

On evolution equations described by distributive lattices

By

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Abstract

We investigate about evolution equations which are formulated in terms of distributive lattices. Shift operators are introduced and the evolution rules are rewritten by shift operators. The complexity of evolution rules is defined by the number of initial values contained in the solution. Some results are obtained about evolution rules such that whose complexity is of polynomial order.

§ 1. Introduction

A distributive lattice is an algebraic system with two binary operations, $\wedge$ and $\vee$, that arise from the theory of partially ordered sets [1]. Distributive lattices are fundamental objects in various fields of mathematics. For example, the power set of a set can be treated as a distributive lattice, where the operations of intersection and union are adopted as $\wedge$ and $\vee$, respectively.

Ikegami, Takahashi, and Matsukidaira investigated evolution equations in the form,

$$u_j^{n+1} = f(u_{j-1}^n, u_j^n, u_{j+1}^n),$$

where f consists of arguments u_j^n , operations of distributive lattices, and unary operations $\neg$ introduced by them [2]. In this formulation, it is assumed that f refers 3 neighbors. In other words, it needs the values of three variables u_{j-1}^n , u_j^n , and u_{j+1}^n to determine the value of u_j^{n+1} . They defined the complexity of evolution rules f by

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means of measuring the number of terms of initial values contained in the expression of the solution u_j^n . They pointed out that the number of terms in u_j^n can be reduced to polynomial order by applying the identities of distributive lattice for specific rules.

We investigate the evolution equations described by distributive lattices in more general situation. We consider the evolution equations in the form of

$$u_j^{n+1} = f(u_{j+a_1}^n, \dots, u_{j+a_k}^n),$$

where $a_i \in \mathbb{Z}$ and f consists of arguments and operations $\wedge$, $\vee$, and $\neg$. In our formulation, f may refer more than 3 neighbors. We also define the complexity in the same manner as the reference [2]. We introduced shift operators which shift the space index contained in the variables and express evolution rules by combinations of shift operators. The mechanism of eliminations of terms is clarified by expressing the rules through shift operators. We obtained some results about evolution operators F^n such that the complexity in the solution can be reduced to polynomial orders.

§ 2. Definitions of lattices

In this section, we review the definitions of lattices, distributive lattices and conjugations on lattices. Lattices are defined as algebraic systems on a set L with two binary operations $\wedge$ and $\vee$ satisfying the following properties [1].

Definition 2.1. A lattice is an algebraic system $(L, \wedge, \vee)$ defined by the following identities. For any $a, b, c \in L$,

$$(2.1) \quad a \wedge b = b \wedge a, \quad a \vee b = b \vee a,$$

$$(2.2) \quad (a \wedge b) \wedge c = a \wedge (b \wedge c), \quad (a \vee b) \vee c = a \vee (b \vee c),$$

$$(2.3) \quad a \wedge (a \vee b) = a \vee (a \wedge b) = a.$$

The identities (2.1), (2.2), and (2.3) are called commutative law, associative law, and absorption law respectively. From these identities, we can derive identities,

$$(2.4) \quad a \wedge a = a, \quad a \vee a = a,$$

since we obtain

$$a \wedge a = a \wedge (a \vee (a \wedge a)) = a,$$

by applying (2.3) twice. In the similar way, we can obtain $a \vee a = a$.

Identities (2.4) are called idempotent law. We frequently apply absorption and idempotent law at later sections in order to reduce the number of terms in the expressions.

Remark. Notice that there exists another definition of lattices. It is also able to define lattices as particular class of partially ordered sets. It is known that these two definitions are equivalent. The above definition 2.1 fits our purpose to investigate properties of evolution equations. Therefore, we adopt this definition.

Next, we recall the definition of distributive lattices.

Definition 2.2. A distributive lattice is a lattice which satisfies the distributive laws between $\wedge$ and $\vee$ shown below,

$$(2.5) \quad \begin{aligned} a \wedge (b \vee c) &= (a \wedge b) \vee (a \wedge c), \\ a \vee (b \wedge c) &= (a \vee b) \wedge (a \vee c). \end{aligned}$$

We introduce conjugations on lattices the same as reference [2].

Definition 2.3. Conjugations $\neg$ are unary operations which act on a set L defined by next formulae,

$$(2.6) \quad \neg a \in L \quad \text{for all } a \in L,$$

$$(2.7) \quad \neg(a \wedge b) = (\neg a) \vee (\neg b), \quad \neg(a \vee b) = (\neg a) \wedge (\neg b),$$

$$(2.8) \quad \neg(\neg a) = a.$$

We use a notation $(L, \wedge, \vee, \neg)$ for a distributive lattice with a conjugation in this article. We also call the operations $\wedge$, $\vee$, and $\neg$ as the lattice operations.

§ 3. Evolution equations expressed by lattice operations

In this section, we formulate evolution equations expressed by the lattice operations, which we investigate in this article.

§ 3.1. Formulations of evolution equations

In this article, we treat evolution equations of variables u_j^n , where each u_j^n takes one of the values in L and its indices $j \in \mathbb{Z}$ and $n \in \mathbb{Z}_{\geq 0}$ denote space and time respectively. An evolution rule is given in a form of

$$(3.1) \quad u_j^{n+1} = f(u_{j+a_1}^n, \dots, u_{j+a_k}^n)$$

and f is constructed from lattice operations $\wedge$, $\vee$, and $\neg$ on arguments, and $a_i \in \mathbb{Z}$. Additionally, it is assumed that the form of f and space indices $(a_1, \dots, a_k)$ do not change for j and n .

§ 3.2. Number of terms included in u_j^n

When f is given, we can obtain an explicit solution of u_j^n by substituting

$$\begin{aligned}
 (3.2) \quad u_j^n &= f(u_{j+a_1}^{n-1}, \dots, u_{j+a_k}^{n-1}) \\
 &= f(f(u_{j+2a_1}^{n-2}, \dots, u_{j+a_1+a_k}^{n-2}), \dots, f(u_{j+a_1+a_k}^{n-2}, \dots, u_{j+2a_k}^{n-2})) \\
 &= f(f(f(u_{j+3a_1}^{n-3}, \dots, u_{j+2a_1+a_k}^{n-3}), \dots, f(u_{j+2a_1+a_k}^{n-3}, \dots, u_{j+a_1+2a_k}^{n-3})), \dots, \\
 &\quad f(f(u_{j+2a_1+a_k}^{n-3}, \dots, u_{j+a_1+2a_k}^{n-3}), \dots, f(u_{j+a_1+2a_k}^{n-3}, \dots, u_{j+3a_k}^{n-3}))) \\
 &= \dots
 \end{aligned}$$

recursively until $n = 0$. We define

$$f^{(1)}(u_{j+a_1}^i, \dots, u_{j+a_k}^i) := f(u_{j+a_1}^i, \dots, u_{j+a_k}^i),$$

