

Tightness of free (abelian) topological groups over metrizable spaces

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1. INTRODUCTION

All spaces are assumed to be Tychonoff. Our notations and terminology follow [4].

Let X be a topological space. For $A \subseteq X$ we use $\overline{A}$ to denote the closure of A in X . Recall that the *tightness* $t(X)$ of a space X is the smallest infinite cardinal number κ such that for every point $x \in X$ and each $A \subset X$, if $x \in \overline{A}$, then $x \in \overline{B}$ for some $B \subset A$ with $|B| \leq \kappa$. The *weight* of a space X is denoted by $w(X)$. We note that $w(X)$ is always infinite. In what follows, $F(X)$ and $A(X)$ will denote the free topological group and the free abelian topological group over a space X , respectively.

The main goal of this paper is to completely compute the tightness of $F(X)$ and $A(X)$ when X is a metric space. In particular, we provide a consistent solution to the following

Problem 1. (Arhangel'skiĭ, Okunev and Pestov [1]) *Does $t(A(X)) = w(X')$ for a metrizable space X , where X' is the set of all non-isolated points of X ?*

Let $\{c_n : n \in \omega\}$ be a faithfully indexed sequence converging to a point $c \notin \{c_n : n \in \omega\}$. and we define $C = \{c_n : n \in \omega\} \cup \{c\}$. Thus C is the infinite countable compact space with a single non-isolated point c .

For every cardinal κ , D_κ will denote the discrete space of size κ , and we define $C_\kappa = C \times D_\kappa$. Topologically, C_κ is (homeomorphic to) the disjoint sum of κ -many convergent sequences. Finally, S_κ will denote the *Fréchet-Urysohn fan of size κ* , i.e. the quotient space obtained from C_κ by identifying all non-isolated points (c, d) ($d \in D_\kappa$) of C_κ to a single point which we will denote by $\{x^*\}$. Therefore, all points of S_κ except $\{x^*\}$ are isolated, and $\{W_f : f \in \omega^\kappa\}$ forms a basis of open neighborhoods of $\{x^*\}$, where

$$W_f = \{x^*\} \cup \{(c_n, d) : n \in \omega, d \in D_\kappa, n \geq f(d)\}$$

for $f \in \omega^\kappa$.

Recall that a *successor cardinal* is a cardinal of the form λ^+ for some cardinal λ , where λ^+ denotes the smallest cardinal strictly bigger than λ . A cardinal κ is *regular* provided that for every $\lambda < \kappa$ and each family $\{\kappa_\alpha : \alpha < \lambda\}$ of cardinals such that $\kappa_\alpha < \kappa$ whenever $\alpha < \lambda$, one has $\sup\{\kappa_\alpha : \alpha < \lambda\} < \kappa$. It is well-known that successor cardinals are regular, see [8].

2. EMBEDDINGS OF FINITE POWERS OF FRÉCHET-URYSOHN FANS INTO FREE (ABELIAN) GROUPS OVER METRIC SPACES

We start with a well-known folklore fact.

Fact 2.1. *If $n \in \omega$ and $X = \oplus_{i=1}^n X_i$, then the map $f : \prod_{i=1}^n A(X_i) \rightarrow A(X)$ defined by*

$$f(g_1, g_2, \dots, g_n) = g_1 + g_2 + \dots + g_n$$

is a topological group isomorphism, i.e. both a homeomorphism and a group isomorphism.

Indeed, the above fact can be proved applying the following result.

Theorem 2.2 ([7, Theorem 6.11]). *Let G be a topological group and $N_1, N_2, \dots, N_m$ be normal subgroups of G with the following properties:*

- (1) $N_1 N_2 \dots N_m = G$;
- (2) $(N_1 N_2 \dots N_k) \cap N_{k+1} = \{e\}$ for $k = 1, 2, \dots, m-1$;
- (3) *if U_i is a neighborhood of e in N_i for $i = 1, 2, \dots, m$, then $U_1 U_2 \dots U_m$ contains a neighborhood of e in G .*

Then G is topologically isomorphic to $N_1 N_2 \dots N_m$.

For a set D we define

$$Y_D = \{dc_n c^{-1} d^{-1} : n \in \omega, d \in D\} \subseteq F(C \oplus D)$$

and

$$Z_D = \{(c_n, d) - (c, d) : n \in \omega, d \in D\} \subseteq A(C \times D).$$

Then, applying Lemma 1.2 in [12], we obtain the following.

Lemma 2.3. *For each infinite cardinal κ the subspace Z_{D_κ} of $A(C_\kappa)$ is homeomorphic to S_κ .*

Fix $n \in \omega$. By Lemma 2.3, $A(C_\kappa)^n$ contains a subspace homeomorphic to S_κ^n . By Fact 2.1, $A(C_\kappa)^n$ is topologically isomorphic to $A(C_\kappa \oplus C_\kappa \oplus \cdots \oplus C_\kappa)$, where the direct sum has n summands. Since $C_\kappa \oplus C_\kappa \oplus \cdots \oplus C_\kappa$ is homeomorphic to C_κ , the latter group is topologically isomorphic to $A(C_\kappa)$. Consequently, we have the following theorem.

