

G-Constellations and Resolutions of Quotient Singularities

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1 Background

Consider an affine scheme $\mathbb{C}^n = \text{Spec } R$, where by R we denote the ring $\mathbb{C}[x_1, \dots, x_n]$. By X we denote the quotient space $\mathbb{C}^n/G = \text{Spec } R^G$. By Y we denote a choice of a resolution of X .

$$\begin{array}{ccc} Y & & \mathbb{C}^n \\ & \searrow \pi & \swarrow q \\ & X & \end{array}$$

The singular quotient space X is in a certain sense ([Muk03], Example 11.8) a coarse moduli space for the set-theoretical orbits of G in $\mathbb{C}^n$. A natural question to ask was whether we can refine a concept of an ‘orbit of G in $\mathbb{C}^n$ ’ and state a moduli problem for it which yields a fine moduli space Y which resolves the singularities of X .

The first step was to equip an orbit with an appropriate scheme-theoretic structure:

Definition 1.1. A G -cluster is a G -invariant subscheme Z of $\mathbb{C}^n$ of dimension 0 whose ring $\Gamma(Z, \mathcal{O}_Z)$ is a regular representation of G .

E.g. any free orbit of G supports a unique G -cluster: the reduced induced closed subscheme structure. On the other hand, we find many different G -clusters supported at the fixed point orbit at the origin of $\mathbb{C}^n$.

Following the ideas of Nakamura, Reid introduced in [Rei97] the scheme $G\text{-Hilb}$, the fine moduli space of all G -clusters. It comes equipped with a Hilbert-Chow morphism $G\text{-Hilb } \mathbb{C}^n \rightarrow X$ which sends each G -cluster to its set-theoretic support. The main irreducible component of $G\text{-Hilb } \mathbb{C}^n$ birational to X can be identified (e.g. [IN00], §2) with the scheme $\text{Hilb}^G \mathbb{C}^n$ introduced by Nakamura and Ito in [IN96]. They then proceeded to show that for G a finite subgroup of $\text{SL}_2(\mathbb{C})$, the scheme $\text{Hilb}^G \mathbb{C}^n$ is the unique crepant minimal resolution of $\mathbb{C}^2/G$.

Then Nakamura showed by explicit toric geometry computations [Nak00] that for G a finite abelian subgroup of $\mathrm{SL}_3(\mathbb{C})$, the scheme $\mathrm{Hilb}^G \mathbb{C}^3$ is a crepant resolution of $\mathbb{C}^3/G$. He conjectured that the same is true for the non-abelian case.

This conjecture was settled by Bridgeland, King and Reid in [BKR01]. They use derived category methods and establish a category equivalence $D(Y) \rightarrow D^G(\mathbb{C}^n)$ between the bounded derived categories of coherent sheaves on $Y = \mathrm{Hilb}^G \mathbb{C}^n$ and of G -equivariant coherent sheaves on $\mathbb{C}^n$, respectively. Under a certain assumption on the dimension of the fibers of Y , which holds automatically when $n \leq 3$, they prove that the Fourier-Mukai transform which uses the structure sheaf of the universal G -cluster $\mathcal{U}_G \subset Y \times \mathbb{C}^n$ is the requisite equivalence. In particular, this shows that Y is a crepant resolution of X , proving Nakamura's conjecture. It is then further shown ([BKR01], §8) that in the case of $n = 3$, $\mathrm{Hilb}^G \mathbb{C}^3$ is the only component of $G\text{-Hilb } \mathbb{C}^3$, i.e. $G\text{-Hilb } \mathbb{C}^3$ is connected. In dimension two this was proven by Ishii in [Ish02], while in dimensions four and higher it is known to be false.

For $n \geq 3$ crepant resolutions of $\mathbb{C}^n/G$, if they exist, are not necessarily unique. The question arose whether G -clusters can be generalised further, to obtain the other crepant resolutions by a moduli space construction. Subsequent research had shown that it was not necessary to give an orbit a subscheme structure - it is sufficient to equip an orbit with a coherent sheaf that looks like what we would expect of an image of a skyscraper sheaf of a point under a derived category equivalence as above. This generalisation was a concept of a G -constellation given by Craw in his thesis [Cra01]:

Definition 1.2. A G -constellation is a G -equivariant coherent sheaf $\mathcal{F}$ on $\mathbb{C}^n$, whose global sections $\Gamma(\mathbb{C}^n, \mathcal{F})$ form a regular representation of G .

