

Log prismatic Dieudonné theory: an announcement

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1 Introduction

This is an announcement of a recent work of the author on the generalization of prismatic Dieudonné theory developed in [ALB23] to log p -divisible groups. The goal of our talk at RIMS was to introduce our main result on the log version of prismatic Dieudonné theory and its application to toroidal compactifications of Shimura varieties. In this article, we define some basic notions, state our main theorem (Theorem 6.3) precisely, and explain a sketch of the proof.

For notation and terminologies on log schemes, see Ogus's book ([Ogu18]).

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2 Kfl topology on fs log formal schemes

We recall the definition of Kummer log flat topology (for short, kfl topology). A monoid map $f: P \rightarrow Q$ of saturated monoids is called **Kummer** if the following conditions are satisfied:

1. f is injective;
2. for every $q \in Q$, there exist $p \in P$ and an integer $n \geq 1$ such that $f(p) = q^n$.

Definition 2.1 (Kummer morphisms of log formal schemes) *Let $f: (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \rightarrow (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$ be a morphism of p -adic saturated log formal schemes. The morphism f is called **Kummer** if, for every $x \in \mathfrak{X}$, the monoid map $\overline{\mathcal{M}_{\mathfrak{Y}, \overline{y}}} \rightarrow \overline{\mathcal{M}_{\mathfrak{X}, \overline{x}}}$ is Kummer. Here, we set $y := f(x)$.*

Definition 2.2 (Log flat morphisms of log schemes, [Kat21, 1.10]) *Let $f: (X, \mathcal{M}_X) \rightarrow (Y, \mathcal{M}_Y)$ be a morphism of fs log schemes.*

1. f is called **log flat** if, fppf locally on X and Y , there exist an injective chart $P \rightarrow Q$ of f called a flat chart) such that the induced morphism

$$(X, \mathcal{M}_X) \rightarrow (Y, \mathcal{M}_Y) \times_{(\mathbb{Z}[P], P)^a} (\mathbb{Z}[Q], Q)^a$$

is strict flat.

2. f is called **Kummer log flat** (or **kfl** for short) if f is Kummer and log flat.

Definition 2.3 (Kummer log flat morphisms of log formal schemes) Let $f: (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \rightarrow (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$ be a morphism of p -adic fs log formal schemes. We say that f is **log flat** if the induced morphism of fs log schemes $(X_n, \mathcal{M}_{X_n}) \rightarrow (Y_n, \mathcal{M}_{Y_n})$ is log flat for each $n \geq 0$, where $(X_n, \mathcal{M}_{X_n})$ (resp. $(Y_n, \mathcal{M}_{Y_n})$) is the strict closed subscheme of $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ (resp. $(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$) defined by p^{n+1} . The morphism f is called **Kummer log flat** (or **kfl** for short) if f is Kummer and log flat.

Definition 2.4 A family of morphisms $\{f_i: (\mathfrak{U}_i, \mathcal{M}_{\mathfrak{U}_i}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})\}_{i \in I}$ is called a Kummer log flat covering (or a kfl covering for short) if the following conditions are satisfied.

1. Each f_i is kfl and locally of finite presentation.
2. The family is set-theoretically surjective i.e. $\mathfrak{X} = \bigcup_{i \in I} f_i(\mathfrak{U}_i)$.

Definition 2.5 (Kfl sites, (cf. [Kat21, Definition 2.3])) For a p -adic fs log formal scheme $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$, we let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\text{kfl}}$ denote the category of fs log schemes $(Y, \mathcal{M}_Y)$ over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ such that p is nilpotent on Y equipped with the Grothendieck topology given by kfl coverings. This topology is called **kfl topology**, and the resulting site $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\text{kfl}}$ is called a **kfl site**.

Proposition 2.6 ([Kat21, Theorem 3.1]) Let $(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a morphism of p -adic fs log formal schemes. Then the functor given by

$$(Z, \mathcal{M}_Z) \mapsto \text{Mor}_{(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})}((Z, \mathcal{M}_Z), (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}))$$

is a sheaf on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\text{kfl}}$.

Remark 2.7 Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a p -adic fs log formal scheme. For a p -adic formal scheme $\mathfrak{Y}$ over $\mathfrak{X}$, we regard $\mathfrak{Y}$ as a log formal scheme over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ by equipping $\mathfrak{Y}$ with the pullback log structure of $\mathcal{M}_{\mathfrak{X}}$ by the structure morphism $\mathfrak{Y} \rightarrow \mathfrak{X}$. As a result, we have fully faithful functors

$$\text{FSch}_{/\mathfrak{X}} \hookrightarrow \text{LFSch}_{/(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})} \hookrightarrow \text{Shv}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\text{kfl}}).$$

Here, $\text{FSch}_{/\mathfrak{X}}$ denotes the category of p -adic formal schemes over $\mathfrak{X}$, and $\text{LFSch}_{/(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})}$ denotes the category of p -adic fs log formal schemes over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$.