and

$$\begin{aligned}
 f^{(n)}(u_{j+na_1}^i, \dots, u_{j+na_k}^i) &:= f(f^{(n-1)}(u_{j+na_1}^i, \dots, u_{j+(n-1)a_1+a_k}^i), \dots \\
 &\quad \dots, f^{(n-1)}(u_{j+a_1+(n-1)a_k}^i, \dots, u_{j+na_k}^i)),
 \end{aligned}$$

for $n \geq 2$. We can rewrite the equation (3.2) with $f^{(n)}$ as

$$\begin{aligned}
 u_j^n &= f^{(1)}(u_{j+a_1}^{n-1}, \dots, u_{j+a_k}^{n-1}) \\
 &= f^{(2)}(u_{j+2a_1}^{n-2}, \dots, u_{j+2a_k}^{n-2}) \\
 &= f^{(3)}(u_{j+3a_1}^{n-3}, \dots, u_{j+3a_k}^{n-3}) \\
 &= \dots \\
 &= f^{(n)}(u_{j+na_1}^0, \dots, u_{j+na_k}^0).
 \end{aligned}$$

The obtained expression in this way will contain exponential order of terms with respect to n in general. However, in the cases of particular f , if we rearrange terms in the formal solution of u_j^n using formulae (2.1) to (2.8), the number of terms can become of polynomial order with respect to n . We show examples of the derivation of reduced expression of u_j^n .

Example 3.1. We set an evolution rule as

$$(3.3) \quad u_j^{n+1} = u_{j-1}^n \wedge u_{j+1}^n \wedge u_{j+2}^n.$$

Obviously, it holds that

$$(3.4) \quad u_j^1 = u_{j-1}^0 \wedge u_{j+1}^0 \wedge u_{j+2}^0.$$

To express u_j^2 by u_j^0 , substitute this form of u_j^1 (3.4) into $u_j^2 = u_{j-1}^1 \wedge u_{j+1}^1 \wedge u_{j+2}^1$. Then, we derive that

$$\begin{aligned}
 (3.5) \quad u_j^2 &= u_{j-1}^1 \wedge u_{j+1}^1 \wedge u_{j+2}^1 \\
 &= (u_{j-2}^0 \wedge u_j^0 \wedge u_{j+1}^0) \wedge (u_j^0 \wedge u_{j+2}^0 \wedge u_{j+3}^0) \wedge (u_{j+1}^0 \wedge u_{j+3}^0 \wedge u_{j+4}^0).
 \end{aligned}$$

By the idempotent law (2.4), the multiple terms u_j^0 , u_{j+1}^0 and u_{j+3}^0 in (3.5) can be combined into single terms, and we obtain

$$(3.6) \quad u_j^2 = u_{j-2}^0 \wedge u_j^0 \wedge u_{j+1}^0 \wedge u_{j+2}^0 \wedge u_{j+3}^0 \wedge u_{j+4}^0.$$

As well as u_j^2 , we can express u_j^3 by

$$(3.7) \quad \begin{aligned} u_j^3 &= u_{j-1}^2 \wedge u_{j+1}^2 \wedge u_{j+2}^2 \\ &= u_{j-2}^1 \wedge u_j^1 \wedge u_{j+1}^1 \wedge u_{j+2}^1 \wedge u_{j+3}^1 \wedge u_{j+4}^1 \\ &= (u_{j-3}^0 \wedge u_{j-1}^0 \wedge u_j^0) \wedge (u_{j-1}^0 \wedge u_{j+1}^0 \wedge u_{j+2}^0) \wedge (u_j^0 \wedge u_{j+2}^0 \wedge u_{j+3}^0) \\ &\quad \wedge (u_{j+1}^0 \wedge u_{j+3}^0 \wedge u_{j+4}^0) \wedge (u_{j+2}^0 \wedge u_{j+4}^0 \wedge u_{j+5}^0) \wedge (u_{j+3}^0 \wedge u_{j+5}^0 \wedge u_{j+6}^0) \end{aligned}$$

or

$$(3.8) \quad u_j^3 = u_{j-3}^0 \wedge u_{j-1}^0 \wedge u_j^0 \wedge u_{j+1}^0 \wedge u_{j+2}^0 \wedge u_{j+3}^0 \wedge u_{j+4}^0 \wedge u_{j+5}^0 \wedge u_{j+6}^0.$$

The latter form (3.8) is derived by the idempotent law similarly. The number of terms of (3.6) and (3.8) are smaller than those of (3.5) and (3.7) respectively. The growth of number of terms through (3.4), (3.6), and (3.8) is linear while exponential for the formal recursive substitutions (3.4), (3.5), and (3.7). Actually, it is able to reduce the number of terms in u_j^n to less than $3n + 1$ since u_j^n include terms from u_{j-n}^0 to u_{j+2n}^0 and those appearing twice or more can be combined into single terms by idempotent law.

Another example of reducing terms in the evolution solution is shown below.

Example 3.2. We set an evolution rule as $u_j^{n+1} = u_j^n \wedge (\neg u_{j+1}^n \vee \neg u_{j+2}^n)$. For $n = 1$, it holds that

$$u_j^1 = u_j^0 \wedge (\neg u_{j+1}^0 \vee \neg u_{j+2}^0).$$

By substitutions, we obtain

$$\begin{aligned} u_j^2 &= u_j^1 \wedge (\neg u_{j+1}^1 \vee \neg u_{j+2}^1) \\ &= (u_j^0 \wedge (\neg u_{j+1}^0 \vee \neg u_{j+2}^0)) \\ &\quad \wedge (\neg(u_{j+1}^0 \wedge (\neg u_{j+2}^0 \vee \neg u_{j+3}^0)) \vee \neg(u_{j+2}^0 \wedge (\neg u_{j+3}^0 \vee \neg u_{j+4}^0))). \end{aligned}$$

It is possible to perform the following calculations. By the identities of conjugations (2.7) and (2.8), it holds that

$$\begin{aligned} &(u_j^0 \wedge (\neg u_{j+1}^0 \vee \neg u_{j+2}^0)) \\ &\quad \wedge (\neg(u_{j+1}^0 \wedge (\neg u_{j+2}^0 \vee \neg u_{j+3}^0)) \vee \neg(u_{j+2}^0 \wedge (\neg u_{j+3}^0 \vee \neg u_{j+4}^0))) \\ &= u_j^0 \wedge (\neg u_{j+1}^0 \vee \neg u_{j+2}^0) \\ &\quad \wedge (\neg u_{j+1}^0 \vee \neg(\neg u_{j+2}^0 \vee \neg u_{j+3}^0) \vee \neg u_{j+2}^0 \vee \neg(\neg u_{j+3}^0 \vee \neg u_{j+4}^0)) \\ &= u_j^0 \wedge (\neg u_{j+1}^0 \vee \neg u_{j+2}^0) \\ &\quad \wedge (\neg u_{j+1}^0 \vee (u_{j+2}^0 \wedge u_{j+3}^0) \vee \neg u_{j+2}^0 \vee (u_{j+3}^0 \wedge u_{j+4}^0)). \end{aligned}$$

