

## A Survey of [AJS]

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This is a survey of the work [AJS] by H.H. Andersen, J.C. Jantzen and W. Soergel. There are also excellent expositions by the authors [A2], [S1], [S2], of which [A2] includes the entire aspect of Lusztig's program.

During the AMS Summer Institute 1986 at Arcata I had an opportunity to ask G. Lusztig how he had come to his conjectural formula [L1] that should describe the irreducible characters of simple  $\mathbb{F}_p$ -groups in terms of the Kazhdan-Lusztig polynomials. He kindly explained me the idea, that is in [H], and said it would be easier to relate the conjecture to his analogous conjecture for affine Kac-Moody Lie algebras than to derive the exact formula in the category of modules for the  $\mathbb{F}_p$ -groups or for their infinitesimal subgroups.

Meanwhile, quantized enveloping algebras were discovered by V. G. Drinfeld and Jimbo M. Their representation theory at roots of 1 has subsequently been related to that of affine Kac-Moody Lie algebras by D. Kazhdan and Lusztig [KL1, 2] and [L4], to the former [AJS] has related the representation theory of simple  $\mathbb{F}_p$ -groups, and Lusztig's conjectural formula for affine Kac-Moody Lie algebras has been verified by Kashiwara M. and Tanisaki T. [KT]. Altogether Lusztig's conjectural modular irreducible character formula is now proved to hold for large  $p$  and in type  $A$ ,  $D$ , and  $E$ .

The morphism spaces of modules for simple  $\mathbb{F}_p$ -groups are  $\mathbb{F}_p$ -linear whereas those for quantized enveloping algebras over cyclotomic fields  $\mathbb{Q}(\zeta)$  are  $\mathbb{Q}(\zeta)$ -linear, hence one cannot hope to have an equivalence between these categories. Neither is  $\mathbb{F}_p$  flat over  $\mathbb{Z}$ . In order to overcome the difficulties,

[AJS] works not over  $\mathbb{F}_p$ ,  $\mathbb{Q}(\zeta)$  or  $\mathbb{Z}$ , but over various localizations of the completions of the Cartan part of the universal enveloping algebra of the Lie algebra of the  $\mathbb{F}_p$ -group and of the quantized enveloping algebra over  $\mathbb{Q}(\zeta)$ , introduces certain combinatorial categories over these algebras and finally over the symmetric algebra of the root lattice, then applies some standard techniques of finite dimensional algebras.

### a° The problem

(a1) Let us first fix the notations.

$R$  an irreducible root system with the set of coroots  $R^\vee$

$R^+$  a positive system of  $R$

$\Sigma$  the simple system of  $R^+$

$X$  the weight lattice of  $R$

$X^+$  the set of dominant weights of  $X$

$\geq$  the standard partial order on  $X$  such that  $\lambda \geq \mu$  iff  $\lambda - \mu \in \sum_{\alpha \in R^+} \mathbb{N}\alpha$

$W$  the Weyl group of  $R$

$W_a = W \ltimes \mathbb{Z}$  the affine group of  $W$

$\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$

$\alpha_0$  the dominant short root of  $R^+$

$h = \langle \rho, \alpha_0^\vee \rangle + 1$  the Coxeter number of  $R$

$(d_\alpha)_{\alpha \in \Sigma} \in \{1, 2, 3\}^\Sigma$  minimal such that  $[(d_\alpha \langle \beta, \alpha^\vee \rangle)]_{\alpha, \beta \in \Sigma}$  is symmetric

(a2) Let  $k = \mathbb{F}_p$  the prime field of characteristic  $p > 0$ , and  $\mathfrak{G}_k$  the simply connected simple  $k$ -group with a maximal torus  $\mathfrak{T}_k$  split over  $\mathbb{Z}$  and the associated root system  $R$ . We will identify  $X$  with the weight group  $\text{Grp}_k(\mathfrak{T}_k, \mathfrak{GL}_1)$  of  $\mathfrak{T}_k$ .

If  $M$  is a  $\mathfrak{T}_k$ -module,  $M$  admits a weight space decomposition  $M = \coprod_{\lambda \in X} M_\lambda$  with  $M_\lambda = \{m \in M \mid t(m \otimes 1) = m \otimes \lambda(t) \text{ in } M \otimes A \ \forall A \in \mathbf{Alg}_k \text{ and } t \in \mathfrak{T}_k(A)\}$ , where  $\mathbf{Alg}_k$  denotes the category of commutative  $k$ -algebras. One

calls  $\lambda \in X$  a weight of  $M$  iff  $M_\lambda \neq 0$ . Set  $\text{ch}M = \sum_{\lambda \in X} (\dim M_\lambda) e(\lambda)$ , called the character of  $M$ , in the group algebra  $\mathbb{Z}[X]$  of  $X$  with the natural basis  $e(\lambda)$ ,  $\lambda \in X$ .

There is a bijection, due to C. Chevalley [J], (II.2.4), between  $X^+$  and the set of the isomorphism classes of the simple  $\mathfrak{G}_k$ -modules such that

$$(1) \quad \lambda \longmapsto L(\lambda)_k \quad \text{simple of highest weight } \lambda.$$

The fundamental problem in the representation theory of  $\mathfrak{G}_k$  has been to find all  $\text{ch}L(\lambda)_k$ .

(a3) Let  $\mathfrak{F}_{\mathfrak{G}} : \mathfrak{G}_k \rightarrow \mathfrak{G}_k$  be the Frobenius endomorphism of  $\mathfrak{G}_k$ . Let  $X_k = \{\mu \in X^+ \mid \langle \mu, \alpha^\vee \rangle \leq p-1 \ \forall \alpha \in \Sigma\}$ . If  $\lambda = \lambda^0 + p\lambda^1$  with  $\lambda^0 \in X_k$  and  $\lambda^1 \in X^+$ , Steinberg's tensor product theorem says

$$L(\lambda)_k \simeq L(\lambda^0)_k \otimes_k L(\lambda^1)_k^{[1]} \quad \text{in } \mathfrak{G}_k\mathbf{Mod},$$

where  $L(\lambda^1)_k^{[1]}$  is the composite of the representation  $L(\lambda^1)_k$  with  $\mathfrak{F}_{\mathfrak{G}}$ . Hence we have only to find all  $\text{ch}L(\lambda)_k$ ,  $\lambda \in X_k$ .

(a4) Let  $\mathfrak{G}_1 = \ker \mathfrak{F}_{\mathfrak{G}}$  the Frobenius kernel of  $\mathfrak{G}_k$ . It is an infinitesimal subgroup of  $\mathfrak{G}_k$  defined by the Hopf algebra  $k[\mathfrak{G}]/\mathfrak{m}_k^p$ , where  $k[\mathfrak{G}]$  is the Hopf algebra of  $\mathfrak{G}_k$  with the augmentation ideal  $\mathfrak{m}_k$ . Due to C.W. Curtis [J], (II.3.15),

$$(1) \quad L(\lambda)_k, \lambda \in X_k, \text{ remains simple as } \mathfrak{G}_1\text{-module.}$$

In order to keep track of the weights, however, we will work in the category of  $\mathfrak{G}_1\mathfrak{I}_k$ -modules.

In  $\mathfrak{G}_1\mathfrak{I}_k\mathbf{Mod}$  the simples are still parametrized by their highest weights, varying though over the entire  $X$ . We will denote the simple of highest weight  $\mu \in X$  in  $\mathfrak{G}_1\mathfrak{I}_k\mathbf{Mod}$  by  $L_k(\mu)$ . Then

$$(2) \quad L_k(\mu) \simeq L(\mu^0)_k \otimes_k p\mu^1 \quad \text{with} \quad p\mu^1 = (\mu^1)^{[1]}.$$

(a5) Let  $\lambda \in X$ . If  $\mathfrak{B}_k$  is the Borel subgroup of  $\mathfrak{G}_k$  whose roots are  $-R^+$ , regard  $\lambda$  as a  $\mathfrak{B}_k$ -module via the projection  $\mathfrak{B}_k \rightarrow \mathfrak{I}_k$ , and let

$$\begin{aligned} \hat{Z}_k(\lambda) &= \{f \in \mathbf{Sch}_k(\mathfrak{G}_1\mathfrak{I}_k, \mathbb{A}^1) \mid f(A)(xb) = (\lambda(A)(b))^{-1} f(A)(x) \\ &\quad \forall x \in \mathfrak{G}_1\mathfrak{I}_k(A), b \in \mathfrak{B}_1\mathfrak{I}_k(A), A \in \mathbf{Alg}_k\}, \end{aligned}$$

that is just the  $\mathfrak{G}_1\mathfrak{T}_k$ -module of global sections of the invertible sheaf on the quotient  $\mathfrak{G}_1\mathfrak{T}_k/\mathfrak{B}_1\mathfrak{T}_k$  induced by the  $\mathfrak{B}_1\mathfrak{T}_k$ -module  $\lambda$ , where  $\mathfrak{B}_1$  is the Frobenius kernel of  $\mathfrak{B}_k$  and  $\mathbf{Sch}_k$  denotes the category of  $k$ -schemes. The  $\mathfrak{G}_1\mathfrak{T}_k$ -module structure is given by  $xf = f(x^{-1}?)$ . Regarded as a functor  $\mathfrak{B}_1\mathfrak{T}_k\mathbf{Mod} \rightarrow \mathfrak{G}_1\mathfrak{T}_k\mathbf{Mod}$   $\hat{Z}_k$  is exact, that makes the representation theory of  $\mathfrak{G}_1\mathfrak{T}_k$  more algebraic than that of  $\mathfrak{G}_k$ . One has

$$\mathrm{ch}\hat{Z}_k(\lambda) = e(\lambda) \prod_{\alpha \in R^+} \frac{1 - e(-p\alpha)}{1 - e(-\alpha)},$$

hence the composition factor multiplicity  $[\hat{Z}_k(\lambda) : L_k(\lambda)] = 1$ , and all the other composition factors of  $\hat{Z}_k(\lambda)$  have highest weights  $< \lambda$ . It follows that the determination of  $\mathrm{ch}L_k(\lambda)$  is now reduced to counting the decomposition numbers  $[\hat{Z}_k(\lambda) : L_k(\mu)]$  for all  $\lambda, \mu \in X$ .

(a6) Define a partition of  $X$  into disjoint subsets, called the blocks of  $\mathfrak{G}_1\mathfrak{T}_k\mathbf{Mod}$ , to be the finest partition such that  $\lambda$  and  $\mu$  belong to the same block if  $\mathrm{Ext}_{\mathfrak{G}_1\mathfrak{T}_k}^1(L_k(\lambda), L_k(\mu)) \neq 0$ . The linkage principle [J], (II.6.17) says

(1) *each block is contained in a  $W_a$ -orbit,*

where we let  $W_a$  act on  $X$  by  $\gamma w \cdot_k \lambda = w(\lambda + \rho) - \rho + p\gamma$ ,  $\gamma \in \mathbb{Z}R$ ,  $w \in W$ , and  $\lambda \in X$ .

If  $b$  is a block of  $\mathfrak{G}_1\mathfrak{T}_k\mathbf{Mod}$ , denote by  $\mathfrak{G}_1\mathfrak{T}_k(b)$  the full subcategory of  $\mathfrak{G}_1\mathfrak{T}_k\mathbf{Mod}$  consisting of all modules whose composition factors are of the form  $L_k(\lambda)$ ,  $\lambda \in b$ . If  $\Omega$  and  $\Gamma$  are two  $W_a$ -orbits in  $X$ , one has an exact functor

$$T_{\Omega}^{\Gamma} : \coprod_{b \subseteq \Omega} \mathfrak{G}_1\mathfrak{T}_k(b) \longrightarrow \coprod_{b \subseteq \Gamma} \mathfrak{G}_1\mathfrak{T}_k(b),$$

called the translation functor from  $\Omega$  to  $\Gamma$ , that is both left and right adjoint to the translation functor  $T_{\Gamma}^{\Omega}$  [J], (II.7).

(a7) Let  $\mathfrak{A}_k = \{x \in X \otimes_{\mathbb{Z}} \mathbb{R} \mid 0 < \langle x + \rho, \alpha_0^{\vee} \rangle < p \ \forall \alpha \in R^+\}$ . The  $W_a$ -translates of  $\mathfrak{A}_k$  are called alcoves. In particular,  $\mathfrak{A}_k$  is called the bottom dominant alcove. One has

$$\mathfrak{A}_k \cap X \neq \emptyset \quad \text{iff} \quad 0 \in \mathfrak{A}_k \quad \text{iff} \quad p \geq h.$$

Assume from now on that  $0 \in \mathfrak{A}_k$  throughout the rest of the survey.

Let  $W_a^+ = \{w \in W_a \mid w \cdot_k 0 \in X^+\}$  and  $W_1 = \{w \in W_a \mid w \cdot_k 0 \in X_k\}$ . Note that both  $W_a^+$  and  $W_1$  are independent of  $k$ .

As  $\hat{Z}_k(\lambda)$  is indecomposable, one can write by the linkage principle

$$\text{ch} L_k(\lambda) = \sum_{w \in W_a} a_{\lambda w} \text{ch} \hat{Z}_k(w \cdot_k \lambda), \quad a_{\lambda w} \in \mathbb{Z}.$$

If  $\mu$  belong to the “upper closure” of the alcove of  $\lambda$ , then the translation principle [J], (II.7.17)(b) yields

$$(1) \quad \text{ch} L_k(\mu) = \sum_{w \in W_a} a_{\lambda w} \text{ch} \hat{Z}_k(w \cdot_k \mu).$$

Also  $\hat{Z}_k(\lambda + p\mu) = \hat{Z}_k(\lambda) \otimes_k p\nu \quad \forall \nu \in X$ , hence together with (a4)(2)

$$(2) \quad [\hat{Z}_k(\lambda + p\nu) : L_k(\eta + p\nu)] = [\hat{Z}_k(\lambda) : L_k(\eta)].$$

As any weight belongs to the upper closure of an alcove, for  $p \geq h$  the problem is now reduced to counting all

$$(3) \quad [\hat{Z}_k(w \cdot_k 0) : L_k(w' \cdot_k 0)], \quad w \in W_a, w' \in W_1.$$

(a8) One says a  $\mathfrak{G}_1 \mathfrak{I}_k$ -module admits a  $\hat{Z}_k$ -filtration if it has a filtration in  $\mathfrak{G}_1 \mathfrak{I}_k \mathbf{Mod}$  with the factors of the form  $\hat{Z}_k(\nu)$ ,  $\nu \in X$ .

Let  $Q_k(\lambda)$  be the projective cover of  $L_k(\lambda)$ ,  $\lambda \in X$ , in  $\mathfrak{G}_1 \mathfrak{I}_k \mathbf{Mod}$ . The Brauer-Humphreys reciprocity [J], (II.11.4) says

$$(1) \quad Q_k(\lambda) \text{ admits a } \hat{Z}_k\text{-filtration}$$

and that the multiplicities in the  $\hat{Z}_k$ -filtration are given by

$$(2) \quad [Q_k(\lambda) : \hat{Z}_k(w \cdot_k \lambda)] = [\hat{Z}_k(w \cdot_k \lambda) : L_k(\lambda)],$$

where the factors of the filtration must be of the form  $\hat{Z}_k(w \cdot_k \lambda)$ ,  $w \in W_a$ , by the linkage principle. Hence the problem is further reduced to finding the multiplicities in  $\hat{Z}_k$ -filtrations

$$(3) \quad [Q_k(w \cdot_k \lambda) : \hat{Z}_k(w' \cdot_k \lambda)] \quad \forall w \in W_1, w' \in W_a.$$

(a9) Let  $\Omega_0 = W_a \cdot_k 0$  and  $\lambda \in \Omega_0$ . In one case the  $\hat{Z}_k$ -filtration of  $Q_k(\lambda)$

is well-understood. The Steinberg module  $\hat{Z}_k((p-1)\rho) = L_k((p-1)\rho) = L((p-1)\rho)_k$  is a projective indecomposable [J], (II.10.2), hence also

$$\hat{Z}_k((p-1)\rho + p\nu) \simeq \hat{Z}_k((p-1)\rho) \otimes_k p\nu \quad \forall \nu \in X.$$

If  $\lambda$  lies in the top alcove of the box  $p\lambda^1 + X_k$ , then [J], (II.11.10)

$$(1) \quad Q_k(\lambda) = T_{W_{a \cdot k}((p-1)\rho + p\lambda^1)}^{\Omega_0} \hat{Z}_k((p-1)\rho + p\lambda^1),$$

in a  $\hat{Z}_k$ -filtration of which all  $\hat{Z}_k(w \cdot_k \lambda^0 + p(\rho - w\rho + \lambda^1))$ ,  $w \in W$ , appear exactly once. More generally [J], (II.9.19),

(a10) **Lemma.** *Let  $\lambda, \mu \in X$  belonging to the closure of an alcove. Then  $T_{W_{a \cdot k}\lambda}^{W_{a \cdot k}\mu} \hat{Z}_k(\lambda)$  has a  $\hat{Z}_k$ -filtration with the factors*

$$\hat{Z}_k(w \cdot_k \mu), \quad w \in C_{W_a}(\lambda)/C_{W_a}(\lambda) \cap C_{W_a}(\mu),$$

*each appearing exactly once.*

(a11) Let  $\Sigma_a$  be the set of reflexions of  $W_a$  in a wall of  $\mathfrak{A}_k$ , that is independent of  $k$ . If  $s \in \Sigma_a$ , choose  $\mu_s \in X \cap \overline{\mathfrak{A}_k}$  with  $C_{W_a}(\mu_s) = \{1, s\}$ , and set  $T_s = T_{\Omega_0}^{W_{a \cdot k}\mu_s}$ ,  $T'_s = T_{W_{a \cdot k}\mu_s}^{\Omega_0}$ , and  $\Theta_s = T_s \circ T'_s$ .

For  $\lambda \in \Omega_0$  define a sequence  $I = (s_1, \dots, s_r)$  of elements of  $\Sigma_a$  inductively as follows. If  $\lambda$  lies in the top alcove of the box  $p\lambda^1 + X_k$ , take  $I = \emptyset$ . Otherwise choose  $s_1 \in \Sigma_a$  such that  $\lambda < ws_1 \cdot_k 0$  if  $\lambda = w \cdot_k 0$ ,  $w \in W_a$ , and that  $ws_1 \cdot_k 0 \in p\lambda^1 + X_k$ . Now set

$$Q_k^I(\lambda) = \Theta_{s_1} \circ \dots \circ \Theta_{s_r} Q_k^\emptyset(\lambda)$$

with  $Q_k^\emptyset(\lambda) = T_{W_{a \cdot k}((p-1)\rho + p\lambda^1)}^{\Omega_0} \hat{Z}_k((p-1)\rho + p\lambda^1)$ . From (a10) we know the  $\hat{Z}_k$ -filtration of  $Q_k^I(\lambda)$ . On the other hand, if  $\hat{\lambda} = w_0 \cdot_k \lambda^0 + p(\lambda^1 + 2\rho)$ ,

$$(1) \quad Q_k^I(\lambda) = \prod_{\substack{\nu \in \Omega_0 \\ \lambda \uparrow \nu \uparrow \hat{\lambda}}} Q_k(\nu)^{m_k(\lambda, \nu)} \quad \text{with} \quad m_k(\lambda, \lambda) = 1,$$

where  $\uparrow$  is a partial order on  $X$  such that  $\nu \uparrow \nu'$  if  $\nu' = s_\beta \cdot_k \nu + pm\beta \geq \nu$  for some  $\beta \in R^+$  and  $m \in \mathbb{Z}$  [J], (II.11.6).