3 Log p -divisible groups

In this section, we review basics on log finite group schemes and log p -divisible groups over p -adic fs log formal schemes.

Definition 3.1 (Log finite group schemes, cf. [Kat23, Definition 1.3 and §1.6]) *Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a p -adic fs log formal scheme.*

1. *Let G be a sheaf of abelian groups on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\text{kfl}}$. We call G a **weak log finite group scheme** if there exists a kfl covering $\{(\mathfrak{U}_i, \mathcal{M}_{\mathfrak{U}_i}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})\}_{i \in I}$ such that the restriction of G to $(\mathfrak{U}_i, \mathcal{M}_{\mathfrak{U}_i})_{\text{kfl}}$ is represented by a finite and locally free group scheme over $\mathfrak{U}_i$. We let $\text{wFin}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ denote the category of weak log finite group schemes over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. The category $\text{Fin}(\mathfrak{X})$ of finite and locally free group schemes over $\mathfrak{X}$ is regarded as the full subcategory of $\text{wFin}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$.*
2. *For a weak log finite group scheme G over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$, we define*

$$G^* := \mathcal{H}om_{(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\text{kfl}}}(G, \mathbb{G}_m)$$

*(which we call the **Cartier dual** of G), where $\mathbb{G}_m$ is regarded as a sheaf on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\text{kfl}}$ via the inclusion functors in Remark 2.7. We say that G is a **log finite group scheme** if G and G^* are representable by finite and kfl log formal schemes over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. We let $\text{Fin}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ denote the category of log finite group schemes over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$.*

As the following lemma implies, weak log finite group schemes are described as kfl descents of usual finite and locally free group schemes.

Lemma 3.2 ([Ino25a, Lemma 3.2]) *Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a quasi-compact p -adic fs log formal scheme and G be a weak log finite group scheme over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. Suppose that we are given an fs chart $P \rightarrow \mathcal{M}_{\mathfrak{X}}$. Then the pullback of G along a kfl covering*

$$(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \times_{(\mathbb{Z}[P], P)^a} (\mathbb{Z}[P^{1/n}], P^{1/n})^a \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$$

belongs to $\text{Fin}(\mathfrak{X} \times_{\mathbb{Z}[P]} \mathbb{Z}[P^{1/n}])$ for some $n \geq 1$.

It is important to treat weak log finite group schemes though our main interest is in log finite group schemes. One primary reason of this is the following descent property for weak log finite group schemes (which does not hold in the category of log finite group schemes).

Proposition 3.3 (“Pro-kfl” descent for weak log finite group schemes)

Let $f: (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a surjection of p -adic saturated log formal schemes. Suppose that there exists an fs chart $\alpha: P \rightarrow \mathcal{M}_{\mathfrak{X}}$ such that f admits a factorization

$$(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \rightarrow (\mathfrak{X}_{\infty, \alpha}, \mathcal{M}_{\mathfrak{X}_{\infty, \alpha}}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}),$$

where $(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \rightarrow (\mathfrak{X}_{\infty, \alpha}, \mathcal{M}_{\mathfrak{X}_{\infty, \alpha}})$ is a strict, quasi-compact, and flat. Let $(\mathfrak{Y}^{(\bullet)}, \mathcal{M}_{\mathfrak{Y}^{(\bullet)}})$ denote the Čech nerve of f in the category of p -adic saturated log formal schemes. Then there exists a natural bi-exact equivalences

$$\text{wFin}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \simeq \varprojlim_{\bullet \in \Delta} \text{Fin}(\mathfrak{Y}^{(\bullet)}).$$

The notion of (weak) log finite group schemes allows us to define (weak) log p -divisible groups in a usual way.

Definition 3.4 (Log p -divisible groups, cf. [Kat23, (4.1)]) Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a p -adic fs log formal scheme. Let G be a sheaf of abelian groups on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\text{kfl}}$. We call G a **weak log p -divisible group** if the following conditions are satisfied.

1. A map $\times p: G \rightarrow G$ is surjective.
2. For every $n \geq 1$, the sheaf $G[p^n] := \text{Ker}(\times p^n: G \rightarrow G)$ is a weak log finite group scheme over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$.
3. $G = \bigcup_{n \geq 1} G[p^n]$.

The category of weak log p -divisible groups over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is denoted by $\text{wBT}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. The category $\text{BT}(\mathfrak{X})$ of p -divisible groups over $\mathfrak{X}$ is regarded as the full subcategory of $\text{wBT}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$.

A weak log p -divisible group G over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is called a **log p -divisible group** if $G[p^n]$ is a log finite group scheme for each $n \geq 1$. The category of log p -divisible groups over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is denoted by $\text{BT}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$.

Lemma 3.5 (cf. [Kat23, Lemma 4.2]) Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a p -adic fs log formal scheme. A weak log p -divisible group G over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ belongs to $\text{BT}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ if $G[p^n]$ belongs to $\text{Fin}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ for some $n \geq 1$.