Note that a priori a definition of G -constellation doesn't exclude sheaves supported at more than one orbit of G . However a *gnat*-family consists only of those supported at a single orbit.

Observe that, tautologically, the structure sheaf of any G -cluster is a G -constellation. In fact on a free orbit this all we get: the concepts of a G -constellation, a G -cluster and a set-theoretic orbit coincide where G acts freely. At the origin, however, there are many G -constellations which do not arise as structure sheaves of G -clusters. Too many in fact: the moduli space of all G -constellations is non-separated at the origin, suggesting that some sort of stability conditions are needed.

These came to us courtesy of a natural 1-to-1 correspondence existing between G -constellations and representations of the McKay quiver of G into the regular representation of G . This allows for the use of an earlier result of King [Kin94] on GIT construction of moduli spaces of quiver representations to introduce the stability conditions known as θ -stability on G -constellations and to construct for any given stability condition θ a moduli space M_θ of θ -stable G -constellations together with a projective morphism to X and a

universal θ -stable G -constellation $\mathcal{U}_\theta$ in $\mathbf{Coh} Y \times \mathbb{C}^n$. In a quiver-theoretic context, Kronheimer [Kro89] had already considered these moduli spaces and have studied the chamber structure in the space Π of stability parameters θ , where all values of θ in the same chamber yield the same M_θ . The methods of [BKR01] can be then extended to show that, under the same assumptions on the fiber dimensions of M_θ , the Fourier-Mukai transform $D(M_\theta) \rightarrow D^G(\mathbb{C}^n)$ is an equivalence of categories, which makes the main irreducible component of M_θ a crepant resolution of $\mathbb{C}^n/G$. In case of an abelian G , an explicit description of this coherent component is provided in toric terms by Craw, MacLagan and Thomas in [CMT05a], [CMT05b].

Craw in his thesis conjectured that when G is a finite subgroup of $\mathrm{SL}_3(\mathbb{C})$ every crepant resolution projective over $\mathbb{C}^3/G$ can be realised as a moduli space M_θ of θ -stable G -constellations for some chamber in Π . In the case of G being abelian, this was proved by Craw and Ishii in [CI04].

Thus one motivation for the study of families of G -constellations on a fixed resolution Y is an observation that, as evident from [CI04], there exist stability parameters θ for which the GIT construction yields isomorphic moduli spaces M_θ , but equips them with different tautological families of G -constellations $\mathcal{U}_\theta$. Another is the desire to obtain for a given crepant resolution Y a direct construction of the derived McKay equivalence $D(Y) \xrightarrow{\sim} D^G(\mathbb{C}^n)$ as a Fourier-Mukai functor using an appropriate G -constellation family. Finally, the question of a moduli construction of non-projective (over X) crepant resolutions still remains open.

2 Gnat-Families

Rather than constructing a resolution as a moduli space of G -constellations, we take an arbitrary (not necessarily projective or crepant) resolution of X and study the flat families of G -constellations that it can parametrise.

We would like for a family of G -constellations to be a flat $\mathcal{O}_Y$ -module, whose restriction to any point of Y would give us the respective G -constellation. From this point of view, it would be better to consider, instead of the whole G -constellation $\mathcal{F}$, just its space of global sections $\Gamma(\mathbb{C}^n, \mathcal{F})$. It is a vector space V with G and R actions, satisfying

$$g.(f.v) = (g.f).(g.v) \tag{2.1}$$

As $\mathbb{C}^n$ is affine, functor $(\bullet) \otimes_R \mathcal{O}_{\mathbb{C}^n}$ recovers $\mathcal{F}$ from $\Gamma(\mathbb{C}^n, \mathcal{F})$, and (2.1) defines the G -equivariant structure.