As the  $Q_k(\nu)$  are linearly independent, the  $m_k(\lambda, \nu)$  are uniquely determined. Then by induction on  $\hat{\lambda} - \lambda$  finding all  $m_k(\lambda, \nu)$  will determine the

$\hat{Z}_k$ -filtration of each  $Q_k(\nu)$ ,  $\nu \in \Omega_0$ .

(a12) The set of  $w \in W_a$  with

$$0 \uparrow w \cdot_k 0 \uparrow \widehat{w \cdot_k 0} \uparrow \hat{0} = 2(p-1)\rho$$

is finite and independent of  $k$ . Enumerate those  $w_1, \dots, w_{n_0}$  such that if  $w_i \cdot_k 0 \uparrow w_j \cdot_k 0 \uparrow \widehat{w_j \cdot_k 0} \uparrow \widehat{w_i \cdot_k 0}$ , then  $j \leq i$ . Note that

$$W_1 \subseteq \{w_1, \dots, w_{n_0}\}.$$

For each  $w_i \cdot_k 0$ ,  $i \in [1, n_0]$ , choose a sequence  $I(i)$  as in (a11) and set  $Q^{[i]}(k) = Q_k^{I(i)}(w_i \cdot_k 0)$ . Then

$$(1) \quad Q^{[i]}(k) = \prod_{j=1}^i Q_k(w_j \cdot_k 0)^{m_k(j,i)} \quad \text{with} \quad m_k(i,i) = 1.$$

Set  $Q(k) = \prod_{i=1}^{n_0} Q^{[i]}(k)$  and let

$$\mathcal{E}_{[i],[j]}(k) = \mathfrak{G}_1 \mathfrak{T}_k \mathbf{Mod}(Q^{[i]}(k), Q^{[j]}(k)), \quad \mathcal{E}(k) = \mathfrak{G}_1 \mathfrak{T}_k \mathbf{Mod}(Q(k), Q(k)).$$

Then  $\mathcal{E}(k) = \prod_{i,j \in [1, n_0]} \mathcal{E}_{[i],[j]}(k)$ . Under the composition each  $\mathcal{E}(k)_{[i],[i]}$  and  $\mathcal{E}(k)$  form finite dimensional  $k$ -algebras.

Let  $1 = \sum_{n \in E_k(i)} e_k^n(i)$  be a decomposition into orthogonal primitive idempotents in  $\mathcal{E}(k)_{[i],[i]}$ , where  $E_k(i)$  is an indexing set with  $e_k^0(i)$  corresponding to  $Q_k(w_i \cdot_k 0)$ , i.e.,  $Q_k(w_i \cdot_k 0) \simeq e_k^0(i)Q^{[i]}(k)$ . Then  $1 = \sum_{i=1}^{n_0} \sum_{n \in E_k(i)} e_k^n(i)$  is a decomposition into orthogonal primitive idempotents in  $\mathcal{E}(k)$ . Now

$$(2) \quad \begin{aligned} &e_k^n(i) \text{ is conjugate to } e_k^m(j) \text{ in } \mathcal{E}(k), \text{ i.e., there is some } u \in \mathcal{E}(k)^\times \\ &\text{with } e_k^n(i) = u e_k^m(j) u^{-1}, \text{ iff } \mathcal{E}(k) e_k^n(i) \simeq \mathcal{E}(k) e_k^m(j) \text{ in } \mathcal{E}(k) \mathbf{Mod} \text{ iff} \\ &e_k^n(i) Q(k) \simeq e_k^m(j) Q(k) \text{ in } \mathfrak{G}_1 \mathfrak{T}_k \mathbf{Mod}. \end{aligned}$$

Hence if  $n \neq 0$ ,  $e_k^n(i)$  is conjugate to some  $e_k^0(j)$  for  $j < i$  while  $e_k^0(i)$  is not conjugate to any of  $e_k^m(j)$ ,  $m \in E_k(j)$  with  $j < i$ . It follows that

$$(3) \quad m_k(j,i) = \#\{s \in E_k(i) \mid E_k^s(i) \text{ is conjugate to } e_k^0(j) \text{ in } \mathcal{E}(k)\}.$$

(a13) By transferring from  $\mathfrak{G}_k \mathbf{Mod}$  to  $\mathfrak{G}_1 \mathfrak{T}_k \mathbf{Mod}$  one has obtained finite

dimensional projectives (in  $\mathfrak{G}_k\mathbf{Mod}$  there are no finite dimensional injectives nor projectives), and the translations in  $W_a$  have been reflected in a simple manner: for each  $\lambda$  and  $\nu \in X$ ,

$$L_k(\lambda + p\nu) \simeq L_k(\lambda) \otimes_k p\nu, \quad \hat{Z}_k(\lambda + p\nu) \simeq \hat{Z}_k(\lambda) \otimes_k p\nu,$$

$$\text{and } Q_k(\lambda + p\nu) \simeq Q_k(\lambda) \otimes_k p\nu.$$

In characteristic 0 similar phenomenon occurs with the quantized enveloping algebra.

Let  $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$  with  $v$  an indeterminate and  $U(\mathcal{A})$  Lusztig's  $\mathcal{A}$ -form of the Drinfeld-Jimbo quantized enveloping algebra over  $\mathbb{Q}(v)$  [L3]. Let  $\ell \in \mathbb{N}^+$  prime to the nonzero entries of the Cartan matrix of  $R$ ,  $\zeta$  a primitive  $\ell$ -th root of 1 in  $\mathbb{C}$ ,  $\kappa = \mathbb{Q}(\zeta)$ , and  $U(\kappa) = U(\mathcal{A}) \otimes_{\mathcal{A}} \kappa$ . Lusztig has discovered a characteristic 0 analogue of the Frobenius kernel in  $U(\kappa)$ , that is an  $\ell^{|R|}(2\ell)^{|\Sigma|}$ -dimensional subalgebra  $\mathfrak{u}(\kappa)$  of  $U(\kappa)$  generated by  $E_{\pm\alpha}$ ,  $K_{\alpha}$ ,  $\alpha \in \Sigma$ . Let  $\mathcal{C}_{U(\kappa)}$  be the category of finite dimensional  $U(\kappa)$ -modules with  $K_{\alpha}^{\ell}$  acting by 1 for each  $\alpha \in \Sigma$ . One has  $K_{\alpha}^{2\ell} = 1$  in  $U(\kappa)$ . Then (cf. [APW1], (9.12); if  $\ell$  is not a prime power, one argues as in [AW]) each  $M \in \mathcal{C}_{U(\kappa)}$  admits a weight space decomposition with respect to the Cartan subalgebra  $U^0(\kappa) = U^0(\mathcal{A}) \otimes_{\mathcal{A}} \kappa$  with  $U^0(\mathcal{A})$  the  $\mathcal{A}$ -subalgebra of  $U(\mathcal{A})$  generated by  $K_{\alpha}^{\pm 1}$  and  $[K_{\alpha}] = \prod_{i=1}^m \frac{K_{\alpha} v^{d_{\alpha}(-i+1)} - K_{\alpha}^{-1} v^{-d_{\alpha}(-i+1)}}{v^{d_{\alpha}i} - v^{-d_{\alpha}i}} \otimes 1$ ,  $\alpha \in \Sigma$ ,  $m \in \mathbb{N}$ :

$$(1) \quad M = \coprod_{\lambda \in X} M_{\lambda} \quad \text{with} \quad M_{\lambda} = \{m \in M \mid um = \lambda(u)m \ \forall u \in U^0(\kappa)\},$$

$$\text{where } \lambda(K_{\alpha}) = \zeta^{d_{\alpha}\langle \lambda, \alpha^{\vee} \rangle} \quad \text{and} \quad \lambda([K_{\alpha}]) = \left[ \begin{smallmatrix} \langle \lambda, \alpha^{\vee} \rangle \\ m \end{smallmatrix} \right]_{d_{\alpha}} \quad \text{with} \quad \left[ \begin{smallmatrix} r \\ m \end{smallmatrix} \right]_{d_{\alpha}} = \prod_{i=1}^m \frac{v^{d_{\alpha}(r-i+1)} - v^{-d_{\alpha}(r-i+1)}}{v^{d_{\alpha}i} - v^{-d_{\alpha}i}} \otimes 1.$$

The simples of  $\mathcal{C}_{U(\kappa)}$  are parametrized by their highest weights in  $X^+$  as in  $\mathfrak{G}_k\mathbf{Mod}$ . Let  $X_{\kappa} = \{\mu \in X^+ \mid \langle \mu, \alpha^{\vee} \rangle \leq \ell - 1 \ \forall \alpha \in \Sigma\}$ . If  $L(\lambda)_{\kappa}$  denotes the simple of  $\mathcal{C}_{U(\kappa)}$  of highest weight  $\lambda \in X^+$  and if  $\lambda = \lambda^0 + \ell\lambda^1$  with  $\lambda^0 \in X_{\kappa}$  and  $\lambda^1 \in X$ , then Lusztig's tensor product theorem [LMR], (7.4) asserts

$$(2) \quad L(\lambda)_{\kappa} \simeq L(\lambda^0)_{\kappa} \otimes_{\kappa} \bar{L}(\lambda^1)_{\kappa}^{[1]} \quad \text{in } \mathcal{C}_{U(\kappa)},$$

where  $\bar{L}(\lambda^1)_{\kappa}^{[1]}$  is the composite of the simple representation  $\bar{L}(\lambda^1)_{\kappa}$  of  $\mathfrak{G}_{\kappa}$ , i.e., of the universal enveloping algebra  $U(\text{Lie}(\mathfrak{G}_{\kappa}))$  of the Lie algebra  $\text{Lie}(\mathfrak{G}_{\kappa})$

of  $\mathfrak{G}_\kappa$ , with Lusztig's lift  $U(\kappa) \rightarrow U(\text{Lie}(\mathfrak{G}_\kappa))$  of the Frobenius morphism [L3], (8.16) such that for each  $\alpha \in \Sigma$  and  $n \in \mathbb{N}$

$$E_{\pm\alpha}^{(n)} \mapsto \begin{cases} \bar{E}_{\pm\alpha}^{(\frac{n}{\ell})} & \text{if } \ell \mid n \\ 0 & \text{otherwise,} \end{cases} \quad K_\alpha^{\pm 1} \mapsto K_\alpha^{\pm 1}, \quad \begin{bmatrix} K_\alpha \\ n \end{bmatrix} \mapsto \begin{cases} \begin{bmatrix} H_\alpha \\ \frac{n}{\ell} \end{bmatrix} & \text{if } \ell \mid n \\ 0 & \text{otherwise,} \end{cases}$$

where  $(\bar{E}_{\pm\beta}, H_\alpha)_{\alpha \in \Sigma, \beta \in R}$  is a basis of  $\text{Lie}(\mathfrak{G}_\kappa)$  obtained from a Chevalley basis, and  $E_{\pm\alpha}^{(r)} = \frac{E_{\pm\alpha}^r}{[r]_{d_\alpha}!}$  in  $U(\kappa)$  with  $[r]_{d_\alpha}! = \prod_{i=1}^r \frac{v^{d_\alpha i} - v^{-d_\alpha i}}{v^{d_\alpha} - v^{-d_\alpha}} \otimes 1$  while  $\bar{E}_{\pm\alpha}^{(r)} = \frac{\bar{E}_{\pm\alpha}^r}{r!}$  in  $U(\text{Lie}(\mathfrak{G}_k))$ . By [AW], (1.9)

$$(3) \quad L(\lambda^0)_\kappa \text{ remains simple as } \mathfrak{u}(\kappa)\text{-module.}$$

Again in order to keep track of the weights, we will consider  $\tilde{\mathfrak{u}}(\kappa) = U^0(\kappa)\mathfrak{u}(\kappa)$  and the category  $\mathcal{C}_{\tilde{\mathfrak{u}}(\kappa)}$  of all finite dimensional  $\tilde{\mathfrak{u}}(\kappa)$ -modules admitting weight space decompositions (1) with  $K_\alpha^\ell$  acting by 1 for each  $\alpha \in \Sigma$ . The category  $\mathcal{C}_{\tilde{\mathfrak{u}}(\kappa)}$  resembles much the category  $\mathfrak{G}_1\mathfrak{T}_k\mathbf{mod}$  of finite dimensional  $\mathfrak{G}_1\mathfrak{T}_k$ -modules [APW2], (4.7/4.10) (again if  $\ell$  is not a prime power, refer to [AW]). In particular, finding the irreducible characters of  $\mathcal{C}_{\tilde{\mathfrak{u}}(\kappa)}$  is reduced for  $\ell \geq h$  to the determination of the multiplicity  $m_\kappa(j, i)$  of the projective cover  $Q_\kappa(w_j \cdot_\kappa 0)$  of  $L_\kappa(w_j \cdot_\kappa 0)$  in the projective  $Q^{[i]}(\kappa)$ :

$$(4) \quad Q^{[i]}(\kappa) = \coprod_{j \leq i} Q_\kappa(w_j \cdot_\kappa 0)^{m_\kappa(j, i)},$$

using the notations of (a12) to define  $Q^{[i]}(\kappa)$ , where  $\cdot_\kappa$  is the  $(\cdot_k)$ -action of  $W_a$  on  $X$  with  $p$  replaced by  $\ell$ . Define  $\mathcal{E}_{[i], [j]}(\kappa)$ ,  $\mathcal{E}(\kappa)$ , and the idempotents as in (a12) with  $k$  replaced by  $\kappa$ . Then

$$(5) \quad m_\kappa(j, i) = \#\{s \in E_\kappa(i) \mid e_\kappa^s(i) \text{ is conjugate to } e_\kappa^0(j) \text{ in } \mathcal{E}(\kappa)\}.$$

(a14) We are not to ask for an equivalence of categories between  $\mathfrak{G}_1\mathfrak{T}_k\mathbf{mod}$  and  $\mathcal{C}_{\tilde{\mathfrak{u}}(\kappa)}$ , but to expect for  $p$  and  $\ell \geq h$

$$(1) \quad m_k(i, j) = m_\kappa(i, j) \quad \forall i, j.$$

Indeed, a morphism space in  $\mathfrak{G}_1\mathfrak{T}_k\mathbf{mod}$  is finite dimensional over  $\mathbb{F}_p$  while that in  $\mathcal{C}_{\tilde{\mathfrak{u}}(\kappa)}$  is finite dimensional over  $\mathbb{Q}(\zeta)$ .

If  $p = \ell < h$ , however, Andersen and Jantzen have found an example [A1], (7.9) that  $\text{ch } L_k(\lambda) \neq \text{ch } L_\kappa(\lambda)$  for some  $\lambda \in X_k = X_\kappa$ .

**b° The theorem**

(b1) Retain the notations of (a12/13).

**Theorem** (cf. [AJS], Corollary 16.8) *There is a  $\mathbb{Z}$ -algebra  $\mathcal{E}$  of finite type as  $\mathbb{Z}$ -module with isomorphisms*

$$\mathcal{E} \otimes_{\mathbb{Z}} k \simeq \mathcal{E}(k) \quad \text{in } k\mathbf{Alg} \quad \text{and} \quad \mathcal{E} \otimes_{\mathbb{Z}} \kappa \simeq \mathcal{E}(\kappa) \quad \text{in } \kappa\mathbf{Alg}.$$

Moreover,  $\mathcal{E}$  admits a decomposition  $\mathcal{E} = \coprod_{i,j \in [1, n_0]} \mathcal{E}_{[i],[j]}$  such that  $\mathcal{E}_{[i],[j]} \mathcal{E}_{[n],[m]} \subseteq \delta_{jn} \mathcal{E}_{[i],[m]}$  for each  $i, j, m$  and  $n$ , and that the above isomorphisms restrict to isomorphisms

$$\mathcal{E}_{[i],[j]} \otimes_{\mathbb{Z}} k \simeq \mathcal{E}_{[i],[j]}(k) \quad \text{and} \quad \mathcal{E}_{[i],[j]} \otimes_{\mathbb{Z}} \kappa \simeq \mathcal{E}_{[i],[j]}(\kappa),$$

respectively.

(b2) **Remark** (cf. [AJS], Corollary 16.11) *One can realize  $\mathcal{E}$  such that  $\mathcal{E} \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{d}]$  is free of finite type over  $\mathbb{Z}[\frac{1}{d}]$  with  $d = (h-1)!$ .*

(b3) For a commutative ring  $A$  let us write  $\mathcal{E}_A = \mathcal{E} \otimes_{\mathbb{Z}} A$ . There is a finite extension field  $F$  of  $\mathbb{Q}$  that is a splitting field of  $\mathcal{E}_{\mathbb{Q}}$  [NT], Theorem 2.3.11. Let  $\mathfrak{o}_F$  be the ring of algebraic integers in  $F$  and let  $1 = \sum_{n \in E_F(i)} e_F^n(i)$ ,  $1 \leq i \leq n_0$ , and  $1 = \sum_{i=1}^{n_0} \sum_{n \in E_F(i)} e_F^n(i)$  be decompositions into orthogonal primitive idempotents in  $(\mathcal{E}_{[i],[i]})_F$  and  $\mathcal{E}_F$ , respectively. One can find  $N \in \mathbb{N}^+$  such that if  $\mathfrak{o} = \mathfrak{o}_F[\frac{1}{N}]$ , then (cf. [NT], Lemma 1.13.14)

$$(2) \quad \mathfrak{o} \text{ is of finite type as } \mathbb{Z}[\frac{1}{N}]\text{-module,}$$

$$(3) \quad \mathcal{E}_{\mathfrak{o}} \text{ is } \mathfrak{o}\text{-free of finite type,}$$

$$(4) \quad \text{all } e_F^n(i) \text{ live in } \mathcal{E}_{\mathfrak{o}},$$

i.e., one can write  $e_F^n(i) = e^n(i) \otimes 1$  with idempotents  $e^n(i)$  in  $\mathcal{E}_{\mathfrak{o}}$ , and

$$(5) \quad e^n(i) \text{ and } e^m(j) \text{ are conjugate in } \mathcal{E}_F \text{ iff they are so in } \mathcal{E}_{\mathfrak{o}} \quad \forall i, j, n, m.$$

If  $\mathfrak{m} \in \text{Max}(\mathfrak{o})$ ,  $\mathfrak{o}_{\mathfrak{m}}$  is a DVR as  $\mathfrak{o}$  is a Dedekind domain [AM], (9.5). Put  $\mathfrak{o}' = \mathfrak{o}_{\mathfrak{m}}$ ,  $\mathfrak{m}' = \mathfrak{m}\mathfrak{o}'$ , and let  $\hat{\mathfrak{o}}'$  be the completion of  $\mathfrak{o}'$  in the  $\mathfrak{m}'$ -adic topology.

