

Hypergeometric System of Contingency Table

By

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Abstract

In this report, we formulate a conjecture for a Gauß-Manin connection of any hyperplane arrangement. The formula relates cohomology intersection form to Gauß-Manin connection. We report that the conjecture is true for a hypergeometric system of contingency table. We also obtain a new formula for the cohomology intersection form.

§ 1. Introduction

A number of studies on Gauß-Manin connections for hypergeometric integrals have been published. An important class of such is Gauß-Manin connection of a hyperplane arrangement ([13]). The description of the Gauß-Manin derivative is given in [13, Theorem 12] in a complicated formula. When the hyperplane arrangement is generic, a formula employing cohomology intersection form is obtained in [5]. An advantage of such a formula is that it does not depend on a choice of a basis of the de Rham cohomology group. In this report, we provide a conjectural formula that generalizes the one obtained in [5] for any hyperplane arrangement (Conjecture 2.3). Moreover, we report that there is a particular class of hyperplane arrangements in which the conjecture is true: hypergeometric system of contingency table. Proofs of the results in the last section are based on studies on integral representations of GKZ systems developed in [6, 7] which will be made available in a separate paper.

§ 2. Gauß-Manin connection of a hyperplane arrangement

§ 2.1. Orlik-Solomon algebra

We recall some terminologies from [11]. Let V be a finite dimensional complex vector space and let $\alpha = (\alpha_1, \dots, \alpha_n) : V \hookrightarrow \mathbb{C}^n$ be a linear injection. We write

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$H_i = \text{Ker} \alpha_i \subset V$ for the hyperplane defined by $\alpha_i \in V^*$. Let L be the intersection lattice of $\mathcal{A}$ and we set $L(\mathcal{A}) = L \setminus \{O\}$ where O is the unique maximal element of L . An element $e \in L(\mathcal{A})$ is called a dense edge if $\mathcal{A}_e = \{H \in \mathcal{A} \mid e \subset H\}$ is irreducible. The symbol $D(\mathcal{A})$ denotes the set of dense edges. Let $E := \Lambda(\mathbb{C}^n)$ be the exterior algebra of the vector space $\mathbb{C}^n$ and let $\{\mathbf{e}_i\}_{i=1}^n$ be the standard basis of $\mathbb{C}^n$. We write $E = \bigoplus_{i=0}^n E^i$ for the graded decomposition of E . Let us define a linear map $\partial : E \rightarrow E$ by the relations $\partial 1 = 0$, $\partial e_i = 1$ and

$$(2.1) \quad \partial(e_{i_1} \wedge \cdots \wedge e_{i_p}) = \sum_{j=1}^p (-1)^{j-1} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_p} \quad (p \geq 2).$$

For an ordered tuple $S = (i_1, \dots, i_p)$, we set $e_S := e_{i_1} \wedge \cdots \wedge e_{i_p}$. The Orlik-Solomon ideal $I = I(\mathcal{A}) \subset E$ is a graded ideal generated by $\{\partial e_S \mid \cap S \neq \emptyset \text{ and } S \text{ is dependent}\}$. Note that $\mathcal{A}$ is a central arrangement.¹ We set $A := A(\mathcal{A}) := E/I(\mathcal{A})$ and call it Orlik-Solomon algebra.

For a fixed element $H \in \mathcal{A}$, we set

$$(2.2) \quad A^p(\mathcal{A}; H) := \bigoplus_{e \in L(\mathcal{A}), e \not\subset H} A^p(\mathcal{A}_e) \hookrightarrow A^p(\mathcal{A})$$

Note that the relation

$$(2.3) \quad A^p(\mathcal{A}; H) \simeq A^p(\mathbf{d}_H \mathcal{A})$$

holds where $\mathbf{d}_H \mathcal{A}$ is the decone of $\mathcal{A}$ along H . Let us fix an index 0 and write H_0 for the corresponding hyperplane. Since $\mathcal{A}$ is a central arrangement, a relation $\partial I \subset I$ holds and we can set $\tilde{A} = \tilde{A}(\mathcal{A}) := \text{Ker}(\partial : E \rightarrow E)$. It is readily seen that $\tilde{A}(\mathcal{A}) \subset A$ is a graded subalgebra.

