

On discrete Morse semi-flow

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1 Introduction.

Set d and D be positive integers greater than 1. Let $\mathbb{B}^d$ and $\mathbb{S}^D$ be the unit ball centered at the origin in $\mathbb{R}^d$, the unit sphere $\mathbb{S}^D$ in $\mathbb{R}^{D+1}$ and T a positive number. Give Q by $(0, T) \times \mathbb{B}^d$. This article studies a certain time-difference space-differential system; We call the solution to it “Discrete Morse Semiflow”, which is abbreviated to “DMS”. This system enables us discuss at least two important problems in Geometric evolutionary problems: Heat flows for harmonic mappings and mean curvature motion. To explain DMS, we introduce a several notation: Let h be a positive number and N_T be $[T/h] + 1$. We put $t_n := nh$ ($n = 0, \dots, N_T$) and set $k_0 = (1 - h/(16T) \log(1/h))$. $\chi(t) \in C^\infty$ with

$$\chi(t) := \begin{cases} t & t \leq 2, \\ 3 & t > 4, \end{cases} \quad (1.1)$$

Give a mapping $u_0 \in H^{1,2}(\mathbb{B}^d; \mathbb{S}^D)$. Then DMS is designated by a sequence of mappings $\{u_n\} (n = 1, \dots, N_T) \subset \{u \in H^{1,2}(\mathbb{B}^d; \mathbb{S}^D); u - u_0 \in \mathring{H}^{1,2}(\mathbb{B}^d; \mathbb{R}^{D+1})\}$ of the solution of the following difference-differential systems:

$$\frac{u_n - u_{n-1}}{h} - \Delta u_n + \frac{k_n}{\sqrt[4]{h}} \dot{\chi}((|u_n|^2 - 1)^2)(|u_n|^2 - 1)u_n = 0 \quad (1.2)$$

in $\mathbb{B}^d$,

$$u_n = u_0 \quad \text{on} \quad \partial\mathbb{B}^d. \quad (1.3)$$

An interpolational convention $u_{\bar{h}}(t, x)$ and $u_h(t, x)$ ($t > 0$) respectively indicates

$$\begin{aligned} u_{\bar{h}}(t, x) &:= u_n(x) && \text{for } t_{n-1} < t \leq t_n, \\ u_h(t, x) &:= \frac{t - t_{n-1}}{h} u_n(x) + \frac{t_n - t}{h} u_{n-1}(x) && \text{for } t_{n-1} < t \leq t_n. \end{aligned}$$

Note $\partial u_h / \partial t(t, x) = (u_n(x) - u_{n-1}(x)) / h$ for $t_{n-1} < t < t_n$. When no confusion may arise, we say a pair of functions $u_{\bar{h}}$ and u_h to be DMS; $u_{\bar{h}}$ and u_h satisfy

$$u_{\bar{h}} \in L^\infty(0, T; H^{1,2}(\mathbb{B}^d; \mathbb{S}^D)), \quad (1.4)$$

$$\int_{\mathbb{B}^d} \left(\left\langle \frac{\partial u_h}{\partial t}, \phi \right\rangle + \langle \nabla u_{\bar{h}}, \nabla \phi \rangle \right) dx = -\frac{k_{\bar{h}}}{\sqrt[4]{h}} \int_{\mathbb{B}^d} (|u_{\bar{h}}|^2 - 1) \langle u_{\bar{h}}, \phi \rangle dx \quad (1.5)$$

$$\text{for all } \phi \in C_0^\infty(\mathbb{B}^d; \mathbb{R}^{D+1}),$$

$$\begin{aligned} u_{\bar{h}}(t, x) - u_0(x) &\in \dot{H}^{1,2}(\mathbb{B}^d; \mathbb{R}^{D+1}) \quad \text{for every } t \quad (0 \leq t \leq N_T h), \\ \lim_{h \searrow 0} \|u_h(t, \circ) - u_0(\circ)\|_{L^2(\mathbb{B}^d)} &= 0. \end{aligned} \quad (1.6)$$

I addict to DMS: We show that DMS satisfies a maximal principle, a few global energy inequalities, a monotonicity inequality for the scaled energy and finally a reverse Poincaré inequality. By using the inequalities above, we prove that DMS converges to a heat flow for harmonic mappings and discuss a partial regularity result on it. Here for any given mapping $u_0 \in H^{1,2}(\mathbb{B}^d; \mathbb{S}^D)$, we call $u \in L^\infty(0, T; H^{1,2}(\mathbb{B}^d; \mathbb{S}^D)) \cap H^{1,2}(0, T; L^2(\mathbb{B}^d; \mathbb{S}^D))$ a heat flow for harmonic mappings provided

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta u + |\nabla u|^2 u \quad \text{in } Q, \\ u(0, x) &= u_0(x) \quad \text{in } \{0\} \times \mathbb{B}^d, \\ u(t, x) &= u_0(x) \quad \text{in } (0, T) \times \partial \mathbb{B}^d. \end{aligned} \quad (1.7)$$

The following fact is well-known

Remark 1 (1.7) is equivalent to

$$\frac{\partial u}{\partial t} \wedge u - \Delta u \wedge u = 0 \quad \text{in } (C_0^\infty(Q; \mathbb{R}^{D+1}))^*, \quad (1.8)$$

$$|u| = 1 \quad \text{in a.e } z \in Q. \quad (1.9)$$

The parabolic system holds in the following weak sense:

$$\int_Q \left(\left\langle \frac{\partial u}{\partial t}, \phi \right\rangle + \langle \nabla u, \nabla \phi \rangle - \langle u, \phi \rangle |\nabla u|^2 \right) dz = 0 \text{ for any } \phi \in C_0^\infty(Q; \mathbb{R}^{D+1}), \quad (1.10)$$

$$u(t, x) - u_0(x) \in \mathring{H}^{1,2}(\mathbb{B}^d; \mathbb{R}^{D+1}) \quad \text{for almost every } t \in (0, T), \quad (1.11)$$

$$\lim_{t \rightarrow +0} u(t, \circ) = u_0(\circ) \quad \text{in } L^2(\mathbb{B}^d; \mathbb{R}^{D+1}). \quad (1.12)$$

My main result of this article is

Theorem 1 (Partial Regularity) *There exists a heat flow for harmonic mappings and it is smooth on a relative open set in Q whose compliment has 0 d -dimensional Hausdorff measure with respect to the parabolic metric.*

The proof of Theorem 1 can be performed by combining Theorem 8 with Theorem 9.

