

Non-Equilibrium Quantum Spin Transport Theory Based on Schwinger-Keldysh Formalism

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Abstract

In this thesis, we investigate the non-equilibrium quantum transport phenomena of magnetization on the basis of the (so-called) Schwinger-Keldysh formalism.

Firstly, we reveal a microscopic origin of spin pumping which is well-established experimentally and used as the standard method of generating spin currents. Beyond a classical phenomenological argument, we microscopically clarify the relevant physical quantities to the experimental detection of spin currents by using the inverse spin-Hall effects. Consequently, we present two microscopic spin pumping theories based on the Schwinger-Keldysh formalism, quantum spin pumping theory and thermal spin pumping theory; in the former, spin currents are generated by quantum fluctuations induced by applied microwaves. On the other hand, in the latter case, spin currents are produced owing to the inhomogeneous thermal fluctuations (i.e. temperature difference). We also analyze microscopically the features of each spin-pumping system.

Then, we seek for the possibility of Bose-Einstein condensation (BEC) of magnons in each spin-pumping system. In quantum spin-pumping systems, we closely analyze the effects of applied microwaves, which drive the systems dynamically out of equilibrium, on magnon BEC. Specifically, we investigate the time-evolution of the macroscopic condensate order parameter, which is associated with the recently attractive phenomenon called quasi-equilibrium (dynamical) magnon condensation. On the other hand, in the case of the usual spin-pumping system (i.e. the one without microwaves), we analyze the effects of the exchange interaction at the interface between metals and magnetic insulators, which is the key to (i.e. the source of) spin pumping. In doing so, we clarify, on the basis of a non-perturbative quantum field theories, the condition for the standard magnon condensation associated with a spontaneous symmetry breaking to occur. The magnon condensate is a macroscopic state with quantum coherence and hence, it is robust against the loss of information. Therefore the active manipulation of magnon condensates is expected to lead to the application to quantum spintronics devices.

Lastly, as the synthesis of the above works, we utilize magnons in BEC and theoretically propose a new kind of methods for the generation of the spin currents arising from the macroscopic quantum effects; the Josephson effects in magnon BEC through the quantum-mechanical phase called the Aharonov-Casher phase. In contrast to the usual spin pumping, this is a qualitatively new method in the sense that transverse Josephson spin currents are carried by magnons in BEC in the ferromagnetic insulators through the influence of via the Aharonov-Casher phase. The Aharonov-Casher phase is characterized by the applied electric field, which is under our control. Then, in addition to the microwave pumping method, the required experimental techniques have already been established. Therefore, by using the method we propose in this thesis, the Josephson effect in magnon BEC through the Aharonov-Casher phase, the first observation of the Josephson effects in magnon BEC and the resultant indirect detection of the A-C phase are now possible. Thus, we theoretically open a new door to experimentally exploring the ferromagnetic insulators.

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Contents

I	Microscopic Spin Pumping Theory	13
1	Introduction	15
1.1	Background	15
1.1.1	Spintronics	15
1.1.2	Spin Pumping	16
1.1.3	Motivations for the Reformulation	18
1.1.4	Magnon BEC	19
1.1.5	Josephson Effects of Magnon BEC through Aharonov-Casher Phase	21
1.2	Organization of Thesis	23
2	Quantum Spin Pumping Theory	27
2.1	Quantum Spin-Pumping Systems	27
2.2	Dephasing Effects	30
2.3	Breaking of Spin Conservation Law	30
2.4	Breaking of Magnon Conservation Law	31
2.5	Net Pumped Spin Current	32
2.6	Spin Relaxation Torque	33
2.6.1	Perturbative Evaluation	33
2.6.2	Quantum Spin Pumping by FMR and ESR	35
2.6.3	Discussion	36
2.7	Temperature Dependence	36
2.8	Distinction from Classical Theory	37
2.8.1	Classical Spin Pumping Theory	38
2.8.2	Pumped Spin Current Based on Classical Theory	38
2.8.3	Distinction; microwaves	39
2.8.4	Discussion; Quantization of Classical Theory	39
2.8.5	Correspondence to the Preceding Classical Theory	41
2.9	Conclusion	43
3	Thermal Spin Pumping Theory	45
3.1	Thermal Spin Relaxation Torque	45
3.2	Net Pumped Thermal Spin Current	46
3.2.1	The Spin Continuity Equation of the Whole System	47
3.3	Schwinger-Keldysh Formalism	47
3.3.1	Spin-flip Condition	48
3.3.2	TSRT	49
3.4	Thermal Spin Pumping Effect	50
3.5	Summary	51
3.6	Supplement; Coherent State	52

II	Magnon BEC Theory and Related Phenomena	55
4	Magnon BEC Theory; Microwaves	57
4.1	Hamiltonian; Effects of Microwaves	57
4.2	Schwinger-Keldysh Formalism	58
4.3	Quasi-equilibrium Magnon BEC	60
4.4	Dynamics after Switching off Microwaves	63
4.4.1	The Interaction Between Magnons and Phonons	64
4.4.2	An Effective Hamiltonian via the Procedure by Bogoliubov et al.	65
4.4.3	An Effective Hamiltonian Including Magnon-Magnon Interactions	67
4.5	Conclusion	68
4.6	Supplement; Protection of Information	68
5	Magnon BEC Theory; SSB	71
5.1	SSB and Magnon BEC	71
5.2	Goldstone Model	71
5.2.1	Minimally Generalized Goldstone Model	73
5.2.2	Minimal Model	74
5.2.3	Stability of Vacuum	75
5.3	Theoretical Proposal	77
5.3.1	Anisotropic Exchange Interaction	78
5.3.2	Correspondence to Spin-pumping System	78
5.3.3	Stable Magnon BEC	79
5.3.4	Discussion; Field Theory and One-body Quantum Mechanics	79
5.3.5	Generalized Goldstone Model	80
5.4	Summary and Outlook	81
6	Dissipation Theory	83
6.1	Caldeira-Leggett Model	83
6.2	LLG Equation	85
6.3	Conclusion	86
7	Theory on Josephson Effects in Magnon BEC through Aharonov-Casher Phase	89
7.1	System and Hamiltonian	90
7.2	Aharonov-Casher Phase	91
7.3	Josephson Spin Current	92
7.4	Features	93
7.4.1	Dissipation Effects	94
7.4.2	Josephson Effects	94
7.4.3	Distinction	95
7.5	Summary	96
7.6	Supplements; Berry's Phase	97
III	Summary and Appendices	99
8	Summary; Concluding Remarks	101

9	Appendix; Review of Schwinger-Keldysh Formalism	103
9.1	Fundamental of Schwinger-Keldysh Formalism	103
9.1.1	Gell-Mann and Low Theorem	103
9.1.2	Emergence of Two-branch Contour	105
9.1.3	Schwinger-Keldysh Closed-time Path	107
9.1.4	Langreth Method	109
9.2	Concrete Example	112
9.2.1	Bosonic Keldysh Green's Function	113
9.2.2	Fermionic Keldysh Green's Function	114
9.3	Detail of Calculation	115
9.3.1	Explicit Form of SRT	115
9.3.2	Detail of the Procedure	117
10	Appendix; Magnon Pumping and Resultant Magnon BEC	123
10.1	Macroscopic Condensate Order Parameter	123
10.2	Pumping Process	125
10.3	Quasi-equilibrium Magnon BEC through Thermalization Processes	125
10.4	Effects of Magnon-magnon Interactions	126
10.4.1	Pumping Processes	128
10.4.2	Thermalization Processes and Resultant Quasi-equilibrium Magnon BEC 128	
10.5	Outlook	129
11	Appendix; Dissipation Theory	131
11.1	Open and Closed Systems	131
11.2	Caldeira-Leggett Model; Quantum Brownian Motion	132
11.3	Caldeira-Leggett Model and Magnon	134
11.3.1	Mapping	135
11.3.2	The Time-development of the Zero-mode of Magnons	136
11.3.3	The Time-development under Microwaves	136
11.3.4	The Time-development after Switching off Microwaves	137
11.3.5	Relaxation Processes	138
11.3.6	Relaxation Processes (I)	138
11.3.7	Relaxation Processes (II)	139
11.3.8	Emergence of Plateaus	141
11.4	LLG Equation	143
11.5	Outlook; Thermalization Processes	145

List of Abbreviations

- A-C; Aharonov-Casher
- BEC; Bose-Einstein condensation
- ESR; electron spin resonance
- FMR; ferromagnetic resonance
- ISHE; inverse spin-Hall effects
- LLG; Landau-Lifshitz-Gilbert
- SRT; spin relaxation torque
- SSB; spontaneous symmetry breaking
- TSRT; thermal spin relaxation torque
- YIG; yttrium iron garnet

Part I

Microscopic Spin Pumping Theory

Chapter 1

Introduction

1.1 Background

In this thesis, we discuss the non-equilibrium quantum transport phenomena of magnetizations and construct microscopic theories based on the Schwinger-Keldysh formalism. In particular, we focus on the microscopic mechanism for the generation of spin currents and clarify the features.

1.1.1 Spintronics

Recently a new branch of physics and nanotechnology called spintronics has emerged and has been attracting special attention from viewpoints of both fundamental science and application.[1, 2] The aim of spintronics is the control and active manipulation of the spin as well as charge degrees of freedom of electrons, in particular spin currents[3] (Fig. 1.1); spintronics avoids the dissipation from Joule heating by replacing charge currents with spin currents. Therefore establishing methods for generation and observation of spin currents is urgent and also significant from viewpoints of fundamental science and potential applications to information and communication technologies.

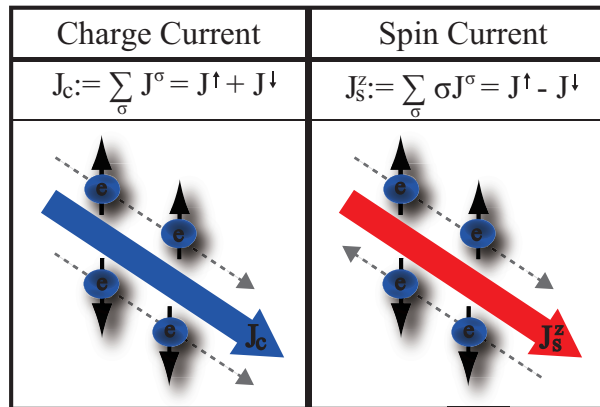


Figure 1.1: Schematic pictures of the charge current J_c and the spin current J_s^z .

It should be remembered that there has been the concept of spin currents[4, 5] simply as the theoretical idea for a long time and it was qualified as the physical quantity owing to the indirect measurement by using inverse spin-Hall effects (Fig. 1.2).[6] Although spin currents have thus been recognized as a physical quantity, there remains an unsolved urgent

theoretical issue of what the proper definition of the spin current is, in particular in the situation where the spin conservation law is broken.[7] In this thesis, we will give an answer to this issue by taking the experimental results of spin pumping into account. We believe firmly that we have presented the proper definition of the spin current generated by spin pumping effects.¹

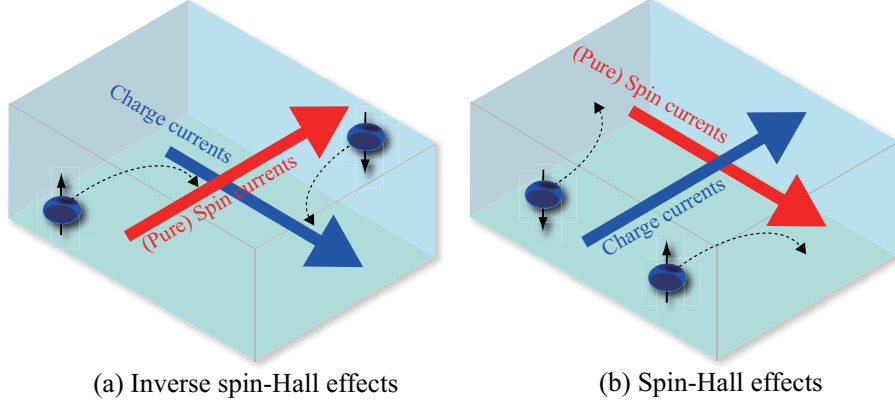


Figure 1.2: Schematic pictures of the inverse spin-Hall effect[6] (a) and the spin-Hall effect (b). The origin of these phenomena is the quantum relativistic effect, namely the spin-orbit interaction, $\vec{p} \cdot \vec{\sigma}$, which is symmetric under the simultaneous transformation; $\vec{p} \mapsto -\vec{p}$ and $\vec{\sigma} \mapsto -\vec{\sigma}$. Therefore, each conduction electron which has different kinds of spins, up- and down- spins, move in the opposite direction. The spin current is converted into the charge current (a), and vice versa (b). Although the spin-orbit interaction creates a variety of interesting phenomena, it is beyond the scope of this thesis; we treat the system in the absent of the spin-orbit interaction in order to focus on the microscopic mechanism for the generation of spin currents, which is produced by the spin pumping effects. The pumped spin currents is converted into the charge currents due to the spin-orbit interaction and it is experimentally measured by the inverse spin-Hall effect.

1.1.2 Spin Pumping

A standard way to generate a (pure) spin current is the spin pumping[9, 10, 6, 11, 12, 13]² at the interface between a ferromagnetic material and a non-magnetic metal (Fig. 1.3). There the precession of the magnetization induces a spin current pumped into a non-magnetic metal, which is proportional to the rate of precession of the magnetic moments. The precessing moments in the ferromagnet act as a source of spin angular momentum; a spin battery.[14] This method was theoretically proposed first by Silsbee et al.[15] and later has been developed

¹Lastly, let us remark that regarding this issue, an ambitious attempt from the viewpoint of holography has been recently made in Ref.[8], where the authors have defined the (so-called) spin currents by using the Nöther's theorem and have investigated the spin transport on the basis of the gauge-gravity correspondence. On the other hand, from the viewpoint of condensed matter physics, it should be remembered that as far as one respects the symmetry on which Nöther's theorem is based, the total angular momentum introduced in Ref.[8] cannot be split into two parts; the contribution arising from orbital and the one from intrinsic spin (, with which we are familiar). This might imply that we should enjoy more and more exchange between two research areas.

²This is the experimentally established method.

by Tserkovnyak et al.,[9] (see also [10]³) which is now briefly explained as follows.⁴

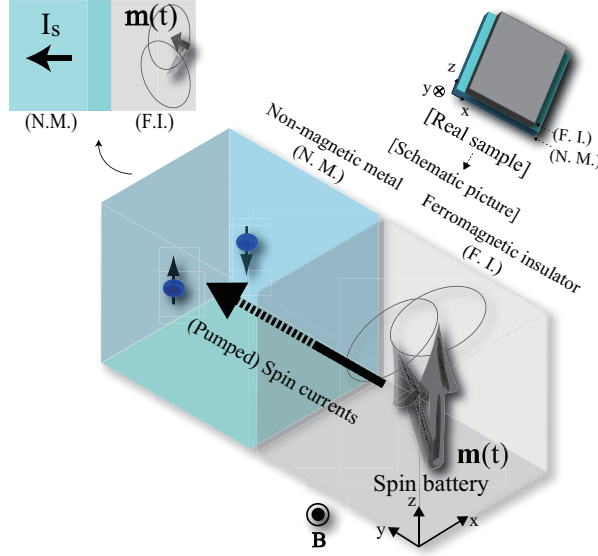


Figure 1.3: Schematic pictures of spin pumping. Localized spins in ferromagnetic insulators works as the spin battery in the sense that they supply spin angular momentum to the conduction electrons.

According to the classical[9, 10, 13] (phenomenological) spin pumping theory by Tserkovnyak et al.,⁵ the pumped spin current $\mathbf{I}_{\text{s-pump}}$ reads, in terms of their notations,[19, 9] as

$$\mathbf{I}_{\text{s-pump}} = G_{\perp}^{(R)} \mathbf{m} \times \dot{\mathbf{m}} + G_{\perp}^{(I)} \dot{\mathbf{m}}, \quad (1.1)$$

where the dot denotes the time derivative. We have taken the electron charge $e = 1$, and $\mathbf{m}(\mathbf{x}, t)$ denotes a unit vector along the magnetization direction; they have treated $\mathbf{m}(\mathbf{x}, t)$ as classical variables. The variable $G_{\perp} = G_{\perp}^{(R)} + iG_{\perp}^{(I)}$ is the complex-valued mixing conductance that depends on the materials in question.[20, 21] Then they have assumed that the magnetization dynamics of the ferromagnets can be described by the Landau-Lifshitz-Gilbert (LLG) equation;[22, 23, 24]

$$\dot{\mathbf{m}} = \gamma \mathbf{H}_{\text{eff}} \times \mathbf{m} + \alpha \mathbf{m} \times \dot{\mathbf{m}}, \quad (1.2)$$

where γ is the gyro-magnetic ratio and α is the Gilbert damping constant that determines the magnetization dissipation rate.[25, 9] The Gilbert damping constant,⁶ α , was originally introduced phenomenologically.[26]

³Regarding the detail of the spin pumping theory proposed by Tserkovnyak et al.,[9] it will be very useful to read the extremely sophisticated master thesis (2007) by T. Taniguchi submitted to Tohoku University.[10]

⁴It was confirmed experimentally by Mizukami et al.[16] Let us remark on a point; in a spin Hall system, i.e., in a nonmagnetic semiconductor, Kato et al.[17] reported an observation of a spin current by measuring optically the spin accumulation which appears as a result of spin currents at the edge of samples (GaAs and InGaAs). A critical issue in the observation of a spin current, however, is that a spin current is not generally conserved and therefore measuring spin accumulation does not necessarily indicate the detection of a spin current, in sharp contrast to the case of charge. Non-conservation of spins is represented by a spin relaxation torque which appears in the spin continuity equation. For a clear interpretation of experimental results on a spin current, to understand the relaxation torque[18] is essential. We clarify and stress this point in this thesis.

⁵Regarding the detail of the spin pumping theory proposed by Tserkovnyak et al.,[9] it will be very useful to read the extremely sophisticated master thesis (2007) by T. Taniguchi submitted to Tohoku University.[10]

⁶They have introduced the isotropic damping constant α (i.e. the value of α does not depend on the spin

1.1.3 Motivations for the Reformulation

The above theory proposed by Tserkovnyak et al. has now been widely used for interpreting vast experimental results despite its phenomenological treatment of spin-flip scattering processes.[9]⁷ Now, let us remark on the issue by which we were originally motivated to reformulate their spin pumping theory. In their formalism, the pumped spin currents carried by the conduction electrons (i.e. fermions) is represented only by using the degrees of freedoms of localized spins (i.e. classical magnetization vectors $\mathbf{m}(t)$) with the help of the mixing conductance, and the relaxation torque term or Gilbert-damping term (i.e. α -term) phenomenologically introduced plays the key role in their classical theory. That is, the spin currents carried by fermionic quantum particles are represented by bosonic ones. At first glance, this looks extremely strange to us. A possible excuse for this treatment might be that localized spins in ferromagnetic insulators precess coherently and respond macroscopically to the applied microwaves. Hence, in the classical limit, their theory (i.e their expression of the pumped spin current) is justified. Nevertheless, we would like to point out that the spin conservation laws of each the conduction electrons and the localized spins (i.e. magnons) do not hold separately due to the exchange interaction, which is expressed by the spin relaxation torque emerging in spin continuity equations. Therefore in order to obtain the exact expression of the pumped spin currents, we need to microscopically analyze the torque term which includes both degrees of freedoms, the conduction electrons and localized spins (i.e. magnons). In addition, the spin-flip phenomenon, which is the most important process of spin pumping, arises essentially (i.e. originally) quantum effects. Therefore we recognize that to construct a quantum spin pumping theory beyond the classical phenomenological one is an urgent issue.

