

The behavior of the interfaces in the fast reaction limits of some reaction-diffusion systems with unbalanced interactions

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1 Introduction

Let Ω be a bounded domain in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$. Hilhorst-Hout-Peletier [2, 3] investigated a simple reaction-diffusion system with a huge positive parameter k

$$\begin{cases} u_t = \Delta u - k u w & \text{in } \Omega, \\ w_t = -k u w & \text{in } \Omega \end{cases} \quad (1)$$

which describes a “fast reaction” between a diffusive reactant u and a non-diffusive one w . Assuming that the initial values of u and w are non-negative and fixing a positive number T , they derived the singular limit as $k \rightarrow \infty$ of an initial-boundary value problem in $\Omega \times (0, T)$ for a class of reaction-diffusion systems with a parameter k such as (1). Their results are summarized as follows: the solution (u_k, w_k) of their initial-boundary value problem possesses its singular limit (u_*, w_*) as $k \rightarrow \infty$ such that $u_* w_* \equiv 0$; therefore, when we use the notation

$$\begin{aligned} \Omega^u(t) &= \{x \in \Omega \mid u_*(x, t) > 0\}, & \Omega^w(t) &= \overline{\text{Int}\{x \in \Omega \mid w_*(x, t) > 0\}}, \\ \Gamma(t) &= \Omega \setminus (\Omega^u(t) \cup \Omega^w(t)), \end{aligned} \quad (2)$$

the region $\Omega^u(t)$ and the region $\Omega^w(t)$ are divided by an “interface” $\Gamma(t)$; moreover u_* satisfies the one-phase Stefan problem

$$\begin{cases} u_{*,t} = \Delta u_* & \text{in } \Omega^u(t), \\ w_*|_{\Gamma(t)+0\mathbf{n}} V_{\mathbf{n}} = -\frac{\partial u_*}{\partial \mathbf{n}}|_{\Gamma(t)-0\mathbf{n}}, & u_*|_{\Gamma(t)} = 0 \end{cases} \quad (3)$$

in a weak sense. Here $\mathbf{n}$ is the unit normal vector to $\Gamma(t)$ oriented from $\Omega^u(t)$ to $\Omega^w(t)$, and $V_{\mathbf{n}}$ is the velocity of $\Gamma(t)$ in the direction of $\mathbf{n}$.

In this article we consider generalized “fast reactions” between u and w :

$$\begin{cases} u_t = \Delta u - k u^{m_1} w^{m_3} & \text{in } \Omega, \\ w_t = -k u^{m_2} w^{m_4} & \text{in } \Omega, \end{cases} \quad (4)$$

where $m_j \geq 1$ ($j = 1, 2, 3, 4$). We are particularly interested in the situations where $(m_1, m_3) \neq (m_2, m_4)$, while Hilhorst-Hout-Peletier [2, 3] investigated situations where $(m_1, m_3) = (m_2, m_4)$. Even in the situations where $(m_1, m_3) \neq (m_2, m_4)$ the corresponding

singular limit (u_*, w_*) of (u_k, w_k) as $k \rightarrow \infty$, if it exists, must formally satisfies $u_* w_* \equiv 0$. However, the rapid dynamics of (4) in such situations are very different from that in the situations where $(m_1, m_3) = (m_2, m_4)$. The rapid dynamics of (4) is essentially determined by the two-dimensional dynamical system

$$\begin{cases} u_t = -u^{m_1} w^{m_3}, \\ w_t = -u^{m_2} w^{m_4}. \end{cases} \quad (5)$$

Note that all the trajectories of (5) are straight and that the trajectories toward the axis $u = 0$ intersect it slantwise if $(m_1, m_3) = (m_2, m_4)$. If $(m_1, m_3) \neq (m_2, m_4)$, then the trajectories toward the axis $u = 0$ intersect it vertically in some situations; those trajectories touch the axis $u = 0$ tangentially in other situations; in some situations among the other ones no trajectories possess intersections with the axis $u = 0$. When $(m_1, m_3) \neq (m_2, m_4)$, these various structures of the trajectories in (5) may cause any different behavior of the interface $\Gamma(t)$ in the singular limit of (4). Related problems were investigated in [6] from the aspect of numerical simulation (see also [4]).