2 DMS.

In this chapter, we state a discrete maximal principle and a few global energy-estimates. Thereafter we establish a monotonicity inequality for the scaled energies and a reverse Poincaré inequality, which are the main technical tools of this sort of study. The first is

Theorem 2 (Discrete maximal Principle) *Each of DMS $\{u_n\}$ ($n = 1, \dots, N_T$) implies*

$$|u_n| \leq 1 \quad \text{for all point } x \in \mathbb{B}^d.$$

Theorem 3 (Energy Estimate). *For any given mapping $u_0 \in H^{1,2}(\mathbb{B}^d; \mathbb{S}^D)$, DMS $\{u_n\}$ ($n = 1, 2, \dots, N_T$) satisfies*

$$\int_{\mathbb{B}^d} \left(\frac{1}{2} |\nabla u_n|^2 + \frac{k_n}{4\sqrt[4]{h}} (|u_n|^2 - 1)^2 \right) dx$$

$$\leq \frac{1}{2} \int_{\mathbb{B}^d} |\nabla u_0|^2 dx \quad \text{for any integer } n \quad (n = 1, \dots, N_T), \quad (2.1)$$

$$\begin{aligned} & \frac{h}{2} \sum_{n=1}^{N_T} \int_{\mathbb{B}^d} \left| \frac{u_n - u_{n-1}}{h} \right|^2 dx \\ & + \frac{\log(1/h)}{16T} h \sum_{n=1}^{N_T} \frac{k_{n-1}}{4\sqrt[4]{h}} \int_{\mathbb{B}^d} (|u_{n-1}|^2 - 1)^2 dx \leq \frac{1}{2} \int_{\mathbb{B}^d} |\nabla u_0|^2 dx. \end{aligned} \quad (2.2)$$

Lemma 1 (Global Pokhojaev Identity). *DMS u_n ($n = 1, 2, \dots, N_T$) have the following property:*

$$\begin{aligned} & \frac{1}{2} \int_{\partial \mathbb{B}^d} \left| \frac{\partial u_n}{\partial |x|} \right|^2 d\mathcal{H}_x^{d-1} \\ & + \frac{d-2}{2} \int_{\mathbb{B}^d} |\nabla u_n| dx + \frac{dk_n}{4\sqrt[4]{h}} \int_{\mathbb{B}^d} (|u_n|^2 - 1)^2 dx \\ & = \frac{1}{2} \int_{\partial \mathbb{B}^d} |\nabla_{\tan} u_0|^2 d\mathcal{H}_x^{d-1} + \int_{\mathbb{B}^d} \left\langle \frac{u_n - u_{n-1}}{h}, \langle x, \nabla \rangle u_n \right\rangle dx. \end{aligned} \quad (2.3)$$

Corollary 1 (The first derivatives estimates at $\partial \mathbb{B}^d$).

$$\begin{aligned} & \frac{1}{2} \int_h^T dt \int_{\mathbb{B}^d} \left| \frac{\partial u_{\bar{h}}}{\partial |x|} \right|^2 d\mathcal{H}_x^{d-1} \\ & \leq 2T \int_{\mathbb{B}^d} |\nabla_{\tan} u_0|^2 d\mathcal{H}_x^{d-1} + 2(T+1) \int_{\mathbb{B}^d} |\nabla u_0|^2 dx. \end{aligned} \quad (2.4)$$

Corollary 2 (The rate of the convergence). *If $\Delta u_0 \in L^{p_0}(\mathbb{B}^d; \mathbb{R}^{D+1})$ for some $p_0 > 1$,*

$$\int_{\mathbb{B}^d} |\nabla(u_1 - u_0)|^2 dx \leq 2^{1-2/p'_0} \|\nabla u_0\|_{L^2(\mathbb{B}^d)}^{2/p'_0} \|\Delta u_0\|_{L^{p_0}(\mathbb{B}^d)} \cdot h^{1-1/p_0}, \quad (2.5)$$

holds with $1/p_0 + 1/p'_0 = 1$.

Remark 2 The typical example of map from $\mathbb{B}^d$ to $\mathbb{S}^{d-1} \subset \mathbb{R}^d$ may be the equator map given by $x/|x|$. If $u_0(x) = x/|x|$, then $\Delta u_0 \in L^{q_0}(\mathbb{B}^d)$ as long as $1 < q_0 < d/2$. We refer to F.Bethuel and X.Zheng [1]. Namely the assumption on L^{p_0} -integrability about Δu_0 is just peril.

Lemma 2 (Higher Order Differential Estimates). *DMS u_n ($n = 1, 2, \dots, N_T$) satisfies*

$$\begin{aligned} & h \sum_{n=2}^{N_T} \int_{\mathbb{B}^d} |\Delta(u_n - u_{n-1})|^2 dx \\ & \leq Ch^{3/2} \int_{\mathbb{B}^d} |\nabla u_0|^2 dx + \frac{1}{2} \int_{\mathbb{B}^d} |\nabla(u_1 - u_0)|^2 dx. \end{aligned} \quad (2.6)$$

Now, we are in the position to state a monotonicity inequality for the scaled energy; For $z_0 = (t_{n_0}, x_0) \in Q$ and a positive number R , the scaled energy is denoted by

$$\begin{aligned} E_h(R; z_0) &:= \frac{1}{2R^d} \int_{t_{n_0}-\theta_0(2R)^2}^{t_{n_0}-\theta_0 R^2} dt \int_{\mathbb{B}^d} \left(|\nabla u_h|^2 + \frac{k_{\bar{h}}}{2\sqrt[4]{h}} (|u_h|^2 - 1)^2 \right) \\ &\quad \times \exp\left(\frac{|x - x_0|^2}{4(t - t_{n_0})}\right) dx. \end{aligned} \quad (2.7)$$

Lemma 3 (Monotonicity for the Scaled Energy). *For any point $z_0 = (t_{n_0}, x_0)$ and any positive number R ,*

$$\begin{aligned} \frac{dE_h}{dR}(R; z_0) &\geq -\frac{1}{R^{d-1}} \int_{t_{n_0}-\theta_0(2R)^2}^{t_{n_0}-\theta_0 R^2} \frac{t - t_{n_0}}{R^2} dt \int_{\mathbb{B}^d} \\ &\quad \times \left| \frac{\partial u_h}{\partial t} + \left\langle \frac{x - x_0}{2(t - t_{n_0})}, \nabla \right\rangle u_h \right|^2 \exp\left(\frac{|x - x_0|^2}{4(t - t_{n_0})}\right) dx \\ &\quad + \frac{1}{2R^{d+1}} \int_{t_{n_0}-\theta_0(2R)^2}^{t_{n_0}-\theta_0 R^2} \frac{k_{\bar{h}} dt}{\sqrt[4]{h}} \int_{\mathbb{B}^d} (|u_h|^2 - 1)^2 \exp\left(\frac{|x - x_0|^2}{4(t - t_{n_0})}\right) dx \\ &\quad - C_M(R, R_0; h). \end{aligned} \quad (2.8)$$

where

$$\begin{aligned}
C_M(R, R_0; h) &:= C_{M,1} + C_{M,2} \\
C_{M,1} &:= \frac{CR}{\text{dist}^{d+1}(x_0, \partial \mathbb{B}^d)} \left(\int_{\mathbb{B}^d} |\nabla u_0|^2 dx + \int_{\partial \mathbb{B}^d} |\nabla_{\tan} u_0|^2 d\mathcal{H}_x^{d-1} \right), \\
C_{M,2} &:= -C \|\nabla(u_1 - u_0)\|_{L^2(\mathbb{B}^d)} \|\nabla u_0\|_{L^2(\mathbb{B}^d)} + \frac{C\sqrt[4]{h}}{R^{d+1}} \|\nabla u_0\|_{L^2(\mathbb{B}^d)}.
\end{aligned}$$

Hereafter we state a point-wise estimate and the inequality of “a hybrid type”. The latter part of the chapter will be devoted to saying these inequalities. We supposed θ_0, θ_1 and R be positive numbers with

$$0 < \theta_0 < 1, \quad 1 < \theta_1, \quad R > 0, \quad \max\left(\frac{2\theta_1}{3}, 2\right) < \frac{\theta_0 R^2}{h}, \quad (2.9)$$

and we set

$$N_1 := \left\lceil \frac{\theta_0 R^2}{h} \right\rceil, \quad N_2 := \left\lceil \frac{\theta_0 (2R)^2}{h} \right\rceil.$$

We must remark that all N_i ($i = 1, 2, 3, 4$) are positive integers by assumption (2.9).