Then  $\hat{\mathfrak{o}}'$  is a complete DVR with the maximal ideal  $\hat{\mathfrak{m}}' = \mathfrak{m}'\hat{\mathfrak{o}}'$  (cf. [B1], (VI.5.3), Proposition 5) and with  $\hat{\mathfrak{o}}'/\hat{\mathfrak{m}}' \simeq \mathfrak{o}'/\mathfrak{m}' \simeq \mathfrak{o}/\mathfrak{m}$  [AM], (10.16). In fact, if  $\hat{\mathfrak{o}}$  is the completion of  $\mathfrak{o}$  in the  $\mathfrak{m}$ -adic topology, then  $\hat{\mathfrak{o}} \simeq \hat{\mathfrak{o}}'$  [B1], Exercise III.2.27(a). As  $F$  is a splitting field of  $\mathcal{E}_{\mathbb{Q}}$ , the  $e^n(i)$  remain primitive in  $\mathcal{E}_{\text{Frac}(\hat{\mathfrak{o}}')}$ , hence in  $\mathcal{E}_{\hat{\mathfrak{o}}'}$ . Also  $e^n(i)$  and  $e^m(j)$  are conjugate in  $\mathcal{E}_{\hat{\mathfrak{o}}'}$  iff they are so in  $\mathcal{E}_{\mathfrak{o}'}$ . Hence (cf. [NT], Theorem 1.14.2(ii))

$$(6) \quad \text{the } e^n(i) \text{ remain primitive in } \mathcal{E}_{\mathfrak{o}/\mathfrak{m}},$$

and (cf. [NT], Theorem 1.14.2(iii))

$$(7) \quad e^n(i) \text{ and } e^m(j) \text{ are conjugate in } \mathcal{E}_{\mathfrak{o}/\mathfrak{m}} \text{ iff they are so in } \mathcal{E}_{\mathfrak{o}}.$$

Rearrange the index sets  $E(i)$  of the primitive idempotents in  $\mathcal{E}_{\mathfrak{o}}$  so that  $e^0(i)$  is not conjugate in  $\mathcal{E}_{\mathfrak{o}}$  to any of  $e^m(j)$ ,  $m \in E(j)$ ,  $j < i$ .

(b4) As the simples of  $\mathcal{C}_{\tilde{\mathfrak{u}}(\kappa)}$  are absolutely simple, any indecomposable projective of  $\mathcal{C}_{\tilde{\mathfrak{u}}(\kappa)}$  remains indecomposable projective under field extensions. Hence

$$(1) \quad m_{\kappa}(j, i) = \#\{s \in E(i) \mid e^s(i) \text{ is conjugate to } e^0(j) \text{ in } \mathcal{E}_{\mathfrak{o}}\}.$$

Also if  $p \gg 0$  so that  $p \notin \mathfrak{o}^{\times}$ , then considering  $\mathfrak{m} \in \text{Max}(\mathfrak{o})$  with  $p \in \mathfrak{m}$  yields

$$(2) \quad m_k(j, i) = \#\{s \in E(i) \mid e^s(i) \text{ is conjugate to } e^0(j) \text{ in } \mathcal{E}_{\mathfrak{o}}\}.$$

Hence for  $p \gg 0$

$$(3) \quad m_k(j, i) = m_{\kappa}(j, i).$$

(b5) Let  $\mathfrak{u}^{-}(\kappa)$  be the  $\kappa$ -subalgebra of  $\mathfrak{u}(\kappa)$  generated by  $E_{-\alpha}$ ,  $\alpha \in \Sigma$ , and let  $\tilde{\mathfrak{u}}^b(\kappa) = \mathfrak{u}^{-}(\kappa)U^0(\kappa)$ . Define a category  $\mathcal{C}_{\tilde{\mathfrak{u}}^b(\kappa)}$  of finite dimensional  $\tilde{\mathfrak{u}}^b(\kappa)$ -modules just like  $\mathcal{C}_{\tilde{\mathfrak{u}}(\kappa)}$ . In analogy to the functor  $\hat{Z}_k : \mathfrak{B}_1\mathfrak{T}_k\mathbf{Mod} \rightarrow \mathfrak{G}_1\mathfrak{T}_k\mathbf{Mod}$  one has an induction functor  $\tilde{Z}_{\kappa} : \mathcal{C}_{\tilde{\mathfrak{u}}^b(\kappa)} \rightarrow \mathcal{C}_{\tilde{\mathfrak{u}}(\kappa)}$  defined by  $\tilde{Z}_{\kappa}(M) = \tilde{\mathfrak{u}}^b(\kappa)\mathbf{Mod}(\tilde{\mathfrak{u}}(\kappa), M)$  [APW2], (1.2). Then

$$(1) \quad \text{ch } \tilde{Z}_{\kappa}(\lambda) = e(\lambda) \prod_{\alpha \in R^+} \frac{1 - e(-\ell\alpha)}{1 - e(-\alpha)} \quad \forall \lambda \in X,$$

and [APW2], (4.10)

$$(2) \quad \text{the Brauer-Humphreys reciprocity carries over to } \mathcal{C}_{\tilde{\mathfrak{u}}(\kappa)}.$$

**Corollary** (cf. [AJS], Corollary 16.23) Assume  $\ell \geq h$  and  $p \gg 0$  relative to  $R$ . Then for each  $w, w' \in W_a$  there is  $d(w, w') \in \mathbb{N}$  independent of  $\ell$  and  $p$  such that

$$[\hat{Z}_k(w \cdot_k 0) : L_k(w' \cdot_k 0)] = d(w, w') = [\tilde{Z}_\kappa(w \cdot_\kappa 0) : L_\kappa(w' \cdot_\kappa 0)].$$

In particular, if  $p = \ell$ , then

$$ch L_k(w \cdot_k 0) = ch L_\kappa(w \cdot_\kappa 0) \quad \forall w \in W,$$

hence together with the translation principle

$$ch L(\lambda)_k = ch L(\lambda)_\kappa \quad \forall \lambda \in X_k = X_\kappa.$$

(b6) It follows that the irreducible characters of  $\mathfrak{G}_k \mathbf{Mod}$  are obtained from that of  $\mathcal{C}_{U(\kappa)}$  if  $p \gg 0$ . Hence from [KL1, 2], [L4] and [KT] Lusztig's conjectural irreducible character formula in  $\mathfrak{G}_k \mathbf{Mod}$  holds if  $p \gg 0$  and if  $R$  is of type  $A$ ,  $D$  or  $E$ .

## c° Reformulation of categories

(c1) In order to treat much alike categories  $\mathfrak{G}_1 \mathfrak{T}_k \mathbf{mod}$  and  $\mathcal{C}_{\tilde{u}(\kappa)}$  simultaneously, we will reformulate these categories as follows.

**Case 1.** Let  $k[\mathfrak{G}]$  be the Hopf algebra defining  $\mathfrak{G}_k$  and  $\mathfrak{m}_k$  the augmentation ideal of  $k[\mathfrak{G}]$ . Let  $\text{Dist}(\mathfrak{G}_k) = \varinjlim_{n \geq 0} \mathbf{Mod}_k(k[\mathfrak{G}]/\mathfrak{m}_k^{n+1}, k)$  the algebra of distributions of  $\mathfrak{G}_k$ , that inherits the structure of Hopf algebra from  $k[\mathfrak{G}]$ .

Any  $\mathfrak{G}_k$ -module  $M$  is a  $k[\mathfrak{G}]$ -comodule, hence a  $\text{Dist}(\mathfrak{G}_k)$ -module : if  $\Delta_M = id_{k[\mathfrak{G}]} \in \mathfrak{G}_k(k[\mathfrak{G}]) : M \rightarrow M \otimes_k k[\mathfrak{G}]$  is the comodule map, then each  $x \in \text{Dist}(\mathfrak{G}_k)$  acts on  $M$  by  $(M \otimes_k x) \circ \Delta_M$ . Conversely, any finite dimensional  $\text{Dist}(\mathfrak{G}_k)$ -module carries a structure of  $\mathfrak{G}_k$ -module [J], (II.1.20).

The Hopf algebra of  $\mathfrak{G}_1$  is  $k[\mathfrak{G}]/\mathfrak{m}_k^p$ , hence  $\text{Dist}(\mathfrak{G}_1) = (k[\mathfrak{G}]/\mathfrak{m}_k^p)^*$ . Then  $\mathfrak{G}_1 \mathbf{Mod} = \text{Dist}(\mathfrak{G}_1) \mathbf{Mod}$  : if  $M$  is a  $\text{Dist}(\mathfrak{G}_1)$ -mod, one gets the comodule map by the commutative diagram

$$\begin{array}{ccccc} m & & M & \dashrightarrow & M \otimes_k k[\mathfrak{G}]/\mathfrak{m}_k^p \\ \downarrow & & \downarrow & & \parallel \\ x \mapsto xm & \mathbf{Mod}_k(\text{Dist}(\mathfrak{G}_1), M) & \xrightarrow{\sim} & & M \otimes_k \text{Dist}(\mathfrak{G}_1)^*. \end{array}$$

Let  $\mathfrak{g} = \text{Lie}(\mathfrak{G}_k) = \text{Mod}_k(\mathfrak{m}_k/\mathfrak{m}_k^2, k) \leq \text{Dist}(\mathfrak{G}_k)$ , and  $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$  the triangular decomposition with  $\mathfrak{h} = \text{Lie}(\mathfrak{T}_k)$ . For each  $x \in \mathfrak{g}$  one has  $x^p \in \mathfrak{g}$  in  $\text{Dist}(\mathfrak{G}_k)$  [DG], (II.7.2.3), which we will denote by  $x^{[p]}$ . In particular [DG], (II.7.2.2), if  $x \in \mathfrak{n}^\pm$ , then  $x^{[p]} = 0$  while if  $x \in \mathfrak{h}$ , then  $x^{[p]} = x$ .

If  $U(\mathfrak{g})$  is the universal enveloping algebra of  $\mathfrak{g}$ , then

$$(1) \quad x^p - x^{[p]} \in Z(U(\mathfrak{g})),$$

where  $x^p$  is the  $p$ -th power of  $x$  in  $U(\mathfrak{g})$ . One calls  $U^{[p]}(\mathfrak{g}) = U(\mathfrak{g})/(x^p - x^{[p]} \mid x \in \mathfrak{g})$  the restricted enveloping algebra of  $\mathfrak{g}$ . There is a commutative diagram of  $k$ -algebras

$$(2) \quad \begin{array}{ccc} U(\mathfrak{g}) & \xrightarrow{\text{natural}} & \text{Dist}(\mathfrak{G}_k) \\ \downarrow & & \uparrow \\ U^{[p]}(\mathfrak{g}) & \xrightarrow{\sim} & \text{Dist}(\mathfrak{G}_1). \end{array}$$

Fix a  $k$ -basis  $(H_\alpha, \bar{E}_\beta \mid \alpha \in \Sigma, \beta \in R)$  of  $\mathfrak{g}$  with  $H_\alpha = [\bar{E}_\alpha, \bar{E}_{-\alpha}]$  obtained from a Chevalley basis. Let  $I = (\bar{E}_\beta^p \mid \beta \in R) \trianglelefteq U(\mathfrak{g})$  and set  $\bar{U}(\mathfrak{g}) = U(\mathfrak{g})/I$ . The adjoint action of  $\mathfrak{T}_k$  on  $U(\mathfrak{g})$  stabilizes  $I$ , hence  $\bar{U}(\mathfrak{g})$  comes equipped with an  $X$ -gradation given by the  $\mathfrak{T}_k$ -action. As  $\bar{E}_\beta^p \in Z(U(\mathfrak{g}))$ ,  $\bar{U}(\mathfrak{g})$  retains a PBW-type basis  $(\bar{E}^m H^r \bar{F}^n \mid m, n \in [0, p-1]^{R^+}, r \in \mathbb{N}^\Sigma)$  with

$$\bar{E}^m = \prod_{\beta \in R^+} \bar{E}_\beta^{m_\beta}, H_r = \prod_{\alpha \in \Sigma} H_\alpha^{r_\alpha} \text{ and } \bar{F}^n = \prod_{\beta \in R^+} \bar{E}_{-\beta}^{n_\beta}.$$

The degree of  $\bar{E}^m H^r \bar{F}^n$  is  $\sum_{\beta \in R^+} (m_\beta - n_\beta)\beta$ .

**Case 2.** Let  $U_2$  be the De Concini-Kac version [DCK], (1.5) of the quantized enveloping algebra over  $\kappa$ , i.e., the  $\kappa$ -algebra with the generators  $E_{\pm\alpha}$ ,  $K_\alpha^{\pm 1}$ ,  $\alpha \in \Sigma$ , and the same relations as the Drinfeld-Jimbo algebra over  $\mathbb{Q}(v)$  with  $v$  replaced by  $\zeta$ . Let  $U_2^\pm$  (resp.  $U_2^0$ ) be the  $\kappa$ -subalgebra of  $U_2$  generated by  $E_{\pm\alpha}$  (resp.  $K_\alpha^{\pm 1}$ ),  $\alpha \in \Sigma$ . For each  $w \in W$  let  $T_w$  be the endomorphism of  $U_2$  carried over from [LQG]. If  $\beta \in R^+$ , choose  $w \in W$  with  $w^{-1}\beta \in \Sigma$ , and set  $E_\beta = T_w(E_{w^{-1}\beta})$  and  $E_{-\beta} = T_w(E_{-w^{-1}\beta})$ . In case  $\beta \in \Sigma$ , the  $E_{\pm\beta}$  so defined coincide with the old ones. One can then make  $U_2$  into an  $X$ -graded algebra by giving  $E_\beta, \beta \in R$  (resp.  $K_\alpha, \alpha \in \Sigma$ ), degree  $\beta$  (resp. 0). By [DCK], Corollary 3.1

$$E_\beta^\ell, K_\alpha^\ell \in Z(U_2) \quad \forall \beta \in R \text{ and } \alpha \in \Sigma.$$

Let  $I^\pm = (E_\beta^\ell \mid \beta \in \pm R^+) \trianglelefteq U_2^\pm$  and  $I = (I^\pm) \trianglelefteq U_2$ . If  $f \in \kappa \mathbf{Alg}(U_2, U(\kappa))$  with  $E_{\pm\alpha} \mapsto E_{\pm\alpha}$  and  $K_\alpha \mapsto K_\alpha$  for each  $\alpha \in \Sigma$ , then  $f$  induces an isomorphism of  $\kappa$ -algebras

$$U_2/(I, K_\alpha^{2\ell} - 1 \mid \alpha \in \Sigma) \simeq \mathfrak{u}(\kappa).$$

Moreover,  $I^\pm = \ker(f \mid_{U_2^\pm})$ , hence  $I^\pm$  are defined independent of the choice of the  $T_w$ 's.

Under a suitable choice of the  $T_w$ 's and orderings in the products  $U_2/I$  retains a PBW-type  $\kappa$ -basis  $(E^m K^r F^n \mid m, n \in [0, \ell - 1]^{R^+}, r \in \mathbb{Z}^\Sigma)$  with

$$E^m = \prod_{\beta \in R^+} E_\beta^{m_\beta}, K_r = \prod_{\alpha \in \Sigma} K_\alpha^{r_\alpha} \text{ and } F^n = \prod_{\beta \in R^+} E_{-\beta}^{n_\beta}.$$

(c2) In order to treat the two cases simultaneously, we will denote  $(\kappa, \ell)$  also by  $(k, p)$  and set

$$(U, U^\pm, U^0) = \begin{cases} (\bar{U}(\mathfrak{g}), U(\mathfrak{n}^\pm) + I/I, U(\mathfrak{h}) + I/I) & \text{in Case 1} \\ (U_2/I, U_2^\pm + I/I, U_2^0 + I/I) & \text{in Case 2.} \end{cases}$$

Hence as  $k$ -algebras

$$U^0 \simeq \begin{cases} k[H_\alpha \mid \alpha \in \Sigma] & \text{the polynomial algebra in } H_\alpha \text{ in Case 1} \\ k[K_\alpha^{\pm 1} \mid \alpha \in \Sigma] & \text{the Laurent polynomial algebra in } K_\alpha \text{ in Case 2,} \end{cases}$$

and  $U$  has

(3) *a structure of  $k$ -Hopf algebra (nontrivial in Case 2),*

(4) *a triangular decomposition, i.e.,*

*a  $k$ -linear bijection  $U^- \otimes_k U^0 \otimes_k U^+ \rightarrow U$  under the multiplication,*

and

(5) *an  $X$ -gradation, indicated by subscripts, such that*

$$U^0 \subseteq U_0, U^+ \subseteq \coprod_{\nu \geq 0} U_\nu, U^- \subseteq \coprod_{\nu \leq 0} U_\nu, \text{ and } (U^+)_0 = k \cdot 1 = (U^-)_0.$$

Define a group homomorphism  $\tilde{\cdot} : X \rightarrow \mathbf{Alg}_k(U^0, U^0)^\times$  by

$$\begin{aligned} \tilde{\lambda}(H) &= H + \lambda(H) \quad \forall H \in \mathfrak{h} \quad \text{in Case 1} \\ \tilde{\lambda}(K_\alpha) &= \zeta^{d_\alpha \langle \lambda, \alpha^\vee \rangle} K_\alpha \quad \forall \alpha \in \Sigma \quad \text{in Case 2.} \end{aligned}$$

Then for each  $s \in U^0$  and  $u \in U_\lambda$  one has  $su = u\tilde{\lambda}(s)$ .

(c3) Let  $A$  be a noetherian domain over  $U^0$  with a structure homomorphism  $\pi : U^0 \rightarrow A$  (the assumption that  $A$  be a domain is only for convenience in the present survey). We define a category  $\mathcal{C}_A$  as follows. An object of  $\mathcal{C}_A$  is a  $U \otimes_k A$ -module  $M$ , which is as  $A$ -module of finite type and  $X$ -graded. We regard  $U$  and  $A$  imbedded in  $U \otimes_k A$  as  $U \otimes 1$  and  $1 \otimes A$ , respectively, and write  $(u \otimes a)m = uma$ . We require

$$(1) \quad U_\nu M_\lambda \subseteq M_{\lambda+\nu} \quad \forall \nu \in X$$

and

$$(2) \quad sm = m\pi(\tilde{\lambda}(s)) \quad \forall s \in U^0 \text{ and } m \in M_\lambda.$$

A morphism of  $\mathcal{C}_A$  is a morphism of  $U \otimes_k A$ -modules that preserves the  $X$ -gradings.

The category  $\mathcal{C}_A$  is equipped with a duality operation. There is an involutory antiautomorphism  $\tau$  of  $U$  [AJS], (1.6) such that

$$E_\alpha \longmapsto E_{-\alpha} \quad \forall \alpha \in \Sigma \quad \text{and} \quad s \longmapsto s \quad \forall s \in U^0.$$

If  $M \in \mathcal{C}_A$ , define  $M^\tau$  to be  $\mathbf{Mod}A(M, A)$  with  $U$  acting by  $(uf)(m) = f(\tau(u)m)$  and with the  $X$ -gradation given by

$$(M^\tau)_\lambda = \{f \in M^\tau \mid f(M_\mu) = 0 \quad \forall \mu \neq \lambda\} \simeq \mathbf{Mod}A(M_\lambda, A).$$

If  $M$  is  $A$ -projective,  $(M^\tau)^\tau \simeq M$  in  $\mathcal{C}_A$ .

Replacing  $U$  by  $U^0 U^+$  (resp.  $U^0$ ) one defines likewise the categories  $\mathcal{C}_A^{\geq 0}$  and  $\mathcal{C}_A^0$ .