As the first attempt to solve the behavior of the interface $\Gamma(t)$ in the situations where $(m_1, m_3) \neq (m_2, m_4)$, we will investigate typical four cases of such “unbalanced interactions” between u and w : $(m_1, m_2, m_3, m_4) = (1, 1, 1, m)$, $(1, 1, m, 1)$, $(1, m, 1, 1)$ and $(m, 1, 1, 1)$, where m is a constant larger than 1. In each case we would like to reveal the interfacial dynamics in the fast reaction limit of (4) as $k \rightarrow \infty$. Hereafter we denote $\Omega \times (0, T)$ by Q_T and consider (4) under the initial condition

$$u|_{t=0} = u_0, \quad w|_{t=0} = w_0 \quad \text{in } \Omega \quad (6)$$

and a boundary condition

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \quad (7)$$

where ν denotes the unit outer normal vector of $\partial\Omega$.

2 Singular limits in Case $(m_1, m_2, m_3, m_4) = (1, 1, 1, m)$ or $(1, 1, m, 1)$: moving interfaces

In these cases we can respectively reduce (4) into a reaction-diffusion system with a “balanced interaction”; namely into a system with $(m_1, m_3) = (m_2, m_4)$ by some transformations of variables. When $(m_1, m_2, m_3, m_4) = (1, 1, 1, m)$ with $1 \leq m < 2$, we put $W_k = w_k^{2-m}$ for any solution (u_k, w_k) to (4). Then (u_k, W_k) becomes a solution to

$$\begin{cases} u_t = \Delta u - kuW^{1/(2-m)} & \text{in } \Omega, \\ W_t = -(2-m)kuW^{1/(2-m)} & \text{in } \Omega. \end{cases} \quad (8)$$

The singular limits of (8) with appropriate initial-boundary conditions were studied by Hilhorst, Hout and Peletier [2, 3]. They showed that u_* of the singular limit $(u_*, W_*) =$

$\lim_{k \rightarrow \infty} (u_k, W_k)$ satisfies a one-phase Stefan problem with a finite normal velocity of the interface. In the same manner as the proofs in [2, 3], we can derive the singular limit of (8) with an initial condition

$$u|_{t=0} = u_0, \quad W|_{t=0} = w_0^{2-m} \quad \text{in } \Omega \quad (9)$$

and a boundary condition (7).

Throughout this section, we impose the following assumption on the initial datum (u_0, w_0) :

(H1) $(u_0, w_0) \in C(\bar{\Omega}) \times L^\infty(\Omega)$, w_0 is continuous in $\text{supp } w_0$ and there exist positive constants M and m_w such that

$$\begin{aligned} u_0 w_0 &= 0, \quad 0 \leq u_0, w_0 \leq M \quad \text{in } \Omega, \\ m_w &\leq w_0 \quad \text{in } \text{supp } w_0. \end{aligned}$$

Under the assumption (H1), there exists a unique solution (u_k, W_k) of the initial-boundary value problem (8), (9) and (7) satisfying

$$\begin{aligned} u_k &\in C([0, T]; C(\bar{\Omega})) \cap C^1((0, T]; C(\bar{\Omega})) \cap C((0, T]; W^{2,p}(\Omega)) \quad (\forall p > 1), \\ w_k &\in C^1([0, T]; L^\infty(\Omega)) \end{aligned} \quad (10)$$

(see [1]). We obtain the following theorem in the same manner as the proofs in [2, 3].

Theorem 2.1 (Hilhorst, Hout and Peletier [2, 3]) *Let (u_k, W_k) be the solution of (8) under the initial and boundary conditions (9) and (7), where $1 \leq m < 2$. Then there exist subsequences $\{u_{k_n}\}$, $\{W_{k_n}\}$ and functions $(u_*, W_*) \in L^2(0, T; H^1(\Omega)) \times L^2(Q_T)$ such that*

$$\begin{aligned} u_{k_n} &\rightarrow u_* \quad \text{strongly in } L^2(Q_T) \text{ and weakly in } L^2(0, T; H^1(\Omega)), \\ W_{k_n} &\rightarrow W_* \quad \text{strongly in } L^2(Q_T), \end{aligned}$$