Theorem 4 (A Point-wise Estimate) *There exists a positive number ϵ_0 depending only on d , such that if $w_{\bar{h}}$ satisfies*

$$\int_{t_{n_1-2N_1}}^{t_{n_1}} dt \int_{B_{2R}(x_0)} (1 - \langle u_{\bar{h}}, K \rangle) dx < \epsilon_0 \quad (2.10)$$

for any cylinder $Q_{2R, 2N_1 h}(t_{n_1}, x_0) := (t_{n_1-2N_1}, t_{n_1}) \times B_{2R}(x_0) \subset\subset Q$, then

$$\begin{aligned}
&|\{z \in Q_{2R, N_1 h}(t_{n_1}, x_0); \langle u_{\bar{h}}, K \rangle \leq 1 - \delta_0\}| \\
&\leq C \frac{h \log(1/h)}{\delta_0^3} \int_{\mathbb{B}^d} |\nabla u_0|^2 dx
\end{aligned} \quad (2.11)$$

with $n_1 = n_0 + N_2 - N_1$ and K is any vector in $\mathbb{R}^{D+1}$.

We must remark that all N_i ($i = 1, 2, 3, 4$) are positive integers by assumption (2.9).

Theorem 5 (Discrete Hybrid Inequality) *DMS $u_{\bar{h}}$ and u_h have the following inequality: There exists positive constant C_H depending only on d such that for any numbers θ_0, θ_1, R satisfying the condition (2.9), for any cylinders $(t_{n_0}, t_{n_0+N_2}) \times B_{2R}(x_0) \subset\subset Q$,*

$$\begin{aligned}
& \int_{t_{n_0+N_2-N_1}}^{t_{n_0+N_2}} dt \int_{B_R(x_0)} \left(\frac{1}{2} |\nabla u_{\bar{h}}|^2 + \frac{k_{\bar{h}}}{4\sqrt[4]{h}} (|u_{\bar{h}}|^2 - 1)^2 + \frac{\theta_0 R^2}{2} \left| \frac{\partial u_h}{\partial t} \right|^2 \right) dx \\
& \quad + \frac{\theta_0 R^2}{10} \int_{B_R(x_0)} \left(\frac{1}{2} |\nabla u_{\bar{h}}|^2 + \frac{k_{\bar{h}}}{4\sqrt[4]{h}} (|u_{\bar{h}}|^2 - 1)^2 \right) dx \Bigg|_{t=t_{n_0+N_2}} \\
& \leq C_H \max \left(\left(1 - \frac{\theta_1}{N_1} \right)^{N_1}, \theta_0, \delta(\epsilon_0) \right) \tag{2.12} \\
& \times \int_{t_{n_0+N_2-3N_1-1}}^{t_{n_0+N_2}} dt \int_{B_{2R}(x_0)} \left(\frac{1}{2} |\nabla u_{\bar{h}}|^2 + \frac{k_{\bar{h}}}{4\sqrt[4]{h}} (|u_{\bar{h}}|^2 - 1)^2 + \frac{\theta_0 R^2}{2} \left| \frac{\partial u_h}{\partial t} \right|^2 \right) dx \\
& \quad + \left(1 + \theta_0 + \frac{1}{\theta_0} \right) \frac{C_H}{\log(1/\theta_1)^2 R^2} \int_{t_{n_0+N_2-3N_1}}^{t_{n_0+N_2}} dt \int_{B_{3R/2}(x_0)} |u_{\bar{h}} - K|^2 dx \\
& \quad + O(h).
\end{aligned}$$

where $R_0 = \min(\sqrt{t_{n_0}}/2\theta_0, \text{dist}(x_0, \partial\mathbb{B}^d))$, ϵ_0 is a certain positive constant appeared in Theorem 4, respectively and $\delta_0(\epsilon_0) = \epsilon_0^{1/d \cdot 1/(1+2/d)(1+4/d)}$.

Remark 3 *If one takes θ_1 being sufficiently large and θ_0, ϵ_0 being sufficiently small, then the coefficient of the first term on the right-hand side above is small.*

3 Heat Flows for Harmonic Mapping.

This chapter establishes the existence and a partial regularity on a heat flow for harmonic mappings that are obtained as the limit of DMS. The existence theorem is a slight modification of Y.Chen [3] and see also L.C.Evans [5, p.48, 5.A.1] and J.Shatah [9]. On the other hand the regularity result will be established by means of a blow-up technique. For

the blow-up technique used here, we refer to R.Hardt, D.Kinderlehrer and F.H.Lin [7] and R.Schöen and K.Uhlenbeck [11]. First of all we mention two convergence theorems directly derived from Theorem 2 and Theorem 3:

Theorem 6 (Convergence) *There exists a subsequence $\{u_{\bar{h}_k}\}, \{u_{h_k}\}$ ($k = 1, 2, \dots$) of $\{u_{\bar{h}}\}, \{u_h\}$ ($h > 0$) respectively and a mapping $u \in L^\infty(0, T; H^{1,2}(\mathbb{B}^d; \mathbb{S}^D)) \cap H^{1,2}(0, T; L^2(\mathbb{B}^d; \mathbb{S}^D))$ such that $u_{\bar{h}_k}$ and u_{h_k} respectively converges weakly-
* and weakly to u in $L^\infty(0, T; H^{1,2}(\mathbb{B}^d; \mathbb{R}^{D+1}))$
and $H^{1,2}(0, T; L^2(\mathbb{B}^d; \mathbb{R}^{D+1}))$, so does $u_{\bar{h}_k}$ strongly to u in $L^2(Q)$ and $u_{\bar{h}_k}$ point-wisely to u as $k \nearrow \infty$.*

Theorem 6 enables us state the following existence theorem:

Theorem 7 (Existence) *Each of DMS: $u_{\bar{h}}$ and u_h respectively converges to a heat flow for harmonic mappings u in $L^\infty(0, T; H^{1,2}(\mathbb{B}^d; \mathbb{R}^{D+1}))$ and $H^{1,2}(0, T; L^2(\mathbb{B}^d; \mathbb{R}^{D+1}))$ as $h \searrow 0$ (modulo a subsequence of h).*

