \dots$ be regular in the unit disk U such

that $(p(z), zp'(z)) \in D$ for all $z \in U$. If

$$\operatorname{Re}\{\phi(p(z), zp'(z))\} > 0 \quad (z \in U),$$

then $\operatorname{Re}\{p(z)\} > 0 \quad (z \in U)$.

Applying the above lemma, we prove

Theorem 1. Let a function $f(z)$ defined by (1.1) be in the class $A(p, n)$.

If

$$\operatorname{Re}\left\{\frac{f^{(j)}(z)}{z^{p-j}}\right\} > \alpha \quad \left(0 \leq \alpha < \frac{p!}{(p-j)!}; z \in U\right),$$

then we have

$$\operatorname{Re}\left\{\frac{f^{(j-1)}(z)}{z^{p-j+1}}\right\} > \frac{1}{(p-j+1)!} \frac{(p-j+1)!2\alpha + np!}{\{2(p-j+1) + n\}} \quad (z \in U),$$

where $1 \leq j \leq p$.

Proof. We define the function $p(z)$ by

$$(2.1) \quad \frac{(p-j+1)!}{p!} \frac{f^{(j-1)}(z)}{z^{p-j+1}} = \beta + (1-\beta)p(z)$$

with $\beta = \frac{(p-j+1)!2\alpha + np!}{p!\{2(p-j+1) + n\}}$. Then $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \dots$ is

regular in U . Differentiating both sides in (2.1), we obtain

$$(2.2) \quad \frac{(p-j+1)!}{p!} f^{(j)}(z) = (p-j+1)\beta z^{p-j} + (p-j+1)(1-\beta)z^{p-j} p(z) \\ + (1-\beta)z^{p-j+1} p'(z)$$

and, by using (2.1) and (2.2), we have

$$(2.3) \quad (p-j+1)! \left\{ \frac{f^{(j)}(z)}{z^{p-j}} - \alpha \right\} = p!(p-j+1)\beta - (p-j+1)!\alpha \\ + p!(p-j+1)(1-\beta)p(z) + p!(1-\beta)zp'(z).$$

Hence, in view of $\operatorname{Re} \{f^{(j)}(z)/z^{p-j}\} > \alpha$, we have

$$(2.4) \quad \operatorname{Re} \{\phi(p(z), zp'(z))\} > 0,$$

where $\phi(u, v)$ is defined by

$$(2.5) \quad \phi(u, v) = p!(p-j+1)\beta - (p-j+1)!\alpha + p!(p-j+1)(1-\beta)u + p!(1-\beta)v$$

with $u = u_1 + iu_2$, $v = v_1 + iv_2$. Then we see that

- (i) $\phi(u, v)$ is continuous in $D = \mathbb{C} \times \mathbb{C}$,
- (ii) $(1, 0) \in D$ and $\operatorname{Re}\{\phi(1, 0)\} = (p-j+1)!\{p/(p-j)! - \alpha\} > 0$,
- (iii) for all $(iu_2, v_1) \in D$ such that $v_1 \leq -n(1+u_2^2)/2$,

$$\operatorname{Re}\{\phi(iu_2, v_1)\} = p!(p-j+1)\beta - (p-j+1)!\alpha + p!(1-\beta)v_1 \\ \leq p!(p-j+1)\beta - (p-j+1)!\alpha - \frac{np!(1-\beta)(1+u_2^2)}{2} \leq 0$$

for $\beta = \frac{(p-j+1)!2\alpha + np!}{p!\{2(p-j+1) + n\}} < 1$. Consequently, $\phi(u, v)$ satisfies the conditions

in lemma 1. Therefore, we have $\operatorname{Re} \{p(z)\} > 0$ ($z \in U$), that is,

$$\operatorname{Re} \left\{ \frac{f^{(j-1)}(z)}{z^{p-j+1}} \right\} > \frac{p!}{(p-j+1)!} \beta = \frac{1}{(p-j+1)!} \frac{(p-j+1)!2\alpha + np!}{\{2(p-j+1) + n\}}$$

which completes the proof of Theorem 1.

Taking $n = 1$ in Theorem 1, we have

Corollary 1. Let $f(z) \in A(p) = A(p, 1)$ and suppose

$$\operatorname{Re} \left\{ \frac{f^{(j)}(z)}{z^{p-j}} \right\} > \alpha \quad \left(0 \leq \alpha < \frac{p!}{(p-j)!} ; z \in U \right).$$

Then we have

$$\operatorname{Re} \left\{ \frac{f^{(j-1)}(z)}{z^{p-j+1}} \right\} > \frac{1}{(p-j+1)!} \frac{(p-j+1)!2\alpha + p!}{2(p-j) + 3} \quad (z \in U),$$

where $1 \leq j \leq p$.