as k_n tends to infinity, where

$$u_* W_* = 0, \quad u_* \geq 0, \quad W_* \geq 0 \quad \text{a.e. in } Q_T.$$

Moreover, u_* and W_* satisfy

$$\iint_{Q_T} \{-(u_* - \lambda W_*) \zeta_t + \nabla u_* \cdot \nabla \zeta\} dx dt = \int_{\Omega} (u_0 - \lambda w_0^{2-m}) \zeta(\cdot, 0) dx \quad (11)$$

for all functions $\zeta \in C^\infty(\bar{Q}_T)$ such that $\zeta(x, T) = 0$, where $\lambda = 1/(2-m)$.

Since $u_* W_* \equiv 0$, we can rewrite (11) as a classical one-phase Stefan problem with a finite propagation speed. Here we use $\Omega^u(t)$, $\Omega^w(t)$ and $\Gamma(t)$ defined by (2) where $w_* = W_*^{1/(2-m)}$ with $1 \leq m < 2$. Also we use the following notation:

$$Q_T^u = \bigcup_{0 < t < T} \Omega^u(t) \times \{t\}, \quad Q_T^w = \bigcup_{0 < t < T} \Omega^w(t) \times \{t\}, \quad \Gamma = \bigcup_{0 < t < T} \Gamma(t) \times \{t\}. \quad (12)$$

Theorem 2.2 Set $(m_1, m_2, m_3, m_4) = (1, 1, 1, m)$ where $1 \leq m < 2$. Let (u_k, w_k) be the solution of (4) under the initial-boundary conditions (6)-(7) and set $W_k = w_k^{2-m}$. Namely (u_k, W_k) is the solution of (8) satisfying (9) and (7). Let (u_*, W_*) be the limit given in Theorem 2.1 and set $w_* = W_*^{1/(2-m)}$. Suppose that $\Gamma(t)$ is a smooth, closed and orientable hypersurface satisfying $\Gamma(t) \cap \partial\Omega = \emptyset$ for all $t \in [0, T]$. Also assume that $\Gamma(t)$ smoothly moves with a normal velocity V_n from $\Omega^u(t)$ to $\Omega^w(t)$, and u_* is continuous in Q_T and smooth on $\overline{Q_T^u}$, and w_* is smooth on $\overline{Q_T^w}$. Then the following relations hold.

$$\begin{aligned} w_*(t) &= w_0, & \text{in } Q_T^w; \\ \begin{cases} u_{*,t} = \Delta u_* & \text{in } Q_T^u, \\ u_* = 0, \quad \frac{w_0^{2-m}}{2-m} V_n = -\frac{\partial u_*}{\partial n} & \text{on } \Gamma, \\ \frac{\partial u_*}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, T), \\ u_* = u_0 & \text{on } \Omega^u(0) \times \{0\}. \end{cases} \end{aligned}$$

When $(m_1, m_2, m_3, m_4) = (1, 1, m, 1)$ with $m \geq 1$, we put $W_k = w_k^m$ for any solution (u_k, w_k) to (4). Then (u_k, W_k) becomes a solution to

$$\begin{cases} u_t = \Delta u - kuW & \text{in } \Omega, \\ W_t = -mkuW & \text{in } \Omega. \end{cases} \quad (13)$$

Taking the fast reaction limit of (13) under the boundary condition (7) and an initial condition

$$u|_{t=0} = u_0, \quad W|_{t=0} = w_0^m \quad \text{in } \Omega, \quad (14)$$

we can similarly derive the same conclusions as those of Theorem 2.1 where $\lambda = 1/m$. Thus we obtain the following theorem. Here we use the notation $\Omega^u(t)$, $\Omega^w(t)$, $\Gamma(t)$, Q_T^u , Q_T^w and Γ defined by (2) and (12) where $w_* = W_*^{1/m}$ with $m \geq 1$.