On top of this, it should be noted that the Gilbert-damping term (i.e. α -term) was originally phenomenologically introduced in order to take the dissipation effects (i.e. spin frictions) into account and we here would like to stress that dissipation arises from the interaction with the conduction electrons as well as phonons. Therefore, we suspect that the spin angular momentum, that localized spins have lost, will be transformed to phonons as well as the conduction electrons. That is, even when one employs the α -term to represent the exchange of spin-angular momenta, one should not assume that all of the spin angular momentum the localized spins have lost has been given only to the conduction electrons; the amount of the spin angular momentum localized spins have lost is not equal to the one which the conduction electrons have obtained because phonons also have gained a part of the spin angular momentum. Lastly, let us emphasize that the crucial flaw of their theory[9] is that the contributions of microwaves, which are essential to the experimental realization of spin pumping, are not directly included at all.

Points of Our Theory Based on Schwinger-Keldysh Formalism

Therefore, in order to overcome these theoretical issues, we microscopically analyze spin-

component S^x , S^y , and S^z) in order to guarantee that spin length is conserved. One can easily shown the relation, $dS_i^2/(dt) = 0$, by using the LLG equation shown by eq. (2.44). Otherwise (i.e. in anisotropic cases), the spin length cannot be conserved in general.

⁷This point has been explicitly explained in their preceding review article[9] (see VII. SUMMARY AND OUTLOOK, pp. 43-44) Let us cite from their review article. [pp.43] “*Except for the phenomenological treatment of spin-flip scattering processes, the theory is derived from first principle. The main subject in this context is the concept of spin pumping due to moving magnetization vectors. The magnetization dynamics is affected by the spin-transfer torque.*” [pp.44] “*Theoretical challenges for the future include a proper treatment of spin-orbit interactions, the coupling of magnetization degrees of freedoms to the lattice, and effects beyond semi-classical regime.*”

flip processes⁸ originating fully from quantum effects⁹ at the interface, where spin angular momenta are exchanged between the conduction electrons and the magnons. Then we reformulate[27, 28, 29, 30] the spin pumping theory on the basis of the Schwinger-Keldysh formalism,[31, 32, 33, 22, 34, 35] which can explicitly analyze non-equilibrium situations beyond classical phenomenological treatments. More in detail, we reconstruct the spin pumping theory on the basis of the spin continuity equation; in sharp contrast to the case of charges, the spin conservation laws of each the conduction electrons and the localized spins (i.e. magnons) do not hold separately due to the exchange interaction, which is expressed by the spin relaxation torque term[18] emerging in spin continuity equations. Then, by analyzing microscopically the torque term characterized by the degrees of freedoms of both the conduction electrons and the localized spins (i.e. magnons), we obtain the exact expression of the pumped spin currents carried by the conduction electrons. This is the main outcome of our reformulation. In particular, we show that the time-average of the net pumped spin current is expressed only in terms of the torque term. Note that the time-derivative of the spin density has nothing to do with the net pumped spin current even when the total spin angular momentum is conserved. That is, the spin density for the conduction electrons is not relevant to quantum spin pumping mediated by magnon. This is one of the main results from our formalism.

Virtues of Using Schwinger-Keldysh Formalism

Here, it will be useful to remark on the strong points of the Schwinger-Keldysh formalism.[31, 32, 33, 22, 34, 35, 36, 37] Thanks to the Schwinger-Keldysh closed time path, the Schwinger-Keldysh formalism (i.e. closed time path formalism or the real-time formalism) is free from the assumption of adiabaticity and the corresponding theorem of Gell-Mann and Low.[38, 39] Therefore, within the perturbative theory via the Schwinger-Keldysh (or contour-ordered) Green's functions, the formalism can deal with an arbitrary time-dependent Hamiltonian[40, 41] and treat the system mechanically out of the equilibrium.¹⁰ On top of this, this formalism is applicable to systems at finite temperature; the well-known Matsubara formalism (i.e. the imaginary-time formalism), which also can deal with thermodynamic average values, can be regarded as a simple corollary[35] of the Schwinger-Keldysh formalism. That is, the Schwinger-Keldysh formalism includes the Matsubara formalism and information about finite temperature is contained in the greater and the lesser Green's functions. Consequently, we can treat non-equilibrium phenomena at finite temperature thanks to the Schwinger-Keldysh formalism. These are the strong points of the formalism.¹¹

1.1.4 Magnon BEC

After that, we construct magnon Bose-Einstein condensation (BEC) theories[45] where magnon BEC is generated by two principles; one is the standard mechanism based on the spontaneous symmetry breaking.[39] The other is the dynamical way called microwave pumping

⁸We have regarded that non-equilibrium spin-flip processes resulting from quantum effects at the interface due to the exchange interaction is the most important effects on spin pumping.

⁹They should seriously take the fact into account that spins themselves are the production of quantum effects.

¹⁰I have employed the terminology mechanically so as to mean that the non-equilibrium situations are induced by a time-dependent Hamiltonian. An attempt to generalize the Schwinger-Keldysh formalism so as to treat systems thermally out of equilibrium has been very recently reported by Dr. A. Shitade,[42] whose guiding principle is the gauge covariance.

¹¹Regarding the Floquet's theorem, which is another powerful tool to analyze (periodic) non-equilibrium dynamics, please see pp. 29 in the review article by Aoki et al.[43]

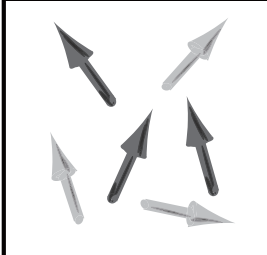
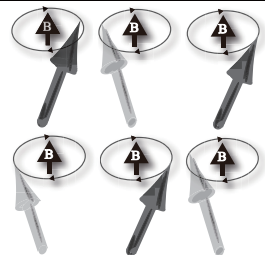
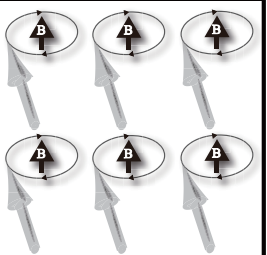
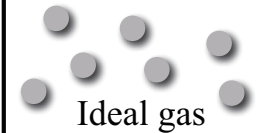
Paramagnet	Spin waves	Magnon BEC (Coherent precession)
		
 Ideal gas	 Quantum gas	 BEC

Figure 1.4: (Color online) (Rough) Schematic pictures[44] of magnon Bose-Einstein condensation, which corresponds to the coherent precession in the language of spins.

method,[46, 47, 48, 49, 50, 51, 52, 53] which is often called quasi-equilibrium magnon BEC. On the basis of these two principles, we investigate the possibility for the emergence of magnon BEC in spin-pumping systems.

As you know, the experimental realizations of magnon BEC in variety kinds of materials have been recently reported; TlCuCl_3 ,[54] Cs_2CuCl_4 ,[55] YIG,[46, 47, 48, 50, 49, 56] and $\text{BaCuSi}_2\text{O}_6$ [57] et al. This fact implies the universal aspects of this phenomenon and magnon BEC is presently one of the most attractive phenomena of condensed matter physics. Magnon BEC, which corresponds to the coherent precession[44] in the language of localized spins (Fig. 1.4), is characterized[45] by the non-zero value of the expectation value of the magnon annihilation operator $\langle a \rangle \neq 0$ under an $U(1)$ -symmetric Hamiltonian.¹² That is, $\langle a \rangle$ deserves the macroscopic condensate order parameter[54, 61] for the occurrence of magnon BEC and $\langle a \rangle \neq 0$ under an $U(1)$ -symmetric Hamiltonian means magnon BEC. By adopting this criterion, motivated by the experimental breakthrough achieved by Demokritov et al.,[46] we investigate the possibility for the occurrence of quasi-equilibrium magnon BEC in quantum spin-pumping systems.

In 2006, Demokritov et al.[46] have reported that they have achieved the observation of quasi-equilibrium magnon BEC in YIG at finite (room) temperature by using the method called microwave pumping. Briefly speaking (to the best of my knowledge),[51] applied microwaves excite the zero-mode of magnons and after thermalization processes, the population of the lowest-energy mode becomes much larger than that of other modes; they have observed that after switching off microwaves where the $U(1)$ -symmetry of the system is recovered, although the population (intensity) of each mode actually decreases due to thermalization or relaxation effects,[52, 53, 62] the lowest-energy mode of magnons survives for much longer time compared with other modes. Then, they have concluded that the lowest-energy mode is stable under the time-development compared with other modes and hence, the mode belongs to the condensation.

¹²Regarding the issue on the criterion for BEC in interacting systems, we in part agree[41] with the argument by Leggett in his textbook[58] (see pp. 31-40); in addition to this commonly used macroscopic condensate order parameter,[45] he has introduced (i.e. presented) the discussion by Penrose-Onsager[59] and Yang[60] about the generalization of original (noninteracting) BEC to interacting cases. We have been tackling this issue.[41]

Stimulated by this experimental progress, we investigate the possibility for the emergence of quasi-equilibrium magnon BEC in spin-pumping systems. In addition, we also study the dissipation effects on magnon BEC with the help of the famous Caldeira-Leggett model.[63, 64]

Remember that BEC of magnon is the macroscopic quantum coherent state (Fig. 1.4),[44, 45] which is robust against the loss of information[61] (see also Sec. 4.6). Therefore to construct the BEC theory will be significant also from the viewpoint of potential applications to information and communication technologies, namely the development of quantum information spintronics.

1.1.5 Josephson Effects of Magnon BEC through Aharonov-Casher Phase

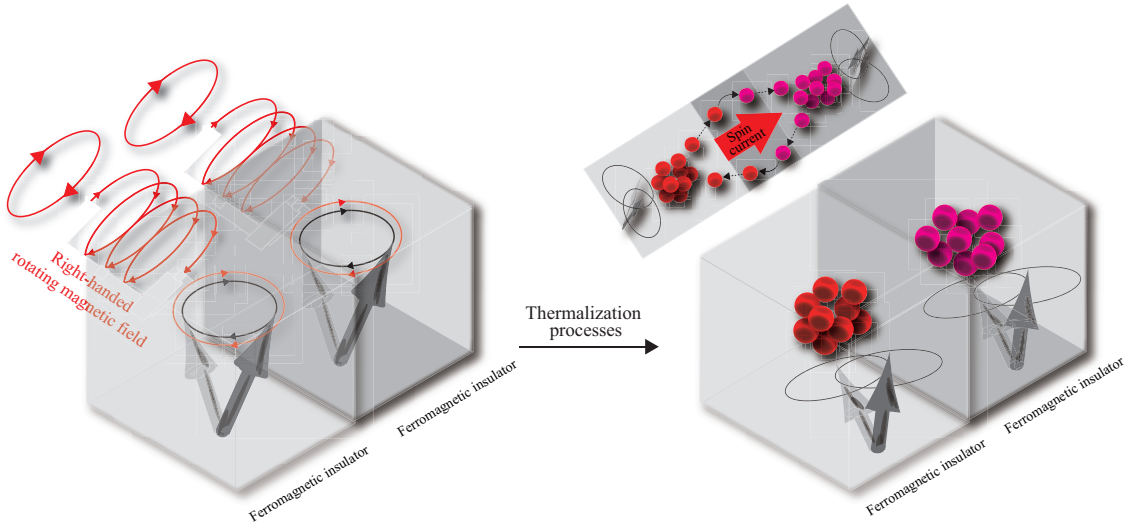


Figure 1.5: (Color online) Schematic pictures of Josephson effects of magnon BEC. Red circles represent magnons and the cloud shows magnon BEC, which corresponds to the coherent precession of localized spins.

Then as the synthesis of our works, stimulated by the experimental breakthrough by Demokritov et al.[46] mentioned above and the following one by Kajiwara et al.,[65] we consider the junction in which two ferromagnetic insulators consisting of magnon BEC are weakly connected to each other (see Fig. 1.5) and investigate the transport of magnon BEC from the viewpoint of spin currents carried by magnon condensates. In particular, we clarify the contribution of the quantum-mechanical phase called the Aharonov-Casher phase[66, 67, 68] θ_{A-C} to the transport; as a charged particle feels a magnetic flux (i.e. a magnetic vector potential) and obtain a quantum-mechanical phase called the Aharonov-Bohm phase,[69, 4, 5] a moving magnetic dipole is affected by electric fields and acquires a quantum-mechanical phase called the Aharonov-Casher phase (see Fig. 1.6); the Aharonov-Casher phase is peculiar to the magnetically polarized particles and it is dual to the Aharonov-Bohm phase. Each quantum-mechanical phase is just a special case of geometric phase called Berry's phase, which causes phase shifts. Under the influence of the Aharonov-Casher phase, the macroscopic condensate order parameter $\langle a(t) \rangle$ is reduced to the form $\langle a(t) \rangle \rightarrow \langle a(t) \rangle_{A-C}$; $\arg(\langle a(t) \rangle_{A-C}) - \arg(\langle a(t) \rangle) = \theta_{A-C}$. The quantum-mechanical phase acts as a driving force on the transport. Therefore this phenomenon can be regarded as the Josephson

effect[70, 71, 72, 73, 74, 75, 76, 77] of magnon BEC[78] through quantum-mechanical phases. We then construct the microscopic theory.

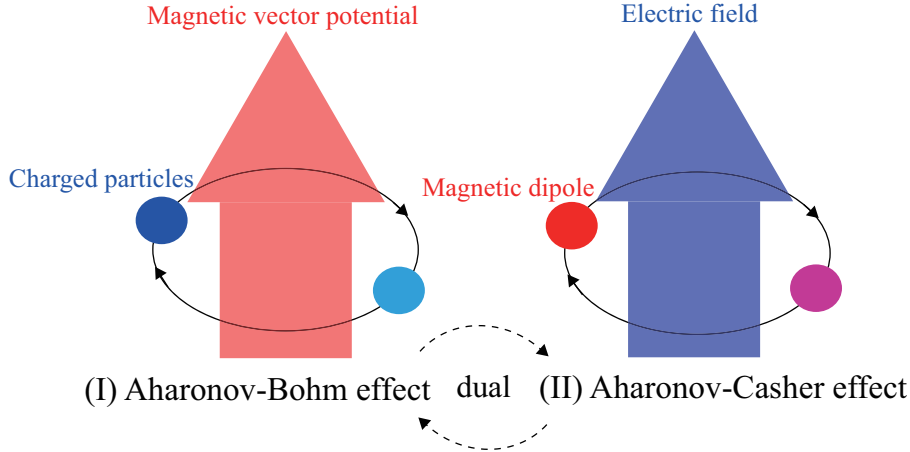


Figure 1.6: (Color online) Schematic pictures of the (I) Aharonov-Bohm effect and (II) Aharonov-Casher effect. A charged particle feels a magnetic flux (i.e. a magnetic vector potential) and obtains the Aharonov-Bohm phase. In the similar way, a moving magnetic dipole is affected by electric fields and acquires the Aharonov-Casher phase, which is dual to the Aharonov-Bohm phase. Each quantum-mechanical phase is just a special case of geometric phase called Berry's phase.

Note that although the method to produce a spin current and detect it has been established by spin pumping and ISHE, there remains a crucial issue; the pumped spin current carried by the conduction electrons in metals disappears within very short distance (typically a few micrometers). This has been the obstacle to the practical use and the application of spin currents. Regarding this issue, Kajiwara et al.[65] have recently achieved an experimental breakthrough by employing ferromagnetic insulators. Note that there is no conduction electrons in insulators, but there exists another kind of carrier, namely spin-waves, which are collective motions of magnetic moments in ferromagnetic insulators. Then, they have shown that a spin-wave spin current, which is a kind of spin currents carried by spin-waves, persists for much greater distance than the usual spin current carried by the conduction electrons in metals.¹³ As an example, the spin-wave decay length of the magnetic insulator yttrium iron garnet $\text{Y}_3\text{Fe}_5\text{O}_{12}$ (YIG) amounts to several centimeters, which is about 10^6 times compared with the case in metals. This is the remarkable feature of using ferromagnetic insulators. Thus, they have opened a new door to exploring spin currents in insulators.

Stimulated by the experimental progress by Kajiwara et al.[65] and Demokritov et al.[46] mentioned above, we utilize magnon BEC, which is the macroscopic effect of magnons corresponding to the quantization of spin-waves, and microscopically analyze the transport of magnon condensates through Aharonov-Casher phases.[66, 67] We then theoretically propose a new mechanism to generate spin currents in ferromagnetic insulators; the Josephson effect[73, 70] in magnon BEC[78] through the A-C phase.[66, 67, 68] To the best of our knowledge, although the macroscopic quantum self-trapping of atomic BEC has been observed by Albiez et al.[76] and the experiment of the Josephson effects in the Bose Josephson junction (i.e. atomic BEC) has been reported by Levy et al.,[77] the Josephson effects in magnon

¹³By using ISHE, they have revealed the conversion of an electric signal into spinwaves and its subsequent transmission through an insulator over a macroscopic distance.

BEC has not been experimentally achieved. Then, our method by using the A-C phase is helpful to the first observation of the Josephson effects in magnon BEC, which deserves also the indirect detection of the A-C phase.[79, 80] We consider that, as mentioned, the required experimental techniques have already been ensured[46, 47, 48, 65] and by using the method we propose in this work, the first observation of the Josephson effects in magnon BEC and the resultant indirect detection of the A-C phase are now possible.

Lastly, let us again stress that magnon BEC[45] is the macroscopic state in the sense that a macroscopic numbers of magnons have formed a coherent quantum state described by a common wavefunction, which is in sharp contrast to spin-waves. Therefore our method is robust against the loss of quantum information[61] and the decay-length of the spin current carried by magnon BEC will be much longer than that of the one by spin-waves. On top of this, the Aharonov-Casher phase acting as a driving force to generate spin currents is actually under our control via electric fields. These points deserve the virtue of our method and lead to the potential application to quantum spintronics devices.

1.2 Organization of Thesis

This thesis is organized as follows.