Proposition 3.6 (“Pro-kff” descent for weak log p -divisible groups) Let $f: (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be as in Proposition 3.3. Let $(\mathfrak{Y}^{(\bullet)}, \mathcal{M}_{\mathfrak{Y}^{(\bullet)}})$ denote the Čech nerve of f in the category of p -adic saturated log formal schemes. Then there exists a natural equivalence

$$\text{wBT}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \simeq \varprojlim_{\bullet \in \Delta} \text{BT}(\mathfrak{Y}^{(\bullet)}).$$

4 Log prismatic crystals

First, let us review some basics of prisms introduced by [BS22]. A δ -ring is a ring A with a map $\delta: A \rightarrow A$ satisfying the following conditions:

1. $\delta(1) = 0$
2. $\delta(xy) = \delta(x)y^p + x^p\delta(y) + p\delta(x)\delta(y)$;
3. $\delta(x+y) = \delta(x) + \delta(y) - \sum_{i=1}^{p-1} \frac{(p-1)!}{i!(p-i)!} x^i y^{p-i}$.

For a δ -ring A , a map $\phi_A: A \rightarrow A$ mapping a to $a^p + p\delta(a)$ is a ring map lifting the Frobenius map on A/p . A *prism* is a pair (A, I) consisting of a derived (p, I) -complete ring A and a locally principal ideal I of A such that $p \in I + \phi_A(I)A$. We say that a prism (A, I) is *bounded* if A/I has bounded p^∞ -torsion. Throughout this paper, we use the following lemma with no reference.

For a bounded p -adic formal scheme $\mathfrak{X}$, the *absolute prismatic site* $\mathfrak{X}_{\Delta}$ is the category of prisms (A, I) equipped with maps $\text{Spf}(A/I) \rightarrow \mathfrak{X}$. We equip $\mathfrak{X}_{\Delta}$ with completely flat

topology. We simply write R_Δ for $\mathrm{Spf}(R)_\Delta$. Let $\mathcal{O}_\Delta$ (resp. $\mathcal{I}_\Delta$) denote the sheaf on $\mathfrak{X}_\Delta$ given by $(A, I) \mapsto A$ (resp. I). The Frobenius lift ϕ_A for a prism (A, I) induces a map $\phi : \mathcal{O}_\Delta \rightarrow \mathcal{O}_\Delta$. A *prismatic crystal* on $\mathfrak{X}$ is a vector bundle on the ringed site $(\mathfrak{X}_\Delta, \mathcal{O}_\Delta)$, and a *prismatic F -crystal* on $\mathfrak{X}$ is a pair $(\mathcal{E}, \varphi_\mathcal{E})$ consisting of a prismatic crystal $\mathcal{E}$ and an isomorphism $\varphi_\mathcal{E} : (\phi^*\mathcal{E})[1/\mathcal{I}_\Delta] \xrightarrow{\sim} \mathcal{E}[1/\mathcal{I}_\Delta]$. When no confusion occurs, we simply write $\mathcal{E}$ for $(\mathcal{E}, \varphi_\mathcal{E})$. Let $\mathrm{Vect}(\mathfrak{X}_\Delta)$ (resp. $\mathrm{Vect}^\varphi(\mathfrak{X}_\Delta)$) denote the category of prismatic crystals (resp. prismatic F -crystals) on $\mathfrak{X}$. For integers $a \leq b$, we define $\mathrm{Vect}_{[a,b]}^\varphi(\mathfrak{X}_\Delta)$ as the full subcategory of $\mathrm{Vect}^\varphi(\mathfrak{X}_\Delta)$ consisting of objects $(\mathcal{E}, \varphi_\mathcal{E})$ such that

$$\mathcal{I}_\Delta^b \mathcal{E} \subset \varphi_\mathcal{E}(\phi^*\mathcal{E}) \subset \mathcal{I}_\Delta^a \mathcal{E}.$$

In particular, we write $\mathrm{DM}(\mathfrak{X}_\Delta) := \mathrm{Vect}_{[0,1]}^\varphi(\mathfrak{X}_\Delta)$.

In [BS22, Proposition 7.11], the following quasi-syntomic descent for prismatic crystals is proved.

Lemma 4.1 (Quasi-syntomic descent for prismatic crystals) *Let $R \rightarrow S$ be a quasi-syntomic cover of bounded p -complete rings and $S^{(\bullet)}$ be the Čech nerve of S over R . Then the natural functor*

$$\mathrm{Vect}(R_\Delta) \rightarrow \varprojlim_{\bullet \in \Delta} \mathrm{Vect}(S_\Delta^{(\bullet)})$$

is bi-exact equivalent. Analogous assertions for Vect^φ and $\mathrm{Vect}_{[a,b]}^\varphi$ also hold.