Theorem 2.3 Set $(m_1, m_2, m_3, m_4) = (1, 1, m, 1)$ where $m \geq 1$. Let (u_k, w_k) be the solution of (4) under the initial-boundary conditions (6)-(7) and set $W_k = w_k^m$. Namely (u_k, W_k) is the solution of (13) satisfying (14) and (7). Set $w_* = W_*^{1/m}$ for the limit (u_*, W_*) given in Theorem 2.1 where (8), (9) and (11) are replaced by (13), (14) and

$$\iint_{Q_T} \{-(u_* - \lambda W_*) \zeta_t + \nabla u_* \cdot \nabla \zeta\} dx dt = \int_{\Omega} (u_0 - \lambda w_0^m) \zeta(\cdot, 0) dx \quad (15)$$

with $\lambda = 1/m$, respectively. Suppose that $\Gamma(t)$ is a smooth, closed and orientable hypersurface satisfying $\Gamma(t) \cap \partial\Omega = \emptyset$ for all $t \in [0, T]$. Also assume that $\Gamma(t)$ smoothly moves with a normal velocity V_n from $\Omega^u(t)$ to $\Omega^w(t)$, and u_* is continuous in Q_T and smooth

on $\overline{Q_T^u}$, and w_* is smooth on $\overline{Q_T^w}$. Then the following relations hold.

$$\begin{aligned} w_*(t) &= w_0, & \text{in } Q_T^w; \\ \begin{cases} u_{*,t} = \Delta u_* & \text{in } Q_T^u, \\ u_* = 0, \quad \frac{w_0^m}{m} V_n = -\frac{\partial u_*}{\partial n} & \text{on } \Gamma, \\ \frac{\partial u_*}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, T), \\ u_* = u_0 & \text{on } \Omega^u(0) \times \{0\}. \end{cases} \end{aligned}$$

3 Singular limits in Case $(m_1, m_2, m_3, m_4) = (1, m, 1, 1)$: immovable interfaces

A free boundary appears in the fast reaction limit also in this case; however, this free boundary does not move.

Throughout this section, we impose (H1) on the initial datum (u_0, w_0) again, and assume $m > 1$. Under the assumption (H1), there exists a unique solution (u_k, w_k) of the initial-boundary value problem (4), (6) and (7) satisfying (10).

We give a result on the convergence of (u_k, w_k) .

Theorem 3.1 *Set $(m_1, m_2, m_3, m_4) = (1, m, 1, 1)$ where $m > 1$. Let (u_k, w_k) be the solution of (4) under the initial and boundary conditions (6) and (7). Then there exist subsequences $\{u_{k_n}\}$ and $\{w_{k_n}\}$ of $\{u_k\}$ and $\{w_k\}$, respectively, and functions u_* , w_* and a distribution U_* such that*

$$u_*, u_*^{\frac{m}{2}} \in L^\infty(Q_T) \cap L^2(0, T; H^1(\Omega)), \quad w_* \in L^\infty(Q_T), \quad U_* \in H^{-1}(Q_T), \quad (16)$$

$$0 \leq u_*, w_* \leq M, \quad u_* w_* = 0 \quad \text{a.e. in } Q_T, \quad U_* \geq 0 \quad \text{in } H^{-1}(Q_T), \quad (17)$$

$$\begin{aligned} u_{k_n} \rightarrow u_* & \quad \text{strongly in } L^p(Q_T) (\forall p \geq 1), \quad \text{a.e. in } Q, \\ & \quad \text{weakly in } L^2(0, T; H^1(\Omega)) \text{ and weakly } * \text{ in } L^\infty(Q_T), \end{aligned} \quad (18)$$

$$w_{k_n} \rightarrow w_* \quad \text{weakly in } L^p(Q_T) (\forall p \geq 1) \text{ and weakly } * \text{ in } L^\infty(Q_T), \quad (19)$$

$$\left| \nabla u_{k_n}^{\frac{m}{2}} \right|^2 \rightarrow U_* \quad \text{weakly in } H^{-1}(Q_T) \quad (20)$$

as k_n tends to infinity. Moreover u_* , w_* and U_* satisfy

$$\begin{aligned} \iint_{Q_T} \left\{ - \left(\frac{1}{m} u_*^m - w_* \right) \zeta_t + \frac{2}{m} u_*^{\frac{m}{2}} \nabla u_*^{\frac{m}{2}} \cdot \nabla \zeta \right\} dx dt \\ + \frac{4(m-1)}{m^2} \langle U_*, \zeta \rangle_{H_0^1(Q_T)} = 0 \end{aligned} \quad (21)$$

for all $\zeta \in H_0^1(Q_T)$.