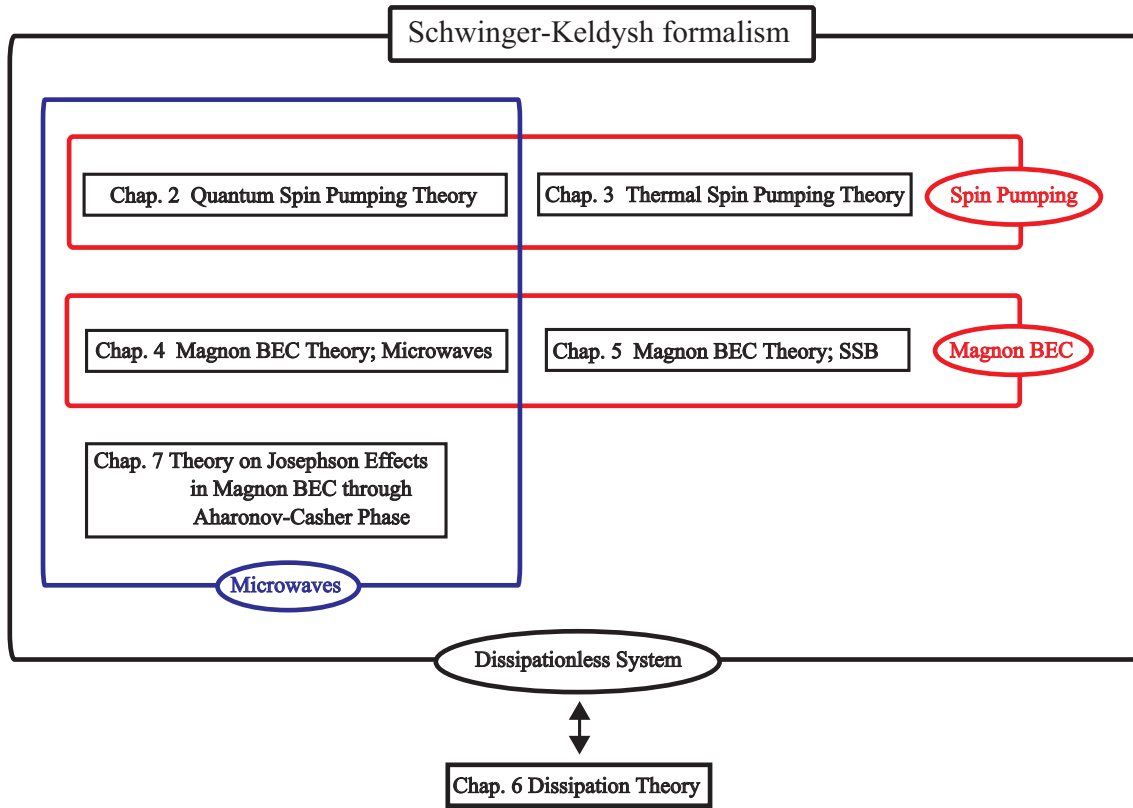


Figure 1.7: (Color online) The organization of this thesis. In addition, the Appendices read as follows. (Appendix) Chap. 9; a review of the Schwinger-Keldysh Formalism. (Appendix) Chap. 10; a magnon pumping theory with the resultant magnon BEC, where a method to phenomenologically include the effects of the thermalization process is shown. (Appendix) Chap. 11; a dissipation theory based on the Caldeira-Leggett model.

We reformulate the spin pumping theory on the basis of the Schwinger-Keldysh formal-

ism¹⁴ in Chap. 2. To clarify the origin (i.e. source) of the pumped spin current, beyond the classical phenomenological theory, we explicitly include the quantum effects arising from applied microwaves and microscopically describe the non-equilibrium spin-flip process resulting from the quantum effects at the interface. For the purpose, taking dephasing effects into account, we investigate spin pumping on the basis of the spin continuity equation at the interface, where the spin angular momenta are exchanged between the conduction electrons and magnons and the spin conservation law is broken due to the interaction arising from proximity effects. We microscopically analyze the origin of the non-conservation and clarify its relation to the pumped spin currents. After that, on the basis of this microscopic analysis, we theoretically propose a new method to generate spin pumping. Lastly, we clarify the distinction between our microscopic quantum theory and the preceding classical (phenomenological) theory.

By using the same procedure, we construct the thermal spin pumping theory in Chap. 3. In particular, by using the Schwinger-Keldysh formalism, we microscopically capture the spin-flip process at the interface and clarify the spin-flip condition between the conduction electrons and the magnons. We discuss in detail the resultant novel features of our thermal spin pumping theory and clarify the distinction from the preceding theories.

At this stage, we have closely investigated quantum and thermal spin pumping and have completed the construction of each microscopic theory based on the Schwinger-Keldysh formalism. Next, we turn our focus on magnon BEC, which is the macroscopic quantum coherent state and robust against the loss of information, in each spin-pumping system.

To this end, in Chap. 4, we investigate the possibility for the emergence of quasi-equilibrium (i.e. dynamical) magnon BEC in the quantum spin-pumping systems. We evaluate the macroscopic condensate order parameter on the basis of the Schwinger-Keldysh formalism and analyze the time-development under the interaction with phonons after switching off microwaves.¹⁵ We also discuss the features of quasi-equilibrium magnon BEC in quantum spin-pumping systems.

After that, in Chap. 5, we clarify the condition for the experimental realization of magnon BEC associated with spontaneous symmetry breaking (SSB) in spin-pumping systems (i.e. in the absence of microwaves). For the purpose, we use a non-perturbative theory and closely analyze beyond a perturbative theory.

Now, recall that we have been treating dissipation-less systems until now. On the other hand, in real materials, the dissipation effects are inevitable. Therefore, in Chap. 6, we investigate the dissipation effects described by the Caldeira-Leggett model and the LLG equation and clarify the distinction from the viewpoint of magnon BEC.¹⁶

Lastly, as the synthesis of our works, we present a theory of the Josephson effect in the magnon condensate through quantum-mechanical phases called the A-C phases in Chap. 7. By using quasi-equilibrium magnon BEC, we theoretically propose a new way to generate spin currents in insulators; the spin current is carried by magnon condensates and our method results fully from quantum effects, magnon BEC and quantum-mechanical phases. We also qualitatively point out the possibility for the experimental realization of persistent spin currents in a ring.

This is the plan of this thesis.

¹⁴Regarding the Schwinger-Keldysh formalism for itself, it will be helpful to see (Appendix) Chap. 9, in which we have quickly reviewed the fundamental.

¹⁵Regarding thermalization processes, please see (Appendix) Chap. 10, in which we have presented the phenomenological procedure to include the contribution arising from thermalization processes.

¹⁶Regarding the detail, please see (Appendix) Chap. 11.

For the reader's convenience, we present three topics associated with the main issue of the thesis in Appendices. In (Appendix) Chap. 9, we invite you to the short trip to the Schwinger-Keldysh formalism, which is the main theoretical tool of the thesis. Firstly, we recapitulate the implicit assumption on which the standard (traditional) equilibrium quantum field theory is based. We then explain how it is eliminated in the framework of the Schwinger-Keldysh formalism. Lastly, from the viewpoint of the practical use, we show the detailed procedure of the formalism and stress the points with employing concrete examples.

In (Appendix) Chap. 10, we present a method to phenomenologically include the effects of the thermalization process, which is relevant to the (dynamical) quasi-equilibrium magnon BEC in YIG due to the peculiar dispersion relation (i.e. the double degeneracy of the ground state). Magnons are produced (i.e. pumped) by the applied microwaves and after thermalization processes, they condense. That is, there are three stages in BEC; pumping, thermalization, and condensation. On the basis of the experimental results, we phenomenologically include the influence of thermalization processes and show a method to connect pumping with BEC.

In (Appendix) Chap. 11, on the basis of the (well-known) Caldeira-Leggett model,[63, 64] we investigate the influence of dissipation due to phonons on magnon BEC. In particular, we closely analyze the time-development of the macroscopic condensate order parameter and clarify the features of the relaxation processes.

Chapter 2

Quantum Spin Pumping Theory

We reformulate the spin pumping theory on the basis of the Schwinger-Keldysh formalism. We have microscopically describe the non-equilibrium spin process arising from quantum effects and include the contribution originating from microwaves. Then, we theoretically propose a new method to generate spin pumping. We also clarify the distinction from the preceding theory.

2.1 Quantum Spin-Pumping Systems

System

We[27, 28, 29]¹ treat a weakly connected junction between a ferromagnetic insulator and a non-magnetic metal shown in Fig. 2.1.[27] At the interface, due to proximity effects, the conduction electrons couple with localized spins $\mathbf{S}(\mathbf{x}, t)$, $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$;

$$\mathcal{H}_{\text{ex}} = -2Ja_0^3 \int_{\mathbf{x} \in (\text{interface})} d\mathbf{x} \, \mathbf{S}(\mathbf{x}, t) \cdot \mathbf{s}(\mathbf{x}, t), \quad (2.1)$$

where the lattice constant of the ferromagnet is denoted as a_0 . The magnitude of the interaction is supposed to be constant and it is expressed as $2J$. Note that owing to this exchange interaction arising from proximity effects $\mathcal{H}_{\text{ex}}$ at the interface, the spin angular momentum can be interchanged between the conduction electrons and the ferromagnet. That is, this exchange interaction arising from proximity effects[81, 82] $\mathcal{H}_{\text{ex}}$ at the interface is the key to spin pumping.[83]² Therefore we identify the system characterized by the exchange interaction between the conduction electrons and the ferromagnet due to proximity effects $\mathcal{H}_{\text{ex}}$ (Hamiltonian (2.1)) with the spin-pumping system. From now on, we exclusively³ focus on the dynamics at the interface to clarify the microscopic mechanism for the generation of spin currents, called spin pumping.

Hamiltonian

¹Please see also the references cited in these articles.[27, 28, 29]

²In our preceding collaborative work with Prof. G. Tatara,[84] we have omitted this exchange interaction in order to focus on the effects of applied microwaves on magnons and then, we have phenomenologically discussed the spin transport. Now, we have improved[27] this issue and have achieved the construction of the rigorous spin pumping theory.

³This procedure can be justified also from the viewpoint of dephasing effects discussed in Sec. 2.2.

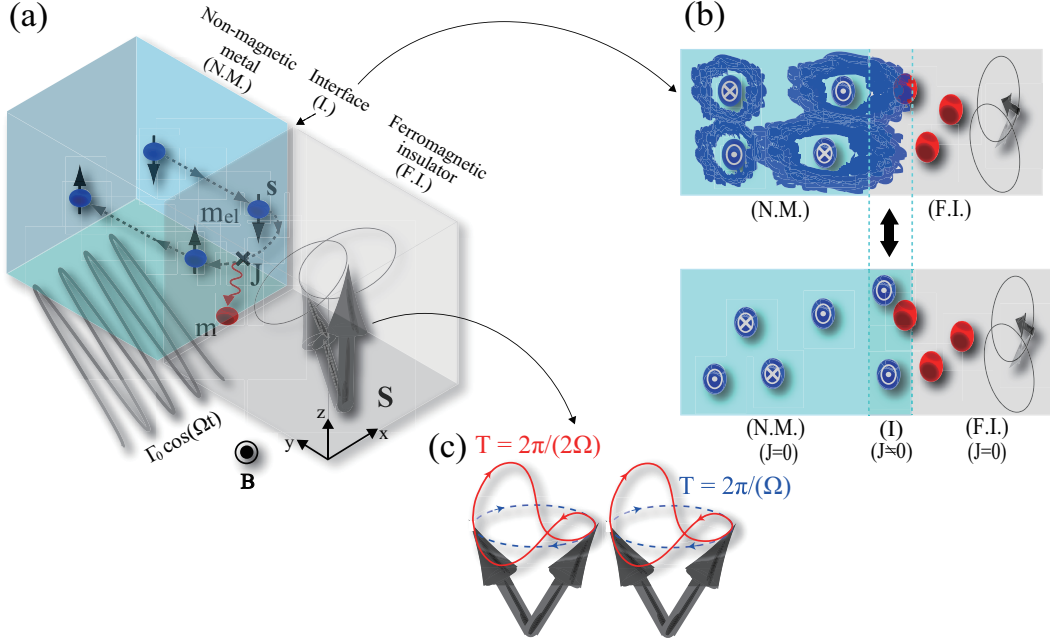


Figure 2.1: (a) The schematic picture of the quantum spin-pumping system. Spheres represent magnons and those with arrows are the conduction electrons. The wavy line denotes the time-dependent transverse magnetic field $\Gamma(t)$ (i.e. the pumping magnetic field or microwave). (b) The interface is defined as an effective area where the Fermi gas (the conduction electrons) and the Bose gas (the magnons) coming from each subsystem coexist to interact due to proximity effects; [81, 82] $J \neq 0$. The width of the interface might be supposed to be of the order of the lattice constant. [85] The interface can be regarded also as a ferromagnetic metal. [81] Conduction electrons cannot enter the ferromagnet, which is an insulator. (c) The schematic picture of localized spins (i.e. coherent precession) under microwaves.

Conduction electron spin variables are represented as

$$s^j = \sum_{\eta, \zeta=\uparrow, \downarrow} c_{\eta}^{\dagger} (\sigma^j)_{\eta\zeta} c_{\zeta} / 2 \quad (2.2)$$

$$=: c^{\dagger} \sigma^j c / 2 \quad (2.3)$$

with the 2×2 Pauli matrices; $[\sigma^j, \sigma^k] = 2i\epsilon_{jkl}\sigma^l$, ($j, k, l = x, y, z$). Operators $c^{\dagger}/c$ are creation/annihilation operators for the conduction electrons satisfying the (fermionic) anticommutation relation; $\{c_{\eta}(\mathbf{x}, t), c_{\zeta}^{\dagger}(\mathbf{x}', t)\} = \delta_{\eta, \zeta} \delta(\mathbf{x} - \mathbf{x}')$. We suppose the uniform magnetization and thus, localized spin degrees of freedom can be mapped into magnon [86, 65, 87] ones via the Holstein-Primakoff transformation;

$$S^+(\mathbf{x}, t) = \sqrt{2\tilde{S}} \sqrt{1 - \frac{a^{\dagger}(\mathbf{x}, t)a(\mathbf{x}, t)}{2\tilde{S}}} a(\mathbf{x}, t) \quad (2.4)$$

$$= \sqrt{2\tilde{S}} \left[1 - \frac{a^{\dagger}(\mathbf{x}, t)a(\mathbf{x}, t)}{4\tilde{S}} \right] a(\mathbf{x}, t) + \mathcal{O}(\tilde{S}^{-3/2}) \quad (2.5)$$

$$= (S^-)^{\dagger} \quad (2.6)$$

$$S^z(\mathbf{x}, t) = \tilde{S} - a^{\dagger}(\mathbf{x}, t)a(\mathbf{x}, t), \quad (2.7)$$

with $\tilde{S} := S/a_0^3$. Operators $a^{\dagger}/a$ are magnon creation/annihilation operators satisfying the (bosonic) commutation relation; $[a(\mathbf{x}, t), a^{\dagger}(\mathbf{x}', t)] = \delta(\mathbf{x} - \mathbf{x}')$. Up to the $\mathcal{O}(\tilde{S})$ terms, lo-

calized spins are reduced to free boson degrees of freedoms. Consequently, in the quadratic dispersion (i.e. long wavelength) approximation, the dynamics of localized spins with the applied magnetic field along the quantization axis (i.e. z-axis) B is described by the Hamiltonian $\mathcal{H}_{\text{mag}}$;

$$\mathcal{H}_{\text{mag}} = \int_{\mathbf{x} \in (\text{interface})} d\mathbf{x} a^\dagger(\mathbf{x}, t) \left(-\frac{\nabla^2}{2m} + B \right) a(\mathbf{x}, t), \quad (2.8)$$

where the effective mass of magnons is denoted by m . In addition, Hamiltonian $\mathcal{H}_{\text{ex}}$ can be rewritten as $\mathcal{H}_{\text{ex}} = \mathcal{H}_{\text{ex}}^S + \mathcal{H}'_{\text{ex}}$ with

$$\mathcal{H}_{\text{ex}}^S = -JS \int_{\mathbf{x} \in (\text{interface})} d\mathbf{x} c^\dagger(\mathbf{x}, t) \sigma^z c(\mathbf{x}, t), \quad (2.9)$$

$$\begin{aligned} \mathcal{H}'_{\text{ex}} = & -Ja_0^3 \sqrt{\frac{\tilde{S}}{2}} \int_{\mathbf{x} \in (\text{interface})} d\mathbf{x} \left\{ a^\dagger(\mathbf{x}, t) \left[1 - \frac{a^\dagger(\mathbf{x}, t)a(\mathbf{x}, t)}{4\tilde{S}} \right] c^\dagger(\mathbf{x}, t) \sigma^+ c(\mathbf{x}, t) \right. \\ & \left. + \left[1 - \frac{a^\dagger(\mathbf{x}, t)a(\mathbf{x}, t)}{4\tilde{S}} \right] a(\mathbf{x}, t) c^\dagger(\mathbf{x}, t) \sigma^- c(\mathbf{x}, t) \right\} + \mathcal{O}(\tilde{S}^{-3/2}). \end{aligned} \quad (2.10)$$

Note that we have adsorbed the Bohr magneton and the g -factors into the definition of the magnetic field B and have taken $\hbar = 1$ for convenience.

Here it should be stressed that according to Hamiltonian (2.9), the localized spin S acts as an effective magnetic field along the quantization axis for the conduction electrons;

$$\mathcal{H}_{\text{ex}}^S = -JS \int_{\mathbf{x} \in (\text{interface})} d\mathbf{x} c^\dagger(\mathbf{x}, t) \sigma^z c(\mathbf{x}, t) \quad (2.11)$$

$$= -2JS \int_{\mathbf{x} \in (\text{interface})} d\mathbf{x} s^z(\mathbf{x}, t). \quad (2.12)$$

Therefore, the diagonal part of the conduction electrons is written as

$$\mathcal{H}_{\text{el}} = \int_{\mathbf{x} \in (\text{interface})} d\mathbf{x} c^\dagger(\mathbf{x}, t) \left[-\frac{\nabla^2}{2m_{\text{el}}} - (JS + \frac{B}{2}) \sigma^z \right] c(\mathbf{x}, t), \quad (2.13)$$

with the effective mass of the conduction electron m_{el} .

From the viewpoint of the experiment,[86, 12, 6, 11, 87] microwaves represented by $\Gamma(t)$ under an angular frequency Ω is applied into the system as a driving energy; $\Gamma(t) := \Gamma_0 \cos(\Omega t)$. This applied periodic transverse magnetic field acts as quantum fluctuations[88, 89] and drives the system into a non-equilibrium steady state.[45] Thus we identify the system described by the exchange interaction $\mathcal{H}_{\text{ex}}$ (Hamiltonian (2.1)) under the microwave $\Gamma(t)$ with the quantum spin-pumping system. Note that the applied time-dependent transverse magnetic field couples with the conduction electrons as well as the localized spins;

$$V_{\text{el}}^\Gamma = \frac{\Gamma(t)}{4} \int_{\mathbf{x} \in (\text{interface})} d\mathbf{x} c^\dagger(\mathbf{x}, t) (\sigma^+ + \sigma^-) c(\mathbf{x}, t) \quad (2.14)$$

$$\begin{aligned} V_{\text{mag}}^\Gamma = & \Gamma(t) \sqrt{\frac{\tilde{S}}{2}} \int_{\mathbf{x} \in (\text{interface})} d\mathbf{x} \left\{ \left[1 - \frac{a^\dagger(\mathbf{x}, t)a(\mathbf{x}, t)}{4\tilde{S}} \right] a(\mathbf{x}, t) \right. \\ & \left. + a^\dagger(\mathbf{x}, t) \left[1 - \frac{a^\dagger(\mathbf{x}, t)a(\mathbf{x}, t)}{4\tilde{S}} \right] \right\}. \end{aligned} \quad (2.15)$$

Therefore spin pumping can be generated⁴ also by electron spin resonance (ESR) ($\Omega = 2JS + B$)[27] as well as the standard method via ferromagnetic resonance (FMR) ($\Omega = B$).[86, 12, 6, 11, 87, 84]

Finally, the total Hamiltonian of the quantum spin-pumping system $\mathcal{H}$ (i.e. the spin-pumping system with $\Gamma(t)$) is arranged as

$$\mathcal{H} := \mathcal{H}_{\text{mag}} + \mathcal{H}'_{\text{ex}} + \mathcal{H}_{\text{el}} + V_{\text{el}}^{\Gamma} + V_{\text{mag}}^{\Gamma}. \quad (2.16)$$

We investigate the features of quantum spin pumping described by this Hamiltonian (Hamiltonian (2.16)). This is the main aim of this chapter.