Lemma 2.1. *The correspondence $A(\mathbf{d}_{H_0} \mathcal{A}) \ni e_S = e_{i_1} \wedge \cdots \wedge e_{i_p} \mapsto (e_{i_1} - e_0) \wedge \cdots \wedge (e_{i_p} - e_0) \in \tilde{A}(\mathcal{A})$ gives rise to an algebra isomorphism $A(\mathbf{d}_{H_0} \mathcal{A}) \simeq \tilde{A}(\mathcal{A})$.*

Proof. A simple computation shows that an identity $e_S = (e_{i_1} - e_0) \wedge \cdots \wedge (e_{i_p} - e_0) + e_0 \wedge \partial e_S$ holds. Thus, the map is a surjection. Injectivity is clear by definition. $\square$

Recall that A is isomorphic to the Brieskorn algebra B generated by $d \log$ forms $d \log \alpha_{i_1} \wedge \cdots \wedge d \log \alpha_{i_p} \in H^0(V \setminus \mathcal{A}, \Omega_{V \setminus \mathcal{A}}^p)$ over $\mathbb{C}$. Under this identification, it is easy to see that $\tilde{A}$ is isomorphic to a subalgebra of B generated by differential forms $\omega_S := d \log(\alpha_{i_1}/\alpha_{i_0}) \wedge \cdots \wedge d \log(\alpha_{i_p}/\alpha_{i_{p-1}})$ ($S = \{i_0, \dots, i_p\}$). Note that any element

¹For an arrangement $\mathcal{A}$ which is not necessarily central, the Orlik-Solomon ideal is an ideal of E generated by $\{e_S \mid \cap S = \emptyset\} \cup \{\partial e_S \mid \cap S \neq \emptyset \text{ and } S \text{ is dependent}\}$.

of $\tilde{A}$ defines an element of $H^0(\mathbb{P}(V \setminus \mathcal{A}), \Omega_{\mathbb{P}(V \setminus \mathcal{A})}^p)$. Hereafter, we often identify e_S with ω_S .

§ 2.2. Formal Gauß-Manin connection on combinatorial strata

Let $k < n$ be a pair of non-negative integers and let $\text{Gr}_{k+1, n+1}$ denote the $(k+1)$ -Grassmannian variety of $\mathbb{C}^{n+1} = \mathbb{C}^{\llbracket 0, n \rrbracket}$. Here, we set $\llbracket 0, n \rrbracket := \{0, 1, \dots, n\}$. We consider a realization of $\text{Gr}_{k+1, n+1}$ as a moduli space of arrangements. Let $Z_{k+1, n+1}$ be the set of complex $(k+1) \times (n+1)$ matrices $z = (z_{ij})_{ij}$ which has rank $k+1$. For each element $z \in Z_{k+1, n+1}$, the i -th column vector α_i can be regarded as an element of the dual space $(\mathbb{C}^{k+1})^*$. Thus, z gives rise to a linear embedding $z : \mathbb{C}^{k+1} = \text{Spec } \mathbb{C}[x_0, \dots, x_n] \hookrightarrow \mathbb{C}^{n+1}$ which in turn defines a matroid M_z on $\llbracket 0, n \rrbracket$. Namely, a subset $B \subset \llbracket 0, n \rrbracket$ is a basis of M_z if and only if $\{\alpha_i\}_{i \in B}$ is a basis of $(\mathbb{C}^{k+1})^*$. Let us consider a left action of the general linear group $\text{GL}(k+1, \mathbb{C})$ on $Z_{k+1, n+1}$ given by the left multiplication of a matrix. The Grassmannian variety $\text{Gr}_{k+1, n+1}$ is identified with the quotient $\text{GL}(k+1, \mathbb{C}) \backslash Z_{k+1, n+1}$. Clearly, any representative z of $\xi \in \text{Gr}_{k+1, n+1} = \text{GL}(k+1, \mathbb{C}) \backslash Z_{k+1, n+1}$ defines the same matroid M_z , for which we write M_ξ . For a matroid M on $\llbracket 0, n \rrbracket$, we set

$$(2.4) \quad \mathcal{R}(M) := \{\xi \in \text{Gr}_{k+1, n+1} \mid M_\xi = M\}.$$