If  $M \in \mathcal{C}_A^0$  is projective in the category of right  $A$ -modules  $\mathbf{Mod}A$ , define the character of  $M$  by

$$\text{ch } M = \sum_{\lambda \in X} \text{rk}_A(M_\lambda) e(\lambda) \quad \text{in } \mathbb{Z}[X].$$

(c4) **Case 1.** Take  $A = k$  with the structure homomorphism  $\pi : U^0 \rightarrow k$  annihilating  $\mathfrak{h}$ . Then for each  $\lambda \in X$  and  $u \in \mathfrak{h}$

$$\tilde{\lambda}(u^p) = \tilde{\lambda}(u)^p = \lambda(u)^p = \lambda(u) = \tilde{\lambda}(u).$$

Hence the  $U$ -module structure on  $M \in \mathcal{C}_k$  factors through  $U^{[p]}(\mathfrak{g})$ . Consequently,  $M$  comes equipped with a structure of  $\text{Dist}(\mathfrak{G}_1)$ -module. Moreover, the  $X$ -gradation on  $M$  makes  $M$  into a  $\mathfrak{T}_k$ -module such that

$$t(xm \otimes 1) = (\text{Ad}(t)(x \otimes 1))t(m \otimes 1) \text{ in } M \otimes_k A' \quad \forall t \in \mathfrak{T}_k(A'), A' \in \mathbf{Alg}_k,$$

hence into a  $\mathfrak{G}_1\mathfrak{T}_k$ -module. One can thus identify  $\mathcal{C}_k$  with  $\mathfrak{G}_1\mathfrak{T}_k\mathbf{mod}$  the category of finite dimensional  $\mathfrak{G}_1\mathfrak{T}_k$ -modules.

**Case 2.** Take  $A = k$  with  $\pi : U^0 \rightarrow k$  such that  $K_\alpha \mapsto 1 \quad \forall \alpha \in \Sigma$ . Then for each  $\lambda \in X$  and  $\alpha \in \Sigma$

$$\tilde{\lambda}(K_\alpha^p) = \tilde{\lambda}(K_\alpha)^p = \zeta^{pd_\alpha \langle \lambda, \alpha^\vee \rangle} = 1.$$

Hence together with the  $X$ -gradation one can identify  $\mathcal{C}_k$  with  $\mathcal{C}_{\tilde{u}(k)}$ .

(c5) The forgetful functor gives an equivalence of categories from  $\mathcal{C}_A^0$  to the category of  $X$ -graded  $A$ -modules of finite type, hence

(1)  $\mathcal{C}_A^0$  has enough projectives.

Define a functor  $\Phi_A : \mathcal{C}_A^0 \rightarrow \mathcal{C}_A$  by setting  $\Phi_A(M) = U \otimes_{U^0} M$ ,  $M \in \mathcal{C}_A^0$ , with  $U$  acting by the left multiplication on  $U$  while  $A$  acting as given on  $M$ . The  $X$ -gradation on  $\Phi_A(M)$  is defined by  $\Phi_A(M)_\lambda = \sum_{\nu \in X} U_\nu \otimes_{U^0} M_{\lambda-\nu}$ .

Define likewise a functor  $\Phi_A^{\geq 0} : \mathcal{C}_A^0 \rightarrow \mathcal{C}_A^{\geq 0}$  by  $\Phi_A^{\geq 0}(M) = U^0 U^+ \otimes_{U^0} M$ . Then

(2)  $\Phi_A$  (resp.  $\Phi_A^{\geq 0}$ ) is exact and

left adjoint to the forgetful functor from  $\mathcal{C}_A$  (resp.  $\mathcal{C}_A^{\geq 0}$ ) to  $\mathcal{C}_A^0$ .

Hence from (1)

(3) both  $\mathcal{C}_A$  and  $\mathcal{C}_A^{\geq 0}$  have enough projectives.

(c6) Define likewise a functor  $Z_A : \mathcal{C}_A^{\geq 0} \rightarrow \mathcal{C}_A$  by setting

$$Z_A(M) = U \otimes_{U^0 U^+} M, \quad M \in \mathcal{C}_A^{\geq 0},$$

with the  $X$ -gradation on  $Z_A(M)$  defined by  $Z_A(M)_\lambda = \coprod_{\nu \in X} (U^-)_\nu \otimes_k M_{\lambda-\nu}$ , using an  $A$ -linear isomorphism  $Z_A(M) \simeq U^- \otimes_k M$ . Then

(1)  $Z_A$  is exact and left adjoint to the forgetful functor  $\mathcal{C}_A \rightarrow \mathcal{C}_A^{\geq 0}$

and

$$(2) \quad \Phi_A = Z_A \circ \Phi_A^{\geq 0}.$$

An object of  $\mathcal{C}_A^0$  can be made into an object of  $\mathcal{C}_A^{\geq 0}$  through an isomorphism  $U^0 U^+ / \coprod_{\nu > 0} (U^0 U^+)_{\nu} \simeq U^0$ . In particular, if  $\lambda \in X$ , define  $A^{\lambda} \in \mathcal{C}_A^0$  by

$$(A^{\lambda})_{\nu} = \begin{cases} A & \text{if } \nu = \lambda \\ 0 & \text{otherwise.} \end{cases}$$

Regarding  $A^{\lambda}$  as an object of  $\mathcal{C}_A^{\geq 0}$ , set  $Z_A(\lambda) = Z_A(A^{\lambda})$ . Then

$$(3) \quad \text{ch } Z_A(\lambda) = e(\lambda) \prod_{\beta \in R^+} \frac{1 - e(-p\beta)}{1 - e(-\beta)},$$

that coincides with  $\text{ch } \hat{Z}_k(\lambda)$  of §a.

In case  $A = F$  is a field

$$(4) \quad Z_F(\lambda) \text{ has a simple head of highest weight } \lambda,$$

which we will denote by  $L_F(\lambda)$ . All simples of  $\mathcal{C}_F$  arise in this way.

(c7) A  $Z$ -filtration of  $M \in \mathcal{C}_A$  is a chain in  $\mathcal{C}_A$  with the successive subquotients isomorphic to some  $Z_A(\lambda)$ ,  $\lambda \in X$ . By (c6)(3)

- (1) *the multiplicity of  $Z_A(\lambda)$  in a  $Z$ -filtration is independent of the choice of the  $Z$ -filtrations.*

As  $\Phi_A = Z_A \circ \Phi_A^{\geq 0}$  and as both  $Z_A$  and  $\Phi_A^{\geq 0}$  are exact,

- (2) *any  $M \in \mathcal{C}_A$  admits an epi  $Q \rightarrow M$  in  $\mathcal{C}$  with  $Q$  projective having a  $Z$ -filtration.*

Moreover,

(c8) **Lemma** (cf. [AJS], Lemma 2.16) *If  $A$  is local, any direct summand of an object of  $\mathcal{C}_A$  with a  $Z$ -filtration admits a  $Z$ -filtration. In particular, any projective of  $\mathcal{C}_A$  has a  $Z$ -filtration.*

*Proof.* One has [AJS], (2.14)

- (1)  $\text{Ext}_{\mathcal{C}_A}^1(Z_A(\lambda), Z_A(\mu)) \neq 0$ ,  $\lambda, \mu \in X$ , then  $\mu > \lambda$ .

Let  $M = M' \oplus M''$  in  $\mathcal{C}_A$  with  $M$  having a  $Z$ -filtration. If  $A = F$  is a field, the standard argument applies: if  $\lambda$  is a maximal weight of  $M$  with  $r = \dim_F M_\lambda$ , then by (1) there is  $V \leq M$  with  $V \simeq Z_F(\lambda)^{\oplus r}$  such that  $M/V$  has a  $Z$ -filtration with  $[M/V : Z_F(\lambda)] = 0$ . If  $m \in M'_\lambda \setminus 0$ , let  $\hat{m} \in \mathcal{C}_F(Z_F(\lambda), M')$  induced by the adjunction from a morphism  $F^\lambda \rightarrow M'$  in  $\mathcal{C}^{\geq 0}$  such that  $1 \mapsto m$ . Then  $\text{im}(\hat{m}) \leq V$ . Denote by  $\hat{m}'$  the morphism  $Z_F(\lambda) \rightarrow V$  induced from  $\hat{m}$ . As  $\mathcal{C}_F(Z_F(\lambda), Z_F(\lambda)) \simeq \mathcal{C}_F^{\geq 0}(Z_F(\lambda), \lambda) \simeq F$ ,  $\mathcal{C}_F(Z_F(\lambda), Z_F(\lambda)) = \text{Fid}_{Z_F(\lambda)}$ . Hence

$$(2) \quad \hat{m}' \text{ is a split mono with } \text{coker}(\hat{m}') \simeq Z_F(\lambda)^{\oplus r-1}.$$

Then  $M'/\text{im}(\hat{m}) \oplus M'' \simeq M/\text{im}(\hat{m}')$  retains a  $Z_F$ -filtration. The assertion follows by induction on the length of a  $Z$ -filtration on  $M$ .

In general, let  $\mathfrak{p} \in \text{Spec} A$  with  $\kappa(\mathfrak{p})$  the residue field of  $A_{\mathfrak{p}}$ . As  $Z_A(\lambda) \otimes_A \kappa(\mathfrak{p}) \simeq Z_{\kappa(\mathfrak{p})}(\lambda)$  in  $\mathcal{C}_{\kappa(\mathfrak{p})}$ , it suffices to check by above that

$$(3) \quad \text{if } L \in \mathcal{C}_A \text{ is } A\text{-free with } L_{\kappa(\mathfrak{p})} = L \otimes_A \kappa(\mathfrak{p}) \text{ admitting a } Z\text{-filtration} \\ \text{in } \mathcal{C}_{\kappa(\mathfrak{p})} \text{ for each } \mathfrak{p} \in \text{Spec} A, \text{ then } L \text{ admits a } Z\text{-filtration in } \mathcal{C}_A.$$

Let  $\lambda$  be a maximal weight of  $L$ . By (1) again if  $s = \dim_{\kappa(\mathfrak{p})}(L_{\kappa(\mathfrak{p})})_\lambda$ , there is  $L' \leq L_{\kappa(\mathfrak{p})}$  with  $L' \simeq Z_{\kappa(\mathfrak{p})}(\lambda)^{\oplus s}$  and such that  $L_{\kappa(\mathfrak{p})}$  has a  $Z$ -filtration with  $[L_{\kappa(\mathfrak{p})}/L' : Z_{\kappa(\mathfrak{p})}(\lambda)] = 0$ . As  $A$  is local,  $L_\lambda$  remains  $A$ -free, say  $L_\lambda = Ae_1 \oplus \dots \oplus Ae_s$ . If  $\hat{e}_1 \in \mathcal{C}_A(Z_A(\lambda), L)$  with  $1 \otimes 1 \mapsto e_1$ , then as in (2)

$$(4) \quad \hat{e}_1 \otimes_A \kappa(\mathfrak{p}) \text{ is injective and } \text{coker}(\hat{e}_1 \otimes_A \kappa(\mathfrak{p})) \text{ admits a } Z\text{-filtration}.$$

On the other hand, using the duality operator  $\tau$  of (c3) one has a commutative diagram

$$\begin{array}{ccc} \text{Mod} A(L, A) \otimes_A \kappa(\mathfrak{p}) & \xrightarrow{(\hat{e}_1)^\tau \otimes_A \kappa(\mathfrak{p})} & \text{Mod} A(Z_A(\lambda), A) \otimes_A \kappa(\mathfrak{p}) \\ \wr \downarrow & & \downarrow \wr \\ \text{Mod} \kappa(\mathfrak{p})(L_{\kappa(\mathfrak{p})}, \kappa(\mathfrak{p})) & \xrightarrow{(\hat{e}_1 \otimes_A \kappa(\mathfrak{p}))^\tau} & \text{Mod} \kappa(\mathfrak{p})(Z_{\kappa(\mathfrak{p})}(\lambda), \kappa(\mathfrak{p})). \end{array}$$

By (4)  $(\hat{e}_1 \otimes_A \kappa(\mathfrak{p}))^\tau$  is surjective, hence  $(\hat{e}_1)^\tau \otimes_A A_{\mathfrak{p}}$  is surjective for each  $\mathfrak{p} \in \text{Spec} A$  by NAK. Then  $(\hat{e}_1)^\tau$  is surjective [AM], (3.9). As  $Z_A(\lambda)$  is  $A$ -free, the short exact sequence in  $\mathcal{C}_A$

$$0 \rightarrow \ker((\hat{e}_1)^\tau) \rightarrow L^\tau \xrightarrow{(\hat{e}_1)^\tau} Z_A(\lambda)^\tau \rightarrow 0$$

splits in  $\mathbf{Mod} A$ . Then  $\ker((\hat{e}_1)^\tau)$  is  $A$ -free as  $A$  is local.. Hence  $(\ker((\hat{e}_1)^\tau))^\tau$  is  $A$ -free in the short exact sequence of  $\mathcal{C}_A$

$$0 \rightarrow Z_A(\lambda) \xrightarrow{\hat{e}_1} L \rightarrow (\ker((\hat{e}_1)^\tau))^\tau \rightarrow 0.$$

By (4)  $(\ker((\hat{e}_1)^\tau))^\tau \otimes_A \kappa(\mathfrak{p}) \simeq \text{coker}(\hat{e}_1 \otimes_A \kappa(\mathfrak{p}))$  has a  $Z$ -filtration in  $\mathcal{C}_{\kappa(\mathfrak{p})}$  for each  $\mathfrak{p} \in \text{Spec} A$ . Then by induction on  $\text{rk}_A L$   $(\ker((\hat{e}_1)^\tau))^\tau$  admits a  $Z$ -filtration, and (3) follows.

The second assertion follows from (c7)(2).

(c9) Define a partition of  $X$  into disjoint subsets, called the blocks over  $A$ , by taking a finest partition such that  $\lambda$  and  $\mu$  belong to the same block if either  $\mathcal{C}_A(Z_A(\lambda), Z_A(\mu)) \neq 0$  or  $\text{Ext}_{\mathcal{C}_A}^1(Z_A(\lambda), Z_A(\mu)) \neq 0$ . Let  $\mathcal{B}_A$  be the set of blocks over  $A$ . Let  $\mathcal{D}_A$  be the full subcategory of  $\mathcal{C}_A$  consisting of all objects with a  $Z$ -filtration. If  $b$  is a block over  $A$ , let  $\mathcal{D}_A(b)$  be the full subcategory of  $\mathcal{D}_A$  consisting of all objects such that the subquotients of a  $Z$ -filtration are  $Z_A(\lambda)$ ,  $\lambda \in b$ . Let  $\mathcal{C}_A(b)$  be the full subcategory of  $\mathcal{C}_A$  consisting of all that are the images of objects of  $\mathcal{D}_A(b)$ .

(c10) **Theorem** (cf. [AJS], Theorem 6.10) (i) *If  $b, b'$  are disjoint blocks over  $A$ , then*

$$\text{Ext}_{\mathcal{C}_A}(M, M') = 0 \quad \forall M \in \mathcal{C}_A(b) \text{ and } M' \in \mathcal{C}_A(b').$$

(ii) *Each  $M \in \mathcal{C}_A$  admits a block decomposition  $M = \coprod_{b \in \mathcal{B}_A} M_b$  with  $M_b$  the largest subobject of  $M$  belonging to  $\mathcal{C}_A(b)$ .*

(iii) *For each block  $b$  over  $A$  the category  $\mathcal{C}_A(b)$  is closed under taking homomorphic images, submodules, extensions, and finite direct sums.*

(c11) Relative to the structure homomorphism  $\pi : U^0 \rightarrow A$ , let

$$R_\pi = \begin{cases} \{\beta \in R \mid \prod_{j=1}^p (\pi(H_\beta) + j) \notin A^\times\} & \text{in Case 1} \\ \{\beta \in R \mid \prod_{j=1}^p (\pi([K_\beta : j])) \notin A^\times\} & \text{in Case 2,} \end{cases}$$

where  $[K_\beta : j] = \begin{bmatrix} K_\beta & j \\ 1 & \end{bmatrix} = \frac{K_\beta \zeta^{jd_\beta} - K_\beta^{-1} \zeta^{-jd_\beta}}{\zeta^{d_\beta} - \zeta^{-d_\beta}} (\neq [K_\beta])$  and  $d_\beta = d_\alpha$  if  $\alpha \in \Sigma$  with  $\beta \in W\alpha$ . Then  $R_\pi$  forms a root system with the Weyl group

$W_\pi = \langle s_\beta \mid \beta \in R_\pi \rangle$  and a positive system of roots  $R_\pi^+ = R_\pi \cap R^+$ . Let  $W_{\pi,a} = W_\pi \ltimes \mathbb{Z}R_\pi \leq W_a$ . It will be convenient to introduce

$$B = \begin{cases} U^0[\frac{1}{\prod_{j=1}^{p-1}(H_\beta+j)} \mid \beta \in R^+] & \text{in Case 1} \\ U^0[\frac{1}{\prod_{j=1}^{p-1}([K_\beta:j])} \mid \beta \in R^+] & \text{in Case 2.} \end{cases}$$

**Proposition** (cf. [AJS], Proposition 6.13) *Suppose  $A$  is a  $B$ -algebra. If  $b \in \mathcal{B}_A$  and  $\lambda \in b$ , then  $b \subseteq W_{\pi,a} \cdot_k \lambda$ .*

(c12) Regard  $k$  as a  $U^0$ -algebra via the augmentation. For each  $E \in \mathcal{C}_k$  and  $M \in \mathcal{C}_A$  one can make  $E \otimes_k M$  into an object of  $\mathcal{C}_A$  by letting  $U$  (resp.  $A$ ) act via the comultiplication (resp. only on  $M$ ). The gradation is defined by  $(E \otimes_k M)_\lambda = \coprod_{\nu \in X} E_\nu \otimes_k M_{\lambda-\nu}$ .

Assume  $A$  is a  $B$ -algebra. Let  $W'$  be a reflexion subgroup of  $W_a$  with  $W_{\pi,a} \leq W'$ . An alcove for  $W'$  is a connected component of  $X \otimes_{\mathbb{Z}} \mathbb{R}$  with the hyperplanes in  $W'$  deleted. Let  $\Omega$  and  $\Gamma$  be two  $W'$ -orbits in  $X$ . The closure of an alcove for  $W'$  contains exactly one element  $\lambda \in \Omega$  and  $\mu \in \Gamma$ . Then  $W(\mu - \lambda)$  is independent of the choice of the alcove. Let  $\nu$  be the unique dominant weight of  $W(\mu - \lambda)$ . Choose a simple  $E$  of highest weight  $\nu$  in  $\mathfrak{G}_k \mathbf{Mod}$  (resp.  $\mathcal{C}_{U(k)}$ ) in Case 1 (resp. Case 2). Let  $\mathcal{C}_A(\Omega) = \coprod_{b \subseteq \Omega} \mathcal{C}_A(b)$  and  $\mathcal{C}_A(\Gamma) = \coprod_{b \subseteq \Gamma} \mathcal{C}_A(b)$ . If  $\text{pr}_\Gamma : \mathcal{C}_A \rightarrow \mathcal{C}_A(\Gamma)$  is the functor such that  $\text{pr}_\Gamma M = \coprod_{b \subseteq \Gamma} M_b$ , one gets an exact functor

$$T_\Omega^\Gamma = \text{pr}_\Gamma \circ (E \otimes_k ?) : \mathcal{C}_A(\Omega) \longrightarrow \mathcal{C}_A(\Gamma),$$

called the translation functor from  $\Omega$  to  $\Gamma$ . In case  $A = k$  the functor recovers the translation functor in  $\mathfrak{G}_k \mathbf{Mod}$  and  $\mathcal{C}_k$ . As usual [AJS], (7.6),

$$(1) \quad T_\Omega^\Gamma \text{ is both left and right adjoint to } T_\Gamma^\Omega.$$

Denote the adjunctions by  $\text{adj}_1 : \mathcal{C}_A(\Omega)(?, T_\Gamma^\Omega ?') \rightarrow \mathcal{C}_A(\Gamma)(T_\Omega^\Gamma ?, ?')$  and  $\text{adj}_2 : \mathcal{C}_A(\Gamma)(?, T_\Omega^\Gamma ?') \rightarrow \mathcal{C}_A(\Omega)(T_\Gamma^\Omega ?, ?')$ .