We can derive

$$\begin{aligned}
& u_j^0 \wedge (\neg u_{j+1}^0 \vee \neg u_{j+2}^0) \wedge (\neg u_{j+1}^0 \vee (u_{j+2}^0 \wedge u_{j+3}^0) \vee \neg u_{j+2}^0 \vee (u_{j+3}^0 \wedge u_{j+4}^0)) \\
&= u_j^0 \wedge (\neg u_{j+1}^0 \vee \neg u_{j+2}^0) \wedge ((\neg u_{j+1}^0 \vee \neg u_{j+2}^0) \vee (u_{j+2}^0 \wedge u_{j+3}^0) \vee (u_{j+3}^0 \wedge u_{j+4}^0)) \\
&= u_j^0 \wedge (\neg u_{j+1}^0 \vee \neg u_{j+2}^0),
\end{aligned}$$

by applying the absorption law (2.3) to the underlined parts. Then, we obtain that $u_j^2 = u_j^0 \wedge (\neg u_{j+1}^0 \vee \neg u_{j+2}^0)$. Similarly, it can be verified that $u_j^n = u_j^0 \wedge (\neg u_{j+1}^0 \vee \neg u_{j+2}^0)$ for $n \geq 1$.

§ 4. Evolutions expressed by shift operators

In this section, we introduce the shift operators acting on variables u_j^n , and rewrite the expressions of the evolution solutions u_j^n using shift operators.

§ 4.1. Definitions of shift operators

Definition 4.1. We define a shift operator $a[\cdot]$ ($a \in \mathbb{Z}$) which acts on u_j^n and it shifts all space indices of u_j^n included in the expression $\cdot$ by a . For example,

$$\begin{aligned}
a[u_j^n] &= u_{j+a}^n, \\
1[u_j^n \wedge (u_k^{n+1} \vee \neg u_l^{n+2})] &= u_{j+1}^n \wedge (u_{k+1}^{n+1} \vee \neg u_{l+1}^{n+2}),
\end{aligned}$$

and so on. Moreover, we define the extended usage of lattice operations, $\wedge$, $\vee$, and $\neg$, on shift operators as follows,

$$\begin{aligned}
(a \wedge b)[u_j] &:= a[u_j] \wedge b[u_j], \\
(a \vee b)[u_j] &:= a[u_j] \vee b[u_j], \\
(\neg a)[u_j] &:= \neg(a[u_j]),
\end{aligned}$$

and recursively,

$$\begin{aligned}
(F_1 \wedge F_2)[u_j] &:= F_1[u_j] \wedge F_2[u_j], \\
(F_1 \vee F_2)[u_j] &:= F_1[u_j] \vee F_2[u_j], \\
(\neg F_1)[u_j] &:= \neg(F_1[u_j]),
\end{aligned}$$

where F_1 and F_2 are operations constructed from the combinations of shift operators and lattice operations, $\wedge$, $\vee$, and $\neg$. We use lower cases (a , b , and so on) to refer shift operators that is confirmed to be a single shift operator, and upper cases (F , X , and so on) for single or combined shift operators.

The following example shows how combined shift operators act on a variable.

Example 4.2. It holds that

$$\begin{aligned}
 (-1 \wedge (\neg 0 \vee 1))[u_j^n] &= (-1[u_j^n]) \wedge ((\neg 0 \vee 1)[u_j^n]) \\
 &= (u_{j-1}^n) \wedge ((\neg 0[u_j^n]) \vee (1[u_j^n])) \\
 &= (u_{j-1}^n) \wedge (\neg(0[u_j^n]) \vee u_{j+1}^n) \\
 &= (u_{j-1}^n) \wedge (\neg u_j^n \vee u_{j+1}^n).
 \end{aligned}$$

Same formulae of lattice operations hold for shift operators. For example, the absorption law for shift operators can be verified as follows.

Example 4.3. It holds that

$$\begin{aligned}
 ((a \vee b) \wedge a)[u_j^n] &= ((a \vee b)[u_j^n]) \wedge (a[u_j^n]) \\
 &= (u_{j+a} \vee u_{j+b}) \wedge u_{j+a} = u_{j+a} = a[u_j^n].
 \end{aligned}$$

Thus, we obtained $(a \vee b) \wedge a = a$. Similarly, we can easily prove that the formulae (2.1) to (2.8) hold for shift operators.

Define a composition $\circ$ of single or combined shift operators F_1 and F_2 by $(F_2 \circ F_1)[u_j^n] := F_2[F_1[u_j^n]]$, then next formulae hold.

Proposition 4.4. We have the following formulae using the composition $\circ$.

$$(4.1) \quad b \circ a = a \circ b = (a + b),$$

$$(4.2) \quad F_1 \circ a = a \circ F_1,$$

$$(4.3) \quad (F_3 \vee F_2) \circ F_1 = (F_3 \circ F_1) \vee (F_2 \circ F_1),$$

$$(4.4) \quad (F_3 \wedge F_2) \circ F_1 = (F_3 \circ F_1) \wedge (F_2 \circ F_1),$$

$$(4.5) \quad \neg(F_2 \circ F_1) = (\neg F_2) \circ F_1.$$

Proof. We show the derivation of (4.3). The others can be proved in the similar way.

$$\begin{aligned}
 ((F_3 \vee F_2) \circ F_1)[u_j^n] &= ((F_3 \vee F_2))[F_1[u_j^n]] \\
 &= (F_3[F_1[u_j^n]]) \vee (F_2[F_1[u_j^n]]) \\
 &= ((F_3 \circ F_1)[u_j^n]) \vee ((F_2 \circ F_1)[u_j^n]) \\
 &= ((F_3 \circ F_1) \vee (F_2 \circ F_1))[u_j^n].
 \end{aligned}$$

□

Note that left-distributive laws corresponding to the right-distributive laws (4.13) and (4.14) do not hold. However, the following left-distributive law restricted to the single shift operator holds.

$$a \circ (F_1 \wedge F_2) = (F_1 \wedge F_2) \circ a = (F_1 \circ a) \wedge (F_2 \circ a) = (a \circ F_1) \wedge (a \circ F_2).$$

Compositions can describe substitutions of variables as follows.