Next, we recall some basics of log prisms introduced in [Kos22]. A $\delta_{\log}$ -ring is a prelog ring (A, M) with a δ -ring structure on A and a map $\delta_{\log} : M \rightarrow A$ satisfying the following conditions:

1. $\delta_{\log}(e) = 0$;
2. $\alpha(m)^p \delta_{\log}(m) = \delta(\alpha(m))$;
3. $\delta_{\log}(mm') = \delta_{\log}(m) + \delta_{\log}(m') + p\delta_{\log}(m)\delta_{\log}(m')$,

where $\alpha : M \rightarrow A$ is the structure map. A *bounded prelog prism* (A, I, M) (resp. *bounded log prism* $(A, I, \mathcal{M}_A)$) is a bounded prism (A, I) equipped with a prelog structure $M \rightarrow A$ (resp. log structure $\mathcal{M}_A$ on $\mathrm{Spf}(A)$) and a $\delta_{\log}$ -structure on the prelog ring (A, M) (resp. the log formal scheme $(\mathrm{Spf}(A), \mathcal{M}_A)$). Here, a $\delta_{\log}$ -structure on the log formal scheme $(\mathrm{Spf}(A), \mathcal{M}_A)$ is a sheaf map $\mathcal{M}_A \rightarrow \mathcal{O}_{\mathrm{Spf}(A)}$ that defines a $\delta_{\log}$ -structure on each section. We can attach to a prelog prism (A, I, M) a log prism $(A, I, M)^a := (A, I, \mathcal{M}_A := M^a)$. For a log prism $(A, I, \mathcal{M}_A)$, the ring map ϕ_A and the map of log structures $\phi_A^* \mathcal{M}_A \rightarrow \mathcal{M}_A$ given by $m \mapsto m^p(1 + p\delta_{\log}(m))$ for $m \in \phi_A^{-1} \mathcal{M}_A$ (here, $1 + p\delta_{\log}(m) \in \mathcal{O}_{\mathrm{Spf}(A)}^\times$) is regarded as a section of $\mathcal{M}_A$ define a morphism of log formal schemes $\phi_{(A, \mathcal{M}_A)} : (\mathrm{Spf}(A), \mathcal{M}_A) \rightarrow (\mathrm{Spf}(A), \mathcal{M}_A)$. When no confusion occurs, we write simply ϕ_A for $\phi_{(A, \mathcal{M}_A)}$.

Definition 4.2 (Absolute log prismatic sites, [Kos22, Remark 4.6]) *Let $(\mathfrak{X}, \mathcal{M}_\mathfrak{X})$ be a bounded p -adic log formal scheme. Let $(\mathfrak{X}, \mathcal{M}_\mathfrak{X})_\Delta$ be the site with objects described as follows: an object consists of*

- a log prism $(A, I, \mathcal{M}_A)$ such that $(\mathrm{Spf}(A), \mathcal{M}_A)$ is quasi-coherent,

- a map of log formal schemes $(\mathrm{Spf}(A/I), \mathcal{M}_{A/I}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$, where $\mathcal{M}_{A/I}$ is the restriction of $\mathcal{M}_A$.

We simply write $(A, I, \mathcal{M}_A)$ for an object when there exists no confusion. A morphism $(A, I, \mathcal{M}_A) \rightarrow (B, J, \mathcal{M}_B)$ in $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}$ is a covering if the map $(\mathrm{Spf}(B), \mathcal{M}_B) \rightarrow (\mathrm{Spf}(A), \mathcal{M}_A)$ is strict (p, I) -completely faithfully flat. The resulting site $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}$ is called the **(absolute) log prismatic site** of $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$.

The structure sheaf $\mathcal{O}_{\Delta}$ is defined by $(A, I, \mathcal{M}_A) \mapsto A$. Similarly, the sheaf $\overline{\mathcal{O}}_{\Delta}$ is defined by $(A, I, \mathcal{M}_A) \mapsto A/I$, and $\mathcal{I}_{\Delta}$ by $(A, I, \mathcal{M}_A) \mapsto I$. The ring map $\mathcal{O}_{\Delta} \rightarrow \overline{\mathcal{O}}_{\Delta}$ induced by ϕ_A for log prisms $(A, I, \mathcal{M}_A)$ is denoted by ϕ .

Definition 4.3 (Log prismatic crystals) Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a bounded p -adic log formal scheme. We let $\mathrm{Vect}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$ denote the category of vector bundles on the ringed site on $((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}, \mathcal{O}_{\Delta})$. Objects of $\mathrm{Vect}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$ are called **log prismatic crystals**.

We let $\mathrm{Vect}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$ denote the category of pairs $(\mathcal{E}, \varphi_{\mathcal{E}})$ where $\mathcal{E}$ is an object of $\mathrm{Vect}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$ and $\varphi_{\mathcal{E}}$ is an isomorphism $(\phi^*\mathcal{E})[1/\mathcal{I}_{\Delta}] \xrightarrow{\sim} \mathcal{E}[1/\mathcal{I}_{\Delta}]$. Objects of the category $\mathrm{Vect}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$ are called **log prismatic F -crystals**. For integers $a \leq b$, we define $\mathrm{Vect}_{[a,b]}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$ as the full subcategory of $\mathrm{Vect}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$ consisting of objects $(\mathcal{E}, \varphi_{\mathcal{E}})$ such that

$$\mathcal{I}_{\Delta}^b \mathcal{E} \subset \varphi_{\mathcal{E}}(\phi^* \mathcal{E}) \subset \mathcal{I}_{\Delta}^a \mathcal{E}.$$

In particular, we write $\mathrm{DM}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) := \mathrm{Vect}_{[0,1]}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$.