Note that M is linearly realizable if and only if $\mathcal{R}(M) \neq \emptyset$. The combinatorial stratification of $\text{Gr}_{k+1, n+1}$ introduced in [4] is the following decomposition:

$$(2.5) \quad \text{Gr}_{k+1, n+1} = \bigcup_{M: \text{ linearly realizable matroid on } \{0, \dots, n\}} \mathcal{R}(M).$$

Note that each stratum $\mathcal{R}(M)$ is a constructible subset of $\text{Gr}_{k+1, n+1}$ in Zariski topology. However, $\mathcal{R}(M)$ can be highly non-trivial due to the universality theorem of Mnëv ([10]):

Theorem 2.2 ([10]). *Given any affine algebraic variety V over $\mathbb{Q}$, there exists a rank 3 realizable matroid M whose projective stratum $\mathcal{R}(M)/H$ is isomorphic to V .*

Let us take a basis set M of a linearly realizable matroid without any loop on $\llbracket 0, n \rrbracket$. Let $\tilde{\mathcal{R}}(M) \subset Z_{k+1, n+1}$ be the preimage of $\mathcal{R}(M)$ by the quotient map $Z_{k+1, n+1} \rightarrow \text{Gr}_{k+1, n+1}$. For a subset $T \subset \llbracket 0, n \rrbracket$ of cardinality $k+1$, we set $C_T := \{z \in Z_{k+1, n+1} \mid \det(z_{ij})_{\substack{i=0, \dots, k \\ j \in T}} = 0\}$. Let the symbol $\binom{\llbracket 0, n \rrbracket}{k+1}$ denote the set of subsets of $\llbracket 0, n \rrbracket$ with cardinality $k+1$. It is easy to see that the identity

$$(2.6) \quad \tilde{\mathcal{R}}(M) = \bigcap_{T \in \binom{\llbracket 0, n \rrbracket}{k+1} \setminus M} C_T \setminus \bigcup_{T \in M} C_T$$

holds. Moreover, setting $D_T := C_T \cap \overline{\tilde{\mathcal{R}}(M)}$ one has an identity

$$(2.7) \quad \overline{\tilde{\mathcal{R}}(M)} \setminus \tilde{\mathcal{R}}(M) = \bigcup_{T \in M} D_T.$$

For any $z \in \tilde{\mathcal{R}}(M)$ and $j \in \llbracket 0, n \rrbracket$, we set $\ell_j(x; z) := \sum_{i=0}^{k+1} z_{ij} x_i$. Let $\tilde{R}(M)^{\text{reg}}$ denote the smooth locus of $\tilde{R}(M)$. We set $X := \{(z, x) \in \tilde{\mathcal{R}}^{\text{reg}}(M) \times \mathbb{P}^k \mid \ell_i(x; z) \neq 0\}$. Let $\pi : X \rightarrow \tilde{\mathcal{R}}(M)^{\text{reg}}$ denote the natural projection. We may assume that M defines an essential arrangement, otherwise the identity $\tilde{\mathcal{R}}(M) = \emptyset$ holds. We write $A = A(M)$ for the corresponding Orlik-Solomon algebra. For any element $B \in M$, we write $z_B = \det(z_{ij})_{\substack{i=0, \dots, k \\ j \in B}}$. For each index $i \in \llbracket 0, n \rrbracket$, we associate a complex number $\lambda_i \in \mathbb{C}$ such that $\lambda_0 + \dots + \lambda_n = 0$.

For an element $e \in D(M)$, we set $\lambda_e := \sum_{i \in e} \lambda_i$. If none of λ_e is a non-negative integer, the result of [3] (see also [12]) shows that the natural morphism

$$(2.8) \quad \tilde{A}^{k+1}/\omega_\lambda \wedge \tilde{A}^k \rightarrow \mathbb{H}^k \left(\mathbb{P}^k \setminus \mathcal{A}_M, (\Omega_{\mathbb{P}^k \setminus \mathcal{A}_M}^\bullet, \nabla_\lambda) \right) =: H_{\text{dR}}^k(M, \lambda)$$

is an isomorphism where $\nabla_\lambda := d_x + \omega_\lambda \wedge$. For any $z \in \tilde{\mathcal{R}}(M)$, the linear morphism

$$(2.9) \quad \tilde{A}^{p+1} \ni e_{i_0 \dots i_p} \mapsto \omega_{i_0 \dots i_p} \in \mathbb{H}^0 \left(\mathbb{P}^k \setminus \mathcal{A}_z, \Omega_{\mathbb{P}^k \setminus \mathcal{A}_z}^p \right)$$

is well-defined. Note that the following identity holds true:

$$\omega_{i_0 \dots i_p} = d_x \log(\ell_{i_1}(x; z)/\ell_{i_0}(x; z)) \wedge \dots \wedge d_x \log(\ell_{i_p}(x; z)/\ell_{i_{p-1}}(x; z)).$$