Corollary 1 is the result by Saitoh [5].

Next, we prove

Theorem 2. Let a function $F_\lambda(z)$ defined by (1.2) for $\lambda \geq 0$ and $f(z) \in A(p, n)$.

If

$$\operatorname{Re} \left\{ \frac{F_\lambda^{(j)}(z)}{z^{p-j}} \right\} > \alpha \quad \left(0 \leq \alpha < \frac{p!(1-\lambda+p\lambda)}{(p-j)!} ; z \in U \right),$$

then

$$\operatorname{Re} \left\{ \frac{f^{(j)}(z)}{z^{p-j}} \right\} > \frac{(p-j)!2\alpha + np!\lambda}{(p-j)!\{2 + (2p+n-2)\lambda\}} \quad (z \in U),$$

where $0 \leq j \leq p$.

Proof. By the differentiation of $F_\lambda(z)$, we obtain

$$(2.6) \quad F_\lambda^{(j)}(z) = (1-\lambda+\lambda j)f^{(j)}(z) + \lambda z f^{(j+1)}(z).$$

We define the function $p(z)$ by

$$(2.7) \quad \frac{(p-j)!}{p!} \frac{f^{(j)}(z)}{z^{p-j}} = \beta + (1-\beta)p(z)$$

with $\beta = \frac{(p-j)!2\alpha + np!\lambda}{p!\{2+(2p+n-2)\lambda\}}$ ($0 \leq \beta < 1$). Then $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \dots$

is regular in U . Making the differentiation in (2.7), we have

$$(2.8) \quad \frac{z f^{(j+1)}(z)}{z^{p-j}} - \frac{p!}{(p-j+1)!} \{\beta + (1-\beta)p(z)\} = \frac{p!}{(p-j)!} (1-\beta) z p'(z).$$

By using (2.6), (2.7) and (2.8), we obtain

$$(2.9) \quad \frac{F_\lambda^{(j)}(z)}{z^{p-j}} - \alpha = \frac{p!(1-\lambda+p\lambda)}{(p-j)!} \beta - \alpha + \frac{p!(1-\lambda+p\lambda)(1-\beta)}{(p-j)!} p(z) + \frac{p!\lambda(1-\beta)}{(p-j)!} z p'(z).$$

Hence, in view of $\operatorname{Re} \{F_\lambda^{(j)}(z)/z^{p-j}\} > \alpha$, we have

$$(2.10) \quad \operatorname{Re} \{\phi(p(z), z p'(z))\} > 0,$$

where $\phi(u, v)$ is defined by

$$(2.11) \quad \phi(u, v) = \frac{p!(1-\lambda+p\lambda)}{(p-j)!} \beta - \alpha + \frac{p!(1-\lambda+p\lambda)(1-\beta)}{(p-j)!} u + \frac{p!\lambda(1-\beta)}{(p-j)!} v$$

with $u = u_1 + iu_2$ and $v = v_1 + iv_2$. Then we see that

(i) $\phi(u, v)$ is continuous in $\mathbb{D} = \mathbb{C} \times \mathbb{C}$,

(ii) $(1, 0) \in \mathbb{D}$ and $\operatorname{Re}\{\phi(1, 0)\} = \frac{p!(1-\lambda+p\lambda)}{(p-j)!} - \alpha > 0$,

(iii) for all $(iu_2, v_1) \in \mathbb{D}$ such that $v_1 \leq -n(1+u_2^2)/2$

$$\begin{aligned} \operatorname{Re}\{\phi(iu_2, v_1)\} &= \frac{p!(1-\lambda+p\lambda)}{(p-j)!} \beta - \alpha + \frac{p!\lambda(1-\beta)}{(p-j)!} v_1 \\ &\leq \frac{p!(1-\lambda+p\lambda)}{(p-j)!} \beta - \alpha - \frac{np!\lambda(1-\beta)(1+u_2^2)}{2(p-j)!} \leq 0 \end{aligned}$$

for $\beta = \frac{(p-j)!2\alpha + np!\lambda}{p!\{2 + (2p+n-2)\lambda\}}$. Consequently, $\phi(u, v)$ satisfies the conditions

in lemma 1. Therefore, we have

$$\operatorname{Re}\{p(z)\} > 0 \quad (z \in U), \text{ that is,}$$

$$\operatorname{Re}\left\{\frac{f^{(j)}(z)}{z^{p-j}}\right\} > \frac{p!}{(p-j)!} \beta = \frac{(p-j)!2\alpha + np!\lambda}{(p-j)!\{2 + (2p+n-2)\lambda\}}$$

which completes the assertion of Theorem 2.