(c13) **Lemma** (cf. [AJS], Lemma 7.5) *Assume  $A$  is a  $B$ -algebra. Let  $\lambda, \mu \in X$  in the closure of an alcove for  $W'$  and  $\Omega = W' \cdot_k \lambda$ ,  $\Gamma = W' \cdot_k \mu$ . Then  $T_\Omega^\Gamma Z_A(\lambda)$  has a  $Z$ -filtration with factors  $Z_A(w \cdot_k \mu)$ ,  $w \in$*

$C_{W'}(\lambda)/C_{W'}(\lambda) \cap C_{W'}(\mu)$ , each occuring exactly once.

## d° Deformations

(d1) Recall that we are after a characteristic free description of  $\mathcal{C}_k(Q^{[i]}(k), Q^{[j]}(k))$ . By (c6)(3) and (c8) we may replace  $\hat{Z}_k(?)$  of §a by  $Z_k(?)$  in  $\mathcal{C}_k$ . We will study  $\mathcal{C}_k$  by deformations.

Let  $\mathfrak{m} \in \text{Spec}(U^0)$  be the annihilator of the trivial 1-dimensional representation:

$$\mathfrak{m} = \begin{cases} (H_\alpha \mid \alpha \in \Sigma) & \text{in Case 1} \\ (K_\alpha - 1 \mid \alpha \in \Sigma) & \text{in Case 2.} \end{cases}$$

Let  $\hat{A} = \hat{U}^0$  be the completion of  $U^0$  at  $\mathfrak{m}$ , denoted by  $A(k)$  in [AJS]. Then  $\hat{A}$  is a noetherian complete local domain, flat over  $U^0$ , with maximal ideal  $\mathfrak{m}\hat{A}$  and the residue field  $k$ . One may regard  $\text{Spec}\hat{A}$  as a formal neighbourhood of  $\mathfrak{m}$  in  $\text{Spec}(U^0)$  (cf. [K], pp. 315-316). Note (cf. [B1], Exercise III.2.27(a)) that  $\hat{A}$  is also the completion of  $B$  in the  $\mathfrak{m}B$ -adic topology.

(d2) **Lemma** (cf. [AJS], Lemma 14.2) *If  $A$  is a noetherian complete local domain, the Krull-Schmidt theorem holds in  $\mathcal{C}_A$ .*

(d3) Let  $\mathcal{P}_A$  be the full subcategory of  $\mathcal{C}_A$  consisting of all its projectives.

**Theorem** (cf. [AJS], Proposition 3.3/Theorem 4.19)) (i) *If  $P, Q \in \mathcal{P}_A$ , then  $\mathcal{C}_A(P, Q)$  is projective of finite type in  $\mathbf{Mod}_A$ . If  $A'$  is a noetherian domain over  $A$ , then in  $\mathbf{Mod}_{A'}$*

$$\mathcal{C}_A(P, Q) \otimes_A A' \simeq \mathcal{C}_{A'}(P \otimes_A A', Q \otimes_A A').$$

(ii) *If  $A$  is local with the residue field  $F$ , then  $? \otimes_A F : \mathcal{P}_A \rightarrow \mathcal{P}_F$  gives a bijection between the isomorphism classes.*

(d4) In particular,  $Q^{[i]}(k) \in \mathcal{P}_k$  lifts to

$$Q^{[i]}(\hat{A}) = \Theta_{i_1} \circ \dots \circ \Theta_{i_r} \circ T_{\Delta_i}^{\Omega_0} Z_{\hat{A}}(\nu_i)$$

of  $\mathcal{P}_{\hat{A}}$ , where  $\nu_i = (p-1)\rho + p(w_i \cdot_k 0)^1$ ,  $\Omega_0 = W_a \cdot_k 0$ ,  $\Delta_i = W_a \cdot_k \nu_i$ ,  $\Theta_{i_j} = \Theta_{s_{i_j}} = T_{\Gamma_{i_j}}^{\Omega_0} \circ T_{\Omega_0}^{\Gamma_{i_j}}$  with  $\Gamma_{i_j} = W_a \cdot_k \mu_{s_{i_j}}$  (cf. (a11/12)). We will see

the projectivity of  $Z_{\hat{A}}(\nu_i)$  in (d14). Hence we want now a characteristic free description of  $\mathcal{C}_{\hat{A}}(Q^{[i]}(\hat{A}), Q^{[j]}(\hat{A}))$ .

Let  $\hat{A}^\emptyset = \hat{A}[\frac{1}{H_\alpha} \mid \alpha \in R^+]$  and  $\hat{A}^\beta = \hat{A}[\frac{1}{H_\alpha} \mid \alpha \in R^+ \setminus \{\beta\}]$ ,  $\beta \in R^+$ , with  $H_\alpha = [K_\alpha : 0]$  in Case 2. Note that  $\hat{A}^\emptyset$  and all  $\hat{A}^\beta$  are naturally  $B$ -algebras. Put for simplicity  $\mathcal{C}_\wedge = \mathcal{C}_{\hat{A}}$ ,  $\mathcal{C}_\emptyset = \mathcal{C}_{\hat{A}^\emptyset}$ ,  $\mathcal{C}_\beta = \mathcal{C}_{\hat{A}^\beta}$ , and  $M^\emptyset = M \otimes_{\hat{A}} \hat{A}^\emptyset$ ,  $M^\beta = M \otimes_{\hat{A}} \hat{A}^\beta$  if  $M \in \mathcal{C}_\wedge$ . Let also  $Z_\wedge(\lambda) = Z_{\hat{A}}(\lambda)$ ,  $Z_\emptyset(\lambda) = Z_{\hat{A}^\emptyset}(\lambda) \simeq Z_\wedge(\lambda)^\emptyset$ , and  $Z_\beta(\lambda) = Z_{\hat{A}^\beta}(\lambda) \simeq Z_\wedge(\lambda)^\beta$  for each  $\lambda \in X$ .

(d5) By our standing hypothesis that  $p = \text{ch } k \geq h$  in Case 1, we have

**Lemma** (cf. [AJS], Lemma 9.1)  $\hat{A} = \bigcap_{\beta \in R^+} \hat{A}^\emptyset$ .

(d6) Let  $P, Q \in \mathcal{P}_{\hat{A}}$ . As  $Q$  is  $\hat{A}$ -flat, one may regard  $Q \leq Q^\beta \leq Q^\emptyset$  for each  $\beta \in R^+$ . Then

$$\begin{aligned} \mathcal{C}_\emptyset(P^\emptyset, Q^\emptyset) \\ &\simeq \mathcal{C}_\wedge(P, Q^\emptyset) \simeq \mathcal{C}_\wedge(P, Q) \otimes_{\hat{A}} \hat{A}^\emptyset \text{ as } \hat{A}^\emptyset \text{ is flat over } \hat{A} \text{ (cf. [AJS], Lemma 3.2)} \\ &\geq \mathcal{C}_\wedge(P, Q^\beta) \simeq \mathcal{C}_\wedge(P, Q) \otimes_{\hat{A}} \hat{A}^\beta \text{ as } \hat{A}^\beta \text{ is flat over } \hat{A} \\ &\geq \mathcal{C}_\wedge(P, Q). \end{aligned}$$

As  $\mathcal{C}_\wedge(P, Q)$  is  $\hat{A}$ -flat, one gets from (d5)

$$(1) \quad \mathcal{C}_\wedge(P, Q) = \bigcap_{\beta \in R^+} \mathcal{C}_\beta(P^\beta, Q^\beta) \text{ inside } \mathcal{C}_\emptyset(P^\emptyset, Q^\emptyset).$$

(d7) Now  $\mathcal{C}_\emptyset$  has a simple structure. If  $\text{Frac}(\hat{A})$  is the fractional field of  $\hat{A}$ ,  $\mathcal{C}_{\text{Frac}(\hat{A})}$  is semisimple. To explain that, let us resume the general set-up of  $\mathcal{C}_A$ .

Let  $w \in W$ . Twist  $\pi : U^0 \rightarrow A$  by  $T_w^{-1}$  to define another  $U^0$ -algebra  $A[w]$  with the structure homomorphism  $\pi \circ T_w^{-1}$ . If  $M \in \mathcal{C}_A$ , define  $M[w] \in \mathcal{C}_{A[w]}$  to be the  $A$ -module  $M$  with each  $u \in U$  acting by  $T_w^{-1}(u)$  and the gradation given by  $M[w]_\nu = M_{w^{-1}\nu}$ . Then the functor  $M \mapsto M[w]$  is an equivalence of categories from  $\mathcal{C}_A$  to  $\mathcal{C}_{A[w]}$ . If  $M$  is  $A$ -projective, then

$$\text{ch}(M[w]) = w(\text{ch } M).$$

Working with the positive system  $w(R^+)$  instead of  $R^+$ , define

$$Z_A^w(\lambda) = U \otimes_{U^0 T_w(U^+)} A^\lambda \in \mathcal{C}_A \quad \forall \lambda \in X.$$

Then (cf. [AJS], (4.4)(2)) for each  $x \in W$

$$(1) \quad Z_A^x(\lambda)[w] \simeq Z_{A[w]}^{wx}(w\lambda) \quad \text{in } \mathcal{C}_{A[w]},$$

and (cf. [AJS], Lemma 4.10)

$$(2) \quad Z_A(\lambda)^\tau \simeq Z_A^{w_0}(\lambda - 2(p-1)\rho).$$

In particular (cf. [J], (9.2)),

$$(3) \quad Z_k^{w_0}(\lambda) \simeq \hat{Z}_k(\lambda + 2(p-1)\rho) \text{ of } \S a.$$

(d8) Fix  $\alpha \in \Sigma$  and put  $s = s_\alpha \in \Sigma_\alpha$ . Let  $U(-\alpha)$  be the subalgebra of  $U$  generated by  $E_{-\alpha}$ , and let  $P(\alpha) = U(-\alpha)U^0U^+ \leq U$ . Define a full subcategory  $\mathcal{C}_A^\alpha$  of  $(P(\alpha) \otimes_k A)\mathbf{Mod}$  just like  $\mathcal{C}_A$ . Define likewise  $Z_A^\alpha(\lambda) = P(\alpha) \otimes_{U^0U^+} A^\lambda$  and  $(Z_A^\alpha)^s(\lambda) = P(\alpha) \otimes_{U^0T_s(U^+)} A^\lambda \in \mathcal{C}_A^\alpha$  for each  $\lambda \in X$ . As the multiplication  $U(-\alpha) \otimes_k U^0U^+ \rightarrow P(\alpha)$  is bijective,

(1)  $Z_A^\alpha(\lambda)$  (resp.  $(Z_A^\alpha)^s(\lambda)$ ) is  $A$ -free of basis

$$v_i = E_{-\alpha}^{(i)} \otimes 1 \text{ (resp. } v'_i = E_\alpha^{(i)} \otimes 1 \text{),}$$

where  $E_{-\alpha}^{(i)} = \frac{E_{-\alpha}^i}{i!} \otimes 1$  (resp.  $E_\alpha^{(i)} = \frac{E_\alpha^i}{[i]_{d_\alpha}!} \otimes 1$ ) in Case 1 (resp. Case 2).

One has (cf. [AJS], (5.4))

$$(2) \quad P(\alpha) = U(-\alpha)U^0U(\alpha) \oplus Q(\alpha) \text{ with } Q(\alpha) = \coprod_{\nu \notin \mathbb{Z}\alpha} P(\alpha)_\nu,$$

$$(3) \quad T_s \text{ stabilizes all } P(\alpha), U(-\alpha)U^0U(\alpha) \text{ and } Q(\alpha),$$

and that

$$(4) \quad Q(\alpha) \text{ annihilates both } Z_A^\alpha(\lambda) \text{ and } (Z_A^\alpha)^s(\lambda).$$

Hence one can describe the  $P(\alpha)$ -action on both  $Z_A^\alpha(\lambda)$  and  $(Z_A^\alpha)^s(\lambda)$  explicitly (cf. [AJS], (5.5)). In particular, there is unique

$$(5) \quad \phi_\alpha \in \mathcal{C}_A^\alpha(Z_A^\alpha(\lambda), (Z_A^\alpha)^s(\lambda - (p-1)\alpha)) \text{ such that } v_0 \mapsto v'_{p-1}.$$

Then  $\phi_\alpha$  forms an  $A$ -basis of  $\mathcal{C}_A^\alpha(Z_A^\alpha(\lambda), (Z_A^\alpha)^s(\lambda - (p-1)\alpha))$  and one has (cf. [AJS], (5.6))

$$(6) \quad \phi_\alpha(v_i) = \begin{cases} (-1)^i v'_{p-1-i} \left( \pi^{(H_\alpha) + \langle \lambda, \alpha^\vee \rangle} \right)_i & \text{in Case 1} \\ (-1)^i v'_{p-1-i} \pi \left( [K_{\alpha i} \langle \lambda, \alpha^\vee \rangle] \right)_i & \text{in Case 2.} \end{cases}$$

It follows that

(7) *if  $\alpha \notin R_\pi$ , then  $\phi_\alpha$  is bijective.*

If  $\alpha \in R_\pi$ , let  $n_\alpha(\lambda) \in [1, p]$  such that  $\pi(H_\alpha) + \langle \lambda + \rho, \alpha^\vee \rangle = n_\alpha(\lambda) \cdot 1$  in Case 1 (resp.  $\pi(K_\alpha)^2 \zeta^{2d_\alpha \langle \lambda + \rho, \alpha^\vee \rangle} = \zeta^{2d_\alpha n_\alpha(\lambda)}$  in Case 2). One has (cf. [AJS], (5.9)) that

(8) *if  $n_\alpha(\lambda) = p$ , then  $\phi_\alpha$  is still bijective.*

(d9) If  $w \in W$ , from  $\phi_\alpha$  over  $A[w^{-1}]$  one gets

$$(1) \quad \phi \in \mathcal{C}_A(Z_A^w(w\lambda), Z_A^{ws}(w\lambda - (p-1)w\alpha))$$

such that the diagram

$$\begin{array}{ccc} Z_A^w(w\lambda) & \xrightarrow{\phi} & Z_A^{ws}(w\lambda - (p-1)w\alpha) \\ \wr \downarrow & & \downarrow \wr \\ Z_{A[w^{-1}]}(\lambda)[w] & & Z_{A[w^{-1}]}^s(\lambda - (p-1)\alpha)[w] \\ \wr \downarrow & & \downarrow \wr \\ (U \otimes_{P(\alpha)} Z_{A[w^{-1}]}^\alpha(\lambda))[w] & \xrightarrow{(U \otimes_{P(\alpha)} \phi_\alpha)[w]} & \{U \otimes_{P(\alpha)} (Z_{A[w^{-1}]}^\alpha)^s(\lambda - (p-1)\alpha)\}[w]. \end{array}$$

commutes. As  $\phi$  sends the standard generator of  $Z_A^w(w\lambda)$  to an  $A$ -basis element of  $Z_A^{ws}(w\lambda - (p-1)w\alpha)_{w\lambda}$ ,

(2)  *$\phi$  is an  $A$ -basis of  $\mathcal{C}_A(Z_A^w(w\lambda), Z_A^{ws}(w\lambda - (p-1)w\alpha))$ .*

One may compare the construction of  $\phi$  with the intertwining homomorphism

$$H^i(\mathfrak{G}_k/\mathfrak{B}_k, \mathcal{L}(s_\alpha \cdot_k \nu)) \longrightarrow H^{i-1}(\mathfrak{G}_k/\mathfrak{B}_k, \mathcal{L}(\nu))$$

for  $\alpha \in \Sigma$  and  $\nu \in X$  with  $\langle \nu + \rho, \alpha^\vee \rangle \geq 0$  in  $\mathfrak{G}_k \mathbf{Mod} [J]$ , (II.5/6).

Choose a reduced expression  $w_0 = s_1 s_2 \dots s_N$  of  $w_0$ . If  $w_i = s_1 s_2 \dots s_{i-1}$ ,  $1 \leq i \leq N+1$ , with  $w_1 = 1$ , and if  $\lambda \langle w_i \rangle = \lambda + (p-1)(w_i \rho - \rho)$ , one gets an  $A$ -basis  $\phi_i$  of  $\mathcal{C}_A(Z_A^{w_i}(\lambda \langle w_i \rangle), Z_A^{w_{i+1}}(\lambda \langle w_{i+1} \rangle))$  like  $\phi$  of (1). One gets from (d8)(7)

(3) *if  $R_\pi = \emptyset$ , then  $Z_A(\lambda) \simeq Z_A^w(\lambda \langle w \rangle) \forall w \in W$ ,*

i.e., the “Borel-Weil-Bott” theorem holds in  $\mathcal{C}_A$  if  $R_\pi = \emptyset$ .

(d10) Let  $\Phi = \phi_N \circ \dots \circ \phi_1 \in \mathcal{C}_A(Z_A(\lambda), Z_A^{w_0}(\lambda - 2(p-1)\rho))$ .

**Lemma** (cf. [AJS], Lemma 5.13) *The morphism  $\Phi$  is nonzero and forms an  $A$ -basis of  $\mathcal{C}_A(Z_A(\lambda), Z_A^{w_0}(\lambda - 2(p-1)\rho))$ .*

(d11) **Lemma** (cf. [AJS], Lemma 4.9) *If  $A = F$  is a field, then*

$$L_F(\lambda) = \text{im } \Phi = \text{soc}_{\mathcal{C}_F} Z_F^{w_0}(\lambda - 2(p-1)\rho).$$

(d12) For each  $\beta \in R_\pi$  define  $n_\beta \in [1, p]$  as in (d8). One now obtains

**Lemma** (cf. [AJS], Lemma 6.3) *Assume  $A = F$  is a field with the structure homomorphism  $\pi$ .*

- (i) *If  $\lambda \in X$  with  $n_\beta(\lambda) = p$  for each  $\beta \in R_\pi^+$ , then  $Z_F(\lambda) \simeq L_F(\lambda) \simeq Q_F(\lambda)$  in  $\mathcal{C}_F$ .*
- (ii) *If  $R_\pi^+ = \emptyset$ , then  $Z_F(\lambda) \simeq L_F(\lambda) \simeq Q_F(\lambda)$  for each  $\lambda \in X$ , i.e.,  $\mathcal{C}_F$  is a semisimple category.*

*Proof.* As  $\phi$  is bijective,  $L_F(\lambda) \simeq Z_F(\lambda)$  for each  $\lambda \in X$  by (d11). If  $\mu \in X$ , then (cf. [AJS], Proposition 4.6)

$$\begin{aligned} \text{Ext}_{\mathcal{C}_F}^1(L_F(\lambda), L_F(\mu)) &\simeq \text{Ext}_{\mathcal{C}_F}^1(L_F(\mu), L_F(\lambda)) \quad \text{using the duality } \tau \\ &\simeq \mathcal{C}_F(\text{rad}_{\mathcal{C}_F} Z_F(\lambda), L_F(\mu)) \quad \text{if } \mu \neq \lambda \\ &= 0. \end{aligned}$$

Hence  $L_F(\lambda)$  is both projective and injective in  $\mathcal{C}_F$ .

(d13) **Proposition** (cf. [AJS], Corollary 3.5) *Let  $M \in \mathcal{C}_A$  with a  $Z$ -filtration. Then  $M$  is projective in  $\mathcal{C}_A$  iff  $M \otimes_A (A/\mathfrak{m})$  is projective in  $\mathcal{C}_{A/\mathfrak{m}}$  for each maximal ideal  $\mathfrak{m}$  of  $A$ .*

(d14) We conclude from (d12/13) that for each  $\lambda \in X$

- (1)  $Z_A((p-1)\rho + p\lambda)$  is projective in  $\mathcal{C}_A$ ,
- (2) the block of  $\lambda$  over  $\hat{A}^\emptyset$  is a singleton  $\{\lambda\}$ ,

and that

(3)  $Z_\emptyset(\lambda)$  is a progenerator of  $\mathcal{C}_\emptyset(\{\lambda\})$ .