Proof of Theorem 7. Since $\nabla u_{\bar{h}}$ and $\partial_t u_h$ is uniform bounded in $L^\infty(0, T; L^2(\mathbb{B}^d; \mathbb{R}^{D+1}))$ and $L^2(0, T; L^2(\mathbb{B}^d; \mathbb{R}^{D+1}))$ respectively, and a subsequence of $u_{\bar{h}}$ and u_h also converges weakly-* and weakly to a map u in $L^\infty(0, T; H^{1,2}(\mathbb{B}^d; \mathbb{R}^{D+1}))$ and $H^{1,2}(0, T; L^2(\mathbb{B}^d; \mathbb{R}^{D+1}))$ respectively, strongly in $L^2(\mathbb{B}^d; \mathbb{R}^{D+1})$, $|u_{\bar{h}}| \leq 1$, almost everywhere in Q as $h \searrow 0$; We show that the map u is indeed a heat flow for harmonic mappings. Since $u_{\bar{h}}$ and u_h satisfy

$$\frac{\partial u_h}{\partial t} - \Delta u_{\bar{h}} = \frac{k_{\bar{h}}}{\sqrt[4]{h}}(1 - |u_{\bar{h}}|^2)u_{\bar{h}},$$

by taking a wedge product, we have

$$\left(\frac{\partial u_h}{\partial t} \wedge u_{\bar{h}} - \Delta u_{\bar{h}} \right) \wedge u_{\bar{h}} = 0 \quad \text{in } (C_0^\infty(Q; \mathbb{R}^{D+1}))^*. \quad (3.1)$$

Thus by virtue of $u \in L^\infty(0, T; H^{1,2}(\mathbb{B}^d; \mathbb{S}^D)) \cap H^{1,2}(0, T; L^2(\mathbb{B}^d; \mathbb{S}^D))$, Remark 1, Theorem 6, we observe that u satisfies (1.10), (1.11) and (1.12), i.e. u is a heat flow for harmonic mappings $\square$

Remark 4 *In the following, we fix a subsequences $\{h_k\}$ ($k = 1, 2, 3, \dots$) of $\{h\}$ ($h > 0$) that makes DMS converge to a heat flow for harmonic mappings u .*

Definition 1 Fix a point $z_0 = (t_0, x_0) \in Q$. We indicate $\mathcal{M}^\star$ by the following rescaled Radon measure:

$$\mathcal{M}^\star(Q_R(z_0)) := \frac{\liminf_{h_k \searrow 0}}{2\theta_0 R^d} \int_{Q_R(z_0)} \left(|\nabla u_{\bar{h}_k}|^2 + \frac{k_{\bar{h}}}{2\sqrt[4]{h_k}} (|u_{\bar{h}_k}|^2 - 1)^2 + \theta_0 R^2 \left| \frac{\partial u_{h_k}}{\partial t} \right|^2 \right) dz,$$

for any positive number θ_0 and any cylinder $Q_R(z_0) \subset\subset Q$.

Remark 5 (Measured Hybrid Inequality) Assume that $u_{\bar{h}_k}$ and u_{h_k} respectively converges weakly-* and weakly in $L^\infty(0, T; H^{1,2}(\mathbb{B}^d; \mathbb{R}^{D+1}))$ and $H^{1,2}(0, T; L^2(\mathbb{B}^d; \mathbb{R}^{D+1}))$ to a heat flow for harmonic mappings $u \in L^\infty(0, T; H^{1,2}(\mathbb{B}^d; \mathbb{R}^{D+1})) \cap H^{1,2}(0, T; L^2(\mathbb{B}^d; \mathbb{R}^{D+1}))$ as $h_k \searrow 0$. Then take the pass to the limit $h_k \searrow 0$ in (2.12) to infer the following: For any positive θ_2 , there exists a positive constant C_{HM} depending only on d , θ_2 such that

$$\mathcal{M}^\star(Q_R(z_0)) \leq \theta_2 \mathcal{M}^\star(Q_{2R}(z_0)) + C_{HM} \int_{Q_{2R}(z_0)} |u - K|^2 dz \quad (3.2)$$

holds for any vector $K \in \mathbb{R}^{D+1}$ and $Q_R(z_0) \subset Q_{2R}(z_0) \subset\subset Q$ with $z_0 = (t_0, x_0)$ and $Q_{2R}(z_0) = (t_0 - \theta_0(2R)^2, t_0) \times B_{2R}(x_0)$.

In the similar way as in L.Simon [10, Lemma 2, p31], we can assert the following reverse Poncaré inequality:

Corollary 3 (Reverse Poincaré inequality). *The rescaled Radon measure implies the reverse Poincaré inequality: whenever $Q_{4R} \subset\subset Q$,*

$$R^{d+2} \mathcal{M}^\star(Q_R(z_0)) \leq C_{PO} \int_{Q_{4R}(z_0)} |u - K|^2 dz \quad (3.3)$$

holds, where C_{PO} is a certain positive constant depending only on θ_2 and d .

Proof of Corollary 3. Let

$$M = \sup_{\{Q_\sigma(z); Q_\sigma(z) \subset Q_{2R}(z_0)\}} \sigma^{d+2} \mathcal{M}^\star(Q_\sigma(z))$$

and then take any cylinder $Q_\sigma(z)$ with $Q_\sigma(z) \subset Q_{2R}(z_0)$. Notice that such a cylinder can be covered by cylinders $Q_{\sigma/4}(z_i)$ ($i = 1, 2, 3, \dots, S$)

with $z_i \in Q_\sigma(z)$ and $Q_\sigma(z_i) \subset Q_{2R}(z_0)$. We can evidently bound the number S by a fixed constant depending only on d . Recall (3.2); Then

$$\begin{aligned}
\sigma^{d+2} \mathcal{M}^\star(Q_\sigma(z)) &\leq 4^{d+2} \sum_{i=1}^S \left(\frac{\sigma}{4}\right)^{d+2} \mathcal{M}^\star(Q_{\sigma/4}(z_i)) \\
&\leq 4^{d+2} \theta_2 \left(\frac{\sigma}{2}\right)^{d+2} \mathcal{M}^\star(Q_{\sigma/2}(z_i)) \\
&+ 4^{d+2} C_{\text{MH}} \sum_{i=1}^S \int_{Q_{2\sigma}(z_i)} |u - K|^2 dz \\
&\leq 4^{d+2} S \theta_2 M + 4^{d+2} C_{\text{MH}} S \int_{Q_{4R}(z_0)} |u - K|^2 dz.
\end{aligned}$$

Taking “sup” on the right-hand side above, we have

$$M \leq 4^{d+2} S \theta_2 M + 4^{d+2} S C_{\text{MH}} S \int_{Q_{4R}(z_0)} |u - K|^2 dz,$$

whereupon $\theta_2 = 1/(24^{d+2} S)$, we infer

$$R^{d+2} \mathcal{M}^\star(Q_R(z_0)) \leq 24^{d+2} S C_{\text{MH}} \int_{Q_{4R}(z_0)} |u - K|^2 dz. \quad (3.4)$$

We can state one of the main assertions:

Theorem 8 (Energy Improvement) *For some positive numbers ϵ_0 , θ_0 and θ_1 , the following holds: for any positive number R and point $z_0 = (t_0, x_0)$ and any measure $\mathcal{M}$, for any cylinder $Q_R(z_0) (= (t_0 - \theta_0 R^2, t_0) \times B_R(x_0)) \subset\subset Q$,*

$$\begin{aligned}
\mathcal{M}^\star(Q_R(z_0)) &< \epsilon_0^2 \quad \text{implies} \\
\mathcal{M}^\star(Q_{\theta_1 R}(z_0)) &< \frac{1}{2} \mathcal{M}^\star(Q_R(z_0)).
\end{aligned} \quad (3.5)$$