Let $\Omega^1(\log D_M)$ denote the vector space spanned by $\{d \log z_B\}_{B \in M}$. One can define the Gauß-Manin connection associated with $\pi : X \rightarrow \tilde{\mathcal{R}}(M)^{\text{reg}}$ in a combinatorial way. We set $\tilde{A}[\lambda] := \tilde{A}(M) \otimes_{\mathbb{C}} \mathbb{C}[\lambda]$ and $\omega_\lambda := \sum_{i=0}^n \lambda_i \omega_i \in \tilde{A}^1(M)[\lambda]$. The result of [13] shows that there is a formal Gauß-Manin derivative $\nabla_\lambda^{GM} : \tilde{A}^k[\lambda]/\omega_\lambda \wedge \tilde{A}^{k-1}[\lambda] \rightarrow (\tilde{A}^k[\lambda]/\omega_\lambda \wedge \tilde{A}^{k-1}[\lambda]) \otimes_{\mathbb{C}[\lambda]} \Omega^1(\log D_M)$ which has the following expression:

$$(2.10) \quad \nabla_\lambda^{GM} = \sum_{B \in M} f_B d \log z_B \wedge,$$

where $f_B \in \text{End}_{\mathbb{C}[\lambda]}(\tilde{A}^k[\lambda]/\omega_\lambda \wedge \tilde{A}^{k-1}[\lambda])$ is a certain endomorphism determined by the combinatorics of M . Note that $d_z \log(g \cdot z)_B = d_z \log z_B$ for any $g \in \text{GL}(k+1, \mathbb{C})$.

§ 2.3. Cohomology intersection form and a universal expression of Gauß-Manin connection

We formulate a general conjecture of a general form of Gauß-Manin connection. To this end, let us briefly recall the construction of the cohomology intersection form. For details, the readers may refer to [1, 8, 9]. Let R be a ring obtained from $\mathbb{C}[\lambda]$ by inverting polynomials $\lambda_e - j$ ($e \in D(M), j \in \mathbb{Z}_{\geq 0}$). We set $H_{\text{dR}}^k(M; R) := (\tilde{A}^{k+1}/\omega_\lambda \wedge \tilde{A}^k) \otimes_{\mathbb{C}[\lambda]} R$. One can define a canonical bilinear pairing

$$(2.11) \quad \langle \bullet, \bullet \rangle_{ch} : H_{\text{dR}}^k(M; R) \otimes_R H_{\text{dR}}^k(M; R) \rightarrow R.$$

Given a tuple of complex numbers $\overset{\circ}{\lambda} \in \mathbb{C}^{n+1}$ such that $\overset{\circ}{\lambda}_e \notin \mathbb{Z}_{\geq 0}$ for any $e \in D(M)$, one can define an evaluation morphism

$$(2.12) \quad \text{ev}(\overset{\circ}{\lambda}) : H_{\text{dR}}^k(M; R) \rightarrow \tilde{A}^{k+1}/\omega_{\overset{\circ}{\lambda}} \wedge \tilde{A}^k.$$

Moreover, the construction is compatible with the usual cohomology intersection form defined over $\mathbb{C}$ ([8]):

$$(2.13) \quad \begin{array}{ccc} H_{\text{dR}}^k(M; R) \otimes_R H_{\text{dR}}^k(M; R) & \xrightarrow{\langle \bullet, \bullet \rangle_{ch}} & R \\ \downarrow \text{ev}(\overset{\circ}{\lambda}) \otimes \text{ev}(-\overset{\circ}{\lambda}) & & \downarrow \text{ev}(-\overset{\circ}{\lambda}) \\ H^k(M, \overset{\circ}{\lambda}) \otimes_{\mathbb{C}} H^k(M, -\overset{\circ}{\lambda}) & \xrightarrow{\langle \bullet, \bullet \rangle_{ch}} & \mathbb{C} \end{array}.$$

