

On some generalizations of q -uniform convexity inequalities

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Abstract. This is an announcement of some recent results of the authors concerning the q -uniform convexity and p -uniform smoothness inequalities.

We shall consider some generalizations of p -uniform smoothness and q -uniform convexity inequalities. In particular we shall characterize these two geometric notions by type- and cotype-like inequalities which are stronger than those of type and cotype, respectively.

1. p -uniformly smooth and q -uniformly convex spaces

Let X be a Banach space with $\dim X \geq 2$. The *modulus of convexity* of X is

$$\delta_X(\varepsilon) = \inf \left\{ 1 - \frac{\|x+y\|}{2} : \|x\| = \|y\| = 1, \|x-y\| = \varepsilon \right\}, \quad 0 \leq \varepsilon \leq 2.$$

X is called *uniformly convex* if $\delta_X(\varepsilon) > 0$ for all $\varepsilon > 0$, and *q -uniformly convex* ($2 \leq q < \infty$) if there exists a constant $C > 0$ such that $\delta_X(\varepsilon) \geq C\varepsilon^q$ for all $\varepsilon > 0$. The *modulus of smoothness* of X is

$$\rho_X(\tau) = \sup \left\{ \frac{\|x+\tau y\| + \|x-\tau y\|}{2} - 1 : \|x\| = \|y\| = 1 \right\}, \quad \tau > 0.$$

X is called *uniformly smooth* if $\rho_X(\tau)/\tau \rightarrow 0$ as $\tau \rightarrow 0$, and *p -uniformly smooth* ($1 < p \leq 2$) if there exists a constant $K > 0$ such that $\rho_X(\tau) \leq K\tau^p$ for all $\tau > 0$. These moduli have the best values with a Hilbert space H (cf. [8, p. 68]): For any Banach space X

$$\delta_X(\varepsilon) \leq \delta_H(\varepsilon) = 1 - \sqrt{1 - \varepsilon^2/4},$$

$$\rho_X(\tau) \geq \rho_H(\tau) = \sqrt{1 + \tau^2} - 1.$$

In view of these facts *no Banach space is q -uniformly convex for $q < 2$ and p -uniformly smooth for $p > 2$* . In fact, if $q < 2$, since

$$\frac{\delta_X(\varepsilon)}{\varepsilon^q} \leq \frac{1 - \sqrt{1 - \varepsilon^2/4}}{\varepsilon^q} = \frac{\varepsilon^{2-q}}{4(1 + \sqrt{1 - \varepsilon^2/4})},$$

we have $\lim_{\varepsilon \rightarrow +0} \delta_X(\varepsilon)/\varepsilon^q = 0$. When $p > 2$,

$$\frac{\rho_X(\tau)}{\tau^p} \geq \frac{\sqrt{1+\tau^2}-1}{\tau^p} = \frac{1}{\tau^{p-2}(\sqrt{1+\tau^2}+1)} \rightarrow \infty \text{ as } \tau \rightarrow 0.$$

Also every Banach space is 1-uniformly smooth as $\rho_X(\tau) \leq \tau$ for all $\tau > 0$. It is clear that p -uniformly smooth spaces are r -uniformly smooth if $1 < r \leq p \leq 2$, and q -uniformly convex spaces are r -uniformly convex if $2 \leq q \leq r < \infty$.

p -uniformly smooth and q -uniformly convex spaces are characterized by the following *p-uniform smoothness* and *q-uniform convexity inequalities*:

Lemma 1 ([1], [2]). (i) Let $1 < p \leq 2$. Then X is p -uniformly smooth if and only if there exists $K > 0$ such that

$$(1) \quad \frac{\|x+y\|^p + \|x-y\|^p}{2} \leq \|x\|^p + \|Ky\|^p \text{ for all } x, y \in X.$$

(ii) Let $2 \leq q < \infty$. Then X is q -uniformly convex if and only if there exists $C > 0$ such that

$$(2) \quad \frac{\|x+y\|^q + \|x-y\|^q}{2} \geq \|x\|^q + \|Cy\|^q \text{ for all } x, y \in X.$$