Proof of Theorem 8. The proof can be proceeded by a contradiction: If the statement would be false, then for any positive number θ_1 less than $1/2$, there would exist sequences of positive numbers R_i , of points $z_i =$

$(t_i, x_i) \subset Q$, of measures $\mathcal{M}_i^\dagger$ and of heat flow for harmonic mappings u_i ($i = 1, 2, \dots$) such that for any θ_0 with $Q_{R_i}(z_i) \subset\subset Q$,

$$\mathcal{M}_i^\dagger(Q_{R_i}(z_i)) =: \epsilon_i^2 < \frac{1}{i}, \quad (3.6)$$

$$\text{but } \mathcal{M}_i^\dagger(Q_{\theta_1 R_i}(z_i)) \geq \frac{\epsilon_i^2}{2}. \quad (3.7)$$

By rescaling

$$z = (t, x) \rightarrow \bar{z} = (\bar{t}, \bar{x}) = \left(\frac{t - t_i}{\theta_0 R_i^2}, \frac{x - x_i}{R_i} \right),$$

without a loss of generality, we can rewrite (3.6) and (3.7) as

$$\mathcal{M}_i^\dagger(Q_1(0)) = \epsilon_i^2 < \frac{1}{i}, \quad (3.8)$$

$$\text{but } \mathcal{M}_i^\dagger(Q_{\theta_1}(0)) > \frac{\epsilon_i^2}{2}. \quad (3.9)$$

By using the rescaling $\bar{z} = ((t - t_i)/\theta_0 R_i^2, (x - x_i)/R_i)$ and a positive number r , (3.2) becomes

$$\mathcal{M}_i^\dagger(Q_{\theta_1}(0)) \leq C \int_{Q_{2\theta_1}(0)} |u_i - u_{i, Q_{2\theta_1}}|^2 d\bar{z}. \quad (3.10)$$

$$\text{Set } v_i(\bar{z}) := \frac{1}{\epsilon_i}(u_i(\bar{z}) - u_{i, Q_1}) \text{ for any } r \text{ with } \theta_1 \leq r \leq \frac{1}{2}.$$

By assumption (3.6), a subsequence of v_i converges weakly to a mapping $v_\infty \in L^2(0, T; H^{1,2}(B_1(0); \mathbb{R}^{D+1})) \cap H^{1,2}(0, T; L^2(B_1(0); \mathbb{R}^{D+1}))$ as $i \nearrow \infty$ (modulo a subsequence of i). In addition, since v_i satisfies the systems:

$$\frac{1}{\theta_0} \frac{\partial v_i}{\partial \bar{t}} - \Delta v_i = \epsilon_i |\nabla v_i|^2 u_i$$

in the sense of $(C_0^\infty(Q_1(0); \mathbb{R}^{D+1}))^*$, by using L.C.Evans [E1, p.39, Theorem 3] and noting (3.6) again, we find that v_∞ is the solution of

$$\frac{1}{\theta_0} \frac{\partial v_\infty}{\partial \bar{t}} - \Delta v_\infty = 0, \quad (3.11)$$

in the classical sense. From the gradient estimate on the solution of the equation (3.11) by S.Campanato [2] and the Rellich-Kondrachev theorem,

it follows that $\operatorname{ess\,sup}_{\bar{z} \in Q_{\theta_1}(0)} (|\nabla v_\infty| + |\partial v_\infty / \partial t|) \leq C \|\nabla v_\infty\|_{L^2(Q_1(0))}$ and v_i converges strongly to v_∞ in $L^2(Q_1(0))$ as $i \nearrow \infty$. Thus

$$\begin{aligned} \int_{Q_{2\theta_1}} |v_i|^2 d\bar{z} &\leq 2 \int_{Q_{2\theta_1}} |v_\infty|^2 d\bar{z} + 2 \int_{Q_{2\theta_1}} |v_i - v_\infty|^2 d\bar{z} \leq Cr^2, \\ \int_{Q_{2\theta_1}} |u_i - u_{i,Q_{2\theta_1}}|^2 d\bar{z} &\leq \epsilon_i^2 \int_{Q_{2\theta_1}} |v_i|^2 d\bar{z} \leq 2C\theta_1^2 \epsilon_i^2, \end{aligned}$$

holds if i is sufficiently large possibly depending on θ_1 . Consequently we infer

$$\mathcal{M}_i^*(Q_{\theta_1}(0,0)) \leq C\theta_1 \epsilon_i^2. \quad (3.12)$$

If we choose $C\theta_1 < 1/2$, which is $\mathcal{M}^*(Q_{\theta_1}(0)) < \epsilon_i^2/2$, then we find that this is a contradiction of our choice.

Theorem 9 (Singular Set) *Let ϵ_0 be the positive number appeared in Theorem 8. Define*

$$\mathbf{sing} := \bigcap_{R>0} \{z_0 \in Q; \mathcal{M}^*(P_R(z_0)) \geq \epsilon_0\}, \quad (3.13)$$

with $P_R(z_0) = (t_0 - \theta_0 R^2, t_0 + \theta_0 R^2) \times B_R(x_0)$. Then **sing** is a relatively closed set and

$$\mathcal{H}^{(d)}(\mathbf{sing}) = 0. \quad (3.14)$$

Proof of Theorem 9. **sing** is a relatively closed set. Indeed, if $z_0 \in \mathbf{sing} \cap Q$, some sequence $z_\nu = (t_\nu, x_\nu) \in \mathbf{sing} \cap Q$ ($\nu = 1, 2, \dots$) implies $z_\nu \rightarrow z_0$ as $\nu \nearrow \infty$, i.e. for any positive δ , there exists a positive number ν_δ such that $\operatorname{dist}(z_\nu, z_0) \leq \delta$ holds for any positive integer $\nu \geq \nu_\delta$. From definition on **sing**, for any $R > \delta$ and any points z_ν ($\nu = \nu_\delta, \nu_\delta + 1, \dots$), we obtain

$$\begin{aligned} \epsilon_0 &\leq \frac{\liminf_{h_k \searrow 0} \int_{P_{R-\delta}(t_\nu, x_\nu)} \left(|\nabla u_{\bar{h}_k}|^2 + \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}} (|u_{\bar{h}_k}|^2 - 1)^2 + \theta_0(R-\delta)^2 \left| \frac{\partial u_{h_k}}{\partial t} \right|^2 \right) dz \\ &\quad \times \int_{P_{R-\delta}(t_\nu, x_\nu)} \left(|\nabla u_{\bar{h}_k}|^2 + \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}} (|u_{\bar{h}_k}|^2 - 1)^2 + \theta_0(R-\delta)^2 \left| \frac{\partial u_{h_k}}{\partial t} \right|^2 \right) dz \end{aligned}$$

$$\leq \frac{\liminf_{h \searrow 0}}{2\theta_2(R-\delta)^d} \int_{P_R(z_0)} \left(|\nabla u_{\bar{h}_k}|^2 + \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}} (|u_{\bar{h}_k}|^2 - 1)^2 + \theta_0 R^2 \left| \frac{\partial u_{h_k}}{\partial t} \right|^2 \right) dz. \quad (3.15)$$