Back to  $P, Q \in \mathcal{P}_{\hat{A}}$ , one can write  $P^\emptyset = \coprod_{\lambda \in X} Z_\emptyset(\lambda)^{p_\lambda}$  and  $Q^\emptyset = \coprod_{\lambda \in X} Z_\emptyset(\lambda)^{q_\lambda}$  with  $p_\lambda, q_\lambda \in \mathbb{N}$ . Then

$$\mathcal{C}_\emptyset(P^\emptyset, Q^\emptyset) \simeq (\hat{A}^\emptyset)^{\sum_{\lambda \in X} p_\lambda q_\lambda}.$$

In particular, if  $P^\emptyset = Q^{[i]}(\hat{A}^\emptyset) = Q^{[i]}(\hat{A})^\emptyset$  and  $Q^\emptyset = Q^{[j]}(\hat{A}^\emptyset) = Q^{[j]}(\hat{A})^\emptyset$ ,  $p_\lambda$  (resp.  $q_\lambda$ ) are determined independent of  $k$ , hence

(4)  $\mathcal{C}_\emptyset(Q^{[i]}(\hat{A}^\emptyset), Q^{[j]}(\hat{A}^\emptyset))$  is described independent of  $k$ .

(d15) More generally,

**Lemma** (cf. [AJS], E.4) *Let  $\lambda \in X$ . For each  $M, N \in \mathcal{C}_\emptyset(\{\lambda\})$  one has an isomorphism of  $\hat{A}^\emptyset$ -modules*

$$\mathcal{C}_\emptyset(M, N) \longrightarrow \mathbf{Mod}_{\hat{A}^\emptyset}(\mathcal{C}_\emptyset(Z_\emptyset(\lambda), M), \mathcal{C}_\emptyset(Z_\emptyset(\lambda), N)) \quad \text{via} \quad f \longmapsto f \circ ?.$$

*Proof.* Put  $P = Z_\emptyset(\lambda)$  and  $M(\lambda) = \mathcal{C}_\emptyset(Z_\emptyset(\lambda), M)$ , likewise  $N(\lambda)$ . Consider first the case  $M = P^m$  and  $N = P^n$  for  $m, n \in \mathbb{N}^+$ . If  $\pi_s : P^m \rightarrow P$  (resp.  $i_r : P \rightarrow P^n$ ) is the projection onto the  $s$ -th (resp. injection from the  $r$ -th) component, one has a commutative diagram

$$\begin{array}{ccc} \mathcal{C}_\emptyset(P^m, P^n) & \longrightarrow & \mathbf{Mod}_{\hat{A}^\emptyset}(P^m(\lambda), P^n(\lambda)) \\ \mathcal{C}_\emptyset(P^m, i_r) \uparrow & & \uparrow \mathbf{Mod}_{\hat{A}^\emptyset}(P^m(\lambda), \mathcal{C}_\emptyset(P, i_r)) \\ \mathcal{C}_\emptyset(P^m, P) & & \mathbf{Mod}_{\hat{A}^\emptyset}(P^m(\lambda), P(\lambda)) \\ \mathcal{C}_\emptyset(\pi_s, P) \uparrow & & \uparrow \mathbf{Mod}_{\hat{A}^\emptyset}(\mathcal{C}_\emptyset(P, \pi_s), P(\lambda)) \\ \mathcal{C}_\emptyset(P, P) & \longrightarrow & \mathbf{Mod}_{\hat{A}^\emptyset}(P(\lambda), P(\lambda)) \\ f & \longmapsto & f \circ ? \end{array}$$

with the bottom horizontal map bijective as  $P(\lambda) = \mathcal{C}_\emptyset(P, P) \simeq \hat{A}^\emptyset$ . Hence

(1) *the assertion holds with  $M = P^m$  and  $N = P^n$ .*

If  $N$  is arbitrary, as  $P$  is a generator of  $\mathcal{C}_\emptyset(\{\lambda\})$ ,  $N$  admits a finite presentation in  $\mathcal{C}_\emptyset(\{\lambda\})$ :  $P^{n'} \longrightarrow P^n \rightarrow N \rightarrow 0$  exact. Then  $P^{n'}(\lambda) \rightarrow P^n(\lambda) \rightarrow N(\lambda) \rightarrow 0$  remains exact as  $?( \lambda)$  is exact, hence one gets a commutative diagram

$$\begin{array}{ccc}
 \mathcal{C}_\emptyset(P^m, P^{n'}) & \longrightarrow & \mathbf{Mod}_{\hat{A}^\emptyset}(P^m(\lambda), P^{n'}(\lambda)) \\
 \downarrow & & \downarrow \\
 \mathcal{C}_\emptyset(P^m, P^n) & \longrightarrow & \mathbf{Mod}_{\hat{A}^\emptyset}(P^m(\lambda), P^n(\lambda)) \\
 \downarrow & & \downarrow \\
 \mathcal{C}_\emptyset(P^m, N) & \longrightarrow & \mathbf{Mod}_{\hat{A}^\emptyset}(P^m(\lambda), N(\lambda)) \\
 \downarrow & & \downarrow \\
 0 & & 0.
 \end{array}$$

As  $P^m$  (resp.  $P^m(\lambda) \simeq (\hat{A}^\emptyset)^m$ ) is a projective of  $\mathcal{C}_\emptyset$  (resp.  $\mathbf{Mod}_{\hat{A}^\emptyset}$ ), the left and the right vertical sequences are both exact. By (1) the top and the middle horizontal maps are bijective, hence also the bottom by the 5-lemma, i.e.,

(2) *the assertion holds if  $M = P^m$ .*

Finally, write  $P^{m'} \rightarrow P^m \rightarrow M \rightarrow 0$  exact in  $\mathcal{C}_\emptyset$ . One then gets a commutative diagram of exact columns

$$\begin{array}{ccc}
 0 & & 0 \\
 \downarrow & & \downarrow \\
 \mathcal{C}_\emptyset(M, N) & \longrightarrow & \mathbf{Mod}_{\hat{A}^\emptyset}(M(\lambda), N(\lambda)) \\
 \downarrow & & \downarrow \\
 \mathcal{C}_\emptyset(P^m, N) & \longrightarrow & \mathbf{Mod}_{\hat{A}^\emptyset}(P^m(\lambda), N(\lambda)) \\
 \downarrow & & \downarrow \\
 \mathcal{C}_\emptyset(P^{m'}, N) & \longrightarrow & \mathbf{Mod}_{\hat{A}^\emptyset}(P^{m'}(\lambda), N(\lambda)).
 \end{array}$$

As the middle and the bottom horizontal maps are bijective by (2), the top horizontal map is bijective by the 5-lemma again, hence the assertion.

(d16) More detailed examination of the  $\phi_i$  and  $\Phi$  shows that  $\mathcal{C}_\beta$  behaves like  $\mathfrak{sl}_2$ -category. If  $\lambda \in X$ ,  $\beta \in R^+$ , and if  $n \in \mathbb{N}$  minimal with  $\langle \lambda + \rho, \beta^\vee \rangle \equiv -n \pmod p$ , put  $\beta \uparrow \lambda = \lambda + n\beta$ .

**Theorem** (cf. [AJS], Proposition 8.6/Corollary 8.7) *Let  $\lambda \in X$  and  $\beta \in R^+$ .*

- (i) *If  $\beta \uparrow \lambda = \lambda$ , then  $Z_\beta(\lambda)$  is a projective of  $\mathcal{C}_\beta$ .*
- (ii) *Suppose  $\beta \uparrow \lambda > \lambda$ . Then*

$$\mathrm{Ext}_{\mathcal{C}_\beta}^1(Z_\beta(\lambda), Z_\beta(\beta \uparrow \lambda)) \simeq \hat{A}^\beta / H_\beta \hat{A}^\beta \quad \text{in } \mathbf{Mod}_{\hat{A}^\beta}.$$

*Given a short exact sequence  $0 \rightarrow Z_\beta(\beta \uparrow \lambda) \rightarrow Q \rightarrow Z_\beta(\lambda) \rightarrow 0$  in  $\mathcal{C}_\beta$ ,  $Q$  is projective in  $\mathcal{C}_\beta$  iff the sequence generates  $\mathrm{Ext}_{\mathcal{C}_\beta}^1(Z_\beta(\lambda), Z_\beta(\beta \uparrow \lambda))$  over  $\hat{A}^\beta$ .*

## e° Combinatorial categories

(e1) In order to glue together all the  $\mathfrak{sl}_2$ -categories  $\mathcal{C}_\beta$  to recover  $\mathcal{C}_\wedge = \mathcal{C}_{\hat{A}^\emptyset}$ , we introduce a combinatorial category  $\mathcal{K}(\Omega)$  for each  $W_a$ -orbit  $\Omega$ . An object of  $\mathcal{K}(\Omega)$  is a family  $(\mathcal{M}(\lambda))_{\lambda \in \Omega}$  of  $\hat{A}^\emptyset$ -modules of finite type, only finitely many nonzero members, together with  $\hat{A}^\beta$ -submodules  $\mathcal{M}(\lambda, \beta)$  of finite type for each  $\lambda \in \Omega$  and  $\beta \in R^+$  of  $\mathcal{M}(\lambda) \oplus \mathcal{M}(\beta \uparrow \lambda)$  if  $\beta \uparrow \lambda > \lambda$  (resp.  $\mathcal{M}(\lambda)$  if  $\beta \uparrow \lambda = \lambda$ ). A morphism of  $\mathcal{K}(\Omega)$  is  $(\psi_\lambda)_{\lambda \in \Omega} \in \prod_{\lambda \in \Omega} \mathcal{C}_\emptyset(\mathcal{M}(\lambda), \mathcal{M}'(\lambda))$  such that for each  $\lambda \in \Omega$  and  $\beta \in R^+$

$$\begin{aligned} (\psi_\lambda \oplus \psi_{\beta \uparrow \lambda}) \mathcal{M}(\lambda, \beta) &\subseteq \mathcal{M}'(\lambda, \beta) \quad \text{if } \beta \uparrow \lambda > \lambda \\ \psi_\lambda \mathcal{M}(\lambda, \beta) &\subseteq \mathcal{M}'(\lambda, \beta) \quad \text{if } \beta \uparrow \lambda = \lambda. \end{aligned}$$

(e2) We define a functor  $\mathcal{V}_\Omega : \mathcal{C}_\wedge(\Omega) \rightarrow \mathcal{K}(\Omega)$ , that depends on the choice of  $e^\beta(\lambda) \in \mathrm{Ext}_{\mathcal{C}_\beta}^1(Z_\beta(\lambda), Z_\beta(\beta \uparrow \lambda))$  for  $\lambda \in \Omega$  and  $\beta \in R^+$  with  $\beta \uparrow \lambda > \lambda$ . If  $M \in \mathcal{C}_\wedge(\Omega)$ , set  $(\mathcal{V}_\Omega M)(\lambda) = \mathcal{C}_\emptyset(Z_\emptyset(\lambda), M^\emptyset)$ , and if  $\beta \uparrow \lambda = \lambda$ , let  $(\mathcal{V}_\Omega M)(\lambda, \beta) = \mathcal{C}_\beta(Z_\beta(\lambda), M^\beta)$ . If  $\beta \uparrow \lambda > \lambda$ , represent  $e^\beta(\lambda)$  by a short exact sequence in  $\mathcal{C}_\beta$

$$0 \longrightarrow Z_\beta(\beta \uparrow \lambda) \longrightarrow Q^\beta(\lambda) \longrightarrow Z_\beta(\lambda) \longrightarrow 0.$$

Tensoring with  $\hat{A}^\emptyset$ , the sequence splits uniquely to yield an isomorphism  $Q^\beta(\lambda)^\emptyset \simeq Z_\emptyset(\beta \uparrow \lambda) \oplus Z_\emptyset(\lambda)$ . We set  $(\mathcal{V}_\Omega M)(\lambda, \beta)$  to be the image of the composite of the natural maps

$$\begin{aligned} \mathcal{C}_\beta(Q^\beta(\lambda), M^\beta) &\longrightarrow \mathcal{C}_\emptyset(Q^\beta(\lambda)^\emptyset, M^\emptyset) \\ &\xrightarrow{\sim} \mathcal{C}_\emptyset(Z_\emptyset(\lambda) \oplus Z_\emptyset(\beta \uparrow \lambda), M^\emptyset) \xrightarrow{\sim} (\mathcal{V}_\Omega M)(\lambda) \oplus (\mathcal{C}_\Omega M)(\beta \uparrow \lambda). \end{aligned}$$

(e3) Let  $\mathcal{FC}_\wedge(\Omega)$  be the full subcategory of  $\mathcal{C}_\wedge(\Omega)$  consisting of all  $\hat{A}$ -flat objects.

**Theorem** (cf. [AJS], Proposition 9.4) *Choose all  $e^\beta(\lambda)$  as generators of  $\text{Ext}_{\mathcal{C}_\beta}^1(Z_\beta(\lambda), Z_\beta(\beta \uparrow \lambda))$ . Then  $\mathcal{V}_\Omega : \mathcal{FC}_\wedge(\Omega) \rightarrow \mathcal{K}(\Omega)$  is fully faithful.*

*Proof.* Let  $M, N \in \mathcal{FC}_\wedge(\Omega)$ . One must show

$$(\mathcal{V}_\Omega)_{M,N} : \mathcal{C}_\wedge(M, N) \longrightarrow \mathcal{K}(\Omega)(\mathcal{V}_\Omega M, \mathcal{V}_\Omega N)$$

is an isomorphism. By (c10) and (d14)(2)

$$\begin{aligned} \mathcal{C}_\emptyset(M^\emptyset, N^\emptyset) &= \mathcal{C}_\emptyset\left(\prod_{\lambda \in \Omega} (M^\emptyset)_{\{\lambda\}}, \prod_{\lambda \in \Omega} (N^\emptyset)_{\{\lambda\}}\right) \\ &\simeq \prod_{\lambda} \mathcal{C}_\emptyset((M^\emptyset)_{\{\lambda\}}, \prod_{\mu \in \Omega} (N^\emptyset)_{\{\mu\}}) \simeq \prod_{\lambda} \mathcal{C}_\emptyset((M^\emptyset)_{\{\lambda\}}, (N^\emptyset)_{\{\lambda\}}) \\ &\simeq \prod_{\lambda} \mathbf{Mod}_{\hat{A}^\emptyset}(\mathcal{C}_\emptyset(Z_\emptyset(\lambda), (M^\emptyset)_{\{\lambda\}}), \mathcal{C}_\emptyset(Z_\emptyset(\lambda), (N^\emptyset)_{\{\lambda\}})) \quad \text{by (d15)} \\ &\simeq \prod_{\lambda} \mathbf{Mod}_{\hat{A}^\emptyset}(\mathcal{C}_\emptyset(Z_\emptyset(\lambda), M^\emptyset), \mathcal{C}_\emptyset(Z_\emptyset(\lambda), N^\emptyset)). \end{aligned}$$

Hence  $(\mathcal{V}_\Omega)_{M,N}$  is injective:

$$\begin{array}{ccc} \mathcal{C}_\wedge(M, N) & \xrightarrow{(\mathcal{V}_\Omega)_{M,N}} & \mathcal{K}(\Omega)(\mathcal{V}_\Omega M, \mathcal{V}_\Omega N) \\ \wedge | & & \wedge | \\ (3) \quad \mathcal{C}_\wedge(M, N^\emptyset) & \xrightarrow{\quad} & \prod_{\lambda \in \Omega} \mathbf{Mod}_{\hat{A}^\emptyset}((\mathcal{V}_\Omega M)(\lambda), (\mathcal{V}_\Omega N)(\lambda)) \\ \wr | & & \parallel \\ \mathcal{C}_\emptyset(M^\emptyset, N^\emptyset) & \xrightarrow{\sim} & \prod_{\lambda} \mathbf{Mod}_{\hat{A}^\emptyset}(\mathcal{C}_\emptyset(Z_\emptyset(\lambda), M^\emptyset), \mathcal{C}_\emptyset(Z_\emptyset(\lambda), N^\emptyset)) \\ h & \longmapsto & h \circ ?. \end{array}$$

To see the surjectivity, let  $\psi \in \mathcal{K}(\Omega)(\mathcal{V}_\Omega M, \mathcal{V}_\Omega N)$ . By (1) there is  $h \in \mathcal{C}_\emptyset(M^\emptyset, N^\emptyset)$  such that for each  $\lambda \in \Omega$

$$\psi_\lambda = h \circ ? \quad \text{in } \mathbf{Mod}_{\hat{A}^\emptyset}((\mathcal{V}_\Omega M)(\lambda), (\mathcal{V}_\Omega N)(\lambda)).$$

Let  $\beta \in R^+$ . For  $\lambda \in \Omega$  let  $Q^\beta(\lambda)$  be the middle term of the short exact sequence representing  $e^\beta(\lambda)$  if  $\beta \uparrow \lambda > \lambda$  (resp.  $Z^\beta(\lambda)$  if  $\beta \uparrow \lambda = \lambda$ ). If  $\beta \uparrow \lambda > \lambda$ , one gets a commutative diagram

$$\begin{array}{ccc} \mathcal{C}_\beta(Q^\beta(\lambda), M^\beta) & \dashrightarrow & \mathcal{C}_\beta(Q^\beta(\lambda), N^\beta) \\ \wedge | & & \wedge | \\ \mathcal{C}_\beta(Q^\beta(\lambda), M^\emptyset) & \xrightarrow[u \mapsto h \circ u]{} & \mathcal{C}_\beta(Q^\beta(\lambda), N^\emptyset) \\ \wr | & & | \wr \\ \mathcal{C}_\emptyset(Q^\beta(\lambda) \otimes_{\hat{A}^\beta} \hat{A}^\emptyset, M^\emptyset) & & \mathcal{C}_\emptyset(Q^\beta(\lambda) \otimes_{\hat{A}^\beta} \hat{A}^\emptyset, N^\emptyset) \\ \wr | & & | \wr \\ (\mathcal{V}_\Omega M)(\lambda) \oplus (\mathcal{V}_\Omega M)(\beta \uparrow \lambda) & \xrightarrow[\psi_\lambda \oplus \psi_{\beta \uparrow \lambda}]{} & (\mathcal{V}_\Omega N)(\lambda) \oplus (\mathcal{V}_\Omega N)(\beta \uparrow \lambda), \end{array}$$

hence each  $h \circ u$ ,  $u \in \mathcal{C}_\beta(Q^\beta(\lambda), M^\beta)$  factors through  $N^\beta$ . Likewise if  $\beta \uparrow \lambda = \lambda$ .

On the other hand, one can write by definition  $\coprod Z_\beta(\lambda_i) \twoheadrightarrow M^\beta$  with the coproduct running over some  $\lambda_i \in \Omega$ , hence  $\coprod Q^\beta(\lambda_i) \twoheadrightarrow M^\beta$ , i.e., there are  $u_i \in \mathcal{C}_\beta(Q^\beta(\lambda_i), M^\beta)$  such that  $M^\beta = \sum \text{im}(u_i)$ . Then

$$h(M^\beta) = h \sum \text{im}(u_i) = \sum \text{im}(h \circ u_i) \subseteq N^\beta.$$

Hence  $h(M) = h(\bigcap_{\beta \in R^+} M^\beta) \leq \bigcap_{\beta \in R^+} h(M^\beta) \leq \bigcap_{\beta \in R^+} N^\beta = N$ , the last equality following from (d5). Consequently,  $h$  arises from  $\mathcal{C}_\wedge(M, N)$  with  $(\mathcal{V}_\Omega)_{M, N}(h) = \psi$ , as desired.