Remark 1. (i) The validity of the inequality (1) implies $K \geq 1$. Thus (1) with the best constant $K = 1$ is the following *Clarkson inequality*

$$(3) \quad \left(\frac{\|x+y\|^p + \|x-y\|^p}{2} \right)^{1/p} \leq (\|x\|^p + \|y\|^p)^{1/p} \quad (1 < p \leq 2)$$

(ii) In (2) we have necessarily $0 < C \leq 1$ (indeed put $x = 0$), and the inequality (2) with the best constant $C = 1$ is the following *Clarkson inequality*

$$(4) \quad \left(\frac{\|x+y\|^q + \|x-y\|^q}{2} \right)^{1/q} \geq (\|x\|^q + \|y\|^q)^{1/q} \quad (2 \leq q < \infty)$$

2. Generalizations of p -uniform smoothness and q -uniform convexity inequalities

We shall present some generalizations of p -uniform smoothness and q -uniform convexity inequalities which hold to characterize these smoothness and convexity. More precisely, in the first sense we shall give two-element inequalities sharper than (1) and (2) respectively, and in the secondary sense we shall characterize p -uniform smoothness and q -uniform convexity by type-, cotype-like inequalities which are stronger than type, cotype inequalities respectively.

The notions of type and cotype were introduced by Hoffman-Jørgensen [3] (cf. [9]) in the context of the law of large numbers for random variables with values in a Banach space. A Banach space X is called *of type p* , $1 \leq p \leq 2$, if there is $M > 0$ (necessarily $M \geq 1$) such that

$$(5) \quad \left(\frac{1}{2^n} \sum_{\theta_j = \pm 1} \left\| \sum_{j=1}^n \theta_j x_j \right\|^p \right)^{1/p} \leq M \left(\sum_{j=1}^n \|x_j\|^p \right)^{1/p}$$

for all finite systems $x_1, \dots, x_n \in X$. X is called *of cotype q* , $2 \leq q < \infty$, if there is $M > 0$ (necessarily $M \geq 1$) such that

$$(6) \quad \left(\frac{1}{2^n} \sum_{\theta_j = \pm 1} \left\| \sum_{j=1}^n \theta_j x_j \right\|^q \right)^{1/q} \geq \frac{1}{M} \left(\sum_{j=1}^n \|x_j\|^q \right)^{1/q}$$

for all finite systems $x_1, \dots, x_n \in X$.

These probabilistic properties are characterized by Clarkson's inequalities which are of geometric nature. Namely, in 1997 the first and second authors [6] showed that X is of type p with $M = 1$ if and only if Clarkson's inequality (3) holds in X and the corresponding fact for cotype and Clarkson's inequality (4) (their presentations are more general). On the other hand it is well known that

- (i) p -uniformly smooth spaces are of type p ,
- (ii) q -uniformly convex spaces are of cotype q

and there is no converse of these assertions. Indeed there exists a non-reflexive space X having type 2 (James [4]). Then X is of type p for any $1 < p \leq 2$, whereas X is not p -uniformly smooth because uniformly smooth spaces must be reflexive. Also its dual space X^* is of cotype q for any $2 \leq q < \infty$, but not q -uniformly convex as X^* is not reflexive.

Theorem 1 (p -uniform smoothness). Let $1 < p \leq 2$ and $1 \leq s < \infty$. The following are equivalent.

- (i) X is p -uniformly smooth.
- (ii) There exists $K \geq 1$ such that

$$(7) \quad \left(\frac{\|x+y\|^s + \|x-y\|^s}{2} \right)^{1/s} \leq (\|x\|^p + \|Ky\|^p)^{1/p} \quad \forall x, y \in X.$$

If $p \leq s < \infty$, in addition:

- (iii) There exists $K \geq 1$ such that

$$(8) \quad \left(\frac{1}{2^n} \sum_{\theta_j = \pm 1} \left\| \sum_{j=1}^n \theta_j x_j \right\|^s \right)^{1/s} \leq \left(\|x_1\|^p + \sum_{j=2}^n \|Kx_j\|^p \right)^{1/p}$$

for all finite systems $x_1, \dots, x_n \in X$.