By the arbitrariness of δ , passing to the limit $\delta \searrow 0$, we can say $\overline{\mathbf{sing}} \cap Q \subset \mathbf{sing} \cap Q$, which provides us with our first assertion. Next we estimate the size of $\mathbf{sing}$ in the d -dimensional Hausdorff measure with respect to the parabolic metric. Fix a positive $R < 1$ and set a compact set $\mathbf{comp}$ in Q . Let $\{P_{2R_k}(z_k)\}$ ($2R_k < R$), be a cover of $\mathbf{sing}$. Since $\overline{\mathbf{sing}} \cap \mathbf{comp}$ is compact set in Q , we can assume that the cover is finite. Moreover the parabolic version of Vitali covering theorem shows that there is a disjoint finite sub-family $\{P_{R_k}(z_k)\}$, $k \in \mathcal{K}$ with

$$\mathbf{sing} \cap \mathbf{comp} \subset \bigcup_{k \in \mathcal{K}} P_{10R_k}(z_k),$$

$$2\epsilon_0 R_k^d \leq \liminf_{h_k \searrow 0} \int_{P_{R_k}(z_k)} \left(|\nabla u_{\bar{h}_k}|^2 + \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}} (|u_{\bar{h}_k}|^2 - 1)^2 + \theta_0 R_k^2 \left| \frac{\partial u_{h_k}}{\partial t} \right|^2 \right) dz.$$

From Corollary 3, we have

$$\begin{aligned} \epsilon_0 R_k^d &\leq \liminf_{h_k \searrow 0} \int_{P_{R_k}(z_k)} \left(|\nabla u_{\bar{h}_k}|^2 + \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}} (|u_{\bar{h}_k}|^2 - 1)^2 + \theta_0 R_k^2 \left| \frac{\partial u_{h_k}}{\partial t} \right|^2 \right) dz \\ &\leq \frac{C_{\text{PO}}}{R_k^2} \int_{P_{2R_k}(z_k)} |u - u_{P_{2R_k}(z_k)}|^2 dz \\ &\leq C \int_{P_{2R_k}(z_k)} \left(|\nabla u|^2 + \theta_0 R_k^2 \left| \frac{\partial u}{\partial t} \right|^2 \right) dz. \end{aligned} \quad (3.16)$$

Thus we obtain

$$\sum_{k=1}^K (10R_k)^d \leq C \bigcup_{k=1}^K \int_{P_{2R_k}(z_k)} \left(|\nabla u|^2 + \theta_0 R_k^2 \left| \frac{\partial u}{\partial t} \right|^2 \right) dz.$$

From

$$\sum_{k=1}^K (10R_k)^{d+2} \leq CR^2 \int_Q \left(|\nabla u|^2 + \left| \frac{\partial u}{\partial t} \right|^2 \right) dz$$

and the absolute continuity of the Lebesgue integral, we conclude

$$\mathcal{H}^{(d)}(\mathbf{sing} \cap \mathbf{comp}) \leq C \lim_{R \searrow 0} \sum_{k=1}^K (10R_k)^d = 0. \quad (3.17)$$

If we set $\mathbf{comp}_n := \{z \in Q; \text{dist}(z, \mathbb{C}Q) \geq 1/n\}$ ($n = 1, 2, \dots$), by $\lim_{n \rightarrow \infty} \mathcal{H}^{(d)}(\mathbf{sing} \cap \mathbf{comp}_n) = \mathcal{H}^{(d)}(\mathbf{sing})$, we can deduce our assertion.

Theorem 10 (Recursive Inequality) *The heat flow for harmonic mapping u is Hölder continuous on $Q \setminus \mathbf{sing}$.*

Proof of Theorem 10. Fix a point $z_0 = (t_0, x_0) \in Q \setminus \mathbf{sing}$ and choose R so that $\mathcal{M}^\star(P_R(z_0)) < \epsilon_0$ with some θ_0 possibly depending on z_0 and R . Because $Q/\mathbf{sing}$ is an open set, there exists some $P_{R_0}(z_0)$ so that

$$\mathcal{M}^\star(P_R(\bar{z}_0)) < \epsilon_0$$

for all point $\bar{z}_0 \in P_{R_0}(z_0)$. Then by Theorem 8, we obtain

$$\mathcal{M}^\star(P_r(\bar{z}_0)) \leq C \left(\frac{r}{R}\right)^{\alpha_0} \mathcal{M}^\star(P_R(\bar{z}_0)) \quad (3.18)$$

for any positive number $r > 0$ with $\alpha_0 = \log 2 / \log(1/\theta_1)$. This leads to our claim. $\square$

We next collect a few properties of the heat flow for harmonic mappings obtained by the perturbation of DMS:

Corollary 4 *From (2.2) in Theorem 3, we obtain*

$$\limsup_{h_k \searrow 0} \int_Q \frac{k_{\bar{h}_k}}{\sqrt[4]{h_k}} (|u_{\bar{h}_k}|^2 - 1)^2 dz = \limsup_{h_k \searrow 0} \frac{1}{\log 1/h_k} = 0. \quad (3.19)$$

From Lemma 4, we infer that there is a positive number ϵ_0 such that for any positive number ϵ less than ϵ_0 if the heat flow for harmonic mappings u satisfies

$$\int_{Q_{2R}} |u - u_{Q_{4R}}|^2 dz < \epsilon,$$

for any cylinder $Q_{2R}(z_0) \subset\subset Q$, then we infer

$$\limsup_{h_k \searrow 0} \sup_{Q_R(z_0)} |u_{\bar{h}_k} - u_{\bar{h}_k, Q_{2R}}|^2 < C(\epsilon),$$

where $C(\epsilon)$ is a positive number satisfying $C(\epsilon) \searrow 0$ as $\epsilon_0 \searrow 0$.

Finally, we close this section by showing the strong convergence of $u_{\bar{h}_k}$ to a heat flow for harmonic mappings u in $H_{\text{loc}}^{1,2}$ -topology as $h_k \searrow 0$;

Theorem 11 (Strong Convergencity of Gradients) *The gradients of $u_{\bar{h}_k}$ converges strongly to the gradients of u in $L_{\text{loc}}^2(Q)$.*

Proof of Theorem 11. Fix two compact sets $\mathbf{comp} \subset \mathbf{comp}_1 \subset Q$, which are compactly contained each other. Take the difference between (1.10) and (1.5), for a map $\phi \in C_0^\infty(\mathbf{comp}_1; \mathbb{R}^{D+1})$, then we obtain

$$\begin{aligned} & \int_{\mathbf{comp}_1} \left\langle \frac{\partial}{\partial t}(u_{h_k} - u), \phi \right\rangle dz + \int_{\mathbf{comp}_1} \langle \nabla(u_{\bar{h}_k} - u), \nabla \phi \rangle dz \\ &= - \int_{\mathbf{comp}_1} |\nabla u|^2 \langle u, \phi \rangle dz + \int_{\mathbf{comp}_1} \frac{k_{\bar{h}_k}}{\sqrt[4]{h_k}} (1 - |u_{\bar{h}_k}|^2) \langle u_{\bar{h}_k}, \phi \rangle dz. \end{aligned} \quad (3.20)$$