The image of a cohomology class $[\omega] \in H_{\text{dR}}^k(M; R)$ by ev_{λ} (resp. $\text{ev}_{-\lambda}$) is denoted by $[\omega]$ (resp. $[\omega^{\vee}]$) omitting λ . We conjecture that the following universal expression is true:

Conjecture 2.3. *The endomorphism f_B of (2.10) takes the form*

$$(2.14) \quad f_B(\bullet) = \lambda_B \langle \bullet, [\omega_B^{\vee}] \rangle_{ch} [\omega_B],$$

where $\lambda_B = \prod_{i \in B} \lambda_i$ and $\langle \bullet, \bullet \rangle_{ch}$ is the cohomology intersection form².

The simplest case of the conjecture above is the following result.

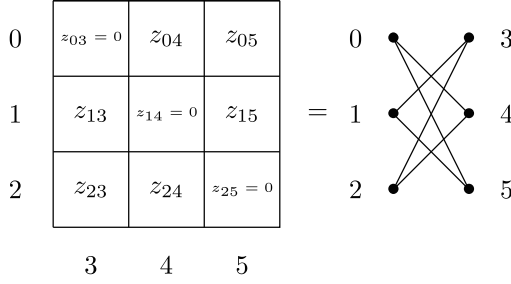
Theorem 2.4 (Theorem 3.12 of [5]). *Conjecture 2.3 is true for generic strata $\mathcal{R}(U_{k+1, n+1})$ where $U_{k+1, n+1}$ is the uniform matroid of rank $k+1$ on $\llbracket 0, n+1 \rrbracket$.*

§ 3. Hypergeometric System of Contingency Table

§ 3.1. Hyperplane arrangement and contingency table

From this subsection, we fix a pair of non-negative integers $k < n$. Let the symbol $\llbracket k, n \rrbracket$ denotes the set of consecutive integers between k and n , i.e., $\llbracket k, n \rrbracket = \{k, k+1, \dots, n\}$. For a subset $\Omega \subset \llbracket 0, k \rrbracket \times \llbracket k+1, n+1 \rrbracket$, we associate a bipartite graph whose vertex set is $\llbracket 0, n+1 \rrbracket$ and whose edge set is Ω . In order to simplify the notation, we write ij for an element $(i, j) \in \Omega$.

²We normalize the intersection form by $(2\pi\sqrt{-1})^k$. This is a good choice when we deal with the field of definition.



A bipartite graph Ω is spanning if any $i \in \llbracket 0, n+1 \rrbracket$ is a vertex of Ω . In the followings, we assume that the graph of Ω is connected and spanning. For any pair of subsets $I \subset \llbracket 0, k \rrbracket$, $J \subset \llbracket k+1, n+1 \rrbracket$ such that $|I| = |J|$, a bijection $m : I \xrightarrow{\sim} J$ is called a matching if $(i, m(i)) \in E(G)$ for any $i \in I$. A pair (I, J) is called a matching pair if there is a matching $m : I \simeq J$. For any $(i, j) \in \llbracket 0, k \rrbracket \times \llbracket k+1, n+1 \rrbracket \setminus \Omega$, we set $z_{ij} = 0$. We set $z = (z_{ij})_{(i,j) \in \llbracket 0, k \rrbracket \times \llbracket k+1, n+1 \rrbracket}$. For any pair of subsets $I \subset \llbracket 0, k \rrbracket$, $J \subset \llbracket k+1, n+1 \rrbracket$ such that $|I| = |J|$, the symbol $z_{I,J}$ denotes the (I, J) -subdeterminant of z . If we treat non-zero z_{ij} as an indeterminate, $z_{I,J}$ is non-zero if and only if (I, J) is a matching pair.