Making $n = 1$ in Theorem 2, we have

Corollary 2. Let a function $F_\lambda(z)$ defined by (1.2) for $\lambda \geq 0$ and $f(z) \in A(p) = A(p, 1)$. If

$$\operatorname{Re}\left\{\frac{F_\lambda^{(j)}(z)}{z^{p-j}}\right\} > \alpha \quad \left(0 \leq \alpha < \frac{p!(1-\lambda+p\lambda)}{(p-j)!}; z \in U\right),$$

then

$$\operatorname{Re}\left\{\frac{f^{(j)}(z)}{z^{p-j}}\right\} > \frac{(p-j)!2\alpha + p!\lambda}{(p-j)!(2-\lambda+2p\lambda)} \quad (z \in U),$$

where $0 \leq j \leq p$.

Corollary 2 is the result by Saitoh [5].

3. Some properties of the class $P(p, \alpha)$.

For giving some results in this part, we need the following lemma of Ruscheweyh and Sheil-Small [4].

Lemma 2. Let $F(z)$ and $G(z)$ be convex in U and

$$f(z) \prec F(z) . \quad \text{Then}$$

$$f * G(z) \prec F * G(z) .$$

With the aid of above Lemma 2, we prove

Theorem 3. Let a function $f(z)$ defined by (1.3) be in the class $P(p, \alpha)$.

Then we have

$$\frac{f^{(p-1)}(z)}{z} \prec 2\alpha - p! - \frac{2(p!-\alpha)}{z} \log(1-z) .$$

Proof. Define the function $G^{(p-1)}(z)$ by

$$G^{(p)}(z) = \frac{p! + (p!-2\alpha)z}{1-z}$$

and $G^{(p-1)}(0) = 0$. Then it follows that

$$\frac{G^{(p-1)}(z)}{z} = 2\alpha - p! - \frac{2(p!-\alpha)}{z} \log(1-z) .$$

Noting $f(z) \in P(p, \alpha)$, we see that

$$f^{(p)}(z) \prec G^{(p)}(z) .$$

Defining the function $k(z)$ by

$$(3.1) \quad k(z) = -\frac{\log(1-z)}{z} = \sum_{n=0}^{\infty} \frac{1}{n+1} z^n ,$$

we have

$$\frac{f^{(p-1)}(z)}{z} = k * f^{(p)}(z) \text{ and } \frac{G^{(p-1)}(z)}{z} = k * G^{(p)}(z) .$$

Further, $k(z)$ is convex and univalent in U , and $G^{(p)}(z)$ is also convex and univalent in U . Therefore, using Lemma 2, we have

$$k * f^{(p)}(z) \prec k * G^{(p)}(z) , \quad \text{that is,}$$

$$\frac{f^{(p-1)}(z)}{z} \prec \frac{G^{(p-1)}(z)}{z} = 2\alpha - p! - \frac{2(p!-\alpha)}{z} \log(1-z) .$$

Thus we complete the proof of Theorem 3.

Next, we prove

Corollary 3. Let $f(z) \in P(p, \alpha)$, Then we have

$$\operatorname{Re} \left\{ \frac{f^{(p-1)}(z)}{z} \right\} > 2\alpha - p! + 2(p! - \alpha) \log 2.$$

Proof. Since the function $k(z)$ defined by (3.1) is convex and univalent in U , the function $G_1(z)$ given by

$$G_1(z) = 2\alpha - p! - \frac{2(p! - \alpha)}{z} \log(1-z)$$

is also convex and univalent in U . Therefore, by the principle of the subordination, we have

$$\operatorname{Re} \left\{ \frac{f^{(p-1)}(z)}{z} \right\} > \inf_{|z| < 1} \operatorname{Re} \{G_1(z)\}.$$

We note that $G_1(U)$ is symmetric with respect to the real axis because all coefficients of $G_1(z)$ are real. Noting that $G_1(U)$ is convex, we obtain

$$\inf_{|z| < 1} \operatorname{Re} \{G_1(z)\} = \inf_{-1 < x < 1} G_1(x) = 2\alpha - p! + 2(p! - \alpha) \log 2,$$

which proves the assertion of Corollary 3.

Putting $p = 1$ in Theorem 3 and Corollary 3, we have the following corollaries which were proved by Owa, Ma and Liu [2].

Corollary 4. Let $f(z) \in A(1)$. If the function $f(z)$ is in the class $P(1, \alpha) = P(\alpha)$, then

$$\frac{f(z)}{z} < 2\alpha - 1 - \frac{2(1-\alpha)}{z} \log(1-z).$$

Corollary 5. If $f(z) \in P(\alpha)$, then

$$\operatorname{Re} \left\{ \frac{f(z)}{z} \right\} > 2\alpha - 1 + 2(1-\alpha) \log 2.$$

References

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