Theorem 2.4. *Let κ be an infinite cardinal. Then, for every $n \in \omega$, $A(C_\kappa)$ contains a subspace homeomorphic to S_κ^n .*

On the other hand, in the non-abelian case, we can show that S_κ^n can be embedded in $F(C \oplus D_\kappa)$, applying an idea in [11, Theorem 2.5].

Lemma 2.5. *For every infinite discrete space D , there exists a continuous group homomorphism $\psi_D : F(C \oplus D) \rightarrow A(C \times F(D))$ such that*

$$\psi_D(dc_n c^{-1} d^{-1}) = (c_n, d) - (c, d)$$

whenever $n \in \omega$ and $d \in D$.

Construction of the map. Let

$$\tau : F(D) \times (C \times F(D)) \rightarrow C \times F(D); (g, (x, h)) \mapsto (x, gh)$$

be the continuous action of $F(D)$ on $C \times F(D)$. Since $F(D)$ is a discrete space, we can get the continuous topological group automorphism

$$\bar{\tau} : F(D) \times A(C \times F(D)) \rightarrow A(C \times F(D)).$$

Let $G = F(D) \ltimes_{\bar{\tau}} A(C \times F(D))$ be the semidirect product formed with respect to $\bar{\tau}$, that is, for $\forall g, h \in F(D)$ and $\forall a, b \in A(C \times F(D))$

$$(g, a) \cdot (h, b) = (gh, a + \bar{\tau}(g, b)).$$

Then, for each $d \in D$ and $x \in C$, we have

$$\begin{aligned} (d^{-1}, 0) \cdot (d, 0) &= (e, 0) \\ (d, 0) \cdot (d^{-1}, 0) &= (e, 0) \\ (e, -(x, e)) \cdot (e, (x, e)) &= (e, 0) \\ (e, (x, e)) \cdot (e, -(x, e)) &= (e, 0) \end{aligned}$$

Define a mapping $\varphi : C \oplus D \rightarrow G$ by

$$\varphi(t) = \begin{cases} (e, (t, e)) \in F(D) \ltimes_{\bar{\tau}} A(C \times F(D)), & \text{if } t \in C \\ (t, 0) \in F(D) \ltimes_{\bar{\tau}} A(C \times F(D)), & \text{if } t \in D \end{cases}$$

and let $\bar{\varphi} : F(C \oplus D) \rightarrow G$ be the continuous homomorphism extension on φ . Finally, put

$$\psi_D = \pi \circ \bar{\varphi} : F(C \oplus D) \rightarrow A(C \times F(D)),$$

where $\pi : G \rightarrow A(C \times F(D))$ is the projection.

Fix $n \in \omega$ and divide the set κ into sets $A_1, A_2, \dots, A_n$ such that $|A_i| = \kappa$ for each $i = 1, 2, \dots, n$. Put

$$X_i = \{d_\alpha x_j x^{-1} d_\alpha^{-1} : j \in \omega, \alpha \in A_i\} \cup \{e\}$$

for each $i = 1, 2, \dots, n$. Then, by Lemma 2.5 and Theorem 2.4, we can show that $X = X_1 X_2 \cdots X_n$ is homeomorphic to S_κ^n . Therefore, we obtain the following.

Theorem 2.6. *Let κ be an infinite cardinal. Then, for every $n \in \omega$, $F(C \oplus D_\kappa)$ contains a subspace homeomorphic to S_κ^n .*

Since tightness is hereditary by subspaces, from Theorem 2.4 and Theorem 2.6 we immediately obtain

Corollary 2.7. *For every infinite cardinal κ , we have*

$$\sup\{t(S_\kappa^n) : n \in \mathbb{N}\} \leq t(A(C_\kappa))$$

and

$$\sup\{t(S_\kappa^n) : n \in \mathbb{N}\} \leq t(F(C \oplus D_\kappa)).$$

3. ZFC RESULTS

Let κ be a regular cardinal with $\omega \leq \kappa \leq w(X')$, where X' is the set of all non-isolated points of X . Then C_κ can be embedded in X as a closed subset. Hence we obtain

Theorem 3.1. *Let X be a metrizable space and X' the set of all non-isolated points of X . Then*

$$t(A(X)) \geq \sup\{t(A(C_\kappa)) : \omega \leq \kappa \leq w(X') \text{ and } \kappa \text{ is a regular cardinal}\}.$$

Our next corollary demonstrates that the general problem of Arhangel'skiĭ, Okunev and Pestov (see Problem 1) is completely reduced to the particular case of metric spaces of type C_κ .

Corollary 3.2. *The following conditions are equivalent:*

- (i) $t(A(X)) = w(X')$ for every metrizable space X , and
- (ii) $t(A(C_\kappa)) = \kappa$ for each successor cardinal $\kappa \geq \omega_1$.