Remark 2. (i) The inequality (7) is sharper than (1) of Lemma 1 if $p \leq s$. Indeed in this case by Lemma 2

$$\left(\frac{\|x+y\|^p + \|x-y\|^p}{2} \right)^{1/p} \leq \left(\frac{\|x+y\|^s + \|x-y\|^s}{2} \right)^{1/s} \leq (\|x\|^p + \|Ky\|^p)^{1/p}.$$

(ii) For the case $K = 1$ the equivalence of the inequalities (7) and (8) is proved in Kato-Takahashi [6].

(iii) The inequality (8) is stronger than the type p inequality (5). Indeed, the space X of James stated above is of type p , whereas (8) fails to hold in X . So we refer to (8) as *strong type p inequality*.

Theorem 2 (q -uniform convexity). Let $2 \leq q < \infty$ and $1 < t \leq \infty$. The following are equivalent.

- (i) X is q -uniformly convex.
- (ii) There exists $0 < C \leq 1$ such that

$$(9) \quad \left(\frac{\|x+y\|^t + \|x-y\|^t}{2} \right)^{1/t} \geq (\|x\|^q + \|Cy\|^q)^{1/q} \quad \forall x, y \in X.$$

If $1 < t \leq q$, in addition:

- (iii) There exists $0 < C \leq 1$ such that

$$(10) \quad \left(\frac{1}{2^n} \sum_{\theta_j = \pm 1} \left\| \sum_{j=1}^n \theta_j x_j \right\|^t \right)^{1/t} \geq \left(\|x_1\|^q + \sum_{j=2}^n \|Cx_j\|^q \right)^{1/q}$$

for all finite systems $x_1, \dots, x_n \in X$.

Remark 3. (i) The inequality (9) is sharper than (2) of Lemma 1 if $q \geq t$. Indeed we have

$$\left(\frac{\|x+y\|^q + \|x-y\|^q}{2} \right)^{1/q} \geq \left(\frac{\|x+y\|^t + \|x-y\|^t}{2} \right)^{1/t} \geq (\|x\|^q + \|Cy\|^q)^{1/q}.$$

(ii) For the case $C = 1$ the equivalence of the inequalities (9) and (10) is proved in Kato-Takahashi [6].

(iii) The inequality (10) is stronger than the cotype q inequality (6). Indeed the dual space X^* of the space X of James is of cotype q , but (10) fails to hold in X . L_1 is also a counter example, since it is of cotype 2 and non-reflexive. So we refer to (10) as *strong cotype q inequality*.

It is well known that if X is of type p , then X^* is of cotype q , where $1/p + 1/q = 1$, and the converse is not true ([2, pp. 309-310]). Indeed, $l_1 = (c_0)^*$ has cotype 2, whereas c_0 has no non-trivial type. Our next theorem asserts that for our strong type and cotype inequalities (8) and (10) the converse is also true if $p \leq s < \infty$.

Theorem 3 (duality). Let $1 \leq p \leq 2$, $1 < s < \infty$ and $1/p + 1/q = 1/s + 1/t = 1$. Let $1 \leq K < \infty$. Then if

$$(8) \quad \left(\frac{1}{2^n} \sum_{\theta_j = \pm 1} \left\| \sum_{j=1}^n \theta_j x_j \right\|^s \right)^{1/s} \leq \left(\|x_1\|^p + \sum_{j=2}^n \|K x_j\|^p \right)^{1/p}$$

holds in X ,

$$(10^*) \quad \left(\frac{1}{2^n} \sum_{\theta_j = \pm 1} \left\| \sum_{j=1}^n \theta_j x_j^* \right\|^t \right)^{1/t} \geq \left(\|x_1^*\|^q + \sum_{j=2}^n \|K^{-1} x_j^*\|^q \right)^{1/q}$$

holds in X^* . If $p \leq s < \infty$ the converse is true.

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