Example 4.5. We set $u_j = v_{j+a} \vee \neg v_{j+b} = (a \vee \neg b)[v_j]$ and $v_j = w_{j+c} \wedge w_{j+d} = (c \wedge d)[w_j]$. If we use a direct substitution for the expression, then we obtain

$$\begin{aligned} u_j &= v_{j+a} \vee \neg v_{j+b} \\ &= (w_{j+a+c} \wedge w_{j+a+d}) \vee \neg(w_{j+b+c} \wedge w_{j+b+d}). \end{aligned}$$

If we use the composition, then we obtain

$$\begin{aligned} u_j &= (a \vee \neg b)[v_j] \\ &= (a \vee \neg b)[(c \wedge d)[w_j]] \\ &= ((a \vee \neg b) \circ (c \wedge d))[w_j] \\ &= ((a \circ (c \wedge d)) \vee \neg(b \circ (c \wedge d)))[w_j] \\ &= (((a \circ c) \wedge (a \circ d)) \vee \neg((b \circ c) \wedge (b \circ d)))[w_j] \\ &= (w_{j+a+c} \wedge w_{j+a+d}) \vee \neg(w_{j+b+c} \wedge w_{j+b+d}). \end{aligned}$$

Thus, we reached to the same expression for both statements.

§ 4.2. Polynomial classes of operators for the evolution equations

Let us consider the general form of evolution equation (3.1). The formal solution can be expressed by the initial data as

$$u_j^n = f^{(n)}(u_{j+na_1}^0, \dots, u_{j+na_k}^0).$$

In the reference [2], the authors discussed the complexity of solutions for the evolution equations defined by lattice operations with 3 neighbors. In this article, we discuss the complexity of evolution solutions by that of operators acting on the variable u_j^n . As previously described, we can rewrite the evolution equation using the combined shift operators;

$$u_j^{n+1} = F[u_j^n].$$

Then the expression of the solution u_j^n becomes

$$u_j^n = F^n[u_j^0] := (F \circ F \circ \dots \circ F)[u_j^0].$$

In this form, the complexity of the evolution solution u_j^n can be measured by the number of shift operators contained in F^n .

Example 4.6. Consider the evolution equation (3.3);

$$u_j^{n+1} = u_{j-1}^n \wedge u_{j+1}^n \wedge u_{j+2}^n.$$

It can be described by the shift operator,

$$u_j^{n+1} = F[u_j^n] = (-1 \wedge 1 \wedge 2)[u_j^n].$$

The solution u_j^n is denoted as

$$u_j^n = F^n[u_j^0]$$

and we can measure the complexity only for the operator part F^n . That is,

$$F = -1 \wedge 1 \wedge 2,$$

$$F^2 = -2 \wedge 0 \wedge 1 \wedge 2 \wedge 3 \wedge 4,$$

$$F^3 = -3 \wedge -1 \wedge 0 \wedge 1 \wedge 2 \wedge 3 \wedge 4 \wedge 5 \wedge 6,$$

and so on.

As shown in the above example, we can define the complexity of evolution solution u_j^n by the complexity of evolution operator F^n acting on u_j^0 . Thus, let us define a polynomial class for evolution operator F^n by the number of shift operators included in F^n . We define the polynomial class P_m for F if F^n contains shift operators of $O(n^m)$ for $n \gg 0$ after reduction of operators using the formulae (2.1) to (2.8).

§ 5. Some results on polynomial class

In this section, we explain some results about polynomial class P_m .

Consider sets of shift operators a_i with a single index and b_{ij} with two indices. Assume the set of shift operators $\{a_i\}_{i \in I}$, $\{b_{jk_j}\}_{j \in J, k_j \in K_j}$ and a combination of shift operators X , where I and J are sets of indices, $\{K_j\}_{j \in J}$ is a family of sets. A notation $\wedge_i a_i$ denotes $a_1 \wedge \cdots \wedge a_{|I|}$, $\wedge_j (\vee_{k_j} \neg b_{jk_j})$ denotes $(\neg b_{11} \vee \cdots \vee \neg b_{1|K_1|}) \wedge \cdots \wedge (\neg b_{|J|1} \vee \cdots \vee \neg b_{|J||K_{|J|}|})$, and so on. Then, the next proposition holds.

Proposition 5.1. For $\{a_i\}_{i \in I}$, $\{b_{jk_j}\}_{j \in J, k_j \in K_j}$ and X , it holds that

$$(5.1) \quad \begin{aligned} & ((\wedge_i a_i) \wedge (\wedge_j (\vee_{k_j} \neg b_{jk_j})) \wedge X)^n \\ &= ((\wedge_i a_i) \wedge X)^{n-1} \circ ((\wedge_i a_i) \wedge (\wedge_j (\vee_{k_j} \neg b_{jk_j})) \wedge X), \end{aligned}$$

where, $\{a_i\}_{i \in I}$ must not be empty, but X may be omitted.

Proof. It is derived by straightforward calculations. At first, we prove the case of $n = 2$. Set $F = ((\wedge_i a_i) \wedge (\wedge_j (\vee_{k_j} \neg b_{jk_j})) \wedge X)$. We write ab for $a \circ b$ in short. Then, it holds that

$$\begin{aligned}
& ((\wedge_i a_i) \wedge (\wedge_j (\vee_{k_j} \neg b_{jk_j})) \wedge X)^2 \\
&= (\wedge_{i'} (a_{i'} \circ ((\wedge_i a_i) \wedge (\wedge_j (\vee_{k_j} \neg b_{jk_j})) \wedge X))) \\
&\quad \wedge (\wedge_{j'} (\vee_{k_{j'}} \neg ((b_{j'k_{j'}}) \circ ((\wedge_i a_i) \wedge (\wedge_j (\vee_{k_j} \neg b_{jk_j})) \wedge X)))) \\
&\quad \wedge (X \circ F) \\
&= (\wedge_{i'} ((\wedge_i a_{i'} a_i) \wedge (\wedge_j (\vee_{k_j} \neg a_{i'} b_{jk_j})) \wedge (a_{i'} \circ X))) \\
&\quad \wedge (\wedge_{j'} (\vee_{k_{j'}} \neg ((\wedge_i b_{j'k_{j'}} a_i) \wedge (\wedge_j (\vee_{k_j} \neg b_{j'k_{j'}} b_{jk_j})) \wedge (b_{j'k_{j'}} \circ X)))) \\
&\quad \wedge (X \circ F) \\
&= (\wedge_{i'} ((\wedge_i a_{i'} a_i) \wedge (\wedge_j (\vee_{k_j} \neg a_{i'} b_{jk_j})) \wedge (a_{i'} \circ X))) \\
&\quad \wedge (\wedge_{j'} (\vee_{k_{j'}} ((\vee_i \neg a_i b_{j'k_{j'}}) \vee (\vee_j (\wedge_{k_j} b_{j'k_{j'}} b_{jk_j})) \vee \neg (b_{j'k_{j'}} \circ X)))) \\
&\quad \wedge (X \circ F) \\
&= (\wedge_{i'} ((\wedge_i a_{i'} a_i) \wedge (\wedge_j (\vee_{k_j} \neg a_{i'} b_{jk_j})) \wedge (a_{i'} \circ X))) \wedge (X \circ F) \\
&= (\wedge_i (a_i \circ F)) \wedge (X \circ F) \\
&= ((\wedge_i a_i) \wedge X) \circ F \\
&= ((\wedge_i a_i) \wedge X) \circ ((\wedge_i a_i) \wedge (\wedge_j (\vee_{k_j} \neg b_{jk_j})) \wedge X).
\end{aligned}$$