Remark 4.4 We have the following canonical bi-exact equivalence.

$$\begin{aligned} \mathrm{Vect}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) &\simeq \varprojlim_{(A, I, \mathcal{M}_A) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}} \mathrm{Vect}(A) \\ \mathrm{Vect}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) &\simeq \varprojlim_{(A, I, \mathcal{M}_A) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}} \mathrm{Vect}^{\varphi}(A, I) \\ \mathrm{Vect}_{[a,b]}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) &\simeq \varprojlim_{(A, I, \mathcal{M}_A) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}} \mathrm{Vect}_{[a,b]}^{\varphi}(A, I) \end{aligned}$$

Here, $\mathrm{Vect}(R)$ is the category of finite projective R -modules for a ring R . For a prism (A, I) , let $\mathrm{Vect}^{\varphi}(A, I)$ denote the category of pairs (N, φ_N) of a finite projective A -module N and an isomorphism $\varphi_N: (\phi_A^* N)[1/I] \xrightarrow{\sim} N[1/I]$, and let $\mathrm{Vect}_{[a,b]}^{\varphi}(A, I)$ denote the full subcategory of $\mathrm{Vect}^{\varphi}(A, I)$ consisting of objects (N, φ_N) such that

$$I^b N \subset \varphi_N(\phi_A^* N) \subset I^a N.$$

For an object $\mathcal{E}$ of a category on the left side in the above equivalences, the image of $\mathcal{E}$ via the projection with respect to $(A, I, \mathcal{M}_A)$ is denoted by $\mathcal{E}_{(A, I, \mathcal{M}_A)}$. When no confusion occurs, we simply write $\mathcal{E}_A$ for this.

To consider the log version of Lemma 4.1 is a natural question. However, such a descent property does not hold for prismatic (F) -crystals. As a remedy, we introduce the notion of **kfl prismatic crystals**. Roughly speaking, the category of kfl prismatic crystals is defined by replacing $\mathrm{Vect}(A)$ on the right hand side in the equivalences in Remark 4.4 with the category of vector bundles on the kfl site of $(\mathrm{Spf}(A), \mathcal{M}_A)$ (called **kfl**

vector bundles). Let $\text{Vect}_{\text{kfl}}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$ denote the category of kfl prismatic crystals on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. We can define $\text{Vect}_{\text{kfl}}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$ and $\text{Vect}_{\text{kfl}, [a, b]}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$ in a similar way. For the precise definition, see [Ino25a, Definition 4.6 and Definition 4.7]. Then the desired descent property is formulated as follows.

Proposition 4.5 (“Kummer quasi-syntomic” descent for kfl prismatic crystals) *Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) = (\text{Spf}(R), \mathcal{M}_R)$ and $(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) = (\text{Spf}(S), \mathcal{M}_S)$ be bounded p -adic saturated log formal schemes, and $f: (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a surjective morphism. Suppose that there exists a **prismatically liftable chart** (in the sense of [Ino25a, Definition 4.16]) $\alpha: \mathbb{N}^r \rightarrow \mathcal{M}_{\mathfrak{X}}$ such that f admits a factorization*

$$(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \rightarrow (\mathfrak{X}_{\infty, \alpha}, \mathcal{M}_{\mathfrak{X}_{\infty, \alpha}}) = (\text{Spf}(R_{\infty, \alpha}), \mathcal{M}_{R_{\infty, \alpha}}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}),$$

where

$$(\mathfrak{X}_{\infty, \alpha}, \mathcal{M}_{\mathfrak{X}_{\infty, \alpha}}) := (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \times_{(\mathbb{Z}[\mathbb{N}^r], \mathbb{N}^r)^a} (\mathbb{Z}[\mathbb{Q}_{\geq 0}^r], \mathbb{Q}_{\geq 0}^r)^a,$$

and the map $(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \rightarrow (\mathfrak{X}_{\infty, \alpha}, \mathcal{M}_{\mathfrak{X}_{\infty, \alpha}})$ is strict quasi-syntomic. We let $(\mathfrak{Y}^{(\bullet)}, \mathcal{M}_{\mathfrak{Y}^{(\bullet)}})$ denote the Čech nerve of f in the category of p -adic saturated log formal schemes. Then there exist natural bi-exact equivalences:

$$\begin{aligned} \text{Vect}_{\text{kfl}}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) &\simeq \varprojlim_{\bullet \in \Delta} \text{Vect}(\mathfrak{Y}_{\Delta}^{(\bullet)}), \\ \text{Vect}_{\text{kfl}}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) &\xrightarrow{\sim} \varprojlim_{\bullet \in \Delta} \text{Vect}^{\varphi}(\mathfrak{Y}_{\Delta}^{(\bullet)}), \\ \text{Vect}_{\text{kfl}, [a, b]}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) &\xrightarrow{\sim} \varprojlim_{\bullet \in \Delta} \text{Vect}_{[a, b]}^{\varphi}(\mathfrak{Y}_{\Delta}^{(\bullet)}). \end{aligned}$$