It is convenient to view such vector spaces as modules for the following non-commutative algebra:

Definition 2.1. A cross-product algebra $R \rtimes G$ is an algebra, which has the vector space structure of $R \otimes_{\mathbb{C}} \mathbb{C}[G]$ and the product defined by setting,

for all $g_1, g_2 \in G$ and $f_1, f_2 \in R$,

$$(f_1 \otimes g_1) \times (f_2 \otimes g_2) = (f_1(g_1 \cdot f_2)) \otimes (g_1 g_2) \quad (2.2)$$

Functors $\tilde{\bullet} = (\bullet) \otimes_R \mathcal{O}_{\mathbb{C}^n}$ and $\Gamma(\mathbb{C}^n, \bullet)$ give an equivalence between the categories of $R \rtimes G$ -modules and of quasi-coherent G -equivariant sheaves on $\mathbb{C}^n$.

This is not a pure formalism - $R \rtimes G$ is one of the *non-commutative crepant resolutions* of $\mathbb{C}^n/G$, a certain class of non-commutative algebras introduced by Michel van den Bergh in [dB02] as an analogue of a commutative crepant resolution for an arbitrary non-quotient Gorenstein singularity. For three-dimensional terminal singularities, van den Bergh shows ([dB02], Theorem 6.3.1) that if a non-commutative crepant resolution Q exists, then it is possible to construct commutative crepant resolutions as moduli spaces of certain stable Q -modules.

Under $\Gamma(\mathbb{C}^n, \bullet)$, to G -constellations correspond $R \rtimes G$ -modules, which are isomorphic, as representations of G , to the regular representation V_{reg} . By abuse of notation, we shall use the term G -constellations to also mean such $R \rtimes G$ -modules. This interpretation allows us to define a family of G -constellations as a locally-free sheaf on Y , instead of $Y \times \mathbb{C}^n$:

Definition 2.2. A family of G -constellations parametrised by Y is a sheaf $\mathcal{F}$ of $(R \rtimes G) \otimes_{\mathbb{C}} \mathcal{O}_Y$ -modules on Y , locally free as an $\mathcal{O}_Y$ -module, such that, for any point $\iota_p : p \rightarrow Y$, the fiber $\mathcal{F}|_p = \iota_p^* \mathcal{F}$ is a G -constellation.

We wish to develop a notion of a geometrically natural family, in which for any $p \in Y$ the G -constellation $\mathcal{F}|_p$ would be geometrically related to the G -orbit $q^{-1}\pi(p)$. For example, the G -constellation $\tilde{\mathcal{F}}|_p$, as a sheaf on $\mathbb{C}^n$, is supported on a finite union of G -orbits. We could ask, mimicking the moduli spaces M_θ of θ -stable G -constellations and their tautological families, for this support to be precisely $q^{-1}\pi(p)$.

This turns out to be enough to warranty a much wider range of naturality properties.

Definition 2.3. A generically natural family of G -constellations parametrised by Y (or a **gnat-family**, for short) is a family $\mathcal{F}$ of G -constellations, such that for every $p \in Y$

$$\text{Supp}_{\mathbb{C}^n}(\mathcal{F}|_p) = q^{-1}\pi(p)$$

Proposition 2.4. Let $\mathcal{F}$ be a family of G -constellations parametrised by Y . Then the following are equivalent:

1. On any $U \subset Y$, such that πU consists of free orbits, $\mathcal{F}$ is equivalent (locally isomorphic) to $\pi^* q_* \mathcal{O}_{\mathbb{C}^n}$.

2. There exists a $(R \rtimes G) \otimes_{\mathbb{C}} K(Y)$ -module isomorphism:

$$\mathcal{F}|_{p_Y} \xrightarrow{\sim} (\pi^* q_* \mathcal{O}_{\mathbb{C}^n})_{p_Y}$$

where p_Y is the generic point of Y .

3. There exists an $(R \rtimes G) \otimes_{\mathbb{C}} \mathcal{O}_Y$ -module embedding

$$F \hookrightarrow K(\mathbb{C}^n)$$

where $\mathcal{O}_Y$ -module structure on $K(\mathbb{C}^n)$ is induced by the map $q : Y \rightarrow X$.

4. $\mathcal{F}$ is a gnat-family.

5. The action of $(R \rtimes G) \otimes_{\mathbb{C}} \mathcal{O}_Y$ on $\mathcal{F}$ descends to the action of $(R \rtimes G) \otimes_{R^G} \mathcal{O}_Y$, where the R^G -module structure on $\mathcal{O}_Y$ is induced by the map $q : Y \rightarrow X$.