Note that we can apply the absorption law to the underlined parts. For any n , it holds that

$$\begin{aligned}
F^n &= \underbrace{(F \circ F) \circ \dots \circ F}_{n \text{ times}} \\
&= ((\wedge_i a_i) \wedge X) \circ \underbrace{(F \circ F) \circ \dots \circ F}_{n-1 \text{ times}} \\
&= ((\wedge_i a_i) \wedge X)^2 \circ \underbrace{(F \circ F) \circ \dots \circ F}_{n-2 \text{ times}} \\
&= \dots \\
&= ((\wedge_i a_i) \wedge X)^{n-1} \circ F.
\end{aligned}$$

Then, the equation holds for any n . □

By this proposition, we obtain the following corollary.

Corollary 5.2. *If $(\wedge_i a_i) \wedge X$ belongs to class P_m , then $(\wedge_i a_i) \wedge (\wedge_j (\vee_{k_j} \neg b_{jk_j})) \wedge X$ also belongs to P_m .*

By changing the order of composition, we can obtain the next proposition.

Proposition 5.3. *Set F_1 and F_2 as combinations of shift operators. If $F_2 \circ F_1 \in P_m$, then $F_1 \circ F_2$ also belongs to P_m .*

Proof. It holds that

$$\begin{aligned} (F_1 \circ F_2)^n &= (F_1 \circ F_2) \circ \cdots \circ (F_1 \circ F_2) \\ &= F_1 \circ (F_2 \circ F_1) \circ \cdots \circ (F_2 \circ F_1) \circ F_2 \\ &= F_1 \circ (F_2 \circ F_1)^{n-1} \circ F_2. \end{aligned}$$

Since $F_2 \circ F_1 \in P_m$, $(F_2 \circ F_1)^{n-1}$ contains shift operators of $O(n^m)$. Therefore, $F_1 \circ (F_2 \circ F_1)^{n-1} \circ F_2$ also, and $F_1 \circ F_2$ belongs to P_m . $\square$

The next condition is a sufficient condition for an operator F to be of class P_{m+1} .

Proposition 5.4. *If F satisfies a recurrence relation,*

$$(5.2) \quad F^{n+1} = G_n \wedge (a_n \circ F^n) \quad \text{or} \quad F^{n+1} = G_n \vee (a_n \circ F^n),$$

where G_n has shift operators of which number is less than or equal to Cn^m for some constant C , then $F \in P_{m+1}$.

Proof. Using the above recurrence relation, F^n can be expressed by F and $G_1, \dots, G_{n-1}$ as follows.

$$\begin{aligned} F^n &= G_{n-1} \wedge (a_{n-1} \circ F^{n-1}) \\ &= G_{n-1} \wedge (a_{n-1} \circ (G_{n-2} \wedge (a_{n-2} \circ F^{n-2}))) \\ &= \dots \\ &= G_{n-1} \wedge (a_{n-1} \circ G_{n-2}) \wedge (a_{n-1} a_{n-2} \circ G_{n-3}) \wedge \\ &\quad \cdots \wedge (a_{n-1} \dots a_2 \circ G_1) \wedge (a_{n-1} \dots a_1 \circ F). \end{aligned}$$

Each G_i are consisted by less than or equal to Ci^m terms, and their sum is less than or equal to $C'n^{m+1}$ for some constant C' . $\square$

We show two examples of F and G_n satisfying (5.2) of proposition 5.4 such that numbers of terms of G_n is constant. Note that the concrete example for $m \geq 1$ has not yet been found.

Example 5.5. Define $F^0 = 0$ for any operators F . It holds that

$$(5.3) \quad \begin{aligned} &(a \wedge ((\vee_i \neg b_i) \vee c))^n \\ &= a^n \wedge (((\vee_i \neg a^{n-1} b_i) \vee (c \circ (a \wedge ((\vee_i \neg b_i) \vee c))^{n-1}))) \quad (n \geq 1), \end{aligned}$$

$$(5.4) \quad \begin{aligned} & (\neg 0 \wedge (0 \vee a))^n \\ & = (0 \vee \neg 0) \wedge ((0 \wedge \neg 0) \vee (a \circ (\neg 0 \wedge (0 \vee a))^{n-1})) \quad (n \geq 2), \end{aligned}$$

for any a , b_i and c .

We can prove the equalities (5.3) and (5.4) by mathematical induction as follows.

Proof of (5.3): Set $F = a \wedge ((\vee_i \neg b_i) \vee c)$. It is obvious that (5.3) holds for $n = 1$. Assume that (5.3) holds for $n = l$, it means that $F^l = a^l \wedge ((\vee_i \neg a^{l-1} b_i) \vee (c \circ (a \wedge ((\vee_i \neg b_i) \vee c))^{l-1}))$, then it is able to calculate F^{l+1} as follows.