The remain of this chapter is organized as follows; we present our quantum spin pumping theory[27] and stress the point in Sec. 2.2-2.6. The temperature dependence[90] of quantum spin pumping by ESR as well as the one by FMR is revealed in Sec. 2.7,[28] in which we qualitatively understand the behavior. Lastly, the distinction[29] from the preceding spin pumping theory by Tserkovnyak et al.[9] is clarified in Sec. 2.8.

2.2 Dephasing Effects

Let us emphasize that on the construction of spin pumping theories, we actively use dephasing effects;[91, 82] we now treat the weakly connected junction⁵ between the non-magnetic metal and the ferromagnetic insulator. Therefore we are allowed to assume dephasing effects[91] significant enough that coherence between the non-magnetic metal and the ferromagnetic insulator is destroyed and the density matrix for the entire system ρ is always in the form;⁶ $\rho = \rho_{\text{met.}} \otimes \rho_{\text{ins.}}$, where $\rho_{\text{met.}}$ denotes the density matrix for the non-magnetic metal as a function of $\mathcal{H}_{\text{el}}$, $\rho_{\text{met.}} = \rho_{\text{met.}}(\mathcal{H}_{\text{el}})$, and $\rho_{\text{ins.}}$ represents the density matrix for the ferromagnetic insulator as a function of $\mathcal{H}_{\text{mag}}$, $\rho_{\text{ins.}} = \rho_{\text{ins.}}(\mathcal{H}_{\text{mag}})$.

Then we focus on the dynamics in the interface, in which the exchange interaction $\mathcal{H}'_{\text{ex}}$ playing the key role on spin pumping works, and construct the (quantum and thermal) spin pumping theories on the basis of the perturbative analysis in terms of the exchange interaction $\mathcal{H}'_{\text{ex}}$.

Thus, the concept of dephasing effects plays the central role on our construction of the spin pumping theory.

2.3 Breaking of Spin Conservation Law

We have formulated the spin pumping theory on the basis of the spin continuity equation of the conduction electrons;[27, 28, 29, 92]

$$\dot{\rho}_s^z + \nabla \cdot \mathbf{j}_s^z = \mathcal{T}_s^z, \quad (2.17)$$

where the dot denotes the time derivative of the z-component for the spin density defined as $\rho_s^z := c^\dagger \sigma^z c / 2$, and $\mathbf{j}_s$ is the spin current density. Let us emphasize that in sharp contrast to the case of charges, the spin conservation law is broken and it is represented by the spin relaxation torque (SRT)[18] $\mathcal{T}_s$, which appears in the spin continuity equation.

⁴See Sec. 2.6.

⁵The magnitude of the exchange interaction J arising from proximity effects is so small that we are allowed to employ perturbative analysis to treat $\mathcal{H}'_{\text{ex}}$.

⁶This procedure has been employed also by the preceding work by Tserkovnyak et al.

Through the Heisenberg equation of motion, the z-component of the SRT can be explicitly written down as

$$\begin{aligned}\mathcal{T}_s^z &= iJa_0^3\sqrt{\frac{\tilde{S}}{2}}\left\{a^\dagger(\mathbf{x},t)\left[1-\frac{a^\dagger(\mathbf{x},t)a(\mathbf{x},t)}{4\tilde{S}}\right]c^\dagger(\mathbf{x},t)\sigma^+c(\mathbf{x},t)\right. \\ &\quad - \left[1-\frac{a^\dagger(\mathbf{x},t)a(\mathbf{x},t)}{4\tilde{S}}\right]a(\mathbf{x},t)c^\dagger(\mathbf{x},t)\sigma^-c(\mathbf{x},t)\Big\} \\ &\quad + \frac{\Gamma(t)}{4i}\left[c^\dagger(\mathbf{x},t)\sigma^+c(\mathbf{x},t)-c^\dagger(\mathbf{x},t)\sigma^-c(\mathbf{x},t)\right].\end{aligned}\quad (2.18)$$

It is clear that this SRT explicitly describes the microscopic mechanism of spin pumping that localized spins at the interface lose spin angular momentum by emitting magnons and the conduction electrons flip from down to up by absorbing the spin angular momentum, and vice versa (Fig. 4.1). Also from this viewpoint, one can easily see that SRT plays the key role on the spin-flip process arising from the non-equilibrium quantum effects at the interface and the resultant spin pumping (rather than ρ_s^z). On top of this, note that the SRT⁷ has arisen from $\mathcal{H}'_{\text{ex}}$ and V_{el}^Γ , which consist of spin-flip operators ($\sigma^\pm$) and quantum fluctuations ($\Gamma(t)$);

$$\mathcal{T}_s^z = [\rho_s^z, \mathcal{H}'_{\text{ex}} + V_{\text{el}}^\Gamma]/i. \quad (2.19)$$

Let me emphasize that though each spin conservation law of the conduction electrons and magnons is broken, the total spin angular momentum is conserved in the spin-pumping systems. We closely investigate it in the next section.

2.4 Breaking of Magnon Conservation Law

The spin conservation law of localized spins (i.e. magnons) is also broken. Then, the magnon continuity equation for localized spins, [84] which corresponds to the equation of motion for localized spins[92] and describes the dynamics, reads

$$\dot{\rho}_m^z + \nabla \cdot \mathbf{j}_m^z = \mathcal{T}_m^z, \quad (2.20)$$

where $\mathbf{j}_m$ is the magnon current density and ρ_m^z represents the z-component of the magnon density. We have defined the magnon density of the system as $\rho_m^z \equiv \langle a^\dagger a \rangle$, and we call $\mathcal{T}_m^z$ the magnon source term [84], which breaks the magnon conservation law. This term arises also from $\mathcal{H}'_{\text{ex}}$ and V_{mag}^Γ ;

$$\mathcal{T}_m^z = [\rho_m^z, \mathcal{H}'_{\text{ex}} + V_{\text{mag}}^\Gamma]/i \quad (2.21)$$

$$\begin{aligned}&= iJa_0^3\sqrt{\frac{\tilde{S}}{2}}\left\{a^\dagger(\mathbf{x},t)\left[1-\frac{a^\dagger(\mathbf{x},t)a(\mathbf{x},t)}{4\tilde{S}}\right]c^\dagger(\mathbf{x},t)\sigma^+c(\mathbf{x},t)\right. \\ &\quad - \left[1-\frac{a^\dagger(\mathbf{x},t)a(\mathbf{x},t)}{4\tilde{S}}\right]a(\mathbf{x},t)c^\dagger(\mathbf{x},t)\sigma^-c(\mathbf{x},t)\Big\} \\ &\quad + i\Gamma(t)\sqrt{\frac{\tilde{S}}{2}}\left\{\left[1-\frac{a^\dagger(\mathbf{x},t)a(\mathbf{x},t)}{4\tilde{S}}\right]a(\mathbf{x},t)\right. \\ &\quad \left.- a^\dagger(\mathbf{x},t)\left[1-\frac{a^\dagger(\mathbf{x},t)a(\mathbf{x},t)}{4\tilde{S}}\right]\right\}.\end{aligned}\quad (2.22)$$

⁷This SRT might be understood as an anomaly.

Within the same approximation⁸ with the SRT, this magnon source term actually satisfies the relation;⁹

$$\langle \mathcal{T}_m^z \rangle = \langle \mathcal{T}_s^z \rangle. \quad (2.23)$$

Then, the z-component of the spin continuity equation for the total system (i.e. the conduction electrons and the magnons) becomes

$$\langle \dot{\rho}_{\text{total}}^z \rangle + \langle \nabla \cdot \mathbf{j}_{\text{total}}^z \rangle = 0, \quad (2.24)$$

where the density of the total spin angular momentum, ρ_{total}^z , is defined as

$$\rho_{\text{total}}^z \equiv \rho_s^z - \rho_m^z, \quad (2.25)$$

and consequently the z-component of the total spin current density, $\mathbf{j}_{\text{total}}^z$, becomes¹⁰

$$\mathbf{j}_{\text{total}}^z = \mathbf{j}_s^z - \mathbf{j}_m^z. \quad (2.26)$$

The spin continuity equation for the whole system, eq. (2.24), means that though each spin conservation law for electrons and magnons is broken (see eqs. (2.17) and (2.56)), the total spin angular momentum is, of course, conserved.

2.5 Net Pumped Spin Current

One can easily see that the expectation value of the spin density for the conduction electrons reads[93]

$$\langle \rho_s^z \rangle = \sum_{n=0, \pm 1} \langle \rho_s^z(n) \rangle e^{2in\Omega t} + \mathcal{O}(\Gamma^4), \quad (2.27)$$

where $\rho_s^z(n)$ represents the (time-independent) expansion coefficient of each angular frequency mode. Thus the time-average of the time-derivative becomes zero;¹¹

$$\overline{\langle \dot{\rho}_s^z \rangle} = 0. \quad (2.28)$$

As the result, the spin continuity equation for the conduction electrons, eq. (2.17), reads $\overline{\langle \nabla \cdot \mathbf{j}_s^z \rangle} = \overline{\langle \mathcal{T}_s^z \rangle}$.

Here it should be noted that the conduction electrons cannot enter the ferromagnet,[65] which is an insulator (see Fig. 2.1 (b)). Consequently by integrating over the volume of the interface and adopting the Gauss's divergence theorem, the time-average of the net spin current pumped into the adjacent non-magnetic metal (i.e. $\int \mathbf{j}_s^z \cdot d\mathbf{S}_{\text{interface}}$ with the surface of the interface $\mathbf{S}_{\text{interface}}$) can be evaluated as

$$\overline{\left\langle \int \mathbf{j}_s^z \cdot d\mathbf{S}_{\text{interface}} \right\rangle} = \int_{\mathbf{x} \in (\text{interface})} d\mathbf{x} \overline{\langle \mathcal{T}_s^z \rangle}. \quad (2.29)$$

⁸We show it at the next section in detail.

⁹On top of this, let us emphasize that the similar relation does not hold in respect to the spin density; $\langle \rho_s^z \rangle \neq \langle \rho_m^z \rangle$. Also from these relations, one can easily see that SRT plays the key role on spin pumping rather than ρ_s^z .

¹⁰Note the relation; $S^z = \tilde{S} - a^\dagger a$ via the Holstein-Primakoff transformation.

¹¹Note the relation; $\langle \dot{\rho}_s^z \rangle := \langle \partial_t \rho_s^z \rangle = \partial_t \langle \rho_s^z \rangle$.

Experimentally, this pumped spin current can be detected via the inverse spin Hall effect[6] in the non-magnetic metal.

Let us emphasize that the time-average of the net pumped spin current, $\overline{\langle \int \mathbf{j}_s^z \cdot d\mathbf{S}_{\text{interface}} \rangle}$, is expressed only in terms of the SRT (see eq. (2.29));[18] note that calculating $\overline{\langle \dot{\rho}_s^z \rangle}$ has no relation with evaluating the net pumped spin current even when the total spin angular momentum is conserved. That is, the spin density for the conduction electrons, ρ_s^z , is not relevant to quantum spin pumping mediated by magnon. This is one of the main results from our formalism.¹² This definition of the pumped spin current, eq. (2.29), is qualitatively the same with the one employed by the prince Dr. H. Adachi[94, 95, 96, 97, 98, 99] for the study on thermal spin pumping (see Chap. 3).

Thus from now on, we focus on $\mathcal{T}_s^z$ and qualitatively clarify the features of quantum spin pumping mediated by magnons.

2.6 Spin Relaxation Torque

It is also easy to see that the expectation value of the SRT reads[93] (see (Appendix) Sec. 9.3)

$$\langle \mathcal{T}_s^z \rangle = \sum_{n=0,\pm 1} \langle \mathcal{T}_s^z(n) \rangle e^{2in\Omega t} + \mathcal{O}(\Gamma^4), \quad (2.30)$$

where $\mathcal{T}_s^z(n)$ represents the (time-independent) expansion coefficient of each angular frequency mode. Thus the time-average becomes

$$\overline{\langle \mathcal{T}_s^z \rangle} = \langle \mathcal{T}_s^z(n=0) \rangle. \quad (2.31)$$

We here again stress that, in quantum spin-pumping systems, although the time-derivative of spin density of the conduction electrons becomes zero in terms of the time-average $\overline{\langle \dot{\rho}_s^z \rangle} = 0$, the expectation value of SRT takes a non-zero value even when we take the time-average; $\overline{\langle \mathcal{T}_s^z \rangle} = \langle \mathcal{T}_s^z(n=0) \rangle$. This fact explicitly shows that SRT plays the key role on quantum spin pumping.

2.6.1 Perturbative Evaluation

The interface is, in general, a weak coupling regime;[12] the exchange interaction J (see Hamiltonian (2.10)) is supposed to be smaller than the Fermi energy and the exchange interaction among ferromagnets. In addition, a weak microwave Γ is applied. Therefore we are allowed to treat $\mathcal{H}'_{\text{ex}}$, V_{el}^Γ , and V_{mag}^Γ as perturbative terms to evaluate the SRT.

Through the standard procedure of the Schwinger-Keldysh (or contour-ordered) Green's function[100, 31, 32] and the Langreth method (see also (Appendix) Chap. 9),[33, 22, 34, 35] the SRT, $\langle \mathcal{T}_s^z(n=0) \rangle$, is evaluated as follows (see also Fig. 2.2 (b)). The detail of the

¹²In other words, this is the significant distinction from the preceding theory by Tserkovnyak et al. Recall the relation; $\langle \mathcal{T}_m^z \rangle = \langle \mathcal{T}_s^z \rangle$. This means that what is relevant to spin pumping is not magnon density ρ_m^z or $\dot{\mathbf{m}}$ in Sec. 2.8.1 but SRT or magnon source term.

calculation had been shown in (Appendix) Sec. 9.3);

$$\begin{aligned}
\langle \mathcal{T}_s^z(n=0) \rangle &= \frac{J}{2} \left(\frac{\Gamma_0}{2} \right)^2 S \left[1 - \frac{i}{\tilde{S}} \int \frac{d\mathbf{k}'}{(2\pi)^3} \int \frac{d\omega'}{2\pi} G_{\mathbf{k}',\omega'}^< \right] \int \frac{d\mathbf{k}}{(2\pi)^3} \int \frac{d\omega}{2\pi} \\
&\times \left[(G_{0,-\Omega}^a + G_{0,-\Omega}^r) (\mathcal{G}_{\downarrow,\mathbf{k},\omega-\Omega}^t \mathcal{G}_{\uparrow,\mathbf{k},\omega}^t - \mathcal{G}_{\downarrow,\mathbf{k},\omega-\Omega}^< \mathcal{G}_{\uparrow,\mathbf{k},\omega}^>) \right. \\
&+ (G_{0,\Omega}^a + G_{0,\Omega}^r) (\mathcal{G}_{\downarrow,\mathbf{k},\omega+\Omega}^t \mathcal{G}_{\uparrow,\mathbf{k},\omega}^t - \mathcal{G}_{\downarrow,\mathbf{k},\omega+\Omega}^< \mathcal{G}_{\uparrow,\mathbf{k},\omega}^>) \\
&- (G_{0,-\Omega}^r + G_{0,-\Omega}^a) (\mathcal{G}_{\uparrow,\mathbf{k},\omega-\Omega}^t \mathcal{G}_{\downarrow,\mathbf{k},\omega}^t - \mathcal{G}_{\uparrow,\mathbf{k},\omega-\Omega}^< \mathcal{G}_{\downarrow,\mathbf{k},\omega}^>) \\
&- (G_{0,\Omega}^r + G_{0,\Omega}^a) (\mathcal{G}_{\uparrow,\mathbf{k},\omega+\Omega}^t \mathcal{G}_{\downarrow,\mathbf{k},\omega}^t - \mathcal{G}_{\uparrow,\mathbf{k},\omega+\Omega}^< \mathcal{G}_{\downarrow,\mathbf{k},\omega}^>) \left. \right] \\
&+ \mathcal{O}(J^0) + \mathcal{O}(J^2) + \mathcal{O}(\Gamma^4) + \mathcal{O}(JS^{-1}). \tag{2.32}
\end{aligned}$$

The variables $\mathcal{G}^{(r,a,<,>)}/G^{(r,a,<,>)}$ are the fermionic/bosonic time-ordered, retarded, advanced, lesser, and greater Green's functions, respectively.[27] We here have taken the extended time (i.e. the contour variable) defined on the Schwinger-Keldysh closed time path,[35, 32, 22, 33, 34] c , on the forward path $c_{\rightarrow}$; $c = c_{\rightarrow} + c_{\leftarrow}$. Even when the time is located on the backward path $c_{\leftarrow}$, the result of the calculation does not change because each Green's function is not independent;[32, 31, 22, 34] $\mathcal{G}^r - \mathcal{G}^a = \mathcal{G}^> - \mathcal{G}^<$. This relation comes into effect also in the bosonic case.[32, 31, 22, 34]

The SRT (eq. (2.32)) has become proportional to Γ_0^2 ; $\langle \mathcal{T}_s^z(n=0) \rangle \propto \Gamma_0^2$. Thus the SRT (i.e. the net pumped spin current) can be interpreted as the non-linear response to the applied microwave (i.e. quantum fluctuations).¹³ This is one of the main features of our quantum spin pumping theory; our theory well describes the experimental features of quantum spin pumping that quantum fluctuations are essential.

Here it should be stressed that the term $\mathcal{O}(J^2)$ in eq. (2.32), which arises from the exchange interaction in the absence of microwaves driving the system out of equilibrium, becomes zero. This result explicitly justifies our formalism. In other words, we can conclude that the pumped spin current is the production of non-equilibrium effects.

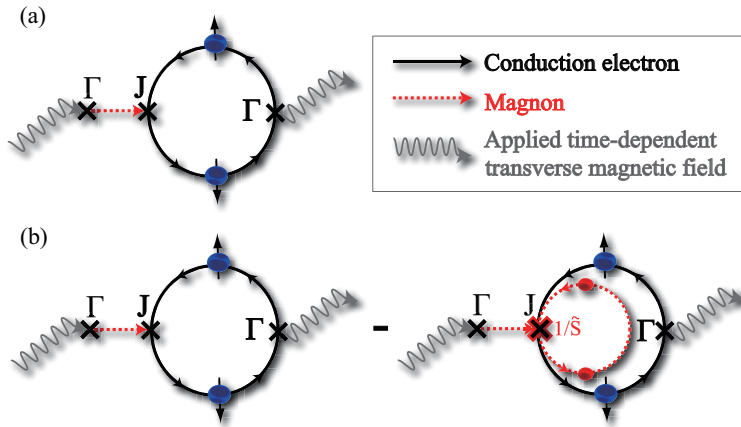


Figure 2.2: (a) A Feynman diagram of the SRT in the large- S limit. (b) A Feynman diagram of the SRT with three-magnon splittings; $\langle \mathcal{T}_s^z(n=0) \rangle$.