Sketch. Implications $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5$ are quite straightforward. The interesting one is $5 \Rightarrow 1$.

Consider a natural algebra homomorphism

$$\Psi : (R \rtimes G) \otimes_{R^G} \mathcal{O}_Y \rightarrow \mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{F})$$

LHS is isomorphic to $\pi^* \mathcal{E}nd_{\mathcal{O}_X}(q_* \mathcal{O}_{\mathbb{C}^n})$. Over U , as q is flat over πU , LHS is further isomorphic to $\mathcal{E}nd_{\mathcal{O}_Y}(\pi^* q_* \mathcal{O}_{\mathbb{C}^n})$. Thus we have

$$\Psi' : \mathcal{E}nd_{\mathcal{O}_U}(\pi^* q_* \mathcal{O}_{\mathbb{C}^n}) \rightarrow \mathcal{E}nd_{\mathcal{O}_U}(\mathcal{F})$$

It is a homomorphism of (split) Azumaya algebras of the same constant rank, which is an isomorphism on the centers. Hence Ψ' must be an isomorphism itself. Then, by Skolem-Noether theorem, Ψ' must locally be induced by isomorphisms $\pi^* q_* \mathcal{O}_{\mathbb{C}^n} \rightarrow \mathcal{F}$.

□

3 G -divisors

Since G is abelian, any family $\mathcal{F}$ of G -constellations on Y splits into invertible eigensheaves: $\mathcal{F} = \bigoplus_{\chi \in G^\vee} \mathcal{F}_\chi$. If $\mathcal{F}$ is also a gnat-family, then it can be embedded into $K(\mathbb{C}^n)$. Now, generally, on a scheme S an invertible sheaf embedded into $K(S)$ defines a Cartier divisor on S .

Therefore, just as the group $K_G^*(\mathbb{C}^n)^*$ of the invertible G -homogeneous elements of $K(\mathbb{C}^n)$ extends $K^*(Y)$:

$$1 \rightarrow K^*(Y) \rightarrow K_G^*(\mathbb{C}^n) \xrightarrow{\rho} G^\vee \rightarrow 1 \quad (3.1)$$

we extend the group of Cartier divisors on Y as follows:

Definition 3.1. A rational function $f \in K^*(\mathbb{C}^n)$ is said to be G -homogeneous (of weight χ), if there exists a character $\chi \in G^\vee$ such that

$$g \cdot f = \chi(g)f \quad \forall g \in G$$

Definition 3.2. A G -Cartier divisor on Y is a global section of the sheaf of multiplicative groups $K_G^*(\mathbb{C}^n)/\mathcal{O}_Y^*$, where the sheaf $K_G^*(\mathbb{C}^n)$ is the constant sheaf on Y of the G -homogeneous elements of $K(\mathbb{C}^n)$ and the sheaf $\mathcal{O}_Y^*$ is the sheaf of invertible regular functions on Y .

Similar to the ordinary Cartier divisors, a G -Cartier divisor can be specified by a set of pairs (U_i, f_i) , where U_i are an open cover of Y and f_i are G -homogeneous rational functions on $\mathbb{C}^n$, such that for any i and j , f_i/f_j defines an invertible regular function on $U_i \cap U_j$.

As with ordinary Cartier divisors, we say that a G -Cartier divisor is principal if it lies in the image of the natural map $K_G^*(\mathbb{C}^n) \rightarrow K_G^*(\mathbb{C}^n)/\mathcal{O}_Y^*$ and call two divisors linearly equivalent if their difference is principal.

Thus, we obtain a short exact sequence of abelian groups:

$$1 \rightarrow \text{Car}(Y) \rightarrow G\text{-Car}(Y) \xrightarrow{\rho} G^\vee \rightarrow 1 \quad (3.2)$$

We call an image of a Cartier divisor D under the map ρ its **weight** and say that D is a $\rho(D)$ -Cartier divisor.

The construction of the invertible subsheaf $\mathcal{L}(D)$ of $K(Y)$ corresponding to a Cartier divisor D , extends naturally to a construction of an invertible subsheaf $\mathcal{L}(D)$ of $K_G^*(\mathbb{C}^n)$ corresponding to a G -Cartier divisor D .