$$\begin{aligned} F^{l+1} &= F^l \circ F = (a^l \wedge ((\vee_i \neg a^{l-1} b_i) \vee c \circ F^{l-1})) \circ F \\ &= a^l \circ F \wedge ((\vee_i \neg a^{l-1} b_i) \circ F \vee c \circ F^l) \\ &= a^{l+1} \wedge ((\vee_i \neg a^l b_i) \vee a^l c) \wedge ((\vee_i \neg a^{l-1} b_i) \circ F \vee c \circ (a^l \wedge ((\vee_i \neg a^{l-1} b_i) \vee c \circ F^{l-1}))) \\ &= a^{l+1} \wedge ((\vee_i \neg a^l b_i) \vee a^l c) \wedge ((\vee_i \neg a^{l-1} b_i) \circ (a \wedge ((\vee_j \neg b_j) \vee c)) \\ &\quad \vee (a^l c \wedge ((\vee_i \neg a^{l-1} b_i c) \vee c^2 \circ F^{l-1}))) \\ &= a^{l+1} \wedge ((\vee_i \neg a^l b_i) \vee a^l c) \wedge (\vee_i ((\neg a^{l-1} b_i) \circ (a \wedge ((\vee_j \neg b_j) \vee c))) \\ &\quad \vee (a^l c \wedge ((\vee_i \neg a^{l-1} b_i c) \vee c^2 \circ F^{l-1}))) \\ &= a^{l+1} \wedge ((\vee_i \neg a^l b_i) \vee a^l c) \wedge ((\vee_i \neg a^l b_i) \vee (\vee_i (\neg a^{l-1} b_i \circ ((\vee_j \neg b_j) \vee c))) \\ &\quad \vee (a^l c \wedge ((\vee_i \neg a^{l-1} b_i c) \vee c^2 \circ F^{l-1}))) \\ &= a^{l+1} \wedge ((\vee_i \neg a^l b_i) \vee a^l c) \wedge (((\vee_i \neg a^l b_i) \vee (\vee_i ((\wedge_j a^{l-1} b_i b_j) \wedge \neg a^{l-1} b_i c))) \\ &\quad \vee (a^l c \wedge ((\vee_i \neg a^{l-1} b_i c) \vee c^2 \circ F^{l-1}))) \\ &= a^{l+1} \wedge ((\vee_i \neg a^l b_i) \vee (a^l c \wedge (\vee_i ((\wedge_j a^{l-1} b_i b_j) \wedge \neg a^{l-1} b_i c) \\ &\quad \vee (a^l c \wedge ((\vee_i \neg a^{l-1} b_i c) \vee c^2 \circ F^{l-1})))))) \\ &= a^{l+1} \wedge ((\vee_i \neg a^l b_i) \vee (a^l c \wedge (\vee_i ((\wedge_j a^{l-1} b_i b_j) \wedge \neg a^{l-1} b_i c) \vee a^l c) \\ &\quad \wedge (\vee_i ((\wedge_j a^{l-1} b_i b_j) \wedge \neg a^{l-1} b_i c) \vee ((\vee_i \neg a^{l-1} b_i c) \vee c^2 \circ F^{l-1})))))) \\ &= a^{l+1} \wedge ((\vee_i \neg a^l b_i) \vee (a^l c \wedge (\vee_i ((\wedge_j a^{l-1} b_i b_j) \wedge \neg a^{l-1} b_i c) \\ &\quad \vee ((\vee_i \neg a^{l-1} b_i c) \vee c^2 \circ F^{l-1})))))) \\ &= a^{l+1} \wedge ((\vee_i \neg a^l b_i) \vee (a^l c \wedge ((\vee_i \neg a^{l-1} b_i c) \vee c^2 \circ F^{l-1}))) \\ &= a^{l+1} \wedge ((\vee_i \neg a^l b_i) \vee c \circ (a^l \wedge ((\vee_i \neg a^{l-1} b_i) \vee c \circ F^{l-1}))) \\ &= a^{l+1} \wedge ((\vee_i \neg a^l b_i) \vee c \circ F^l). \end{aligned}$$

The last form of this calculation corresponds to (5.3) if $n = l + 1$. Therefore, we obtain that (5.3) holds for $n \geq 1$, by the principle of mathematical induction.

Proof of (5.4): Set $F = \neg 0 \wedge (0 \vee a)$. F^2 can be calculated as follows.

$$\begin{aligned} F^2 &= \neg 0 \circ (\neg 0 \wedge (0 \vee a)) \wedge (0 \circ (\neg 0 \wedge (0 \vee a)) \vee a \circ (\neg 0 \wedge (0 \vee a))) \\ &= (0 \vee (\neg 0 \wedge \neg a)) \wedge ((\neg 0 \wedge (0 \vee a)) \vee (\neg a \wedge (a \vee a^2))) \end{aligned}$$

$$\begin{aligned}
&= (0 \vee \neg 0) \wedge (0 \vee \neg a) \wedge ((0 \wedge \neg 0) \vee (\neg 0 \wedge a) \vee (\neg a \wedge (a \vee a^2))) \\
&= (0 \vee \neg 0) \wedge ((0 \wedge \neg 0 \wedge (0 \vee \neg a)) \vee (\neg 0 \wedge a \wedge (0 \vee \neg a)) \vee (\neg a \wedge (a \vee a^2) \wedge (0 \vee \neg a))) \\
&= (0 \vee \neg 0) \wedge ((0 \wedge \neg 0) \vee (\neg 0 \wedge a \wedge (0 \vee \neg a)) \vee (\neg a \wedge (a \vee a^2))) \\
&= (0 \vee \neg 0) \wedge ((0 \wedge \neg 0) \vee (\neg 0 \wedge a \wedge 0) \vee (\neg 0 \wedge a \wedge \neg a) \vee (\neg a \wedge a) \vee (\neg a \wedge a^2)) \\
&= (0 \vee \neg 0) \wedge ((0 \wedge \neg 0) \vee (\neg a \wedge a) \vee (\neg a \wedge a^2)) \\
&= (0 \vee \neg 0) \wedge ((0 \wedge \neg 0) \vee (\neg a \wedge (a \vee a^2))) \\
&= (0 \vee \neg 0) \wedge ((0 \wedge \neg 0) \vee a \circ F)
\end{aligned}$$

Thus, (5.4) holds for $n = 2$. Assume that (5.4) holds for $n = l \geq 2$, it means that $F^l = (0 \vee \neg 0) \wedge ((0 \wedge \neg 0) \vee (a \circ (\neg 0 \wedge (0 \vee a))^{l-1}))$, then it is able to calculate F^{l+1} as follows.