¹³Note the relation; $\Gamma(\bar{t}) = 0$ and $(\Gamma(\bar{t}))^2 \neq 0$.

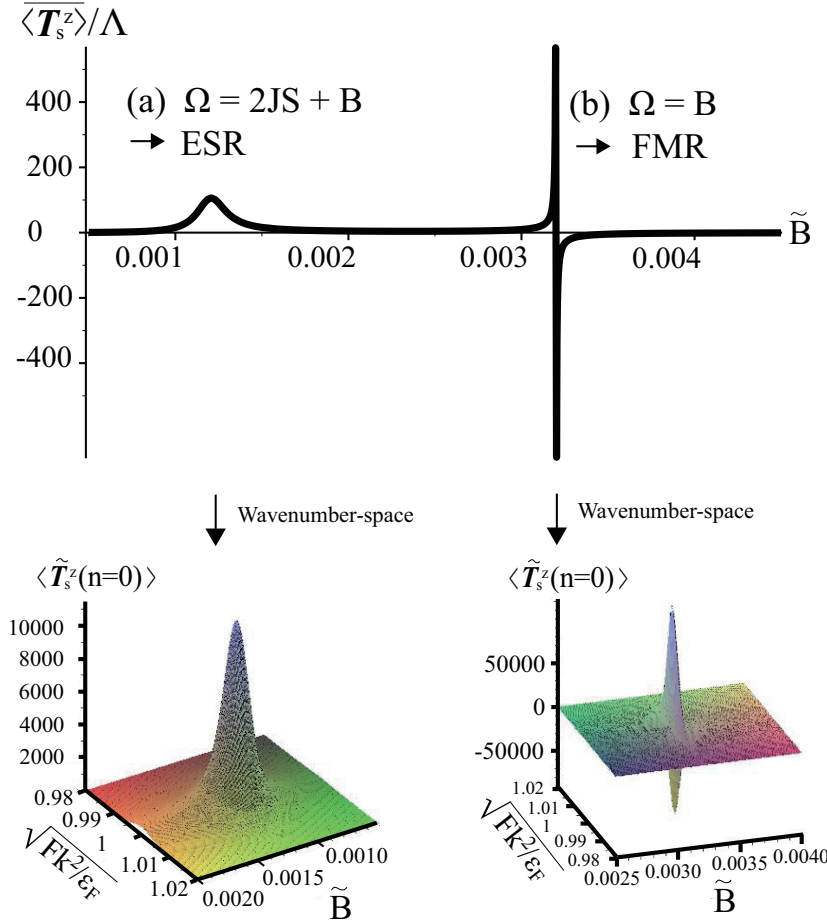


Figure 2.3: Plot of the SRT as a function of $\tilde{B} := B/\epsilon_F$; $\langle \mathcal{T}_s^z \rangle / \Lambda$ and $\langle \tilde{\mathcal{T}}_s^z(n=0) \rangle$. Sharp peaks exist on the point; (a) $\Omega = 2JS + B$ and (b) $\Omega = B$, which has resulted from (a) ESR and (b) FMR.

2.6.2 Quantum Spin Pumping by FMR and ESR

Now, let us introduce the dimensionless SRT, $\langle \mathcal{T}_s^z \rangle / \Lambda$, and the one in the wavenumber-space of the conduction electrons, $\langle \tilde{\mathcal{T}}_s^z(n=0) \rangle$, as follows;

$$\langle \mathcal{T}_s^z \rangle := \Lambda \int_0^\infty \left(\sqrt{\frac{F}{\epsilon_F}} dk \right) \langle \tilde{\mathcal{T}}_s^z(n=0) \rangle \quad (2.33)$$

with

$$\Lambda := \frac{\sqrt{\epsilon_F} S \Gamma_0^2}{4(2\pi\sqrt{F})^3}. \quad (2.34)$$

We have denoted as $F := (2m_{\text{el}})^{-1}$ and the parameters, ϵ_F and k , represent the Fermi energy and the wavenumber of the conduction electrons.

According to Hamiltonian (2.13) and (2.8), the effective magnetic field along the quantization axis for the conduction electrons, $s^z = c^\dagger \sigma^z c/2$, reads $2JS + B$ and that for magnons does B at the interface. On top of this, the applied time-dependent transverse magnetic field $\Gamma(t) = \Gamma_0 \cos(\Omega t)$ (i.e. quantum fluctuations) affects the conduction electrons as well as the magnons at the interface; Fig. 2.1. Thus, the SRT (eq. (2.32)) becomes a non-zero value around the points $\Omega = 2JS + B$ and $\Omega = B$, which are generated by ESR and FMR,

respectively (Fig. 2.3). That is (eq. (2.29)), spin pumping is generated by ESR on

$$\Omega = 2JS + B \text{ (ESR)} \quad (2.35)$$

as well as FMR on

$$\Omega = B \text{ (FMR)}. \quad (2.36)$$

This is our theoretical proposal or prediction; spin pumping is generated also by ESR as well as the standard method via FMR.[86, 12, 6, 11, 87, 84]

2.6.3 Discussion

Here let us remark that we can find the indication of spin pumping by ESR in the experimental data[101] of Dr. Kazuya Ando (Tohoku University); in addition to the peak by FMR, there is an another peak in the measured value of the voltage resulting from ISHE V_{ISHE} (Fig. 1.2). Although they have understood that the peak arises from three-magnon splittings (i.e. non-linear interactions),[87] let us mention that we have already included the three-magnon splittings into formalism and have found that the contribution originating from it is so small even at room temperature and consequently, the position of the peak cannot be changed by three-magnon splittings. Therefore we expect that another peak has been produced not by three-magnon splittings but by ESR.

Lastly it should be remarked that strictly speaking, the shape of our graph of SRT does not exactly agree with that of V_{ISHE} in their experiment. We suspect, however, that it is due to the difference of the quantity we focus on; we now concentrate on the microscopic mechanism (i.e. origin) of the generation of spin currents, in which SRT plays the key role. On the other hand, V_{ISHE} represents the measured charge current which has been converted from the pumped spin currents by ISHE (Fig. 1.2).[6] Therefore we hopefully expect that if we directly evaluate V_{ISHE} by using our SRT with including the effects of ISHE, we can reproduce their experimental data. This is an arguent and important issue, which will be addressed in the near future. Let us, however, stress that our present result actually has captured the essence of their experiment and qualitatively agree with it.

2.7 Temperature Dependence

The SRT at finite temperature, $\overline{\langle \mathcal{T}_s^z \rangle} |_{T=0} = \langle \mathcal{T}_s^z(n=0) \rangle$, can be expressed also as follows (see eq. (2.32));

$$\overline{\langle \mathcal{T}_s^z \rangle} |_T = \overline{\langle \mathcal{T}_s^z \rangle} |_{T=300[\text{K}]} \times \eta^{\text{ratio}}(T) \times S_{\text{eff}}^{\text{ratio}}(T) \quad (2.37)$$

with

$$\eta^{\text{ratio}}(T) := \frac{\mathcal{T}_s^{z\text{-ratio}}(T)}{S_{\text{eff}}^{\text{ratio}}(T)}, \quad (2.38)$$

$$\mathcal{T}_s^{z\text{-ratio}}(T) := \frac{\overline{\langle \mathcal{T}_s^z \rangle} |_T}{\overline{\langle \mathcal{T}_s^z \rangle} |_{T=300[\text{K}]}} \quad (2.39)$$

and

$$S_{\text{eff}}^{\text{ratio}}(T) := \frac{\widetilde{S}_{\text{eff}}(T)}{\widetilde{S}_{\text{eff}}(T = 300[\text{K}])}. \quad (2.40)$$

By using eq. (2.38), the dimensionless SRT at finite temperature can be rewritten as

$$\mathcal{T}_s^{z\text{-ratio}}(T) = \eta^{\text{ratio}}(T) \times S_{\text{eff}}^{\text{ratio}}(T). \quad (2.41)$$

Consequently, it is clear that the variable $\eta^{\text{ratio}}(T)$ represents the contribution of the conduction electrons to spin pumping (see also eq. (2.32)) and thus, it can be interpreted to correspond to the mixing conductance in the spin pumping theory proposed by Tserkovnyak et al (see Sec. 2.8.1).[9, 14, 19]

We have plotted $\eta^{\text{ratio}}(T)$ on ESR point (i.e. $\Omega = 2JS + B$) and one can easily see from Fig. 2.4 that $\eta^{\text{ratio}}(T)$ is little influenced by temperature;

$$\eta^{\text{ratio}}(T) \sim 1. \quad (2.42)$$

This temperature dependence of $\eta^{\text{ratio}}(T)$ qualitatively shows the good agreement with the experimental result by Czeschka et al.[90] (i.e. the measurement of the mixing conductance under the standard spin pumping by FMR[86, 12, 6, 11, 87]) that the mixing conductance is little influenced by temperature. That is, this temperature dependence (eq. (2.42)) is the common properties of spin pumping by FMR and ESR.¹⁴

We consider that this temperature dependence arises from the fact that the Fermi temperature is far higher ($\sim 10^4$ K) than the Curie temperature ($\sim 10^2$ or 10^3 K) and room temperature is so low compared with the Fermi temperature.

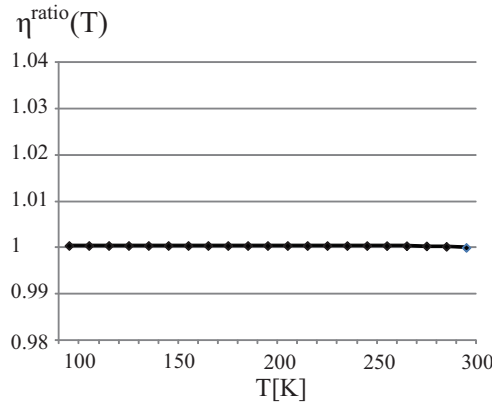


Figure 2.4: Plot of $\eta^{\text{ratio}}(T)$ as a function of temperature T on the ESR point ($\Omega = 2JS + B$). The function $\eta^{\text{ratio}}(T)$ is little influenced by temperature; $\eta^{\text{ratio}}(T) \sim 1$.

2.8 Distinction from Classical Theory

Lastly, let us discuss the distinction[29, 27, 28] between our quantum spin pumping theory and the semi-classical one proposed by Tserkovnyak et al.[9]

It should be noted that, as has been mentioned in their article[9] (see §VIII. SUMMARY AND OUTLOOK in their article[9]), they have phenomenologically treated the spin-flip scattering processes, which we have regarded as the most important processes for spin pumping. Nevertheless, now their spin pumping theory has been widely used for interpreting vast experimental results,[12, 87, 13, 11] in particular by experimentalists. Thus, it would be significant to clarify the difference between our quantum spin pumping theory and the one proposed by Tserkovnyak et al.

¹⁴By using our present quantum spin pumping theory, we have actually reproduced this temperature dependence also in the case of FMR.

2.8.1 Classical Spin Pumping Theory

Let us again explain the preceding spin pumping theory by Tserkovnyak et al. According to the classical[9, 10, 13] (phenomenological) spin pumping theory by Tserkovnyak et al.,¹⁵ the pumped spin current $\mathbf{I}_{\text{s-pump}}$ reads, in terms of their notations,[19, 9] as

$$\mathbf{I}_{\text{s-pump}} = G_{\perp}^{(R)} \mathbf{m} \times \dot{\mathbf{m}} + G_{\perp}^{(I)} \dot{\mathbf{m}}, \quad (2.43)$$

where the dot denotes the time derivative. We have taken $e = 1$, and $\mathbf{m}(\mathbf{x}, t)$ denotes a unit vector along the magnetization direction; they have treated $\mathbf{m}(\mathbf{x}, t)$ as classical variables. The variable $G_{\perp} = G_{\perp}^{(R)} + iG_{\perp}^{(I)}$ is the complex-valued mixing conductance that depends on the material.[20, 21]

Then, they have assumed that the magnetization dynamics of ferromagnets can be described by LLG equation;[22, 23, 24]

$$\dot{\mathbf{m}} = \gamma \mathbf{H}_{\text{eff}} \times \mathbf{m} + \alpha \mathbf{m} \times \dot{\mathbf{m}}, \quad (2.44)$$

where γ is the gyro-magnetic ratio and α is the Gilbert damping constant that determines the magnetization dissipation rate.[25, 9] Here it should be emphasized that though this Gilbert damping constant,¹⁶ α , was phenomenologically introduced,[26] it can be derived microscopically by considering a whole system including spin relaxation;[102] thus the effect of the exchange coupling to the conduction electrons should be considered to have already been included into this Gilbert damping term.

The effective magnetic field is set as

$$\mathbf{H}_{\text{eff}} = (\Gamma(t), 0, B), \quad (2.45)$$

where $\Gamma(t)$ represents a microwave. Then, the LLG eq., eq. (2.44), becomes

$$\begin{pmatrix} \dot{m}^x \\ \dot{m}^y \\ \dot{m}^z \end{pmatrix} = \gamma \begin{pmatrix} -Bm^y \\ Bm^x - \Gamma m^z \\ \Gamma m^y \end{pmatrix} + \alpha \begin{pmatrix} m^y \dot{m}^z - \dot{m}^y m^z \\ m^z \dot{m}^x - \dot{m}^z m^x \\ m^x \dot{m}^y - \dot{m}^x m^y \end{pmatrix}. \quad (2.46)$$

2.8.2 Pumped Spin Current Based on Classical Theory

Eq. (2.46) is substituted into $I_{\text{s-pump}}^z$, eq. (2.43); we include the contribution arising from the Gilbert damping term, which depends on the materials, up to $\mathcal{O}(\alpha)$; $\alpha \sim 10^{-3}, 10^{-2}$ for $\text{Ni}_{81}\text{Fe}_{19}$ (metal),[12] and $\alpha \sim 10^{-5}$ for $\text{Y}_3\text{Fe}_5\text{O}_{12}$ (insulator),[65] as examples. Their theory is applicable to both ferromagnetic metals and insulators.[14]

Consequently, the z-component of the pumped spin current reads

$$\begin{aligned} I_{\text{s-pump}}^z &= G_{\perp}^{(R)} \left\{ \gamma B[(m^x)^2 + (m^y)^2] - \gamma \Gamma m^x m^z \right. \\ &\quad \left. - \alpha \gamma \Gamma [m^x m^y m^x + (m^y)^3 + (m^z)^2 m^y] \right\} \\ &\quad + G_{\perp}^{(I)} \left\{ \alpha \gamma \{ B[(m^x)^2 + (m^y)^2] - \Gamma m^x m^z \} + \gamma \Gamma m^y \right\} + \mathcal{O}(\alpha^2). \end{aligned} \quad (2.47)$$

$$\xrightarrow{\Gamma \rightarrow 0} [G_{\perp}^{(R)} + \alpha G_{\perp}^{(I)}] \gamma B[(m^x)^2 + (m^y)^2]. \quad (2.48)$$

$$\xrightarrow{G_{\perp}^{(I)} \rightarrow 0} G_{\perp}^{(R)} \gamma B[(m^x)^2 + (m^y)^2]. \quad (2.49)$$

¹⁵Regarding the detail of the spin pumping theory proposed by Tserkovnyak et al.,[9] it will be very useful to read the extremely sophisticated master thesis (2007) by T. Taniguchi submitted to Tohoku University.[10]

¹⁶They have introduced the isotropic damping constant α (i.e. the value of α does not depend on the spin component S^x , S^y , and S^z) in order to guarantee that spin length is conserved. One can easily shown the relation, $dS_i^2/(dt) = 0$, by using the LLG equation shown by eq. (2.44). Otherwise (i.e. in anisotropic cases), the spin length cannot be conserved in general.

At finite temperature, the magnetization is thermally activated; $\dot{\mathbf{m}} \neq 0$.^[103] Then the time derivative of the z-component means

$$\dot{m}^z = \gamma \Gamma m^y + \alpha \gamma \{B[(m^x)^2 + (m^y)^2] - \Gamma m^x m^z\} + \mathcal{O}(\alpha^2). \quad (2.50)$$

$$\xrightarrow{\Gamma \rightarrow 0} \alpha \gamma B[(m^x)^2 + (m^y)^2]. \quad (2.51)$$

$$\neq 0. \quad (2.52)$$

2.8.3 Distinction; microwaves

Eqs. (2.48), (2.49) and (2.52) mean that, within the framework by Tserkovnyak et al. with the LLG equation, they may gain spin currents at finite temperature if only the magnetic field along the z-axis, B , is applied;

$$I_{\text{s-pump}}^z \xrightarrow{\Gamma \rightarrow 0 (B \neq 0)} 0. \quad (2.53)$$

That is, the spin pumping theory by Tserkovnyak et al.[9, 19, 14] with the LLG equation concludes that spin currents may be pumped at finite temperature without microwaves.

On the other hand, our approach based on the Schwinger-Keldysh formalism gives different result (see eqs. (2.31) and (2.32));

$$\mathcal{T}_s^z \xrightarrow{\Gamma \rightarrow 0} 0. \quad (2.54)$$

That is, our quantum spin pumping theory means that spin currents mediated by magnons cannot be pumped without quantum fluctuations (i.e. microwaves, $\Gamma(t)$).¹⁷ This our result fully agrees with the experiment.[12, 87, 13, 11]

In conclusion, eqs. (2.54) and (2.53) show the significant distinction between our quantum spin pumping theory and the one proposed by Tserkovnyak et al.

2.8.4 Discussion; Quantization of Classical Theory

We have discussed this issue with Dr. Tserkovnyak at University of Basel during my stay on July 9 (2013). They have argued that the distinction arises from the treatment of localized spins in ferromagnetic insulators (i.e. quantum effects), classical spins or magnons (i.e. quantized spin waves). Thus, they have expected that our present quantum spin pumping theory corresponds to the quantum version of their classical spin pumping theory. On the other hand, they have intensively stressed that localized spins in ferromagnetic insulators precess coherently (i.e. uniformly) due to applied microwaves and hence, their treatment is never incorrect and actually their theory agrees with experiments.

Lastly, let us remark on our opinion. We have fully agreed with them on the point that our spin pumping theory, in which we have focused on the generation of pumped spin currents under spin-flip processes, corresponds to the quantum version of their semi-classical one; by using the Schwinger-Keldysh formalism, which can treat the dynamics out of equilibrium beyond phenomenology, we have microscopically described the non-equilibrium spin-flip processes via magnons at the interface in order to take quantum effects into account, in which ferromagnetic localized spin lose their spin angular momentum by emitting magnons and the conduction electrons flip from down to up by absorbing the momentum. As has been explicitly explained in their preceding review article[9] (see VII. SUMMARY AND OUTLOOK,

¹⁷Quantum fluctuations are essential to spin pumping mediated by magnons as well as the exchange interaction between the conduction electrons and the ferromagnets.

pp. 43-44),¹⁸ they have viewed localized spins as classical moving magnetization vectors and have phenomenologically treated spin-flip scattering processes, which we have regarded as the most important processes in spin pumping. Therefore it can be concluded that we have remarkably progressed their spin pumping theory and our microscopic quantum spin pumping theory will be useful to the analysis for experiments as well as the theoretical prediction of new phenomena. On the quantization of the theory, we have appropriately included quantum effects such as spin-flip processes, magnons (i.e. quantized spin-waves), and microwaves, which affect the magnons and the conduction electrons as quantum fluctuations. The resultant distinction between our quantum spin pumping theory and their semi-classical one has already been addressed in Sec. 2.8.