Proposition 3.3. *The map $D \rightarrow \mathcal{L}(D)$ gives an isomorphism between $G\text{-Car } Y$ and the group of invertible G -subsheaves of $K(\mathbb{C}^n)$. Furthermore, it descends to an isomorphism of the group $G\text{-Cl}$ of G -Cartier divisors up to linear equivalence and the group $G\text{-Pic}$ of invertible G -sheaves on Y .*

We now seek to define a matching notion of a G -Weil divisor. The key notion is: valuations at prime divisors of Y define a unique group homomorphism val_K from $K^*(Y)$ to $\text{Div } Y$, the group of Weil divisors. Looking at the short exact sequence (3.1), we see that val_K must extend uniquely to a homomorphism val_{K_G} from $K_G^*(\mathbb{C}^n)$ to $\mathbb{Q}\text{-Div } Y$, as $G^\vee$ is finite and $\mathbb{Q}$ is injective. We further obtain a quotient homomorphism $\text{val}_{G^\vee}$ from $G^\vee$ to $\mathbb{Q}/\mathbb{Z}\text{-Div } Y$.

The short exact sequence (3.2) now becomes a commutative diagram:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \text{Car } Y & \longrightarrow & G\text{-Car } Y & \xrightarrow{\rho} & G^\vee \longrightarrow 1 \\ & & \text{val}_K \downarrow & & \text{val}_{K_G} \downarrow & & \text{val}_{G^\vee} \downarrow \\ 0 & \longrightarrow & \text{Div } Y & \longrightarrow & \mathbb{Q}\text{-Div } Y & \longrightarrow & \mathbb{Q}/\mathbb{Z}\text{-Div } Y \longrightarrow 0 \end{array} \quad (3.3)$$

Aiming to have a short exact sequence similar to (3.2), we now define the group $G\text{-Div } Y$ of G -Weil divisors to be the subgroup of $\mathbb{Q}\text{-Div } Y$, which consists of the pre-images of $\text{val}_{G^\vee}(G^\vee) \subset \mathbb{Q}/\mathbb{Z}\text{-Div } Y$.

We call a G -Weil divisor principal if it is an image of a single function $f \in K_G^*(\mathbb{C}^n)$ under val_{K_G} , call two G -Weil divisors linearly equivalent if their difference is principal and call a divisor $\sum q_i D_i$ effective if all $q_i \geq 0$.

We now have a following commutative diagram:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \text{Car } Y & \longrightarrow & G\text{-Car } Y & \xrightarrow{\rho} & G^\vee \longrightarrow 1 \\ & & \downarrow \text{val}_K & & \downarrow \text{val}_{K_G} & & \downarrow \text{val}_{G^\vee} \\ 0 & \longrightarrow & \text{Div } Y & \longrightarrow & G\text{-Div } Y & \longrightarrow & \text{val}_{G^\vee}(G^\vee) \longrightarrow 0 \end{array} \quad (3.4)$$

A priori there is no reason for val_{K_G} in (3.4) to be an isomorphism. Indeed, although all the definitions above make sense for a general scheme Y birational to X , simply assuming Y to be smooth is not enough to warranty G -Cartier and G -Weil divisors to be isomorphic or even well-behaved. For an example let Y be the smooth locus of X . It can be shown, that while val_K is an isomorphism, val_{K_G} is not even injective as $G\text{-Car } Y$ has torsion. And $\text{val}_{G^\vee}$ is the zero map, thus $G\text{-Div } Y$ is just $\text{Div } Y$.

Proposition 3.4. *If Y is smooth and proper over X , then val_K , val_{K_G} and $\text{val}_{G^\vee}$ in (3.4) are all isomorphisms.*

4 Classification of the *gnat*-Families

Given a *gnat*-family $\mathcal{F} = \oplus \mathcal{F}_\chi$, we can embed it into $K(\mathbb{C}^n)$. An image of $\mathcal{F}_\chi$ under such an embedding is an invertible subsheaf of $K_G^*(\mathbb{C}^n)$ and therefore the embedding defines a unique G -Weil divisor set $\{D_\chi\}_{\chi \in G^\vee}$ on Y such that the image of $\mathcal{F}$ in $K(\mathbb{C}^n)$ is $\oplus \mathcal{L}(-D_\chi)$.