In the non-abelian case, we obtain the following, similarly.

Theorem 3.3. *Let X be a non-discrete metrizable space. Then*

$$t(F(X)) \geq \sup\{t(F(C \oplus D_\kappa)) : \omega \leq \kappa \leq w(X) \text{ and } \kappa \text{ is a regular cardinal}\}.$$

Corollary 3.4. *The following conditions are equivalent:*

- (i) $t(F(X)) = w(X)$ for every non-discrete metrizable space X , and
- (ii) $t(F(C \oplus D_\kappa)) = \kappa$ for each successor cardinal $\kappa \geq \omega_1$.

Theorem 3.5. *Let X be a non-discrete metric space. Then $F(X)$ has countable tightness if and only if X is separable.*

Theorem 3.6. *If $\sup\{t(S_\kappa^n) : n \in \mathbb{N}\} = \kappa$ for every successor cardinal $\kappa \geq \omega_1$, then:*

- (i) $t(A(X)) = w(X')$ for every metrizable space X , and
- (ii) $t(F(X)) = w(X)$ for every non-discrete metrizable space X .

4. CONSISTENCY RESULTS

Let $\kappa \geq \omega_1$ be a regular cardinal. Let $\square(\kappa)$ denote the statement that there exists a sequence $\{C_\alpha : \alpha \in \kappa\}$ such that:

- (i) for $\alpha < \kappa$, C_α is closed and unbounded in α , and $C_{\alpha+1} = \{\alpha\}$,
- (ii) if α is a limit point of C_β , then $C_\alpha = C_\beta \cap \alpha$,
- (iii) if C is a closed and unbounded subset of κ , then there is a limit point α of C so that $C_\alpha \neq C \cap \alpha$.

Fact 4.1. (Todorćević) *Let $\kappa \geq \omega_1$ be a regular cardinal. Then:*

- (i) $\square(\kappa)$ implies $t(S_\kappa^2) = \kappa$, and
- (ii) if $\square(\kappa)$ fails, then κ is weakly compact in L .

Proof. Indeed, (i) follows from [2, Proposition 4.18], and (ii) coincides with [2, Theorem 4.6]. □

Fact 4.2. $\square_\kappa$ implies $\square_\kappa^T$ for every successor cardinal $\kappa \geq \omega_1$.

Fact 4.3. $\square_\kappa^T$ implies $\square(\kappa)$ for every successor cardinal $\kappa \geq \omega_1$.

Fact 4.4. Under the Axiom of Constructibility $V = L$, $\square(\kappa)$ holds for every successor cardinal $\kappa \geq \omega_1$.

Theorem 4.5. Assume $\square(\kappa)$ for every successor cardinal $\kappa \geq \omega_1$. Then:

- (i) $t(A(X)) = w(X')$ for every metrizable space X , and
- (ii) $t(F(X)) = w(X)$ for every non-discrete metrizable space X .

Proof. From the assumption of our theorem and Fact 4.1(i) we conclude that $\kappa = t(S_\kappa^2) \leq \sup\{t(S_\kappa^n) : n \in \mathbb{N}\} = \kappa$ for every successor cardinal $\kappa \geq \omega_1$. Now the result follows from Theorem 3.6. $\square$

Item (i) of our next corollary provides a positive consistent answer to Problem 1.

Corollary 4.6. Under the Axiom of Constructibility $V = L$, one has:

- (i) $t(A(X)) = w(X')$ for every metrizable space X , and
- (ii) $t(F(X)) = w(X)$ for every non-discrete metrizable space X .

Proof. Combine Theorem 4.5 with Fact 4.4. $\square$

Our next result shows that both statements (i) and (ii) from Theorem 4.5 have “large cardinal strength” in a sense that the failure of either of them implies the existence of large cardinal.

Corollary 4.7. Suppose that either the statement (i) or the statement (ii) from Theorem 4.5 fails. Then there exists a weakly compact cardinal in L .

Proof. The assumption of our corollary implies that the conclusion of Theorem 4.5 fails. Therefore, the assumption of Theorem 4.5 must also fail, i.e. $\square(\kappa)$ must fail for some regular cardinal $\kappa \geq \omega_1$. Now κ is weakly compact in L according to Fact 4.1(ii). $\square$

Corollary 4.8. If there exists a counterexample to the Problem 1, then there exists a weakly compact cardinal in L .

5. OPEN QUESTION

Question 1. *Does Theorem 4.5 hold in ZFC?*

Even a particular version of the above question seems interesting:

Question 2. *Are statements (i) and (ii) of Theorem 4.5 equivalent in ZFC?*

Question 3. *Does $t(F(C \oplus D_\kappa)) = t(A(C_\kappa))$ for every successor cardinal $\kappa \geq \omega_1$?*

Question 4. *Does $t(F(S_\kappa)) = t(A(S_\kappa))$ for every (successor) cardinal $\kappa \geq \omega_1$?*

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