(e4) To get a characteristic free description of  $\mathcal{C}_\wedge(Q^{[i]}(\hat{A}), Q^{[j]}(\hat{A}))$ , it is now enough to find a characteristic free description of

$$\mathcal{K}(\Omega_0)(\mathcal{V}_{\Omega_0}(Q^{[i]}(\hat{A})), \mathcal{V}_{\Omega_0}(Q^{[j]}(\hat{A})))$$

with  $\Omega_0 = W_a \cdot_k 0$ . Define  $\mathcal{Z}_{\nu_i}(\hat{A}) \in \mathcal{K}(\Delta_i)$  by setting for each  $\mu \in \Delta_i =$

$W_a \cdot k \nu_i$

$$\mathcal{Z}_{\nu_i}(\hat{A})(\mu) = \begin{cases} \hat{A}^\emptyset & \text{if } \mu = \nu_i \\ 0 & \text{otherwise,} \end{cases}$$

and for each  $\beta \in R^+$

$$\mathcal{Z}_{\nu_i}(\hat{A})(\mu, \beta) = \begin{cases} \hat{A}^\beta & \text{if } \mu = \nu_i \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$(1) \quad \mathcal{V}_{\Delta_i}(Z_\wedge(\nu_i)) \simeq \mathcal{Z}_{\nu_i}(\hat{A}).$$

(e5) We want next to construct a translation functor  $\mathcal{T} : \mathcal{K}(\Omega) \rightarrow \mathcal{K}(\Gamma)$  for  $W_a$ -orbits  $\Omega$  and  $\Gamma$  such that  $\mathcal{V}_\Gamma \circ T_\Omega^\Gamma \simeq \mathcal{T} \circ \mathcal{V}_\Omega$ . We will consider only the case that for each  $\lambda \in \Omega$  there is a unique  $\mu \in \Gamma$  that lies in the closure of the facet of  $\lambda$ , which we will denote by  $\lambda_\Gamma$ .

Put  $T = T_\Omega^\Gamma$  and  $T' = T_\Gamma^\Omega$ . For each  $\lambda \in \Omega$  choose an isomorphism  $f_\lambda \in \mathcal{C}_\wedge(Z_\wedge(\lambda_\Gamma), TZ_\wedge(\lambda))^\times$ . Let  $\beta \in R^+$  with  $\beta \uparrow \lambda > \lambda$ . Define

$$t[f_\lambda, f_{\beta \uparrow \lambda}] : \text{Ext}_{\mathcal{C}_\wedge}^1(Z_\wedge(\lambda), Z_\wedge(\beta \uparrow \lambda)) \longrightarrow \text{Ext}_{\mathcal{C}_\wedge}^1(Z_\wedge(\lambda_\Gamma), Z_\wedge((\beta \uparrow \lambda)_\Gamma))$$

by sending each short exact sequence  $0 \rightarrow Z_\wedge(\beta \uparrow \lambda) \rightarrow Q \rightarrow Z_\wedge(\lambda) \rightarrow 0$  to the bottom horizontal exact sequence of the commuting diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & TZ_\wedge(\beta \uparrow \lambda) & \longrightarrow & TQ & \longrightarrow & TZ_\wedge(\lambda) \longrightarrow 0 \\ & & \uparrow f_{\beta \uparrow \lambda} & & \parallel & & \uparrow f_\lambda \\ 0 & \longrightarrow & Z_\wedge((\beta \uparrow \lambda)_\Gamma) & \longrightarrow & TQ & \longrightarrow & Z_\wedge(\lambda_\Gamma) \longrightarrow 0, \end{array}$$

where the top horizontal sequence is the one obtained by hitting  $T$  on the first exact sequence.

Assume first  $\beta \uparrow \lambda_\Gamma > \lambda_\Gamma$ , so that  $(\beta \uparrow \lambda)_\Gamma = \beta \uparrow \lambda_\Gamma$ . Let  $W_\beta = \langle s_\beta \rangle \rtimes \mathbb{Z}\beta \leq W_a$  and let  $T_0 = T_{W_{\beta \cdot k\lambda}}^{W_{\beta \cdot k\lambda_\Gamma}}$ ,  $T'_0 = T_{W_{\beta \cdot k\lambda_\Gamma}}^{W_{\beta \cdot k\lambda}}$ . As  $\beta \uparrow \lambda_\Gamma > \lambda_\Gamma$ ,

$$T'_0 Z_\beta(\lambda_\Gamma) \simeq Z_\beta(\lambda) \quad \text{and} \quad T'_0 Z_\beta(\beta \uparrow \lambda_\Gamma) \simeq Z_\beta(\beta \uparrow \lambda).$$

Let  $f'_\lambda = \text{adj}_1^{-1}(f_\lambda^{-1}) \in \mathcal{C}_\beta(Z_\beta(\lambda), T'_0 Z_\beta(\lambda_\Gamma))^\times$  using the adjunction  $\text{adj}_1$  with respect to  $T_0$  and  $T'_0$ , and likewise  $f'_{\beta \uparrow \lambda}$ . Define

$$t[f'_\lambda, f'_{\beta \uparrow \lambda}] : \text{Ext}_{\mathcal{C}_\beta}^1(Z_\beta(\lambda_\Gamma), Z_\beta((\beta \uparrow \lambda)_\Gamma)) \longrightarrow \text{Ext}_{\mathcal{C}_\beta}^1(Z_\beta(\lambda), Z_\beta(\beta \uparrow \lambda))$$

just like  $t[f_\lambda, f_{\beta \uparrow \lambda}]$  replacing  $T$  by  $T'_0$ . Then (cf. [AJS], (10.6)(1))

$$(1) \quad t[f_\lambda, f_{\beta \uparrow \lambda}] \circ t[f'_\lambda, f'_{\beta \uparrow \lambda}] = \text{id}_{\text{Ext}_{\mathcal{C}_\beta}^1(Z_\beta(\lambda_\Gamma), Z_\beta((\beta \uparrow \lambda)_\Gamma))}.$$

Suppose we have chosen an  $\hat{A}^\beta$ -generator  $e^\beta(\lambda)$  (resp.  $e^\beta(\lambda_\Gamma)$ ) of

$$\text{Ext}_{\mathcal{C}_\beta}^1(Z_\beta(\lambda), Z_\beta(\beta \uparrow \lambda)) \quad (\text{resp. } \text{Ext}_{\mathcal{C}_\beta}^1(Z_\beta(\lambda_\Gamma), Z_\beta((\beta \uparrow \lambda)_\Gamma))).$$

Then one can write with some  $a_\lambda^\beta$  and  $b_\lambda^\beta \in \hat{A}^\beta$

$$t[f_\lambda, f_{\beta \uparrow \lambda}]e^\beta(\lambda) = a_\lambda^\beta e^\beta(\lambda_\Gamma) \quad \text{and} \quad t[f'_\lambda, f'_{\beta \uparrow \lambda}]e^\beta(\lambda_\Gamma) = b_\lambda^\beta e^\beta(\lambda).$$

By (1) and (d16)

$$(2) \quad a_\lambda^\beta b_\lambda^\beta \in 1 + H_\beta \hat{A}^\beta \quad \text{in } \hat{A}^\beta / H_\beta \hat{A}^\beta.$$

Assume next  $\beta \uparrow \lambda_\Gamma = \lambda_\Gamma = (\beta \uparrow \lambda)_\Gamma$ . Define an isomorphism

$$\theta[f_\lambda, f_{\beta \uparrow \lambda}] \in \mathbf{Mod}_{\hat{A}^\beta}(\text{Ext}_{\mathcal{C}_\beta}^1(Z_\beta(\lambda), Z_\beta(\beta \uparrow \lambda)), \hat{A}^\beta H_\beta^{-1} / \hat{A}^\beta)^\times$$

as follows (cf. [AJS], Proposition 8.14). Let  $e \in \text{Ext}_{\mathcal{C}_\beta}^1(Z_\beta(\lambda), Z_\beta(\beta \uparrow \lambda))$  represented by a short exact sequence

$$0 \longrightarrow Z_\beta(\beta \uparrow \lambda) \xrightarrow{i} Q \xrightarrow{j} Z_\beta(\lambda) \longrightarrow 0.$$

As  $eH_\beta = 0$  in  $\text{Ext}_{\mathcal{C}_\beta}^1(Z_\beta(\lambda), Z_\beta(\beta \uparrow \lambda))$  by (d16), there is a unique  $j' \in \mathcal{C}_\beta(Z_\beta(\lambda), Q)$  such that  $j \circ j' = H_\beta \text{id}_{Z_\beta(\lambda)}$  (cf. [B2], (X.119) Proposition 4/ (X.120) Corollary 3(ii)): one has a commutative diagram of short exact sequences with the top sequence representing  $eH_\beta$

$$\begin{array}{ccccccc} 0 & \longrightarrow & Z_\beta(\beta \uparrow \lambda) & \longrightarrow & Z_\beta(\beta \uparrow \lambda) \oplus Z_\beta(\lambda) & \longrightarrow & Z_\beta(\lambda) \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow H_\beta \\ 0 & \longrightarrow & Z_\beta(\beta \uparrow \lambda) & \xrightarrow{i} & Q & \xrightarrow{j} & Z_\beta(\lambda) \longrightarrow 0. \end{array}$$

As  $TZ_\beta(\lambda) \simeq Z_\beta(\lambda_\Gamma) \simeq TZ_\beta(\beta \uparrow \lambda)$ ,  $Te$  splits by (d16) to yield  $i' \in \mathcal{C}_\beta(TQ, TZ_\beta(\beta \uparrow \lambda))$  such that  $i' \circ Ti = \text{id}_{TZ_\beta(\beta \uparrow \lambda)}$ . Then one can write

$$f_{\beta \uparrow \lambda}^{-1} \circ i' \circ Tj' \circ f_\lambda = a \text{id}_{Z_\beta(\lambda_\Gamma)} \quad \text{for some } a \in \hat{A}^\beta.$$

One can check  $a$  is independent of the choice of the representative of  $e$  (cf. [AJS], (8.13)). Set

$$\theta[f_\lambda, f_{\beta \uparrow \lambda}]e = a.$$

Then one can write with some  $\bar{a}_\lambda^\beta \in \hat{A}^\beta H_\beta^{-1}$  and  $\bar{b}_\lambda^\beta \in \hat{A}^\beta H_\beta$

$$\theta[f_\lambda, f_{\beta \uparrow \lambda}]e^\beta(\lambda) = \bar{a}_\lambda^\beta + \hat{A}^\beta \quad \text{and} \quad \theta[f_\lambda, f_{\beta \uparrow \lambda}]^{-1}\left(\frac{1}{H_\beta} + \hat{A}^\beta\right) = \bar{b}_\lambda^\beta \frac{1}{H_\beta} e^\beta(\lambda),$$

in which case  $\bar{a}_\lambda^\beta \bar{b}_\lambda^\beta \in 1 + \hat{A}^\beta H_\beta$  in  $\hat{A}^\beta / \hat{A}^\beta H_\beta$ .

Define now  $\mathcal{T} : \mathcal{K}(\Omega) \rightarrow \mathcal{K}(\Gamma)$  and  $\mathcal{T}' : \mathcal{K}(\Gamma) \rightarrow \mathcal{K}(\Omega)$  as follows. If  $\mathcal{M} \in \mathcal{K}(\Omega)$ , set

$$(\mathcal{T}\mathcal{M})(\mu) = \coprod_{\substack{\lambda \in \Omega \\ \lambda_\Gamma = \mu}} \mathcal{M}(\lambda) \quad \forall \mu \in \Gamma,$$

and for each  $\beta \in R^+$  set  $(\mathcal{T}\mathcal{M})(\mu, \beta) =$

$$\begin{cases} \coprod_{\substack{\lambda \in \Omega, \beta \uparrow \lambda = \lambda \\ \lambda_\Gamma = \mu}} \mathcal{M}(\lambda, \beta) \oplus \coprod_{\substack{\lambda \in \Omega, \beta \uparrow \lambda > \lambda \\ \lambda_\Gamma = \mu = (\beta \uparrow \lambda)_\Gamma}} ((\bar{b}_\lambda^\beta, 1)\mathcal{M}(\lambda, \beta) + H_\beta \mathcal{M}(\lambda)_\beta) & \text{if } \beta \uparrow \mu = \mu \\ \coprod_{\substack{\lambda \in \Omega \\ \lambda_\Gamma = \mu}} ((\bar{b}_\lambda^\beta, 1)\mathcal{M}(\lambda, \beta) + H_\beta \mathcal{M}(\lambda)_\beta) & \text{if } \beta \uparrow \mu > \mu, \end{cases}$$

where  $\mathcal{M}(\lambda)_\beta = \mathcal{M}(\lambda) \cap \mathcal{M}(\lambda, \beta)$ . If  $\mathcal{N} \in \mathcal{K}(\Gamma)$ , set

$$(\mathcal{T}'\mathcal{N})(\lambda) = \mathcal{N}(\lambda_\Gamma) \quad \forall \lambda \in \Omega,$$

and for each  $\beta \in R^+$  set  $(\mathcal{T}'\mathcal{N})(\lambda, \beta) =$

$$\begin{cases} \mathcal{N}(\lambda_\Gamma, \beta) & \text{if } \beta \uparrow \lambda = \lambda \\ (a_\lambda^\beta, 1)\mathcal{N}(\lambda_\Gamma, \beta) + \mathcal{N}(\lambda_\Gamma)_\beta & \text{if } \beta \uparrow \lambda_\Gamma > \lambda_\Gamma \\ \mathcal{N}(\lambda_\Gamma, \beta) \oplus \mathcal{N}(\lambda_\Gamma + p\beta, \beta) & \text{if } (\beta \uparrow \lambda)_\Gamma = \lambda_\Gamma + p\beta \\ \{(x + \bar{a}_\lambda^\beta y, y) \mid x, y \in \mathcal{N}(\lambda_\Gamma, \beta)\} & \text{if } (\beta \uparrow \lambda)_\Gamma = \lambda_\Gamma \text{ and } \beta \uparrow \lambda > \lambda. \end{cases}$$

Although  $\mathcal{T}$  and  $\mathcal{T}'$  depend on the choices of  $a_\lambda^\beta$ ,  $b_\lambda^\beta$ , and  $\bar{a}_\lambda^\beta$  in their classes modulo  $\hat{A}^\beta$ , and  $\bar{b}_\lambda^\beta$  modulo  $\hat{A}^\beta H_\beta^2$ , the restriction of  $\mathcal{T}$  (resp.  $\mathcal{T}'$ ) to the image of  $\mathcal{V}_\Omega$  (resp.  $\mathcal{V}_\Gamma$ ) is independent of those choices (cf. [AJS], Remark 10.10).

(e6) **Proposition** (cf. [AJS], Proposition 10.11) *One has natural isomorphisms*

$$\mathcal{V}_\Gamma \circ \mathcal{T} \simeq \mathcal{T} \circ \mathcal{V}_\Omega \quad \text{and} \quad \mathcal{V}_\Omega \circ \mathcal{T}' = \mathcal{T}' \circ \mathcal{V}_\Gamma.$$

(e7) Let  $\mathcal{H}$  be the set of reflexion hyperplanes for the  $\cdot_k$ -action of  $W_a$  on

$X \otimes_{\mathbb{Z}} \mathbb{R}$ . If  $H = \{\nu \in X \otimes_{\mathbb{Z}} \mathbb{R} \mid \langle \nu + \rho, \gamma^\vee \rangle = mp\}$ ,  $\gamma \in R^+$  and  $m \in \mathbb{Z}$ , then we set  $\gamma = \alpha(H)$ . Also we will write for each  $\nu \in X \otimes_{\mathbb{Z}} \mathbb{R}$

$$\nu \geq H \quad \text{iff} \quad \langle \nu + \rho, \gamma^\vee \rangle \geq mp.$$

If  $\beta \in R^+$ , let  $\mathcal{H}(\beta) = \{H \in \mathcal{H} \mid s_\beta(\alpha(H)) < 0\}$ . If  $\lambda, \mu \in X$  with  $\mu$  lying in the closure of the facet of  $\lambda$ , set in the fractional field  $\text{Frac}(\hat{A})$  of  $\hat{A}$

$$C^\beta(\lambda, \mu) = \prod_{\substack{H \in \mathcal{H}(\beta) \\ \mu \in H, \lambda > H}} h_{-\alpha(H)} \prod_{\substack{H \in \mathcal{H}(\beta) \\ \mu \in H, \lambda < H}} \frac{1}{h_{\alpha(H)}},$$

where  $h_\alpha = d_\alpha H_\alpha$  in Case 1 (resp.  $\log K_\alpha = \sum_{j \geq 1} \frac{(-1)^{j+1}}{j} (K_\alpha - 1)^j$  in Case 2) for each  $\alpha \in R$ . If  $H_\beta^\mu \in \mathcal{H}$  with  $\alpha(H_\beta^\mu) = \beta$  and  $\mu \in H_\beta^\mu$ , then

$$C^\beta(\lambda, \mu) \in \begin{cases} (\hat{A}^\beta)^\times & \text{if } \beta \uparrow \mu > \mu \\ \frac{1}{h_\beta} (\hat{A}^\beta)^\times & \text{if } \beta \uparrow \mu = \mu \text{ and } \lambda < H_\beta^\mu \\ h_\beta (\hat{A}^\beta)^\times & \text{if } \beta \uparrow \mu = \mu \text{ and } \lambda > H_\beta^\mu. \end{cases}$$

(e8) Let  $s \in \Sigma_a$ ,  $\mu_s \in X \cap \mathfrak{A}_k$  with  $C_{W_a}(\mu_s) = \{1, s\}$ , and  $\Gamma_s = W_a \cdot_k \mu_s$ . Let  $\mathcal{G} = \{\Gamma_s, \Delta_i \mid s \in \Sigma_a, i \in [1, n_0]\}$ . We can now state a highlight of [AJS], difficult

**Theorem of good choice** (cf. [AJS], Theorem 13.4) *For  $\lambda \in \Omega_0$  let  $\lambda_\Gamma \in \Gamma$  in the closure of the alcove of  $\lambda$ ,  $\Gamma \in \mathcal{G}$ . One can simultaneously choose  $\hat{A}^\beta$ -generators  $e^\beta(\mu)$  of  $\text{Ext}_{\mathcal{C}_\beta}^1(Z_\beta(\mu), Z_\beta(\beta \uparrow \mu))$  for each  $\mu \in \Omega_0 \cup (\cup_{\Gamma \in \mathcal{G}} \Gamma)$  and  $\beta \in R^+$  with  $\beta \uparrow \mu > \mu$ , and  $f_\lambda \in \mathcal{C}_\wedge(Z_\wedge(\lambda_\Gamma), TZ_\wedge(\lambda))^\times$  for each  $\lambda \in \Omega_0$  and  $\Gamma \in \mathcal{G}$  such that for each  $\lambda \in \Omega_0$ ,  $\beta \in R^+$  and  $\Gamma \in \mathcal{G}$*

$$t[f_\lambda, f_{\beta \uparrow \lambda}]e^\beta(\lambda) = C^\beta(\lambda, \lambda_\Gamma)e^\beta(\lambda_\Gamma) \quad \text{if } \beta \uparrow \lambda_\Gamma > \lambda_\Gamma,$$

and

$$\theta[f_\lambda, f_{\beta \uparrow \lambda}]e^\beta(\lambda) = C^\beta(\lambda, \lambda_\Gamma) + \hat{A}^\beta \quad \text{if } \beta \uparrow \lambda_\Gamma = \lambda_\Gamma = (\beta \uparrow \lambda)_\Gamma.$$