Conversely, given a G -divisor set $\{D_\chi\}_{\chi \in G^\vee}$ such that each D_χ is a χ -Weil divisor, we could ask when is $\oplus \mathcal{L}(-D_\chi)$ a *gnat*-family.

Proposition 4.1. *Let $\{D_\chi\}_{\chi \in G^\vee}$ be as above. Then $\oplus \mathcal{L}(-D_\chi)$ is a *gnat*-family if and only if for any G -homogeneous $f \in R$ and any $\chi \in G^\vee$ we have*

$$D_\chi + (f) - D_{\chi\rho(f)} \geq 0 \quad (4.1)$$

where $\rho(f) \in G^\vee$ is the weight of f .

NB: Observe that the condition (4.1) is equivalent to a set of $|G|$ inequalities for each prime Weil divisor, and that these sets are all independent of each other.

We call G -divisor sets $\{D_\chi\}_{\chi \in G^\vee}$ which satisfy (4.1) the **reductor sets**.

Recall that in moduli problems it is a standard practice to consider the families up to equivalence, that is up to a local isomorphism.

Theorem 4.1. *The isomorphism classes of gnat-families on Y are in a one-to-one correspondence with the linear equivalence classes of the reductor sets $\{D_\chi\}$. The equivalence classes of gnat-families on Y are in a one-to-one correspondence with the reductor sets $\{D_\chi\}$, in which $D_{\chi_0} = 0$.*

We say that a reductor set $\{D_\chi\}$ is **normalised**, if $D_{\chi_0} = 0$.

Proposition 4.2 (Canonical family). *Define the divisor set $\{D_\chi\}$ by*

$$D_\chi = \sum_P v(P, \chi) P$$

Then $\{D_\chi\}$ is a normalised reductor set. Moreover, the corresponding family $\oplus \mathcal{L}(-D_\chi)$ is the pushdown to Y of the structure sheaf of the normalization of the reduced fibre product $Y \times_X \mathbb{C}^n$.

Proposition 4.3 (Maximal shift family). *Define the divisor set $\{M_\chi\}$ by*

$$M_\chi = \sum_P \min_{f \in R_\chi} v_P(f) P$$

Then $\{M_\chi\}$ is a normalised reductor set.

NB: It can be shown that, for any $\chi \in G^\vee$, the coefficient of M_χ at a prime Weil divisor P is non-zero if and only if P is exceptional or the image of P in X is the branch divisor of the quotient map $\mathbb{C}^n \rightarrow X$. Therefore, for each $\chi \in G^\vee$, the coefficient of M_χ is non-zero at only finitely many prime divisors in Y .

Proposition 4.4. *Let $\{D_\chi\}$ be any normalised reductor set. Then*

$$-M_{\chi^{-1}} \leq D_\chi \leq M_\chi$$

for any $\chi \in G^\vee$.

Corollary 4.5. *The number of equivalence classes of gnat-families is finite.*

We summarise our results in the following theorem:

Theorem 4.2 (Classification of gnat-families). *Let G be a finite abelian subgroup of $\mathrm{GL}_n(\mathbb{C})$, X the quotient of $\mathbb{C}^n$ by the action of G , Y nonsingular and $\pi : Y \rightarrow X$ a proper birational map. Then isomorphism classes of gnat-families on Y are in 1-to-1 correspondence with linear equivalence classes of G -divisor sets $\{D_\chi\}_{\chi \in G^\vee}$, each D_χ a χ -Weil divisor, which satisfy the inequalities*

$$D_\chi + (f) - D_{\chi\rho(f)} \geq 0 \quad \forall \chi \in G^\vee, G\text{-homogeneous } f \in R$$

Such a divisor set $\{D_\chi\}$ corresponds then to a gnat-family $\oplus \mathcal{L}(-D_\chi)$.

This correspondence descends to a 1-to-1 correspondence between equivalence classes of gnat-families and sets $\{D_\chi\}$ as above and with $D_{\chi_0} = 0$. Furthermore, each divisor D_χ in such a set satisfies inequality

$$-M_{\chi^{-1}} \leq D_\chi \leq M_\chi$$

where $\{M_\chi\}$ is a fixed divisor set defined by

$$M_\chi = \sum_P (\min_{f \in R_\chi} v_P(f)) P$$

As a consequence, the number of equivalence classes of gnat-families is finite.

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