Substituting ϕ for $(u_{h_k} - u)\eta_1$, we obtain

$$\begin{aligned} & \int_{\mathbf{comp}_1} |\nabla(u_{\bar{h}_k} - u)|^2 \eta_1 dz \leq \int_{\mathbf{comp}_1} \left(\left| \frac{\partial u_{h_k}}{\partial t} \right| + \left| \frac{\partial u}{\partial t} \right| \right) |u_{h_k} - u| dz \\ &+ \int_{\mathbf{comp}_1} (|\nabla u_{h_k}| + |\nabla u|) |u_{h_k} - u| |\nabla \eta| dz \\ &+ \int_{\mathbf{comp}_1} |\nabla u|^2 |u_{h_k} - u| dz \\ &+ \int_{\mathbf{comp}_1} \frac{k_{\bar{h}_k}}{\sqrt[4]{h_k}} (1 - |u_{\bar{h}_k}|^2) |u_{h_k} - u| dz, \end{aligned} \quad (3.21)$$

where η_1 is a smooth function with the support of $\mathbf{comp}_1$ and $\eta_1 = 1$ in $\mathbf{comp}$.

By using Schwarz's inequality and recalling the energy inequality (2.1) and (2.2) in Theorem 3 and the strong convergencity of $u_{\bar{h}_k}$: Theorem 6, we can easily estimate the 1st, the 2nd and the 3rd terms on the right-hand side in (3.21). We estimate the last term of the right-hand side in (3.21). Since $\mathcal{H}^{(d)}(\mathbf{sing}) = 0$ and $\mathbf{sing} \cap \mathbf{comp}$ is compact, from definition of Hausdorff measure, for any positive number ϵ , there exists a positive number R_ϵ less than $R_0 (= \text{dist}(\mathbf{comp}, \partial \mathbf{comp}_1))$ and a finite cover

of **sing**: $\{Q_{R_i}(z_i)\}$ ($i = 1, 2, \dots, K_1$) with $R_i < R_\epsilon$ such that

$$\begin{aligned} \mathbf{sing} &\subset \bigcup_{i=1}^{K_1} Q_{R_i}(z_i), \quad \mathcal{H}^{(d)}(\mathbf{sing}) \leq \sum_{i=1}^{K_1} R_i^d + \epsilon, \\ \mathcal{A}^\star(Q_{R_i}(z_i)) &\leq CR^{\alpha_0}. \end{aligned}$$

We decompose the last term as follows:

$$\begin{aligned} &\int_{\mathbf{comp}_1} \frac{k_{\bar{h}}}{2\sqrt[4]{h_k}} (1 - |u_{\bar{h}_k}|^2) |u_{h_k} - u| dz \\ &\leq \int_{\bigcup_{i=1}^{K_1} Q_{R_i}(z_i)} \frac{k_{\bar{h}}}{2\sqrt[4]{h_k}} (1 - |u_{\bar{h}_k}|^2) |u_{h_k} - u| dz \\ &+ \int_{\mathbf{comp}_1 \setminus \bigcup_{i=1}^{K_1} Q_{R_i}(z_i)} \frac{k_{\bar{h}}}{2\sqrt[4]{h_k}} (1 - |u_{\bar{h}_k}|^2) |u_{h_k} - u| dz. \end{aligned} \quad (3.22)$$

Moreover there exists a finite cover $\{Q_{R_j}\}$ ($j = 1, 2, \dots, K_2$) with $z_i \in \mathbf{comp}_1 \setminus \bigcup_{i=1}^{K_1} Q_{R_i}(z_i)$, because it is compact; we can proceed to estimate (3.22) as follows:

$$\begin{aligned} &\int_{\mathbf{comp}_1} \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}} (1 - |u_{\bar{h}_k}|^2) |u_{h_k} - u| dz \\ &\leq \int_{\bigcup_{i=1}^{K_1} Q_{R_i}(z_i)} \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}} (1 - |u_{\bar{h}_k}|^2) |u_{h_k} - u| dz \\ &+ \int_{\bigcup_{i=1}^{K_2} Q_{R_j}(z_j)} \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}} (1 - |u_{\bar{h}_k}|^2) |u_{h_k} - u| dz \\ &\leq 2 \int_{\bigcup_{i=1}^{K_1} Q_{R_i}(z_i)} \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}} (1 - |u_{\bar{h}_k}|^2) |u_{h_k} - u| dz \\ &+ \int_{\bigcup_{i=1}^{K_2} Q_{R_j}(z_j)} \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}} (1 - |u_{\bar{h}_k}|^2) |u_{h_k} - u_{h_k Q_{R_j}(z_j)}| dz \\ &+ \int_{\bigcup_{i=1}^{K_2} Q_{R_j}(z_j)} \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}} (1 - |u_{\bar{h}_k}|^2) |u_{Q_{R_j}(z_j)} - u_{h_k Q_{R_j}(z_j)}| dz \end{aligned}$$

$$+ \int_{\cup_{i=1}^{K_2} Q_{R_j}(z_j)} \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}} (1 - |u_{\bar{h}_k}|^2) |u - u_{Q_{R_j}(z_j)}| dz. \quad (3.23)$$

From now on, we estimate the each term of (3.23). First, we majorize the 1st term as follows: Recall (1.5) as $h = h_k$ and substitute $u_{\bar{h}_k} \eta_i$ for ϕ in (1.5) where η_i is smooth function having only x -variable with the compact support in $B_{2R_i}(x_i)$ satisfying

$$\eta_i = \begin{cases} 1 & \text{in } B_{R_i}(x_i), \\ 0 & \text{outside } B_{2R_i}(x_i) \end{cases}$$

to obtain

$$\begin{aligned} & \int_{Q_{R_i}(z_i)} \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}} (1 - |u_{\bar{h}_k}|^2) dz \\ & \leq \int_{Q_{2R_i}(z_i)} (|\nabla u_{\bar{h}_k}|^2 + \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}} (|u_{\bar{h}_k}|^2 - 1)^2) dz \\ & + \left| \int_{Q_{2R_i}(z_i)} \left(\langle \frac{\partial u_{h_k}}{\partial t}, u_{h_k} \rangle + \frac{1}{2} \langle \nabla |u_{\bar{h}_k}|^2, \nabla \eta_1 \rangle \right) dz \right|. \end{aligned} \quad (3.24)$$

From Lemma 3, for $R_0 = 1/2 \text{ dist}(\text{comp}, \partial \text{comp}_1)$, we obtain

$$\begin{aligned} & \int_{Q_{R_i}(z_i)} \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}} (1 - |u_{\bar{h}_k}|^2) dz \leq CR_i^d \int_Q (|\nabla u_{\bar{h}_k}|^2 + \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}} (|u_{\bar{h}_k}|^2 - 1)^2) dz \\ & + \left| \int_{Q_{2R_i}(z_i)} \left(\langle \frac{\partial u_{h_k}}{\partial t}, u_{h_k} \rangle \eta_1 - \frac{1}{2} \langle |u_{\bar{h}_k}|^2, \Delta \eta_1 \rangle \right) dz \right|. \end{aligned} \quad (3.25)$$