We can prove $U_* = \left| \nabla u_*^{\frac{m}{2}} \right|^2 \in L^1(Q_T)$ under additional conditions. Here we use the notation $\Omega^u(t)$, $\Omega^w(t)$, $\Gamma(t)$, Q_T^u , Q_T^w and Γ defined by (2) and (12). Then we can give an explicit equation of motion for the free boundary.

Theorem 3.2 Let u_*, w_*, U_* be the functions satisfying (16)-(20). Suppose that $\Gamma(t)$ is a smooth, closed and orientable hypersurface satisfying $\Gamma(t) \cap \partial\Omega = \emptyset$ for all $t \in [0, T]$. Also assume that $\Gamma(t)$ smoothly moves with a normal velocity V_n from $\Omega^u(t)$ to $\Omega^w(t)$, and u_* is continuous in Q_T and smooth on $\overline{Q_T^u}$, and w_* is smooth on $\overline{Q_T^w}$. Then the following relations hold.

$$V_n = 0 \text{ on } \Gamma, \quad \text{that is, } \Omega^u(t) \equiv \Omega^u(0), \Omega^w(t) \equiv \Omega^w(0), \Gamma(t) \equiv \Gamma(0);$$

$$w_*(t) = w_0, \quad U_* = |\nabla u^{\frac{m}{2}}|^2 \quad \text{in } Q_T;$$

$$\begin{cases} u_{*,t} = \Delta u_* & \text{in } Q_T^u = \Omega^u(0) \times (0, T), \\ u_* = 0 & \text{on } \Gamma = \Gamma(0) \times (0, T), \\ \frac{\partial u_*}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, T), \\ u_* = u_0 & \text{on } \Omega^u(0) \times \{0\}. \end{cases}$$

See [5] for the proofs of Theorems 3.1 and 3.2.

4 Singular limits in Case $(m_1, m_2, m_3, m_4) = (m, 1, 1, 1)$: vanishing interfaces

In this case the non-diffusive reactant w consumes much faster than diffusive one u in the limit as $k \rightarrow \infty$. This fact makes the propagation speed of $\Gamma(t)$ too rapid. At least if $m > 2$, then $\Omega^u(t)$ spread too rapidly for us to follow its boundary $\Gamma(t)$: actually we cannot observe any free boundary.

Throughout this section, we impose the following assumptions on the initial data:

(H2) $(u_0, w_0) \in C^2(\overline{\Omega}) \times C^\alpha(\overline{\Omega})$ satisfy

$$u_0(x)w_0(x) = 0, \quad 0 \leq u_0(x) \leq M_u, \quad 0 \leq w_0(x) \leq M_w$$

for any $x \in \Omega$, where $\alpha \in (0, 1)$ represents a Hölder exponent and

$$M_u := \max_{x \in \overline{\Omega}} |u_0|, \quad M_w := \max_{x \in \overline{\Omega}} |w_0|.$$

(H3) u_0 holds the homogeneous Neumann boundary condition:

$$\frac{\partial u_0}{\partial \nu} = 0 \quad \text{on } \partial\Omega.$$

We can derive the following result on the singular limit of (4) (see [5]).

Theorem 4.1 Set $(m_1, m_2, m_3, m_4) = (m, 1, 1, 1)$ where $m > 1$. Let (u_k, w_k) be the solution of (4) under the initial and boundary conditions (6) and (7). Then

$$\begin{aligned} u_k &\rightarrow u_* & \text{in } C^0(\overline{Q_T}) & \quad \text{as } k \rightarrow \infty, \\ w_k &\rightarrow 0 & \text{in } C^0(\overline{\Omega} \times [\varepsilon, T]) & \quad \text{as } k \rightarrow \infty \text{ for any } \varepsilon \in (0, T), \end{aligned}$$

where $u_*(x, t)$ belongs to $C^{2,1}(\overline{Q_T})$ and satisfies the heat equation in the whole domain as follows :

$$\begin{cases} u_{*,t} = \Delta u_* & \text{in } Q_T, \\ \frac{\partial u_*}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, T), \\ u_* = u_0 & \text{on } \Omega \times \{0\}. \end{cases}$$

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