$$\begin{aligned}
F^{l+1} &= F^l \circ F = ((0 \vee \neg 0) \wedge ((0 \wedge \neg 0) \vee a \circ F^{l-1})) \circ F \\
&= (0 \vee \neg 0) \circ F \wedge ((0 \wedge \neg 0) \circ F \vee a \circ F^l) \\
&= ((\neg 0 \wedge (0 \vee a)) \vee 0 \vee (\neg 0 \wedge \neg a)) \wedge ((\neg 0 \wedge (0 \vee a) \wedge (0 \vee (\neg 0 \wedge \neg a))) \vee a \circ F^l) \\
&= ((0 \wedge \neg 0) \vee (\neg 0 \wedge a) \vee 0 \vee (\neg 0 \wedge \neg a)) \\
&\quad \wedge ((\neg 0 \wedge (0 \vee a) \wedge (0 \vee \neg 0) \wedge (0 \vee \neg a)) \vee a \circ F^l) \\
&= ((\neg 0 \wedge a) \vee 0 \vee (\neg 0 \wedge \neg a)) \wedge ((\neg 0 \wedge (0 \vee a) \wedge (0 \vee \neg a)) \vee a \circ F^l) \\
&= (0 \vee (\neg 0 \wedge (a \vee \neg a))) \wedge ((\neg 0 \wedge (0 \vee (a \wedge \neg a))) \vee a \circ F^l) \\
&= (0 \vee \neg 0) \wedge (0 \vee a \vee \neg a) \wedge ((\neg 0 \wedge 0) \vee (\neg 0 \wedge a \wedge \neg a) \vee a \circ F^l) \\
&= (0 \vee \neg 0) \wedge ((\neg 0 \wedge 0 \wedge (0 \vee a \vee \neg a)) \\
&\quad \vee (\neg 0 \wedge a \wedge \neg a \wedge (0 \vee a \vee \neg a)) \vee ((0 \vee a \vee \neg a) \wedge a \circ F^l)) \\
&= (0 \vee \neg 0) \wedge ((0 \wedge \neg 0) \vee (\neg 0 \wedge a \wedge \neg a) \vee ((0 \vee a \vee \neg a) \wedge a \circ F^l)) \\
&= (0 \vee \neg 0) \wedge ((0 \wedge \neg 0) \vee ((0 \vee a \vee \neg a) \vee (\neg 0 \wedge a \wedge \neg a)) \\
&\quad \wedge ((\neg 0 \wedge a \wedge \neg a) \vee a \circ F^l)) \\
&= (0 \vee \neg 0) \wedge ((0 \wedge \neg 0) \vee ((0 \vee a \vee \neg a) \wedge ((\neg 0 \wedge a \wedge \neg a) \vee a \circ F^l))) \\
&= (0 \vee \neg 0) \wedge ((0 \wedge \neg 0) \vee) \\
&\quad (((0 \vee a \vee \neg a) \wedge \neg 0 \wedge a \wedge \neg a) \vee ((0 \vee a \vee \neg a) \wedge a \circ F^l))) \\
&= (0 \vee \neg 0) \wedge ((0 \wedge \neg 0) \vee (((\neg 0 \wedge a \wedge \neg a) \vee ((0 \vee a \vee \neg a) \wedge a \circ F^l)))) \\
&= (0 \vee \neg 0) \wedge ((0 \wedge \neg 0) \vee (((\neg 0 \wedge a \wedge \neg a) \\
&\quad \vee ((0 \vee a \vee \neg a) \wedge (a \vee \neg a) \wedge ((a \wedge \neg a) \vee a^2 \circ F^{l-1})))))) \\
&= (0 \vee \neg 0) \wedge ((0 \wedge \neg 0) \vee (((\neg 0 \wedge a \wedge \neg a) \vee ((a \vee \neg a) \wedge ((a \wedge \neg a) \vee a^2 \circ F^{l-1})))))) \\
&= (0 \vee \neg 0) \wedge ((0 \wedge \neg 0) \vee (((((\neg 0 \wedge a \wedge \neg a) \vee a \vee \neg a) \\
&\quad \wedge ((\neg 0 \wedge a \wedge \neg a) \vee (a \wedge \neg a) \vee a^2 \circ F^{l-1}))))))
\end{aligned}$$

$$\begin{aligned}
&= (0 \vee \neg 0) \wedge ((0 \wedge \neg 0) \vee ((a \vee \neg a) \wedge ((a \wedge \neg a) \vee a^2 \circ F^{l-1}))) \\
&= (0 \vee \neg 0) \wedge ((0 \wedge \neg 0) \vee a \circ F^l).
\end{aligned}$$

The last form of this calculation corresponds to (5.4) if $n = l + 1$. Therefore, we obtain that (5.4) holds for $n \geq 2$, by the principle of mathematical induction.

Remark. Ikegami, Takahashi and Matsukidaira investigated evolution equations defined by lattice operators $\wedge$, $\vee$, and $\neg$ which embed Elementary Cellular Automaton (ECA) [2]. They also defined the polynomial class for the complexity of solutions. Their definition is based on the order of the number of terms included in solutions. Our definition is on the order of that in shift operators. As previously mentioned, both are equivalent each other. Moreover, we can derive some part of their results as special cases of our results. Let us consider the equation (5.1) in Proposition 5.1,

$$\begin{aligned}
&((\wedge_i a_i) \wedge (\wedge_j (\vee_{k_j} \neg b_{jk_j})) \wedge X)^n \\
&= ((\wedge_i a_i) \wedge X)^{n-1} \circ ((\wedge_i a_i) \wedge (\wedge_j (\vee_{k_j} \neg b_{jk_j})) \wedge X).
\end{aligned}$$

Assume X is empty, $|I| = |J| = |K_1| = 1$ and $a_1 = 0$, $b_{11} = -1$. Then (5.1) becomes

$$\begin{aligned}
(5.5) \quad &(0 \wedge (\neg(-1)))^n = 0^{n-1} \circ (0 \wedge (\neg(-1))) \\
&= 0 \circ (0 \wedge (\neg(-1))) \\
&= 0 \wedge (\neg(-1)).
\end{aligned}$$

This means $0 \wedge (\neg(-1)) \in P_0$ since the right-hand side, $0 \wedge (\neg(-1))$, of (5.5) does not depend on n . Assume X is empty, $|I| = 2$, $|J| = |K_1| = 1$ and $a_1 = 0$, $a_2 = 1$, $b_{11} = -1$. Then (5.1) becomes

$$(5.6) \quad ((0 \wedge 1) \wedge \neg(-1))^n = (0 \wedge 1)^{n-1} \circ ((0 \wedge 1) \wedge \neg(-1)).$$

This means $(0 \wedge 1) \wedge \neg(-1) \in P_1$ since $(0 \wedge 1)^{n-1}$ in the right-hand side of (5.6) depends on n and we can easily see $0 \wedge 1 \in P_1$.

The above two examples means that the evolution equation

$$u_j^{n+1} = \neg u_{j-1}^n \wedge u_j^n$$

is of class P_0 , and

$$u_j^{n+1} = \neg u_{j-1}^n \wedge u_j^n \wedge u_{j+1}^n$$

is of class P_1 .

Through the similar discussion, we can easily derive

$$\neg(-1) \wedge \neg 0 \wedge 1 \in P_0,$$

$$\begin{aligned}
&(\neg(-1) \vee \neg 0) \wedge 1 \in P_0, \\
&(\neg(-1) \wedge \neg 1) \wedge 0 \in P_0, \\
&-1 \wedge \neg 0 \wedge 1 \in P_1,
\end{aligned}$$

using Proposition 5.1. In the reference [2], the above evolution equations are independently analyzed and obtained the same classifications. It is interesting that Proposition 5.1 unifies the results to be a single equation (5.1). Similarly, from equation (5.3) we can obtain the following rules reported in the reference [2].

$$\begin{aligned}
&1 \wedge (\neg(-1) \vee 0) \in P_1, \\
&1 \wedge (\neg 0 \vee -1) \in P_1, \\
&0 \wedge (\neg(-1) \vee 1) \in P_1.
\end{aligned}$$

§ 6. Concluding remarks

In this article, evolution equations expressed in terms of lattice operations $\wedge$, $\vee$, and $\neg$ are investigated. We introduced shift operators which act on u_j^n and shift a space index j , and we rewrote the evolution solution u_j^n using combined shift operators F . The complexity of F is defined by the number of shift operators contained in F^n . If it is of $O(n^m)$ for sufficiently large n , then F is in class P_m . We obtained some results about shift operators which belong to class P_m .