(e9) With the good choices of the  $e^\beta(\lambda)$ 's redefine functors  $\mathcal{V}_{\Omega_0}$ ,  $\mathcal{V}_{\Delta_i}$ ,  $i \in [1, n_0]$ ,  $\mathcal{V}_{\Gamma_s}$ ,  $s \in \Sigma_a$ , so that the combinatorial functors  $\mathcal{T}_s$ ,  $\mathcal{T}'_s$ , and  $\mathcal{T}'_i$  corresponding to  $T_{\Omega_0}^{\Gamma_s}$ ,  $T_{\Gamma_s}^{\Omega_0}$ , and  $T_{\Delta_i}^{\Omega_0}$ , respectively, involve only the constants

$C^\beta(\lambda, \lambda_\Gamma)$ ,  $\lambda \in \Omega_0$ ,  $\Gamma \in \mathcal{G}$ . If we set  $\mathcal{Q}^{[i]}(\hat{A}) = \mathcal{T}_{i_1} \mathcal{T}'_{i_1} \dots \mathcal{T}_{i_r} \mathcal{T}'_{i_r} \mathcal{T}'_i \mathcal{Z}_{\nu_i}(\hat{A})$  with  $\mathcal{T}_{i_j} = \mathcal{T}_{s_{i_j}}$  and  $\mathcal{T}'_{i_j} = \mathcal{T}'_{s_{i_j}}$  (cf. (d4)), then

$$\mathcal{Q}^{[i]}(\hat{A}) \simeq \mathcal{V}_{\Omega_0}(\mathcal{Q}^{[i]}(\hat{A})) \quad \text{in } \mathcal{K}(\Omega_0).$$

(e10) Let  $S = S(\mathbb{Z}R)$  the symmetric algebra of  $\mathbb{Z}R$ . Put  $S_k = S \otimes_{\mathbb{Z}} k$ . Recall the  $h_\alpha$  from (e7). If  $\hat{S}_k$  is the completion of  $S_k$  with respect to the maximal ideal generated by all  $\alpha \in R$ , one has a  $k$ -algebra isomorphism

$$(1) \quad \hat{S}_k \longrightarrow \hat{A} \quad \text{via} \quad \alpha \longmapsto h_\alpha \quad \forall \alpha \in R.$$

Through the isomorphism one can regard  $C^\beta(\lambda, \lambda_\Gamma)$  living in  $\text{Frac}(S)$  for each  $\lambda \in \Omega_0$  and  $\Gamma \in \mathcal{G}$ . Hence one can define combinatorial categories  $\mathcal{K}(\Omega_0, S)$ ,  $\mathcal{K}(\Gamma_s, S)$ ,  $\mathcal{K}(\Delta_i, S)$ , combinatorial translation functors  $\mathcal{T}_s$ ,  $\mathcal{T}'_s$ ,  $\mathcal{T}'_i$  between them, and  $\mathcal{Z}_{\nu_i}(S)$ ,  $\mathcal{Q}^{[i]}(S)$  by copying the definitions of  $\mathcal{K}(\Omega_0)$ , etc., with  $\hat{A}$  replaced by  $S$  and with  $S^\emptyset = S[\frac{1}{\alpha} \mid \alpha \in R^+]$  and  $S^\beta = S[\frac{1}{\alpha} \mid \alpha \in R^+ \setminus \{\beta\}]$ . Note that  $h_\alpha \in H_\alpha \hat{A}^\times$ .

More generally, let  $A \in \mathbf{Alg}_S$  with  $\alpha \neq 0$  in  $A$  for any  $\alpha \in R$ . For a  $W_a$ -orbit  $\Omega$  define  $\mathcal{K}(\Omega, A)$  likewise. If  $A' \in \mathbf{Alg}_A$  with  $\alpha \neq 0$  in  $A'$  for all  $\alpha \in R$ , define a functor of extension of scalars  $\mathcal{K}(\Omega, A) \rightarrow \mathcal{K}(\Omega, A')$ , written  $\mathcal{M} \mapsto \mathcal{M}_{A'}$ , by

$$\mathcal{M}_{A'}(\lambda) = \mathcal{M}(\lambda) \otimes_{A^\emptyset} A'^\emptyset \simeq \mathcal{M}(\lambda) \otimes_A A' \quad \forall \lambda \in \Omega$$

and for each  $\beta \in R^+$  by setting  $\mathcal{M}_{A'}(\lambda, \beta)$  equal to the image of

$$\mathcal{M}(\lambda, \beta) \otimes_{A^\beta} A'^\beta \simeq \mathcal{M}(\lambda, \beta) \otimes_A A'$$

in  $\mathcal{M}_{A'}(\lambda) \oplus \mathcal{M}_{A'}(\beta \uparrow \lambda)$  if  $\beta \uparrow \lambda > \lambda$  (resp.  $\mathcal{M}_{A'}(\lambda)$  if  $\beta \uparrow \lambda = \lambda$ ). The translation functors  $\mathcal{T}_s$ ,  $\mathcal{T}'_s$ ,  $\mathcal{T}'_i$  commute with functors of extension of scalars. In particular,

$$(2) \quad \mathcal{Q}^{[i]}(S)_{S_k} \simeq \mathcal{Q}^{[i]}(S_k) \quad \text{and} \quad \mathcal{Q}^{[i]}(S)_{\hat{A}} \simeq \mathcal{Q}^{[i]}(\hat{A}).$$

Note that

$$(3) \quad \mathcal{K}(\Omega, A) \text{ is independent of } k.$$

For let  $\mu \in \Omega \cap \bar{\mathfrak{A}}_k$  and  $W_\Omega = C_{W_a}(\mu)$ . Define a category  $\mathcal{K}(W_a/W_\Omega, A)$  just like  $\mathcal{K}(\Omega, A)$ . An object of  $\mathcal{K}(W_a/W_\Omega, A)$  is a family of  $A^\emptyset$ -modules  $(\mathcal{M}(wW_\Omega))_{wW_\Omega \in W_a/W_\Omega}$ , almost all members 0, together with  $A^\beta$ -submodules

$\mathcal{M}(wW_\Omega, \beta)$ ,  $wW_\Omega \in W_a/W_\Omega$  and  $\beta \in R^+$ , of  $\mathcal{M}(wW_\Omega) \oplus \mathcal{M}((\beta \uparrow w)W_\Omega)$  if  $(\beta \uparrow w)W_\Omega \neq wW_\Omega$  (resp.  $\mathcal{M}(wW_\Omega)$  if  $(\beta \uparrow w)W_\Omega = wW_\Omega$ ), where  $\beta \uparrow w \in W_a$  such that  $(\beta \uparrow w) \cdot_k 0 = \beta \uparrow (w \cdot_k 0)$ . Then one has an isomorphism

$$\mathcal{K}(\Omega, A) \longrightarrow \mathcal{K}(W_a/W_\Omega, A) \quad \text{via} \quad \mathcal{M} \longmapsto \mathcal{M}'$$

with  $\mathcal{M}'(wW_\omega) = \mathcal{M}(w \cdot_k \mu)$  and  $\mathcal{M}'(wW_\Omega, \beta) = \mathcal{M}(w \cdot_k \mu, \beta)$  for each  $w \in W_a$  and  $\beta \in R^+$ .

(e11) **Lemma** (cf. [AJS], Lemma 14.8) *If  $A'$  is flat over  $A$ , then for each  $\mathcal{M}, \mathcal{N} \in \mathcal{K}(\Omega, A)$*

$$\mathcal{K}(\Omega, A)(\mathcal{M}, \mathcal{N}) \otimes_A A' \simeq \mathcal{K}(\Omega, A')(\mathcal{M}_{A'}, \mathcal{N}_{A'}).$$

(e12) **Theorem** (cf. [AJS], Lemma 14.9) *Assume  $p \gg 0$  in Case 1. Then for each  $i, j \in [1, n_0]$ ,*

$$\mathcal{K}(\Omega_0, S)(\mathcal{Q}^{[i]}(S), \mathcal{Q}^{[j]}(S)) \otimes_S S_k \simeq \mathcal{K}(\Omega_0, S_k)(\mathcal{Q}^{[i]}(S_k), \mathcal{Q}^{[j]}(S_k)).$$

*Proof.* We first rewrite the left hand side as  $\mathcal{K}(\Omega_0, S)(\mathcal{Q}^{[i]}(S), \mathcal{Q}^{[j]}(S)) \otimes_{\mathbb{Z}} k$ . For each  $\lambda \in \Omega_0$  and  $\beta \in R^+$  let

$$\mathcal{Q}^{[i]}(S)(\lambda, \beta)^0 = \begin{cases} (\mathcal{Q}^{[i]}(S)(\lambda) \oplus \mathcal{Q}^{[i]}(S)(\beta \uparrow \lambda)) / \mathcal{Q}^{[i]}(\lambda, \beta) & \text{if } \beta \uparrow \lambda > \lambda \\ \mathcal{Q}^{[i]}(S)(\lambda) / \mathcal{Q}^{[j]}(S)(\lambda, \beta) & \text{if } \beta \uparrow \lambda = \lambda. \end{cases}$$

One has (cf. [AJS], Lemma 14.15(b)/(14.16)) for each  $\lambda \in \Omega_0$  and  $\beta \in R^+$

$$(1) \quad \mathcal{Q}^{[i]}(S)(\lambda) \text{ is } S^0\text{-free of finite rank,}$$

$$(2) \quad \mathcal{Q}^{[i]}(S)(\lambda, \beta) \text{ is } S^\beta\text{-free of finite rank,}$$

and

$$(3) \quad \mathcal{Q}^{[i]}(S)(\lambda, \beta)^0 \text{ has no } p\text{-torsion.}$$

Consider a natural map

$$\psi : \coprod_{\lambda \in \Omega_0} \text{Mod}_{S^0}(\mathcal{Q}^{[i]}(S)(\lambda), \mathcal{Q}^{[j]}(S)(\lambda)) \longrightarrow \coprod_{\lambda \in \Omega_0} \coprod_{\beta \in R^+} \text{Mod}_{S^\beta}(\mathcal{Q}^{[i]}(S)(\lambda, \beta), \mathcal{Q}^{[j]}(S)(\lambda, \beta)^0).$$

Then  $\mathcal{K}(\Omega_0, S)(\mathcal{Q}^{[i]}(S), \mathcal{Q}^{[j]}(S)) = \ker \psi$ . By (3) the codomain of  $\psi$  has no  $p$ -torsion, hence  $\text{im } \psi$  has no  $p$ -torsion. Then  $\text{Tor}_1^{\mathbb{Z}}(\text{im } \psi, k) = 0$ . On the other hand, for each  $\lambda \in \Omega_0$

$$\begin{aligned} & \mathbf{Mod}_{S_k^\emptyset}(\mathcal{Q}^{[i]}(S_k)(\lambda), \mathcal{Q}^{[j]}(S_k)(\lambda)) \\ & \simeq \mathbf{Mod}_{S_k^\emptyset}(\mathcal{Q}^{[i]}(S)(\lambda) \otimes_{S^\emptyset} S_k^\emptyset, \mathcal{Q}^{[j]}(S)(\lambda) \otimes_{S^\emptyset} S_k^\emptyset) \quad \text{by (e10)(2)} \\ & \simeq \mathbf{Mod}_{S^\emptyset}(\mathcal{Q}^{[i]}(S)(\lambda), \mathcal{Q}^{[j]}(S)(\lambda)) \otimes_{S^\emptyset} S_k^\emptyset \quad \text{by (1)} \\ & \simeq \mathbf{Mod}_{S^\emptyset}(\mathcal{Q}^{[i]}(S)(\lambda), \mathcal{Q}^{[j]}(S)(\lambda)) \otimes_{\mathbb{Z}} k. \end{aligned}$$

Hence if  $\psi_k$  is the analogue of  $\psi$  over  $S_k$ , one gets a commutative diagram of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & (\ker \psi) \otimes_{\mathbb{Z}} k & \longrightarrow & \coprod_{\lambda \in \Omega_0} \mathbf{Mod}_{S^\emptyset}(\mathcal{Q}^{[i]}(S)(\lambda), \mathcal{Q}^{[j]}(S)(\lambda)) \otimes_{\mathbb{Z}} k & & \\ & & \downarrow & & \downarrow \wr & & \\ (4) & 0 & \longrightarrow & \ker(\psi_k) & \longrightarrow & \coprod_{\lambda \in \Omega_0} \mathbf{Mod}_{S_k^\emptyset}(\mathcal{Q}^{[i]}(S_k)(\lambda), \mathcal{Q}^{[j]}(S_k)(\lambda)) & \\ & & & & \longrightarrow & (\text{im } \psi) \otimes_{\mathbb{Z}} k & \longrightarrow 0 \\ & & & & & \downarrow & \\ & & & & & \longrightarrow & \text{im } (\psi_k) \longrightarrow 0. \end{array}$$

Consequently,

$$(5) \quad \mathcal{K}(\Omega, S)(\mathcal{Q}^{[i]}(S)(\lambda), \mathcal{Q}^{[j]}(S)(\lambda)) \otimes_{\mathbb{Z}} k \text{ injects into } \mathcal{K}(\Omega, S_k)(\mathcal{Q}^{[i]}(S_k)(\lambda), \mathcal{Q}^{[j]}(S_k)(\lambda)).$$

Suppose  $\text{Tor}_1^{\mathbb{Z}}(\text{coker } \psi, k) = 0$ . Then from (2) one gets as in (4) a commutative diagram of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & (\text{im } \psi) \otimes_{\mathbb{Z}} k & \longrightarrow & \coprod_{\lambda, \beta} \mathbf{Mod}_{S^\emptyset}(\mathcal{Q}^{[i]}(S)(\lambda, \beta), \mathcal{Q}^{[j]}(S)(\lambda, \beta)^0) \otimes_{\mathbb{Z}} k & & \\ & & \downarrow & & \downarrow \wr & & \\ (6) & 0 & \longrightarrow & \text{im } (\psi_k) & \longrightarrow & \coprod_{\lambda, \beta} \mathbf{Mod}_{S_k^\emptyset}(\mathcal{Q}^{[i]}(S_k)(\lambda, \beta), \mathcal{Q}^{[j]}(S_k)(\lambda, \beta)^0). & \end{array}$$

As the left vertical arrow is surjective by (4),  $(\text{im } \psi) \otimes_{\mathbb{Z}} k \simeq \text{im } (\psi_k)$ . Then the 5-lemma applied to (4) yields  $(\ker \psi) \otimes_{\mathbb{Z}} k \simeq \ker(\psi_k)$ , i.e., the bijectivity in (5).

Finally,  $\text{Tor}_1^{\mathbb{Z}}(\text{coker } \psi, k) = 0$  automatically in Case 2. In Case 1  $\text{coker } \psi$  is of finite type in  $\mathbf{Mod}_S$ , hence (cf. [M], Theorem 6.5)  $|\text{Ass}(\text{coker } \psi)| < \infty$ . Also (cf. [M], Theorem 6.1)  $\text{coker } \psi$  has a  $p$ -torsion iff  $p \in \bigcup_{\mathfrak{p} \in \text{Ass}(\text{coker } \psi)} \mathfrak{p}$ . As each  $\mathfrak{p}$  contains a unique prime of  $\mathbb{Z}$ ,  $\text{coker } \psi$  has no  $p$ -torsion for  $p \gg 0$ , in which case  $\text{Tor}_1^{\mathbb{Z}}(\text{coker } \psi, k) = 0$ .

(e13) **Remark** Theorem e.12 holds, in fact, for  $p \geq h$  (cf. [AJS], Theorem 16.7). Its proof, however, requires introduction of  $\mathbb{Z}$ -graded combinatorial categories  $\tilde{\mathcal{K}}(\Omega_0, S)$  and  $\tilde{\mathcal{K}}(\Omega_0, S_k)$  (cf. [AJS], Lemma 16.6), that are also relevant to the question of the Koszulity of  $U^{[p]}(\mathfrak{g})$  and  $\mathfrak{u}(k)$  in [AJS], §§17/18.

(e14) Regarding  $\mathbb{Z}$  as  $S$ -algebra via  $\alpha \mapsto 0$  for each  $\alpha \in \Sigma$ , let  $\mathcal{E}_{[i],[j]}(\mathbb{Z}) = \mathcal{K}(\Omega_0, S)(\mathcal{Q}^{[i]}(S), \mathcal{Q}^{[j]}(S)) \otimes_S \mathbb{Z}$ .

**Corollary** *Let  $i, j \in [1, n_0]$ .*

(i)  $\mathcal{E}_{[i],[j]}(\mathbb{Z})$  *is independent of  $k$ .*

(ii) *One has a  $k$ -linear isomorphism  $\mathcal{E}_{[i],[j]}(\mathbb{Z}) \otimes_{\mathbb{Z}} k \simeq \mathcal{C}_k(Q^{[i]}(k), Q^{[j]}(k))$ .*

*Proof.* (i) follows from (e10)(3). The left hand side of (ii) is isomorphic to

$$\begin{aligned}
& \mathcal{K}(\Omega_0, S)(\mathcal{Q}^{[i]}(S), \mathcal{Q}^{[j]}(S)) \otimes_S S_k \otimes_{S_k} \hat{A} \otimes_{\hat{A}} k \\
& \simeq \mathcal{K}(\Omega_0, S_k)(\mathcal{Q}^{[i]}(S_k), \mathcal{Q}^{[j]}(S_k)) \otimes_{S_k} \hat{A} \otimes_{\hat{A}} k \quad \text{by (e12/13)} \\
& \simeq \mathcal{K}(\Omega_0, S_k)(\mathcal{Q}^{[i]}(S_k)_{\hat{A}}, \mathcal{Q}^{[j]}(S_k)_{\hat{A}}) \otimes_{\hat{A}} k \quad \text{by (e11) as } \hat{A} \simeq \hat{S}_k \text{ is flat over } S_k \\
& \simeq \mathcal{K}(\Omega_0, \hat{A})(\mathcal{Q}^{[i]}(\hat{A}), \mathcal{Q}^{[j]}(\hat{A})) \otimes_{\hat{A}} k \quad \text{by (e10)(2)} \\
& = \mathcal{K}(\Omega_0)(\mathcal{Q}^{[i]}(\hat{A}), \mathcal{Q}^{[j]}(\hat{A})) \otimes_{\hat{A}} k \\
& \simeq \mathcal{K}(\Omega_0)(\mathcal{V}_{\Omega_0}(Q^{[i]}(\hat{A})), \mathcal{V}_{\Omega_0}(Q^{[j]}(\hat{A}))) \otimes_{\hat{A}} k \\
& \simeq \mathcal{C}_{\wedge}(Q^{[i]}(\hat{A}), Q^{[j]}(\hat{A})) \otimes_{\hat{A}} k \quad \text{by (e3)} \\
& \simeq \mathcal{C}_k(Q^{[i]}(\hat{A}) \otimes_{\hat{A}} k, Q^{[j]}(\hat{A}) \otimes_{\hat{A}} k) \quad \text{by (d3) as } Q^{[i]}(\hat{A}) \text{ is projective} \\
& \simeq \mathcal{C}_k(Q^{[i]}(k), Q^{[j]}(k)).
\end{aligned}$$

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