It might be useful to mention that originally we have tackled this issue in order to deeply understand their preceding spin pumping theory, which has now become the standard theory in the context of spintronics, in particular we would like to have understood the following issues; in their formalism, the pumped spin currents carried by the conduction electrons (i.e. the fermions) is represented by using the degrees of freedoms of only localized spins¹⁹ (i.e. something like Bosons or classical magnetization vectors $\mathbf{m}(t)$) with employing the mixing conductance, where the relaxation torque term or Gilbert-damping term (i.e. α -term) plays the key role. It should be, however, noted that the Gilbert-damping term (i.e. α -term) was originally phenomenologically introduced in order to take dissipation effects (i.e. spin frictions) into account and we here would like to stress that dissipations arise from also the interaction with the conduction electrons as well as phonons. Therefore we suspect that the spin angular momentum localized spins have lost will be supplied to phonons as well as the conduction electrons. That is, even when one employs the α -term to represent the interchange of spin-angular momentum, one should not assume that all of the spin angular momentum localized spins have lost has been exchanged with the conduction electrons; the amount of spin angular momentum localized spins have lost is not equal to that the conduction electrons have obtained because phonons also have gained.

Then, in order to go beyond their semi-classical theory, we have employed the Schwinger-Keldysh formalism and have microscopically described the spin-flip processes (i.e. the interchange of spin angular momentum between the conduction electrons and the magnons) beyond phenomenology and have found that SRT plays the key role. In addition, we have noticed that the dissipation effects (i.e. the interactions with infinite phonons) can be taken into account by using the famous Caldeira-Leggett model.[63, 64]²⁰ These are the strong point of our approach.

On the other hand, we have shared the awareness of the issues with them; to microscopically include the proximity effects at the interface, which have been phenomenologically included into also our work beyond phenomenology has been left and it is an important future task. In addition, we should tackle how the spin current pumped into the adjacent non-magnetic metal flow and the microscopic description of the convert processes from pumped spin currents to charge currents under the spin-orbit interaction.[104]

¹⁸Let us again cite from their review article.[9] [pp.43] “Except for the phenomenological treatment of spin-flip scattering processes, the theory is derived from first principle. The main subject in this context is the concept of spin pumping due to moving magnetization vectors. The magnetization dynamics is affected by the spin-transfer torque.” [pp.44] “Theoretical challenges for the future include a proper treatment of spin-orbit interactions, the coupling of magnetization degrees of freedoms to the lattice, and effects beyond semi-classical regime.”

¹⁹That is, the physical quantity carried by fermions is represented by using the bosonic degrees of freedoms. This looks extremely strange to us. In the fully classical limit, this treatment might be reasonable.

²⁰The Brownian particles can be mapped into magnons (see (Appendix) Chap. 11).

2.8.5 Correspondence to the Preceding Classical Theory

The preceding classical theory is based on the LLG equation, which describes classical dynamics of the ferromagnetic spins;

$$\dot{\mathbf{m}} = \gamma \mathbf{H}_{\text{eff}} \times \mathbf{m} + \alpha \mathbf{m} \times \dot{\mathbf{m}}, \quad (2.55)$$

where the α -term (i.e. the Gilbert damping term)[26] originally introduced phenomenologically to take the dissipation effects into account plays the key role.

On the other hand, our microscopic theory is based on the spin continuity equation. Then, we have microscopically found that the SRT term $\mathcal{T}_s^z$, which breaks the spin conservation law of the conduction electrons in the interface (Fig. 2.5), is the origin of the pumped spin current; all information about the spin-flip processes mediated by magnons, the applied microwave, and the three-magnon splitting is included into the SRT.

In the similar way, on the basis of the magnon continuity equation, we can microscopically analyze the quantum dynamics of the ferromagnetic spins under the applied microwave at the interface;²¹

$$\dot{\rho}_m^z + \nabla \cdot \mathbf{j}_m^z = \mathcal{T}_m^z, \quad (2.56)$$

where $\mathbf{j}_m$ is the magnon current density and ρ_m^z represents the z-component of the magnon density; $\rho_m^z := a^\dagger a = S - S^z$. It is clear, as in the case of the spin continuity equation of the conduction electrons, that the spin conservation law of the ferromagnetic spins (i.e. the magnon conservation law) at the interface is broken due to the exchange interaction between the conduction electrons and the magnons at the interface, which is represented by the magnon source term[84] $\mathcal{T}_m^z$. Then, the magnon source term $\mathcal{T}_m^z$ will correspond to the α -term (i.e. the Gilbert damping term) in the sense that each (quantum or classical) quantity represents the breaking of the spin conservation law of the ferromagnetic spins and shows the the friction (i.e. the dissipation effect);

$$\mathcal{T}_m^z \leftrightarrow \alpha \mathbf{m} \times \dot{\mathbf{m}}. \quad (2.57)$$

This is the main point of the correspondence between our quantum theory and the classical one.[105]

Here, we stress that in real materials (i.e. the spin pumping systems), there are variety kinds of microscopic causes for the dissipations such as magnetic or non-magnetic impurities, phonons, conduction electrons, and so on. Therefore, even when the ferromagnetic spins lose some spin angular momentum, we cannot assume that all the lost spin angular momentum is given to the conduction electrons and converted into the pumped spin current.[108] That is, in the real materials, the amount of the spin angular momentum that the ferromagnetic spins have lost does not equal to the one which the conduction electrons have gained to be converted into the spin current. That is why, the microscopic analysis of the interaction between the conduction electrons and the magnons at the interface, where the spin angular momentum is exchanged between them, has been an urgent issue to develop the theory.

Our present study has achieved the task; we have microscopically analyzed the non-equilibrium spin-flip processes arising from the applied microwave and the exchange interaction at the interface (Fig. 2.5) and have explicitly evaluated the SRT and the magnon source term. Then, we have found the following relation;

$$\langle \mathcal{T}_m^z \rangle = \langle \mathcal{T}_s^z \rangle = \mathcal{O}(J\Gamma_0^2). \quad (2.58)$$

²¹The magnon continuity equation will be regarded as the counterpart to the LLG equation in the preceding classical theory.[105]

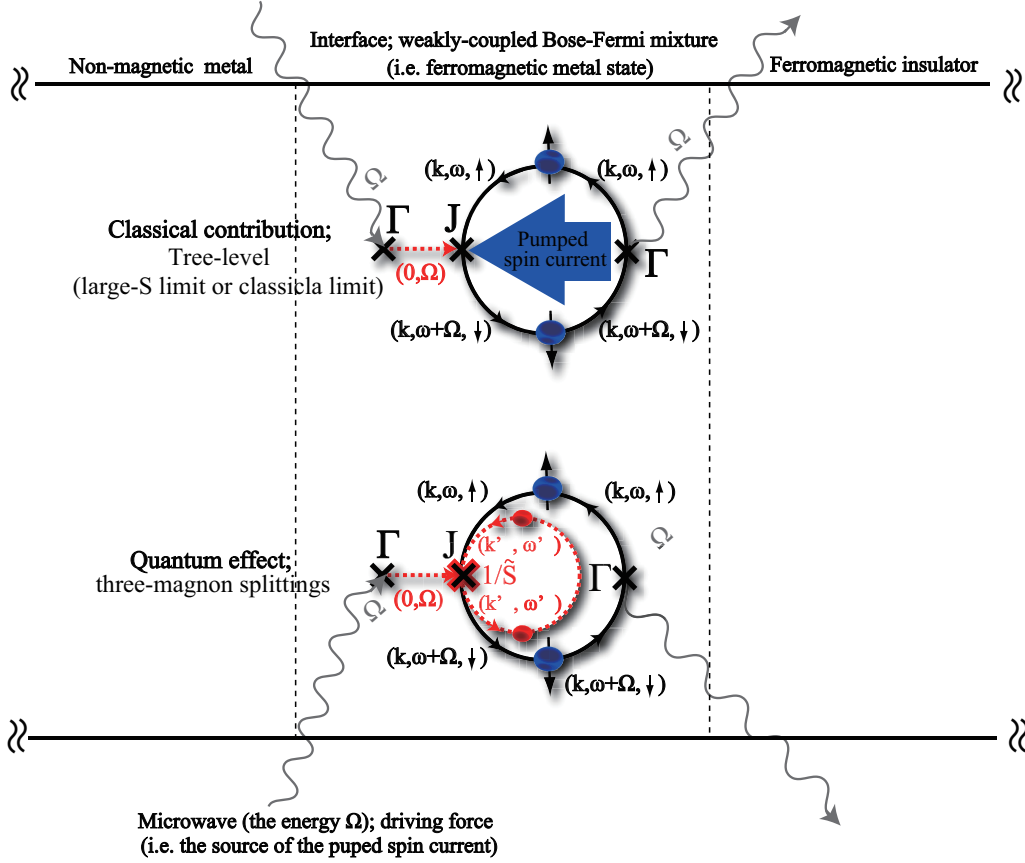


Figure 2.5: (Color online) The schematic picture of the microscopic mechanism of the pumped spin current in the interface, where the ferromagnetic metal state is realized due to the proximity effect.[81] That is, the magnons and the conduction electrons arising from proximity effect coexist to weakly interact via the exchange interaction and hence, the interface can be regarded as a weakly coupled Bose-Fermi mixture.[106] That is why, by applying the microwave to also the interface, spin pumping is generated also by ESR (of the conduction electrons in the interface)[107, 106] as well as the usual FMR. The applied microwave gives the energy $\hbar\Omega$ to the interface, which deserves the (energy) source of the pumped spin current.[106] That is, the energy supplied by the applied microwave is converted into the pumped spin current.

This clearly shows that all the spin angular momentum emitted by the ferromagnetic spins (i.e. magnons) $\langle \mathcal{T}_s^z \rangle$ is converted into the pumped spin current (i.e. the SRT) $\langle \mathcal{T}_m^z \rangle$ due to the exchange interaction J under the applied microwave Γ_0 , which works as the driving force (i.e. the (energy) source) of the pumped spin current,[106] in our microscopic spin pumping theory.

This is the story of the correspondence between our quantum theory and the preceding classical one. We stress that our theoretical prediction²² that spin pumping is generated also by ESR at the interface[107] is the fruits of our microscopic theory, which cannot be imagined within the preceding phenomenological classical theory.²³

²²We believe that the indication has already been gained in the experiments.

²³In addition to our microscopic approach based on the Schwinger-Keldysh formalism, approaches[105] to generalize the LLG equation so as to explicitly include the effect of the (s-d type) exchange interaction with the conduction electrons have been also achieved in the following papers [24, 23].

Here, let us remark that the spin-flip process at the interface, which play the key role on spin pumping, is essentially the quantum-mechanical dynamics. Therefore, we believe that quantum-mechanical procedure is the best way to investigate spin pumping. In addition, from the viewpoint of the quantum-mechanics, in order to generate the spin-flip of the conduction electrons, it costs the spin angular momentum which amounts to $\hbar/2 - (-\hbar/2) = \hbar$; otherwise, the spin-flip does not occur.[108] That is, although the classical spins can take a continuous value, the quantum spins cannot. This fact also strongly shows the necessity of the quantum-mechanical treatment of the ferromagnetic spins as well as the conduction electrons.²⁴

In addition, we stress that our microscopic theory explicitly include the non-equilibrium spin-flip dynamics under the applied microwave, which was phenomenologically discussed in the preceding theory. Therefore, even in the large- S limit[108] (i.e. the classical limit, see also Fig. 2.5), our microscopic theory shows the good agreement with the experiment and corresponds to the formalism which modifies the defect of the preceding theory based on the LLG equation.

On the other hand, on the basis of our quantum theory, to derive an effective theory which has the similar form with the LLG equation[26, 24, 23] remains an important issue to be tackled.[105]

2.9 Conclusion

We have reformulated the spin pumping theory on the basis of the Schwinger-Keldysh formalism, which can explicitly treat the system out of equilibrium at finite temperature, and have constructed the microscopic quantum spin pumping theory. Beyond the preceding classical phenomenological theory, we have microscopically described the non-equilibrium spin-flip process arising from quantum effects and the exchange interaction between the conduction electrons and the magnons at the interface, which is the key (essential) to spin pumping; localized spins lose spin angular momentum by emitting magnons and the conduction electrons flip from down to up by absorbing the momentum, and vice versa. Consequently, we have found that spin pumping is characterized not by spin density but by SRT breaking the spin conservation law of the conduction electrons. As the result, the net pumped spin current is represented in terms of only SRT and it is recognized as the non-linear response to applied microwaves working as quantum fluctuations. This point cannot be captured by the preceding classical theory, where the spin-flip process has been phenomenologically included. Then, we consider that this point we have newly revealed is the fruit of our microscopic description of the essential process on the basis of the Schwinger-Keldysh formalism.

We have also clarified the distinction from the preceding theory based on semi-classical phenomenology and have confirmed that our theory agrees with (well describes) the experiments. Then, it can be concluded that we have modified the failings of the preceding classical theory by the microscopic description based on the Schwinger-Keldysh formalism and hence, our formalism will corresponds to the quantum version of the preceding theory. In addition, we theoretically predict that spin pumping is generated also by ESR as well as the usual method via FMR. This point is the milestone of the validity of our theory.

²⁴Consider the situation where the ferromagnetic classical spin emits the spin angular momentum which is a continuous value, which does not amount to $\hbar$. In that case, although the ferromagnetic spin classically emits the spin angular momentum, from the viewpoint of quantum-mechanics, it does not generate the spin-flip because it does not amount to $\hbar$.

Chapter 3

Thermal Spin Pumping Theory

We construct the theory on spin pumping induced by inhomogeneous thermal fluctuations. By employing the Schwinger-Keldysh formalism, we have microscopically described the spin-flip process at the interface and have clarified the spin-flip condition between the conduction electrons and the magnons. This condition leads to the novel features of thermal spin pumping.

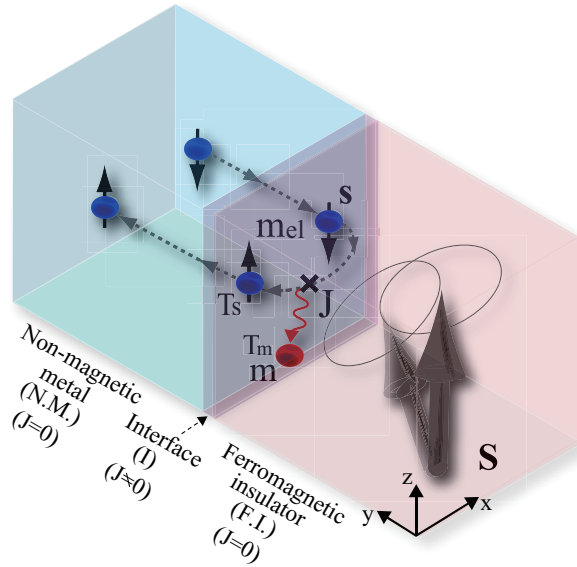


Figure 3.1: A schematic picture of thermal spin pumping mediated by magnons where, in sharp contrast to quantum spin-pumping systems, microwaves are not applied. Spheres represent magnons and those with arrows are the conduction electrons. When the effective temperature of magnons (T_m) is lower than that of the conduction electrons (T_s), localized spins lose spin angular momentum by emitting a magnon and the conduction electrons flip from down to up by absorbing the momentum, and vice versa.

3.1 Thermal Spin Relaxation Torque

We also construct the thermal spin pumping theory microscopically on the basis of the spin continuity equation.

The thermal spin relaxation torque (TSRT), [94, 109, 110] $\mathcal{T}_s^z$, is defined as the term which breaks the spin conservation law of the conduction electrons;

$$\dot{\rho}_s^z + \nabla \cdot \mathbf{j}_s^z = \mathcal{T}_s^z. \quad (3.1)$$

Through the Heisenberg equation of motion, the z-component of the TSRT is defined as¹

$$\mathcal{T}_s^z = iJa_0^3 \sqrt{\frac{\tilde{S}}{2}} [a^\dagger(\mathbf{x}, t)c^\dagger(\mathbf{x}, t)\sigma^+c(\mathbf{x}, t) - a(\mathbf{x}, t)c^\dagger(\mathbf{x}, t)\sigma^-c(\mathbf{x}, t)]. \quad (3.2)$$

This term arises from $\mathcal{H}'_{\text{ex}}$, which consist of electron spin-flip operators;

$$\mathcal{T}_s^z = [\rho_s^z, \mathcal{H}'_{\text{ex}}]/i. \quad (3.3)$$

Thus, eq. (9.103) explicitly shows that the TSRT ($\mathcal{T}_s^z > 0$) can be understood as the number density of the conduction electrons which flip from down to up per a unit of time, [111] and vice versa.²

3.2 Net Pumped Thermal Spin Current

In this section, we clarify the relation between the TSRT and the net pumped spin current.

As discussed in the last section, the spin conservation law of the conduction electrons is broken due to the interaction $\mathcal{H}'_{\text{ex}}$;

$$\dot{\rho}_s^z + \nabla \cdot \mathbf{j}_s^z = \mathcal{T}_s^z. \quad (3.4)$$

Thus one cannot simply view the time derivative of the spin density for the conduction electrons, $\dot{\rho}_s^z$, as the spin current density. A more detailed analysis is required.

As an analysis, we focus on the fact that in terms of Planck's constant (we here partially recover $\hbar$), the time derivative of the spin density and the TSRT satisfy the relation; [114, 115]

$$\frac{\langle \dot{\rho}_s^z \rangle}{\langle \mathcal{T}_s^z \rangle} = \mathcal{O}(\hbar). \quad (3.5)$$

Therefore $\dot{\rho}_s^z$ is negligible in comparison with $\mathcal{T}_s^z$ at the semi-classical regime. As the result, the spin continuity equation, eq. (3.4), is reduced to the form;

$$\mathcal{T}_s^z = \nabla \cdot \mathbf{j}_s^z. \quad (3.6)$$

Then by integrating over the interface, we can evaluate the net pumped thermal spin current, $\int \mathbf{j}_s^z \cdot d\mathbf{S}_{\text{interface}}$;

$$\int_{\mathbf{x} \in (\text{interface})} d\mathbf{x} \mathcal{T}_s^z = \int_{\mathbf{x} \in (\text{interface})} d\mathbf{x} \nabla \cdot \mathbf{j}_s^z \quad (3.7)$$

$$= \int \mathbf{j}_s^z \cdot d\mathbf{S}_{\text{interface}}. \quad (3.8)$$

In addition, the conduction electrons cannot enter the ferromagnet, which is an insulator. [65] Thus the net spin current pumped into the non-magnetic metal can be calculated by integrating the TSRT over the interface, eq. (3.8). This definition of the pumped spin current has been employed by the prince Dr. H. Adachi. [94, 95, 96, 97, 98, 99]

From now on, we focus on $\mathcal{T}_s^z$ and qualitatively clarify the behavior of the thermal spin pumping effect mediated by magnons at room temperature in the semi-classical regime.

¹We here have employed the Holstein-Primakoff transformation, which has been approximated as follows; $S^+ = S^x + iS^y = \sqrt{2\tilde{S}a} + \mathcal{O}((\tilde{S})^{-1/2}) = (S^-)^\dagger$ and $S^z = \tilde{S} - a^\dagger a$.