We set $Z = Z(\Omega) = \mathbb{C}^\Omega \setminus \prod_{(I,J): \text{ matching pair}} \{z_{I,J} = 0\}$, $\ell_i(x; z) := x_i$ ($i \in \llbracket 0, k \rrbracket$) and $\ell_j(x; z) := \sum_{i=0}^k z_{ij} x_i$ ($j \in \llbracket k+1, n+1 \rrbracket$). The Gauß-Manin connection of a contingency table is associated with a family of arrangements $\pi : \mathbb{P}^k \times Z \setminus \{(x, z) \mid \prod_{i=0}^{n+1} \ell_i(x; z) = 0\} \rightarrow Z$. Let us state it more clearly. We identify $z \in Z$ with a $\llbracket 0, k \rrbracket \times \llbracket 0, n+1 \rrbracket$ matrix

$$(3.1) \quad \tilde{z} := \left(\begin{array}{c|ccc} 1 & z_{0k+1} & \cdots & z_{0n+1} \\ \cdot & \vdots & \ddots & \vdots \\ \cdot & z_{kk+1} & \cdots & z_{kn+1} \\ \hline 1 & z_{kk+1} & \cdots & z_{kn+1} \end{array} \right)$$

Clearly, any $\tilde{z}$ defines the same matroid M as long as $z \in Z$ and therefore, we obtain an open embedding $Z \ni z \mapsto [\tilde{z}] \in \tilde{\mathcal{R}}(M)^{\text{reg}}$. This embedding naturally induces a Gauß-Manin connection on Z whose fiber is given by $\tilde{A}^k / \omega_\lambda \wedge \tilde{A}^{k-1}$.

Theorem 3.1. *Conjecture 2.3 is true on Z . More concretely, the formal Gauß-Manin connection takes the form*

$$(3.2) \quad \nabla_\lambda^{GM} = \sum_{(I,J): \text{ matching pair}} \lambda_{I;J} \langle \bullet, [\omega_{I,J}^\vee] \rangle_{ch} [\omega_{I,J}] d \log z_{I,J} \wedge.$$

Here, we set $\lambda_{I,J} := \prod_{i \in \llbracket 0, k \rrbracket \setminus I} \lambda_i \prod_{j \in J} \lambda_j$, $\omega_{I,J} := \omega_{(\llbracket 0, k \rrbracket \setminus I) \cup J}$.

§ 3.2. matching formula of the cohomology intersection form

Given a subset $S \subset \llbracket 0, n+1 \rrbracket$ such that $|S| = k+1$, we set $S_0 := S \cap \llbracket 0, k \rrbracket$, $S_1 = S \cap \llbracket k+1, n+1 \rrbracket$. In view of the identity $|S_1| = k+1 - |S_0|$, setting $I :=$

$\llbracket 0, k \rrbracket \setminus S_0$ and $J := S_1$, we have $|I| = |J|$. For $I = \{i_0 < \cdots < i_l\} \subset \llbracket 0, k \rrbracket$, we set $I^c = \llbracket 0, k \rrbracket \setminus I = \{i'_0 < \cdots < i'_m\}$. We set $\text{sgn}(I^c, I) := \frac{\mathbf{e}_{I^c} \wedge \mathbf{e}_I}{\mathbf{e}_0 \wedge \cdots \wedge \mathbf{e}_k} = \frac{\mathbf{e}_{i'_0} \wedge \cdots \wedge \mathbf{e}_{i'_m} \wedge \mathbf{e}_{i_0} \wedge \cdots \wedge \mathbf{e}_{i_l}}{\mathbf{e}_0 \wedge \cdots \wedge \mathbf{e}_k}$. A simple computation shows the identity

$$(3.3) \quad \omega_{I,J} = \omega_S = \text{sgn}(I^c, I) z_{I,J} \frac{\prod_{i \in I} \ell_i}{\prod_{j \in J} \ell_j} \omega_0.$$

Let us fix a generic weight vector w on variables $(z_{ij})_{ij \in \Omega}$. A matching m corresponds to a monomial $z^m := \prod_{i \in I} z_{im(i)}$. Note that $z_{I,J} = \sum_{m: \text{matching}} \text{sgn}(m) z^m$. Here sgn is the signature of a matching regarded as a permutation. We always align elements of I, J in increasing order. For a given matching pair (I, J) , a matching $m : I \simeq J$ such that $\text{in}_<(z_{I,J}) = \pm z^m$ is called the w -minimal matching. When I, J comes from a subset $S \subset \llbracket 0, n+1 \rrbracket$ of cardinality $k+1$, we write m_S for the w -minimal matching. We set $M(S) := \{(i, m_S(i))\}_{i \in I}$. Note that $M(S) = \emptyset$ if $I = \emptyset$.