We give an important example of p -adic log formal scheme satisfying the assumption of Proposition 4.5. Let K be a complete discrete valuation field with a valuation ring $\mathcal{O}_K$ whose residue field k is a perfect field of characteristic p .

Definition 4.6 (Small affine log formal schemes) *Fix a uniformizer π of $\mathcal{O}_K$ and $E(t) \in W[t]$ be a corresponding Eisenstein polynomial. Let $l, m, n \in \mathbb{Z}_{\geq 0}$, $r := m + n$. Consider the prelog ring $(R^0, \mathbb{N}^r)$ defined as follows:*

$$R^0 := \begin{cases} \mathcal{O}_K \langle x_1^{\pm 1}, \dots, x_l^{\pm 1}, y_1, \dots, y_m \rangle & (n = 0) \\ \mathcal{O}_K \langle x_1^{\pm 1}, \dots, x_l^{\pm 1}, y_1, \dots, y_m, z_1, \dots, z_n \rangle / (\prod_{k=1}^n z_k - \pi) & (n \geq 1) \end{cases}$$

equipped with a prelog structure $\mathbb{N}^r \rightarrow R^0$ given by

$$e_i \mapsto \begin{cases} y_i & (1 \leq i \leq m) \\ z_{i-m} & (m+1 \leq i \leq r). \end{cases}$$

A p -adic affine log formal scheme $(\text{Spf}(R), \mathcal{M}_R)$ over $\mathcal{O}_K$ is called **small affine** if $(\text{Spf}(R), \mathcal{M}_R)$ admits a strict étale map to $(\text{Spf}(R^0), \mathbb{N}^r)^a$ for some l, m, n , called a **framing**.

Remark 4.7 Consider a smooth variety X over K . By de Jong's alteration, there exist a finite extension L of K , a smooth variety Y over L admitting an open immersion into a proper semi-stable scheme $\bar{Y}$ over $\mathcal{O}_L$ with a normal crossings complement, and a dominant proper morphism $Y \rightarrow X_L$. Since some computation on the 'etale cohomology of X can be reduced to one of Y , the study of such $\bar{Y}$ is important.

Let $(\bar{Y}, \mathcal{M}_{\bar{Y}})$ be the fs log scheme equipped with the log structure defined by the normal crossings boundary divisor $\bar{Y} - Y$. Then the p -adic completion of $(\bar{Y}, \mathcal{M}_{\bar{Y}})$ admits a strict étale covering consisting of small affine log formal schemes.

As the following lemma implies, Proposition 4.5 holds for small affine log formal schemes.

Lemma 4.8 ([Ino25a, Lemma 4.17 (1)]) Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a small affine log formal scheme over $\mathcal{O}_K$, and fix a framing on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. Let $\alpha: \mathbb{N}^r \rightarrow \mathcal{M}_{\mathfrak{X}}$ be chart defined from the fixed framing. Then α is prismatically liftable chart.

5 Log crystalline crystals

Definition 5.1 (Absolute log crystalline sites) Let $(X, \mathcal{M}_X)$ be a log scheme over $\mathbb{F}_p$. Let $(X, \mathcal{M}_X)_{\text{crys}}$ be the site defined as follows: an object consists of

- a log scheme $(T, \mathcal{M}_T)$ on which p is nilpotent and a PD-thickening $U \hookrightarrow T$ such that the PD-structure on $\text{Ker}(\mathcal{O}_T \rightarrow \mathcal{O}_U)$ is compatible with the natural PD-structure on $(p) \subset \mathbb{Z}_p$,
- a map of log schemes $f: (U, \mathcal{M}_U) \rightarrow (X, \mathcal{M}_X)$, where $\mathcal{M}_U$ is the inverse image log structure of $\mathcal{M}_T$.

Morphisms are obvious ones. We say that a map $(U_1, T_1, \mathcal{M}_{T_1}) \rightarrow (U_2, T_2, \mathcal{M}_{T_2})$ in $(X, \mathcal{M}_X)_{\text{crys}}$ is **Cartesian** if the natural map $U_1 \rightarrow U_2 \times_{T_2} T_1$ is an isomorphism. A family of Cartesian maps $\{(U_i, T'_i, \mathcal{M}_{T'_i}) \rightarrow (U, T, \mathcal{M}_T)\}_i$ is covering if $\{(T'_i, \mathcal{M}_{T'_i}) \rightarrow (T, \mathcal{M}_T)\}_i$ is a strict fpqc covering. The resulting site $(X, \mathcal{M}_X)_{\text{crys}}$ is called an **absolute log crystalline site** of $(X, \mathcal{M}_X)$. Let $\mathcal{O}_{\text{crys}}$ denote the ring sheaf given by $(T, \mathcal{M}_T) \mapsto \Gamma(T, \mathcal{O}_T)$.