²In addition, the TSRT operates the coherent magnon state (see also Sec. 3.6). [112, 113]

3.2.1 The Spin Continuity Equation of the Whole System

It will be useful to point out that the spin conservation law of localized spins (i.e. magnons) is also broken.

The magnon continuity equation of localized spins[84] reads

$$\dot{\rho}_m^z + \nabla \cdot \mathbf{j}_m^z = \mathcal{T}_m^z, \quad (3.9)$$

where $\mathbf{j}_m$ is the magnon current density, and $\rho_m^z := a^\dagger a$ represents the z-component of the magnon density. In addition, the magnon source term[84] breaking the magnon conservation law $\mathcal{T}_m^z$ arises also from $\mathcal{H}'_{\text{ex}}$:

$$\mathcal{T}_m^z = [\rho_m^z, \mathcal{H}'_{\text{ex}}]/i. \quad (3.10)$$

Through the Heisenberg equation of motion, we obtain the relation;

$$\mathcal{T}_m^z = \mathcal{T}_s^z. \quad (3.11)$$

Then the z-component of the spin continuity equation for the total system (i.e. the conduction electrons and the magnons) becomes

$$\dot{\rho}_{\text{total}}^z + \nabla \cdot \mathbf{j}_{\text{total}}^z = 0, \quad (3.12)$$

where the density of the total spin angular momentum, ρ_{total}^z , is defined as

$$\rho_{\text{total}}^z \equiv \rho_s^z - \rho_m^z, \quad (3.13)$$

and consequently the z-component of the total spin current density, $\mathbf{j}_{\text{total}}^z$, becomes³

$$\mathbf{j}_{\text{total}}^z = \mathbf{j}_s^z - \mathbf{j}_m^z. \quad (3.14)$$

The spin continuity equation for the whole system, eq. (3.12), means that though each spin conservation law for electrons and magnons is broken (see eqs. (3.4) and (3.9)), the total spin angular momentum is, of course, conserved[111] also in thermal spin-pumping systems as well as quantum spin-pumping systems. On top of this, the interchange of spin angular momentum between the magnons and the conduction electrons is characterized by TSRT, which represents the net pumped thermal spin current.

3.3 Schwinger-Keldysh Formalism

The interface is, in general, a weak coupling regime;[12] the exchange interaction, J , is supposed to be smaller than the Fermi energy and the exchange interaction among ferromagnets. Thus $\mathcal{H}'_{\text{ex}}$ can be treated as a perturbative term.

Through the standard procedure of the Schwinger-Keldysh (or non-equilibrium) Green's function,[100, 31, 32] the Langreth method (see also (Appendix) Chap. 9),[33, 22, 34] the TSRT can be evaluated as

$$\begin{aligned} \langle \mathcal{T}_s^z \rangle &= 2iJ^2 a_0^3 S \int \frac{d\mathbf{k}_1}{(2\pi)^3} \int \frac{d\mathbf{k}_2}{(2\pi)^3} \int \frac{d\omega_1}{2\pi} \int \frac{d\omega_2}{2\pi} \\ &\times [\mathcal{G}_{\uparrow, \mathbf{k}_2, \omega_2}^> \mathcal{G}_{\mathbf{k}_1, \omega_1}^> \mathcal{G}_{\downarrow, \mathbf{k}_1 + \mathbf{k}_2, \omega_1 + \omega_2}^< - \mathcal{G}_{\uparrow, \mathbf{k}_2, \omega_2}^< \mathcal{G}_{\mathbf{k}_1, \omega_1}^< \mathcal{G}_{\downarrow, \mathbf{k}_1 + \mathbf{k}_2, \omega_1 + \omega_2}^>] + \mathcal{O}(J^3). \end{aligned} \quad (3.15)$$

³Note that, $S^z = \tilde{S} - a^\dagger a$, via the Holstein-Primakoff transformation.

The variable $\mathcal{G}^{(< >)}$ is the fermionic lesser (greater) Green's function, and $G^{(< >)}$ is the bosonic one. We here have taken the extended time defined on the Keldysh contour[32, 33, 22], c , on the forward path $c_{\rightarrow}$; $c = c_{\rightarrow} + c_{\leftarrow}$. Even when the time is located on the backward path $c_{\leftarrow}$, the result of the calculation does not change because each Green's function is not independent; $\mathcal{G}^r - \mathcal{G}^a = \mathcal{G}^> - \mathcal{G}^<$, where $\mathcal{G}^{r(a)}$ represents the retarded (advanced) Green's function.[84] This relation comes into effect also for the bosonic case.[32]

Each Green's function reads as follows:[31]

$$G_{\mathbf{k},\omega}^< = -2\pi i f_B(\omega) \delta(\omega - \omega_{\mathbf{k}}), \quad (3.16)$$

$$G_{\mathbf{k},\omega}^> = -2\pi i [1 + f_B(\omega)] \delta(\omega - \omega_{\mathbf{k}}), \quad (3.17)$$

$$\mathcal{G}_{\sigma,\mathbf{k},\omega}^< = 2\pi i f_F(\omega) \delta(\omega - \omega_{\sigma,\mathbf{k}}), \quad (3.18)$$

$$\mathcal{G}_{\sigma,\mathbf{k},\omega}^> = -2\pi i [1 - f_F(\omega)] \delta(\omega - \omega_{\sigma,\mathbf{k}}), \quad (3.19)$$

where the variables $f_B(\omega)$ and $f_F(\omega)$ are the Bose distribution function and the Fermi one. The energy dispersion relation reads $\omega_{\mathbf{k}} \equiv Dk^2 + B$ and $\omega_{\sigma,\mathbf{k}} \equiv Fk^2 - (JS + B/2)\sigma - \mu$, where $D \equiv 1/(2m)$, $F \equiv 1/(2m_{\text{el}})$, $\sigma = +1, -1 (= \uparrow, \downarrow)$, and μ denotes the chemical potential; $\mu(T) = \epsilon_F - (\pi k_B T)^2 / (12\epsilon_F) + \mathcal{O}(T^4)$. The variable ϵ_F represents the Fermi energy.

Consequently, eq.(3.15) can be rewritten as

$$\begin{aligned} \langle \mathcal{T}_s^z \rangle &= 4\pi J^2 a_0^3 S \int \frac{d\mathbf{k}_1}{(2\pi)^3} \int \frac{d\mathbf{k}_2}{(2\pi)^3} \int d\omega_1 \int d\omega_2 \\ &\times \delta(\omega_1 - \omega_{\mathbf{k}_1}) \delta(\omega_2 - \omega_{\uparrow,\mathbf{k}_2}) \delta(\omega_1 + \omega_2 - \omega_{\downarrow,\mathbf{k}_1+\mathbf{k}_2}) \\ &\times \left\{ [1 + f_B(\omega_1)] f_F(\omega_1 + \omega_2) [1 - f_F(\omega_2)] \right. \\ &\left. - f_B(\omega_1) f_F(\omega_2) [1 - f_F(\omega_1 + \omega_2)] \right\} \end{aligned} \quad (3.20)$$

$$\begin{aligned} &= 4\pi J^2 a_0^3 S \int \frac{d\mathbf{k}_1}{(2\pi)^3} \int \frac{d\mathbf{k}_2}{(2\pi)^3} \delta(\omega_{\mathbf{k}_1} + \omega_{\uparrow,\mathbf{k}_2} - \omega_{\downarrow,\mathbf{k}_1+\mathbf{k}_2}) \\ &\times \left\{ f_F(\omega_{\mathbf{k}_1} + \omega_{\uparrow,\mathbf{k}_2}) [1 - f_F(\omega_{\uparrow,\mathbf{k}_2})] + f_B(\omega_{\mathbf{k}_1}) f_F(\omega_{\mathbf{k}_1} + \omega_{\uparrow,\mathbf{k}_2}) \right. \\ &\left. - f_B(\omega_{\mathbf{k}_1}) f_F(\omega_{\uparrow,\mathbf{k}_2}) \right\}. \end{aligned} \quad (3.21)$$

3.3.1 Spin-flip Condition

The delta function in eq. (3.21) represents the spin-flip condition between the conduction electrons and the magnons. The modes (i.e. $\mathbf{k}_1$ and $\mathbf{k}_2$) which do not satisfy this condition cannot contribute to thermal spin pumping.

The delta function reads

$$\delta(\omega_{\mathbf{k}_1} + \omega_{\uparrow,\mathbf{k}_2} - \omega_{\downarrow,\mathbf{k}_1+\mathbf{k}_2}) = \delta\left((D - F)k_1^2 - 2F\mathbf{k}_1 \cdot \mathbf{k}_2 - 2JS\right) \quad (3.22)$$

$$= \frac{1}{2Fk_1k_2} \delta\left(\cos\theta - \frac{(D - F)k_1^2 - 2JS}{2Fk_1k_2}\right), \quad (3.23)$$

where $\cos\theta \equiv \mathbf{k}_1 \cdot \mathbf{k}_2 / (k_1 k_2)$. Eq. (3.23) holds true on the condition; $k_1 \neq 0, k_2 \neq 0$, and $F \neq 0$. This condition can be justified because the zero-mode for the conduction electrons ($k_2 = 0$) originally cannot contribute to spin pumping which is the low energy dynamics; in order to excite the zero-mode so as to become relevant to spin pumping, it costs vast energy which amounts to the Fermi energy. Such a (relatively high energy) dynamics is out of the system we focus on, $\mathcal{H}$. In addition, when the zero mode for magnons ($k_1 = 0$) is substituted

into eq. (3.22), it gives zero because of the finite effective magnetic fields $JS(\neq 0)$. Thus the zero-mode of magnons also originally cannot contribute to spin pumping and are eliminated. Then we are allowed to calculate eq. (3.21) on the condition; $k_1 \neq 0$ and $k_2 \neq 0$.

3.3.2 TSRT

Consequently by using eq. (3.23), the TSRT (eq. (3.21)) can be rewritten as

$$\langle \frac{4\pi^3 DF^2}{a_0^3 S \epsilon_F^4} \mathcal{T}_s^z \rangle = \int_{\searrow 0}^{\infty} d\bar{k}_1 \int_{\searrow 0}^{\infty} d\bar{k}_2 \langle \bar{\mathcal{T}}_s^z \rangle(\bar{k}_1, \bar{k}_2) \quad (3.24)$$

$$\equiv \int_{\searrow 0}^{\infty} d\bar{k}_1 \langle \bar{\mathcal{T}}_s^z \rangle(\bar{k}_1), \quad (3.25)$$

$$\equiv \langle \bar{\mathcal{T}}_s^z \rangle, \quad (3.26)$$

where

$$\begin{aligned} \langle \bar{\mathcal{T}}_s^z(\bar{k}_1, \bar{k}_2) \rangle &\equiv \bar{J}^2 \int_{-1}^1 d\zeta \delta\left(\zeta - \frac{(1 - \frac{F}{D})\bar{k}_1^2 - 2\bar{J}S}{2\sqrt{\frac{F}{D}}\bar{k}_1\bar{k}_2}\right) \cdot \bar{k}_1\bar{k}_2 \\ &\times \left\{ -\frac{1}{e^{(\bar{k}_1^2 + \bar{B})/\bar{T}_m} - 1} \cdot \frac{1}{e^{(\bar{k}_2^2 - \bar{J}S - \bar{B}/2 - 1 + \pi^2 \bar{T}_s^2/12)/\bar{T}_s} + 1} \right. \\ &+ \left[1 - \frac{1}{e^{(\bar{k}_2^2 - \bar{J}S - \bar{B}/2 - 1 + \pi^2 \bar{T}_s^2/12)/\bar{T}_s} + 1} + \frac{1}{e^{(\bar{k}_1^2 + \bar{B})/\bar{T}_m} - 1} \right] \\ &\times \left. \frac{1}{e^{(\bar{k}_1^2 + \bar{k}_2^2 - \bar{J}S + \bar{B}/2 - 1 + \pi^2 \bar{T}_s^2/12)/\bar{T}_s} + 1} \right\}. \end{aligned} \quad (3.27)$$

We here have defined a variable, $\zeta \equiv \cos\theta$, and have introduced dimensionless variables; $\bar{k}_1 \equiv \sqrt{D/\epsilon_F}k_1$, $\bar{k}_2 \equiv \sqrt{F/\epsilon_F}k_2$, $\bar{B} \equiv B/\epsilon_F$, $\bar{J} \equiv J/\epsilon_F$, $\bar{T}_{m(s)} \equiv T_{m(s)}/T_F \equiv k_B T_{m(s)}/\epsilon_F$, where k_B denotes the Boltzmann constant. The variable $T_{m(s)}$ is the effective local temperature of the magnons (the conduction electrons), [98, 94, 103, 116, 117] and

$$\langle \bar{\mathcal{T}}_s^z(\bar{k}_1, \bar{k}_2) \rangle \quad (3.28)$$

represents the dimensionless TSRT in the wavenumber space of the magnons and the conduction electrons;

$$\langle \bar{\mathcal{T}}_s^z(\bar{k}_1) \rangle \quad (3.29)$$

denotes the dimensionless TSRT in the wavenumber space of magnons, $\bar{k}_1$, after integrating over the wavenumber space of the conduction electrons, $\bar{k}_2$. Both quantities, $\langle \bar{\mathcal{T}}_s^z(\bar{k}_1, \bar{k}_2) \rangle$ and $\langle \bar{\mathcal{T}}_s^z(\bar{k}_1) \rangle$, characterize the TSRT in terms of the exchange interaction (J) and the temperature ($T_{m(s)}$).

We set each parameter, as a typical case, as follows; [103, 65, 118] $\epsilon_F = 5.6$ eV, $B/\epsilon_F = 0$, $F = 4$ eV $\text{\AA}^2$, $D = 0.3$ eV $\text{\AA}^2$, $S = 1/2$. Here it should be noted that we do not apply any magnetic fields along the quantization axis;

$$B \searrow 0. \quad (3.30)$$

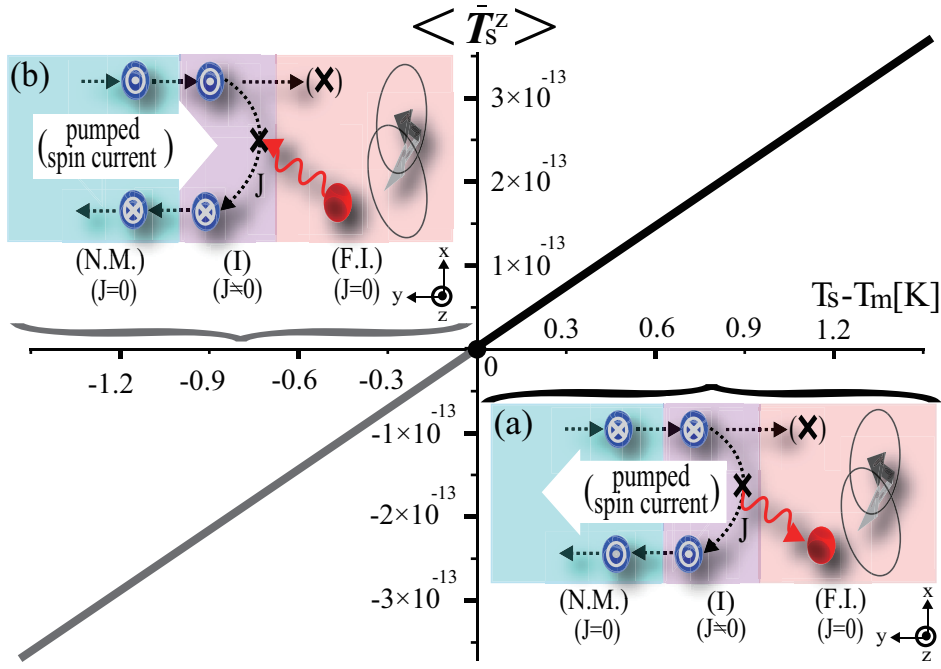


Figure 3.2: The temperature difference dependence of the dimensionless TSRT, $\langle \bar{T}_s^z \rangle$, and the corresponding schematic pictures. Each parameter reads $\bar{J} = 0.002$ and $T_s = 300$ K. When the effective temperature of magnons is lower than that of the conduction electrons, localized spins at the interface lose spin angular momentum by emitting magnons and the conduction electrons flip from down to up by absorbing the momentum (a), and vice versa (b).

3.4 Thermal Spin Pumping Effect

Fig. 3.2 shows that under the thermal equilibrium condition where temperature difference does not exist between ferromagnet and non-magnetic metal, spin currents cannot be pumped because of the balance between thermal fluctuations in ferromagnet and those in non-magnetic metal.[98, 94, 103] In addition, it can be concluded that the pumped spin current is proportional to the temperature difference between the magnon and the conduction electron temperatures (i.e. $T_s - T_m$); when the effective temperature of magnons is lower than that of the conduction electrons (see Fig. 3.2 (a)), localized spins at the interface lose spin angular momentum by emitting magnons and the conduction electrons flip from down to up by absorbing all the emitted momentum,[111] and vice versa (see Fig. 3.2 (b)). This result shows the good agreement with the work by Xiao et al.:[103] they have reached this result by the approach in the combination of the spin pumping theory proposed by Tserkovnyak et al.[9] and the LLG equation.⁴

Figs. 3.3 (a) and (c) show that magnons at (near) the zero-mode cannot contribute to thermal spin pumping because they do not satisfy the spin-flip condition between the

⁴About the distinction from other preceding works:[103, 94] let us mention that we have set $B = 0$. That is, a spin current can be generated via the thermal spin pumping effect without any applied magnetic fields. This point cannot be obtained by Xiao et al.[103] The pumped spin current is proportional to the temperature difference between the magnon and the conduction electron temperatures; inhomogeneous thermal fluctuations induce a net spin current.[98] This is the main difference from the quantum spin pumping effect.[27]

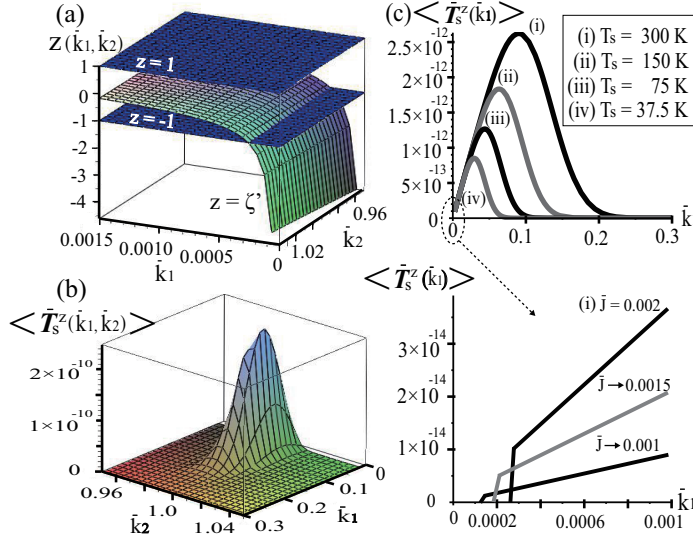


Figure 3.3: (a) The spin-flip condition via magnons; $z(\bar{k}_1, \bar{k}_2) = \zeta' \equiv [(1 - F/D)\bar{k}_1^2 - 2\bar{J}S](2\sqrt{F/D}\bar{k}_1\bar{k}_2)^{-1}$, where $\bar{J} = 0.002$. Magnons at (near) the zero-mode cannot contribute to thermal spin pumping because they do not satisfy the spin-flip condition, eq. (3.22). (b) The TSRT in the wavenumber space for the conduction electrons and magnons, $\langle \bar{T}_s^z(\bar{k}_1, \bar{k}_2) \rangle$. Each parameter reads $\bar{J} = 0.002$, $T_s = 300$ K, and $T_s - T_m = 1.2$ K. A sharp peak exists on the Fermi wavenumber. (c) The TSRT in the wavenumber space for magnons, $\langle \bar{T}_s^z(\bar{k}_1) \rangle$; the condition is the same with (b). The higher the effective magnon temperature becomes, the longer wavenumber of magnons becomes relevant to thermal spin pumping.

conduction electrons and the magnons, due to the finite effective magnetic field JS (see eqs. (3.22), (3.27), and Fig. 3.3 (a)).