Let $\{\mathbf{e}(i)\}_{i=0}^{n+1}$ be the standard basis of $\mathbb{Z}^{\llbracket 0, n+1 \rrbracket}$, i.e., all the entries of $\mathbf{e}_i$ but the i -th one is zero and the i -th entry is 1. We set $\mathbf{a}(i, j) = \mathbf{e}(i) + \mathbf{e}(j)$ ($i, j \in \Omega$). For any spanning tree $\sigma \subset \Omega$, the equation

$$(3.4) \quad \sum_{i=0}^k \sum_{j=k+1}^{n+1} v_{ij} (\mathbf{e}_i + \mathbf{e}_j) = \sum_{i=0}^k \lambda_i \mathbf{e}_i - \sum_{j=k+1}^{n+1} \lambda_j \mathbf{e}_j$$

has a unique solution $v = v(\sigma; \lambda)$ under the condition $v_{ij} = 0$ ($(i, j) \notin \sigma$). If we regard $A = \{\mathbf{a}(i, j)\}_{ij \in \Omega}$ as a lattice configuration of $\mathbb{Z}^{\llbracket 0, n+1 \rrbracket}$, any simplex σ in the convex hull of A is a spanning tree and vice versa.

For a given vector $w = (w_{ij}) \in \mathbb{R}^\Omega$, one defines a w -weight of a monomial $z^e = \prod_{ij \in \Omega} z^{e_{ij}}$ by $\sum_{ij \in \Omega} w_{ij} e_{ij}$. For a given polynomial f in z , let the symbol $\text{in}_w(f)$ denote the w -initial term of f , that is the sum of monomials with minimal w -weight. Finally, for a generic vector $w \in \mathbb{R}^\Omega$, let T_w denote the regular triangulation of A determined by w ([2, Chapter 5]). The following formula expresses the cohomology intersection number in terms of combinatorics of a regular triangulation.

Theorem 3.2. *Let $S_1, S_2 \subset \llbracket 0, n+1 \rrbracket$ be subsets such that $|S_1| = |S_2| = k+1$. Fix a generic weight vector w and set $M_a := M(S_a)$ ($a = 1, 2$), $T(S_1; S_2) = \{\sigma \in T_w \mid$*

$M_1 \cup M_2 \subset \sigma\}$. Then, the following identity holds:

$$\begin{aligned}
 & \langle [\omega_{S_1}], [\omega_{S_2}] \rangle_{ch} \\
 &= \text{sgn}(I_1^c, I_1) \text{sgn}(I_2^c, I_2) \text{sgn}(m_{S_1}) \text{sgn}(m_{S_2}) \frac{(-\lambda_{k+1}) \cdots (-\lambda_{n+1})}{\left(\prod_{j \in J_1} (-\lambda_j)\right) \left(\prod_{j \in J_2} (-\lambda_j)\right)} \times \\
 (3.5) \quad & \sum_{\sigma \in T(S_1; S_2)} \frac{\prod_{(i,j) \in M_1 \cap M_2} v_{i,j}(\sigma; \lambda)}{\prod_{(i,j) \in \sigma \setminus (M_1 \cup M_2)} v_{i,j}(\sigma; \lambda)} \\
 &= \text{sgn}(I_1^c, I_1) \text{sgn}(I_2^c, I_2) \text{sgn}(m_{S_1}) \text{sgn}(m_{S_2}) \frac{(-\lambda_{k+1}) \cdots (-\lambda_{n+1})}{\left(\prod_{j \in J_1} (-\lambda_j)\right) \left(\prod_{j \in J_2} (-\lambda_j)\right)} \times \\
 (3.6) \quad & \sum_{\sigma \in T(S_1; S_2)} \frac{\prod_{(i,j) \in M_1} v_{i,j}(\sigma; \lambda) \prod_{(i,j) \in M_2} v_{i,j}(\sigma; \lambda)}{\prod_{(i,j) \in \sigma} v_{i,j}(\sigma; \lambda)}.
 \end{aligned}$$

Here $v_{i,j}(\sigma; \lambda)$ is the ij -th entry of $v(\sigma; \lambda)$. In particular, the intersection number $\langle [\omega_{S_1}], [\omega_{S_2}] \rangle_{ch}$ is zero if $T(S_1; S_2) = \emptyset$.

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