Remark 5.2 In many references, a certain full subcategory of $(X, \mathcal{M}_X)_{\text{crys}}$ is referred to as an (absolute) log crystalline site. However, this change of sites does not change the categories of crystals. See [Ino25b, Lemma 4.24] for this point.

Definition 5.3 (Crystalline crystals on log schemes)

Let $\text{Vect}((X, \mathcal{M}_X)_{\text{crys}})$ be the category of crystals of vector bundles on the absolute log crystalline site $(X, \mathcal{M}_X)_{\text{crys}}$. An object of $\text{Vect}((X, \mathcal{M}_X)_{\text{crys}})$ is called a **locally free crystalline crystal**. The Frobenius morphism $F: (X, \mathcal{M}_X) \rightarrow (X, \mathcal{M}_X)$ induces a functor

$$F^*: \text{Vect}((X, \mathcal{M}_X)_{\text{crys}}) \rightarrow \text{Vect}((X, \mathcal{M}_X)_{\text{crys}}).$$

Let $\text{Vect}^\varphi((X, \mathcal{M}_X)_{\text{crys}})$ be the category of pairs $(\mathcal{E}, \varphi_{\mathcal{E}})$ consisting of an object $\mathcal{E} \in \text{Vect}((X, \mathcal{M}_X)_{\text{crys}})$ and an isomorphism $\varphi_{\mathcal{E}}: F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ in the isogeny category $\text{Vect}((X, \mathcal{M}_X)_{\text{crys}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Morphisms $(\mathcal{E}_1, \varphi_{\mathcal{E}_1}) \rightarrow (\mathcal{E}_2, \varphi_{\mathcal{E}_2})$ are morphisms $\mathcal{E}_1 \rightarrow \mathcal{E}_2$ that are compatible with $\varphi_{\mathcal{E}_1}$ and $\varphi_{\mathcal{E}_2}$.

Remark 5.4 *We have a canonical bi-exact equivalence*

$$\mathrm{Vect}((X, \mathcal{M}_X)_{\mathrm{crys}}) \simeq \varinjlim_{(U, T, \mathcal{M}_T) \in (X, \mathcal{M}_X)_{\mathrm{crys}}} \mathrm{Vect}(T)$$

In the same way as the prismatic theory, we can define **kfl locally free crystalline crystals**. Roughly speaking, the category of kfl locally free crystalline crystals is defined by replacing $\mathrm{Vect}(T)$ on the right hand side in the equivalences in Remark 5.4 with the category of kfl vector bundles on $(T, \mathcal{M}_T)$. Let $\mathrm{Vect}_{\mathrm{kfl}}((X, \mathcal{M}_X)_{\mathrm{crys}})$ denote the category of kfl locally free crystalline crystals on $(X, \mathcal{M}_X)$. We can define $\mathrm{Vect}_{\mathrm{kfl}}^\varphi((X, \mathcal{M}_X)_{\mathrm{crys}})$ in a similar way to $\mathrm{Vect}^\varphi((X, \mathcal{M}_X)_{\mathrm{crys}})$. For the precise definition, see [Ino25a, Definition 4.25].

6 Log prismatic Dieudonné theory

First, let us recall some results on (non-log) prismatic Dieudonné theory for p -divisible groups established in [ALB23]. Let $\mathfrak{X}$ be a bounded p -adic formal scheme.

Definition 6.1 *Let G be a finite and locally free group scheme or a p -divisible group over $\mathfrak{X}$. We define a sheaf G_Δ on $\mathfrak{X}_\Delta$ by*

$$G_\Delta(A, I) := G(A/I).$$

Here, we consider an $\mathcal{O}_\Delta$ -module

$$\mathcal{M}_\Delta(G) := \mathcal{E}xt_{\mathfrak{X}_\Delta}^1(G_\Delta, \mathcal{O}_\Delta).$$

The morphism $\varphi: \mathcal{O}_\Delta \rightarrow \mathcal{O}_\Delta$ induces a $\mathcal{O}_\Delta$ -linear morphism $\varphi_{\mathcal{M}_\Delta(G)}: \varphi^ \mathcal{M}_\Delta(G) \rightarrow \mathcal{M}_\Delta(G)$. The functor $\mathcal{M}_\Delta$ is called a **prismatic Dieudonné functor**.*

The main theorem of prismatic Dieudonné theory is the following.