That is, noting that $\partial u_{h_k}/\partial t$ and $u_{\bar{h}_k}$ converges weakly to $\partial u/\partial t$ and strongly to u as $k \nearrow \infty$, respectively and $|u| = 1$, a.e,

$$\begin{aligned} & \limsup_{h_k \searrow 0} \int_{\cup_{i=1}^{K_1} Q_{R_i}(z_i)} \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}} (1 - |u_{\bar{h}_k}|^2) dz \leq CR_i^d \\ & + \sum_{i=1}^{K_1} \limsup_{h_k \searrow 0} \left| \int_{Q_{2R_i}(z_i)} \left(\langle \frac{\partial u_{h_k}}{\partial t}, u_{h_k} \rangle \eta_1 - \frac{1}{2} \langle |u_{\bar{h}_k}|^2, \Delta \eta_1 \rangle \right) dz \right| \end{aligned}$$

$$\leq C\mathcal{H}^{(d)}(\mathbf{sing}) + \epsilon = \epsilon. \quad (3.26)$$

Next we estimate the 2nd and the 4th term on the right-hand side: First recall that since $z_i \in Q \setminus \mathbf{sing}$, by using Corollary 4 and Theorem 10, we obtain

$$\begin{aligned} & \limsup_{h_k \searrow 0} \int_{\cup_{j=1}^{K_2} Q_{R_j}(z_j)} \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}} (1 - |u_{\bar{h}_k}|^2) |u_{h_k} - u| dz \\ & \leq \limsup_{h_k \searrow 0} \int_{\cup_{j=1}^{K_2} \widehat{Q}_{R_j}(z_j)} \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}} (1 - |u_{\bar{h}_k}|^2) |u_{h_k} - u_{h_k Q_{R_i}(z_i)}| dz \\ & \leq \sum_{j=1}^{K_2} \limsup_{h_k \searrow 0} \int_{\widehat{Q}_{R_j}(z_j)} \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}} (1 - |u_{\bar{h}_k}|^2) dz \sup_{z \in \widehat{Q}_{R_j}(z_j)} |u_{h_k} - u_{h_k Q_{R_i}(z_i)}| \\ & \leq C(R^{\alpha_0}) \limsup_{h_k \searrow 0} \int_{\cup_{j=1}^{K_2} \widehat{Q}_{R_j}(z_j)} \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}} (1 - |u_{\bar{h}_k}|^2) dz, \end{aligned} \quad (3.27)$$

$$\begin{aligned} & \limsup_{h_k \searrow 0} \int_{\cup_{j=1}^{K_2} Q_{R_j}(z_j)} (1 - |u_{\bar{h}_k}|^2) |u - u_{Q_{R_j}(z_j)}| dz \\ & \leq \sum_{i=1}^{K_2} \limsup_{h_k \searrow 0} \int_{\widehat{Q}_{R_j}(z_j)} \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}} (1 - |u_{\bar{h}_k}|^2) dz \sup_{z \in \widehat{Q}_{R_j}(z_j)} |u - u_{Q_{R_j}(z_i)}| \\ & \leq C(R^{\alpha_0}) \limsup_{h_k \searrow 0} \int_{\cup_{j=1}^{K_2} \widehat{Q}_{R_j}(z_j)} \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}} (1 - |u_{\bar{h}_k}|^2) dz, \end{aligned} \quad (3.28)$$

where we set $\widehat{Q}_{R_1}(z_1) = Q_{R_1}(z_1)$, $\widehat{Q}_{R_j}(z_j) = Q_{R_j}(z_j) \setminus \cup_{i=1}^{j-1} Q_{R_i}(z_i)$ ($j = 2, 3, \dots, K_2$). If we now estimate $\int_{\cup_{j=1}^{K_2} Q_{R_j}(z_j)} k_{\bar{h}_k}/2\sqrt[4]{h_k} (1 - |u_{\bar{h}_k}|^2) dz$ in

the same way as before, we find

$$\begin{aligned} & \limsup_{h_k \searrow 0} \int_{\cup_{j=1}^{K_2} Q_{R_j}(z_j)} \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}} (1 - |u_{\bar{h}_k}|^2) |u - u_{Q_{R_j}(z_j)}| dz \\ & \leq C \int_Q \left(|\nabla u_{\bar{h}_k}|^2 + \frac{k_{\bar{h}_k}}{2\sqrt[4]{h}} (|u_{\bar{h}_k}|^2 - 1)^2 \right) dz \leq C. \end{aligned} \quad (3.29)$$

Finally, we estimate the 3rd term on the right-hand side:

$$\begin{aligned}
& \limsup_{h_k \searrow 0} \int_{\bigcup_{j=1}^{K_2} Q_{R_j}(z_j)} \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}} (1 - |u_{\bar{h}_k}|^2) |u_{\bar{h}_k} - u_{\bar{h}_k Q_{R_j}(z_j)}| dz \\
& \leq \limsup_{h_k \searrow 0} \int_{\text{comp}} \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}} (1 - |u_{\bar{h}_k}|^2) dz \\
& \times \max_j \limsup_{h_k \searrow 0} \frac{1}{|Q_{R_j}|} \int_{Q_{R_j}(z_j)} |u_{\bar{h}_k} - u| dz = 0. \tag{3.30}
\end{aligned}$$

By applying (3.29) into (3.27) and (3.28) and gathering the estimates of (3.26), (3.27), (3.28) and (3.30), we arrive at

$$\limsup_{h_k \searrow 0} \int_{\text{comp}} \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}} (1 - |u_{\bar{h}_k}|^2) dz \leq C\epsilon + C(R^{\alpha_0}).$$

Let $\epsilon \searrow 0$ and recall $C(R^{\alpha_0}) \searrow 0$ as $\epsilon \searrow 0$ to deduce our claim:

$$\limsup_{h_k \searrow 0} \int_{\text{comp}} \frac{k_{\bar{h}_k}}{2\sqrt[4]{h_k}} (1 - |u_{\bar{h}_k}|^2) dz = 0,$$

which implies

$$\limsup_{h_k \searrow 0} \int_{\text{comp}} |\nabla(u_{\bar{h}_k} - u)|^2 dz = 0.$$

4 Final Remarks.

We will mention a few open problems that we should research from now: The 1st is a so-called Federer's dimension reduction argument (For the Federer's dimension reduction argument, we refer to R.Schöen and K.Uhlenbeck [11])

Conjecture 1 $\mathcal{H}^{(d-1)}(\text{sing}) < \infty$.

Note that the estimate above is sharp in the sense that for any fixed time $t > 0$, any mapping $u(t, \cdot) : \mathbb{B}^3 \rightarrow \mathbb{S}^2$ i.e $d = 3$, $D = 2$ must have as least

one point singularity and “the time” has two-dimension in the parabolic metric.

Next we are drawn into the regularity on the singular set:

Problem 1 *sing is rectifiable set.*

Author believes that once we can show the above, we also arrive

Problem 2 *Let $(s, a) \in Q$ be a singular point of the heat flow for harmonic mappings; There exists a matrix $R \in O(3)$ (may be independent of (t, x) , but possibly depending on a) such that the heat flow for harmonic mappings u behaves $R(x - a)/|x - a|$ around the singular points a at each time slice s .*

These will be proved by establishing the parabolic analogue of L.Simon [10]. We do emphasize that a monotonicity for scaled energy is crucially made the best of his theory.

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