The future problems can be considered as follows. The complexity of evolution equations, defined in this article, solely depends on the form of the evolution rule F , but not on the conditions of the initial values $\{u_j^0\}_{j \in \mathbb{Z}}$. As for the general theory of evolution equations, some formulations of complexity which regards both of evolution rules and initial conditions were invented. For example, it is known that Wolfram's classification of ECA depends on not only rule numbers but also initial values [4]. It will be meaningful to generalize the formulation of complexity which we defined in this article to reflect the both of evolution rules and initial conditions.

We assumed that the form of f and the space indices $(a_1, \dots, a_k)$ do not vary for j and n . In other words, f is assumed to be "homogeneous". From this assumption, u_j^n can be described by compositions of the constant operator F , as $u_j^n = F^n[u_j^0]$. It is possible to describe u_j^n by shift operators even if the above assumption is not satisfied. As previously noticed at example 4.5, compositions of operators can describe substitutions of variables. Assume that the evolution rules are defined as

$$u_j^{n+1} = f_j^n(\{u_{j+a_i}^n\}_{a_i \in \mathbb{Z}, i \in I_j^n})$$

for each j and n . Note that the set of space indices I_j^n can change as well as operations

of f_j^n . For example, set u_0^2 , u_0^1 , and u_1^1 as

$$\begin{aligned} u_0^2 &= u_0^1 \wedge u_1^1, \\ u_0^1 &= (u_1^0 \wedge u_2^0) \vee u_3^0, \\ u_1^1 &= \neg u_4^0, \end{aligned}$$

then it holds that

$$\begin{aligned} (6.1) \quad u_0^2 &= u_0^1 \wedge u_1^1 \\ &= ((u_1^0 \wedge u_2^0) \vee u_3^0) \wedge (\neg u_4^0). \end{aligned}$$

Set $F_0^1 = 0 \wedge 1$, $F_0^0 = (1 \wedge 2) \vee 3$, $F_1^0 = \neg 3$, then we can rewrite u_0^2 , u_0^1 , and u_1^1 as

$$\begin{aligned} u_0^2 &= F_0^1[u_0^1] = (0 \wedge 1)[u_0^1], \\ u_0^1 &= F_0^0[u_0^0] = ((1 \wedge 2) \vee 3)[u_0^0], \\ u_1^1 &= F_1^0[u_1^0] = (\neg 3)[u_1^0]. \end{aligned}$$

While, by rewriting second and third sides of the equation (6.1) with shift operators, we obtain

$$u_0^1 \wedge u_1^1 = (0 \wedge 1)[u_0^1] = F_0^1[u_0^1]$$

and

$$\begin{aligned} ((u_1^0 \wedge u_2^0) \vee u_3^0) \wedge (\neg u_4^0) &= (((1 \wedge 2) \vee 3)[u_0^0]) \wedge ((\neg 4)[u_0^0]) \\ &= (((1 \wedge 2) \vee 3) \wedge \neg 4)[u_0^0]. \end{aligned}$$

The shift operator $((1 \wedge 2) \vee 3) \wedge \neg 4$, appearing at the latter equation, can be obtained by the following calculation.

$$((1 \wedge 2) \vee 3) \wedge \neg 4 = (0 \circ F_0^0) \wedge (1 \circ F_1^0).$$

In order to rewrite this calculation with shift operators F_0^1 , F_0^0 , and F_1^0 , we attempted to define “partial” compositions $\circ'$ in the same manner as follows,

$$\begin{aligned} F_0^1 \circ' (F_0^0, F_1^0) &= (0 \wedge 1) \circ' (F_0^0, F_1^0) \\ &:= (0 \circ F_0^0) \wedge (1 \circ F_1^0). \end{aligned}$$

The rigorous definition of partial compositions have not formulated, and it is left to be investigated in future works.

Takahashi, Matsukidaira, Hara, and Feng investigated binary particle cellular automata (PCA) [3]. Through their research, they derived evolution equations which are expressed by max, min, +, and − operations. We show the equations and solutions f_j^n

with initial conditions f_j^0 below. The name of equations in the reference are indicated rightside. The equations are

$$(6.2) \quad f_j^{n+1} = \max(f_{j-1}^n, f_{j+1}^n), \quad (\text{PCA3})$$

$$(6.3) \quad f_j^{n+1} = \max(f_{j-2}^n, f_{j+1}^n), \quad (\text{PCA4-1})$$

$$(6.4) \quad f_j^{n+1} = \max(\min(f_{j-2}^n, f_{j+1}^n + 1/2), f_{j+1}^n - 1/2), \quad (\text{PCA4-2})$$

$$(6.5) \quad f_j^{n+1} = \max(\min(f_{j-2}^n, f_j^n), f_{j+1}^n - 1/2), \quad (\text{PCA4-3})$$

and the solutions are

$$(6.6) \quad f_j^n = \max(f_{j-n}^0, f_{j-n+2}^0, \dots, f_{j+n-2}^0, f_{j+n}^0), \quad (\text{PCA3})$$

$$(6.7) \quad f_j^n = \max(f_{j-2n}^0, f_{j-2n+3}^0, \dots, f_{j+n-3}^0, f_{j+n}^0), \quad (\text{PCA4-1})$$

$$(6.8) \quad f_j^n = \max_{0 \leq k \leq n} (\min(f_{j-2n+3k}^0 - k/2, \min_{k+1 \leq i \leq n} (f_{j-2n+3i}^0 + i/2))), \quad (\text{PCA4-2})$$

$$(6.9) \quad f_j^n = \max_{0 \leq k \leq n} (\min(f_{j-2n+3k}^0 - k/2, \min_{3k/2 \leq i \leq n} (f_{j-2n+2i}^0 + i/2))). \quad (\text{PCA4-3})$$

The evolution rules (6.2), (6.3) and the solutions (6.6), (6.7) are described only by max operations. It is known that $([0, 1], \max, \min)$ forms a distributive lattice, and if we define $\neg a := 1 - a$, then $([0, 1], \max, \min, 1 - a)$ forms a distributive lattice with a conjugation. Therefore, PCA3 and PCA4-1 are included in our formulation of the evolution equations. While, (6.4), (6.5) contain constant terms $\pm 1/2$ and (6.8), (6.9) contain $-k/2$ and $+i/2$. These terms cannot be regarded as lattice operations $\wedge$, $\vee$, and $\neg$. However, we can observe that the growth of the number of terms contained in (6.8) and (6.9) is of $O(n^2)$. This fact suggests that it maybe possible to define the complexity in the same manner as we did in this article, even if other operations that are not lattice operations are included.

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