Theorem 6.2 ([ALB23, Proposition 4.68, Theorem 4.74, and Theorem 5.10])

Let $\mathfrak{X}$ be a quasi-syntomic p -adic formal scheme and G be a p -divisible group over $\mathfrak{X}$. Then $(\mathcal{M}_\Delta(G), \varphi_{\mathcal{M}_\Delta(G)})$ belongs to $\mathrm{DM}(\mathfrak{X}_\Delta)$, and the contravariant functor

$$\mathcal{M}_\Delta: \mathrm{BT}(\mathfrak{X}) \rightarrow \mathrm{DM}(\mathfrak{X}_\Delta)$$

is fully faithful. Moreover, if $\mathfrak{X}$ has an étale cover consisting of an affine formal scheme $\mathrm{Spf} R$ which admits a quasi-syntomic cover $R \rightarrow S$ with S being an integral perfectoid ring, this functor gives an anti-equivalence.

From now on, we turn to log prismatic Dieudonné theory. For simplicity, we are restricted to the case that $(\mathfrak{X}, \mathcal{M}_\mathfrak{X})$ is small affine. In order to construct the log prismatic Dieudonné functor, we begin with working in the larger categories. By “Kummer quasi-syntomic descent” (Proposition 3.6 and Proposition 4.5), the (non-log) prismatic Dieudonné functor descends to a fully faithful functor

$$\mathcal{M}_\Delta: \mathrm{wBT}(\mathfrak{X}, \mathcal{M}_\mathfrak{X}) \rightarrow \mathrm{DM}_{\mathrm{kfl}}((\mathfrak{X}, \mathcal{M}_\mathfrak{X})_\Delta) := \mathrm{Vect}_{[0,1]}^\varphi((\mathfrak{X}, \mathcal{M}_\mathfrak{X})_\Delta),$$

which turns out to be an anti-equivalence because we can take $(\mathfrak{Y}, \mathcal{M}_\mathfrak{Y}) = (\mathrm{Spf}(S), \mathcal{M}_S)$ in Proposition 4.5 with S being integral perfectoid ([Ino25b, Lemma 2.3]). We call the functor $\mathcal{M}_\Delta$ a **log prismatic Dieudonné functor**. Then the main theorem of log prismatic Dieudonné theory is stated in the following way.

Theorem 6.3 (cf. [Ino25a, Theorem 5.14]) *Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a small affine log formal scheme. Then the log prismatic Dieudonné functor $\mathcal{M}_{\Delta}$ defined above restricts to the following anti-equivalence:*

$$\mathcal{M}_{\Delta}: \mathrm{BT}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \rightarrow \mathrm{DM}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}).$$

We end up with a sketch of the proof of Theorem 6.3. First, we prove that the image of $\mathrm{BT}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is contained in $\mathrm{DM}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$. This can be reduced to an analogous statement for log finite group schemes by defining log prismatic Dieudonné functor for log finite group schemes. Strict étale locally on the base, log finite group schemes are written as an extension of two (non-log) finite and locally free group schemes ([Ino25a, Lemma 3.7]). Hence, the claim follows from the exactness of the log prismatic Dieudonné functor and the fact that the category of (non-log) vector bundles is closed under extensions in the category of kfl vector bundles ([Niz08, Proposition 3.29]).

Now, it is enough to prove that, for a weak log p -divisible group G over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$, if $\mathcal{M}_{\Delta}(G)$ belongs to $\mathrm{DM}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$, then G is a log p -divisible group. To do this, it suffices to show that the Lie algebra of G , denoted by $\mathrm{Lie}(G)$ (which is a priori just a kfl vector bundle on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$) is indeed a (non-log) vector bundle on $\mathfrak{X}$ by [Kat23, Proposition 7.3] or [Ino25a, Proposition 3.23]. Let $(X_{p=0}, \mathcal{M}_{X_{p=0}})$ be the mod- p fiber of $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. Since $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is small affine, we have a specialization functor

$$\mathrm{Vect}_{\mathrm{kfl}}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) \rightarrow \mathrm{Vect}_{\mathrm{kfl}}^{\varphi}((X_{p=0}, \mathcal{M}_{X_{p=0}}))$$

sending $\mathrm{Vect}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$ into $\mathrm{Vect}^{\varphi}((X_{p=0}, \mathcal{M}_{X_{p=0}}))$, and $\mathcal{M}_{\Delta}(\mathcal{E})$ is specialized to the log crystalline Dieudonné crystal of G , denoted by $\mathcal{M}_{\mathrm{crys}}(G)$. The evaluation of $\mathcal{M}_{\mathrm{crys}}(G)$ at a log PD-thickening $(X_{p=0}, \mathcal{M}_{X_{p=0}}) \hookrightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ (which is a (non-log) vector bundle on $\mathfrak{X}$) admits a Hodge filtration, and $\mathrm{Lie}(G)$ occurs as the dual of Fil^1 as the non-log theory. Then, by using [Niz08, Proposition 3.29] again, we conclude that $\mathrm{Lie}(G)$ is a (non-log) vector bundle on $\mathfrak{X}$. Note that we critically used the assumption that $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is small affine to use the specialization functor to F -crystals on log crystalline sites.

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