3.5 Summary

On the basis of the Schwinger-Keldysh formalism, we have qualitatively studied thermal spin pumping mediated by magnons in the semiclassical regime and have clarified the distinction from the preceding theories.

Pumped spin currents are proportional to the temperature difference between the conduction electrons and the magnons, which means that inhomogeneous thermal fluctuations induce a net thermal spin current; when the effective temperature of magnons is lower than that of the conduction electrons, localized spins lose spin angular momentum by emitting magnons and the conduction electrons flip from down to up by absorbing the momentum, and vice versa.

On top of this, owing to the Schwinger-Keldysh formalism, we have succeeded in the microscopic description of the non-equilibrium spin-flip processes at the interface and have found the spin-flip condition between the conduction electrons and the magnons. As the result, we have clarified that thermal spin pumping has the advantage that it does not cost any kinds of applied magnetic fields (i.e. $B \searrow 0$), which is in sharp contrast to the case of quantum spin pumping generated by resonances (see Chap. 2), because magnons at the zero mode, which often cause a divergence, are eliminated due to the spin-flip condition. This fact will be useful also from the viewpoint of potential applications to information and

communication technologies.

On the other hand, though the behavior of the thermal spin pumping effect mediated by magnons can be qualitatively captured by calculating the TSRT, we recognize that the theoretical estimation for the width of the interface arising from so called proximity effects is an urgent issue to obtain a quantitative understanding.

In addition, we are also interested in the contribution of phonons (see Sec. 4.4) and that of magnons under a spatially nonuniform magnetization to spin pumping.

3.6 Supplement; Coherent State

A coherent state $|v\rangle$, which is also called Glauber state,[112]⁵ is defined as eigenstate of the annihilation operator a , with eigenvalues $v \in \mathbb{C}$;

$$a |v\rangle = v |v\rangle. \quad (3.31)$$

In particular, it should be remembered that the coherent state can be regarded as the specific quantum state whose dynamics most closely resembles the classical one.

Glauber-Mehta-Sudarshan's Theorem

According to Glauber et al.,[112, 113] if the time derivative of the annihilation operator does not involve a functional dependence on the creation operator, i.e.⁶

$$\dot{a}(t) = \mathcal{F}(a(t), t), \quad (3.32)$$

then the states which are initially coherent remain coherent at all times;⁷ that is the coherent state is stable under the time evolution.

Proof

The Heisenberg's equation of motion reads as follows;

$$\dot{a}(t) = \frac{[a(t), \mathcal{H}(t)]}{i}. \quad (3.33)$$

On the other hand, the differential calculus reads as follows by definition (τ ; a infinitesimal value);

$$\dot{a}(t) = \frac{a(t + \tau) - a(t)}{\tau} + \mathcal{O}(\tau). \quad (3.34)$$

As the result, the following relation is satisfied;

$$\frac{[a(t), \mathcal{H}(t)]}{i} = \frac{a(t + \tau) - a(t)}{\tau} + \mathcal{O}(\tau). \quad (3.35)$$

Now, we assume that the eigenstate $|v\rangle$ of $a(t)$ with the eigenvalue $\zeta(v, t)$, $a(t) |v\rangle = \zeta(v, t) |v\rangle$, is also the eigenstate of $a(t + \tau)$ with the eigenvalue $\zeta(v, t + \tau)$;

$$a(t + \tau) |v\rangle = \zeta(v, t + \tau) |v\rangle. \quad (3.36)$$

⁵Roy J. Glauber who was awarded the 2005 Nobel prize for his contribution to the quantum theory of optical coherence.

⁶Note that this is the sufficient condition.

⁷The function $\mathcal{F}$ is the arbitrary one consisting of only the annihilation operator $a(t)$.

Then, eq. (3.35) can be rewritten as

$$\frac{[a(t), \mathcal{H}(t)]}{i} |v\rangle = \frac{\partial \zeta(v, t)}{\partial t} |v\rangle. \quad (3.37)$$

Note that $\partial \zeta(v, t)/(\partial t)$ is a c-number. Then, eq. (3.37) means that the commutator $[a(t), \mathcal{H}(t)]$ must depend exclusively only on $a(t)$; ⁸

$$[a(t), \mathcal{H}(t)] = \mathcal{F}_2(a(t), t). \quad (3.38)$$

Thus, by taking eq. (3.33) into account, we have obtained the Glauber-Mehta-Sudarshan's theorem shown in eq. (3.32);

$$\dot{a}(t) = \mathcal{F}(a(t), t). \blacksquare \quad (3.39)$$

⁸The function $\mathcal{F}_2$ is an arbitrary one consisting of only the annihilation operator $a(t)$.

Part II

Magnon BEC Theory and Related Phenomena

Chapter 4

Magnon BEC Theory; Microwaves

Stimulated by the experimental breakthrough by Demokritov et al.,[46] we investigate the possibility for the occurrence of quasi-equilibrium magnon BEC (i.e. dynamical magnon condensation) in quantum spin-pumping systems. By employing the Schwinger-Keldysh formalism, we microscopically evaluate the macroscopic condensate order parameter after switching off microwaves. We have found a kind of quasi-equilibrium magnon BEC which is peculiar to quantum spin-pumping systems.

4.1 Hamiltonian; Effects of Microwaves

Motivated by the experimental breakthrough by Demokritov et al.,[46] who has observed quasi-equilibrium magnon BEC at room temperature by using microwave pumping method, we investigate quasi-equilibrium magnon BEC in quantum spin-pumping systems.

For the experimental realization of spin pumping,[86, 12, 6] a pumping magnetic field $\Gamma(t)$ with an angular frequency Ω is applied and it drives the system into a non-equilibrium steady state.[45] In the present study, thanks to the Schwinger-Keldysh formalism, we can take the effects of the external pumping magnetic field, which deserves the driving energy, into account beyond phenomenology.

Then, we begin to apply the pumping magnetic field $\Gamma(t)$ along the transverse axis (i.e. x -axis) to the system at $t = -t_0/2$ and switch off at $t = t_0/2$ (> 0);

$$V_{\text{el}}^{\Gamma} = \frac{\Gamma(t)}{4} \int d\mathbf{x} c^{\dagger}(\mathbf{x}, t)(\sigma^{+} + \sigma^{-})c(\mathbf{x}, t) \quad (4.1)$$

$$V_{\text{mag}}^{\Gamma} = \Gamma(t) \sqrt{\frac{\tilde{S}}{2}} \int d\mathbf{x} [a(\mathbf{x}, t) + a^{\dagger}(\mathbf{x}, t)] \quad (4.2)$$

with

$$\Gamma(t) := [\Gamma_0 \cos(\Omega t)] H_1(t_0/2 - t) H_1(t_0/2 + t), \quad (4.3)$$

where the Heaviside-step function $H_1(\chi)$ is defined as $H_1(\chi; \chi < 0) = 0$ and $H_1(\chi; \chi \geq 0) = 1$. This applied periodic pumping magnetic field $\Gamma(t)$ acts as quantum fluctuations[89] and hence, we identify the system described by the exchange interaction $\mathcal{H}_{\text{ex}}$ under the applied pumping magnetic field $\Gamma(t)$ with the quantum spin-pumping system. Finally, the total Hamiltonian of the quantum spin-pumping system $\mathcal{H}$ is arranged as $\mathcal{H} := \mathcal{H}_{\text{mag}} + \mathcal{H}'_{\text{ex}} + \mathcal{H}_{\text{el}} + V_{\text{el}}^{\Gamma} + V_{\text{mag}}^{\Gamma}$.

Here, let us remark that the applied pumping magnetic field couples with the conduction electrons as well as localized spins. Therefore, magnon pumping[84] can be generated at the interface also by ESR ($\Omega = 2JS + B$)[27, 28] as well as FMR ($\Omega = B$).[6, 86, 84] In addition,

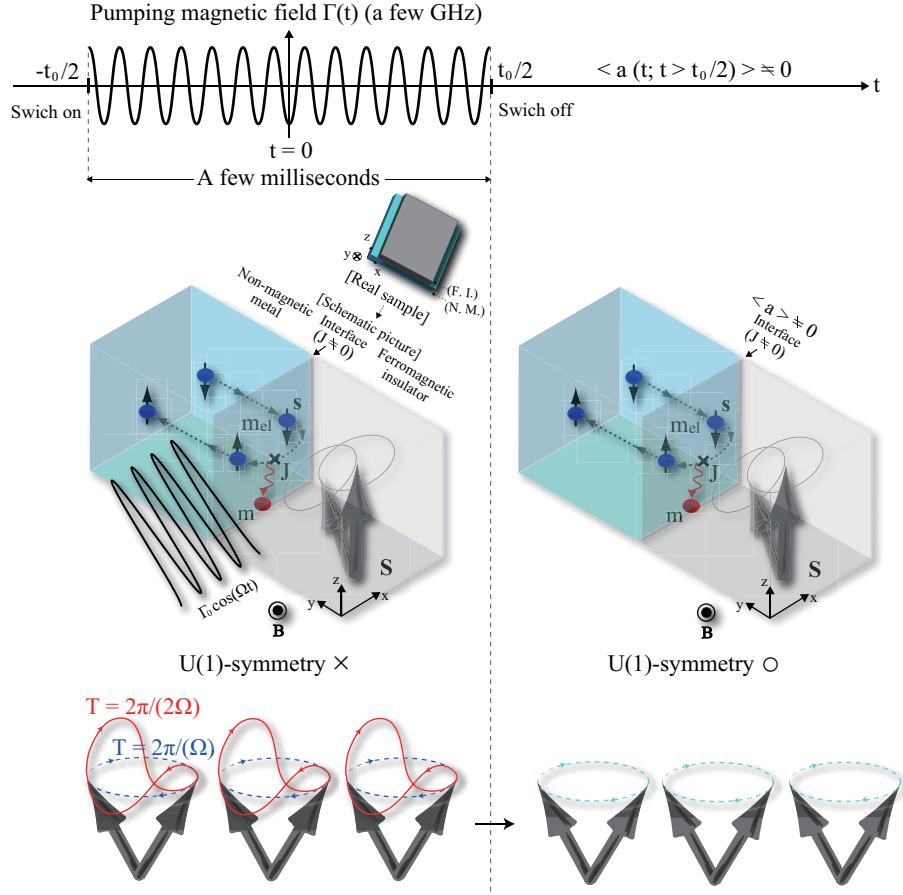


Figure 4.1: (Color online). Schematic pictures of our quantum spin-pumping system. The interface is defined as the effective area where the Fermi gas (the conduction electrons) and the Bose gas (the magnons) coexist to interact via the exchange interaction $\mathcal{H}'_{\text{ex}}$; $J \neq 0$. The interface can be regarded as a ferromagnetic metal state.[81] We begin to apply the pumping magnetic field $\Gamma(t)$ at $t = -t_0/2$ and switch off at $t = t_0/2$. As the result, the $U(1)$ -symmetry of the system is recovered after switching off ($t > t_0/2$). By applying the pumping magnetic field (a few GHz) for an enough time t_0 (a few milliseconds), magnon BEC is realized at the interface; $\langle a(t; t > t_0/2) \rangle \neq 0$. Spheres represent magnons and those with arrows are the conduction electrons. The wavy line denotes the pumping magnetic field.

it is clear from eqs. (4.1)-(4.3) that although the $U(1)$ -symmetry of the systems is explicitly violated due to the applied pumping magnetic field, the $U(1)$ -symmetry is recovered after switching off (i.e. $t \geq t_0/2$).

4.2 Schwinger-Keldysh Formalism

Through the standard procedure of the Schwinger-Keldysh formalism[100, 31, 32] and Wick's theorem (see also (Appendix) Chap. 9), the macroscopic condensate order parameter after

switching off the pumping magnetic field $\langle a(t; t > t_0/2) \rangle$ reads (Fig. 4.3 (a))

$$\begin{aligned} \langle a(t; t > t_0/2) \rangle &= -Ja_0^2 \sqrt{\frac{\tilde{S}}{2}} \int d\mathbf{x}' \int d\mathbf{x}'' \int_c d\tau' \int_c d\tau'' \Gamma(\tau'') \\ &\times \langle T_c a(\mathbf{x}, \tau) a^\dagger(\mathbf{x}', \tau') \rangle \langle T_c c_\uparrow(\mathbf{x}'', \tau'') c_\uparrow^\dagger(\mathbf{x}', \tau') \rangle \\ &\times \langle T_c c_\downarrow(\mathbf{x}', \tau') c_\downarrow^\dagger(\mathbf{x}'', \tau'') \rangle + \mathcal{O}(J^2) + \mathcal{O}(\Gamma^2). \end{aligned} \quad (4.4)$$

Here T_c is the path-ordering operator defined on the Schwinger-Keldysh closed time path,[35, 32, 22, 33] c (Fig. 4.2). We express the Schwinger-Keldysh closed time path as a sum of the forward path, $c_\rightarrow$, and the backward path, $c_\leftarrow$; $c = c_\rightarrow + c_\leftarrow$. The integral on the Schwinger-Keldysh closed time path is executed by taking the relations into account; $\int_c d\tau' = \int_{c_\rightarrow} d\tau' + \int_{c_\leftarrow} d\tau'$, which results in

$$\int_c d\tau' \int_c d\tau'' = \left(\int_{c_\rightarrow} d\tau' + \int_{c_\leftarrow} d\tau' \right) \left(\int_{c_\rightarrow} d\tau'' + \int_{c_\leftarrow} d\tau'' \right) \quad (4.5)$$

$$\begin{aligned} &= \int_{c_\rightarrow} d\tau' \int_{c_\rightarrow} d\tau'' + \int_{c_\rightarrow} d\tau' \int_{c_\leftarrow} d\tau'' \\ &+ \int_{c_\leftarrow} d\tau' \int_{c_\rightarrow} d\tau'' + \int_{c_\leftarrow} d\tau' \int_{c_\leftarrow} d\tau''. \end{aligned} \quad (4.6)$$

Then by employing eq. (4.6) and the Langreth method,[27] eq. (4.4) can be rewritten as

$$\begin{aligned} \langle a(t; t > t_0/2) \rangle &= iJa_0^2 \sqrt{\frac{\tilde{S}}{2}} \int d\mathbf{x}' \int d\mathbf{x}'' \int dt' \int dt'' \Gamma(t'') \\ &\times \left[G^t(t, t') \mathcal{G}_\uparrow^t(t'', t') \mathcal{G}_\downarrow^t(t', t'') - G^t(t, t') \mathcal{G}_\uparrow^>(t'', t') \mathcal{G}_\downarrow^<(t', t'') \right. \\ &\left. - G^<(t, t') \mathcal{G}_\uparrow^<(t'', t') \mathcal{G}_\downarrow^>(t', t'') + G^<(t, t') \mathcal{G}_\uparrow^{\bar{t}}(t'', t') \mathcal{G}_\downarrow^{\bar{t}}(t', t'') \right], \end{aligned} \quad (4.7)$$

where the variables $G^{t(r,a,<,>,\bar{t})} / \mathcal{G}^{t(r,a,<,>,\bar{t})}$ are the bosonic/fermionic time-ordered, retarded, advanced, lesser, greater, and anti-time-ordered Green's functions, respectively.[27] We have taken τ which denotes the contour variable defined on the Schwinger-Keldysh closed time path on forward path, $c_\rightarrow$. Even when τ is located on backward path, $c_\leftarrow$, the result of this calculation is invariant because each Green's function is not independent[32, 31, 34, 22] and they obey the relations; $G^r - G^a = G^> - G^<$ and $\mathcal{G}^r - \mathcal{G}^a = \mathcal{G}^> - \mathcal{G}^<$.

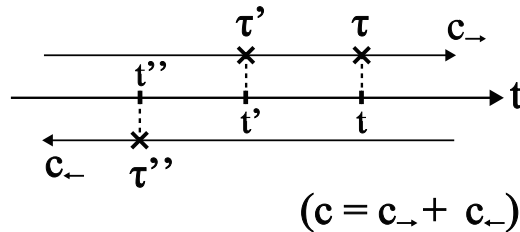


Figure 4.2: The Schwinger-Keldysh closed time path, c . Both the forward ($c_\rightarrow$) and backward ($c_\leftarrow$) paths are actually on the real axis but shifted slightly upwards and downwards, respectively, to distinguish them clearly.

By adopting the Fourier transform, eq. (4.7) reads as follows;

$$\begin{aligned}
\langle a(t; t > t_0/2) \rangle &= iJa_0^2 \sqrt{\frac{\tilde{S}}{2}} \frac{1}{(2\pi)^3} \frac{\Gamma_0}{2} \int_0^\infty k dk \int d\omega_1 \int d\omega_2 \\
&\times \left[\int dt'' H_1(t_0/2 - t'') H_1(t_0/2 + t'') \right] \\
&\times e^{-i\omega_1 t} [e^{-i(-\omega_1 - \Omega)t''} + e^{-i(-\omega_1 + \Omega)t''}] \\
&\times [G_{0,\omega_1}^t \mathcal{G}_{\uparrow,\mathbf{k},\omega_2}^t \mathcal{G}_{\downarrow,\mathbf{k},\omega_1+\omega_2}^t - G_{0,\omega_1}^t \mathcal{G}_{\uparrow,\mathbf{k},\omega_2}^> \mathcal{G}_{\downarrow,\mathbf{k},\omega_1+\omega_2}^< \\
&- G_{0,\omega_1}^< \mathcal{G}_{\uparrow,\mathbf{k},\omega_2}^< \mathcal{G}_{\downarrow,\mathbf{k},\omega_1+\omega_2}^> + G_{0,\omega_1}^< \mathcal{G}_{\uparrow,\mathbf{k},\omega_2}^{\bar{t}} \mathcal{G}_{\downarrow,\mathbf{k},\omega_1+\omega_2}^{\bar{t}}] \quad (4.8)
\end{aligned}$$

$$\begin{aligned}
&= iJa_0^2 \sqrt{\frac{\tilde{S}}{2}} \frac{1}{(2\pi)^3} \frac{\Gamma_0}{2} \int_0^\infty k dk \int d\omega_1 \int d\omega_2 \\
&\times \left[\int_{-t_0/2}^{t_0/2} dt'' \right] \times e^{-i\omega_1 t} [e^{-i(-\omega_1 - \Omega)t''} + e^{-i(-\omega_1 + \Omega)t''}] \\
&\times [G_{0,\omega_1}^t \mathcal{G}_{\uparrow,\mathbf{k},\omega_2}^t \mathcal{G}_{\downarrow,\mathbf{k},\omega_1+\omega_2}^t - G_{0,\omega_1}^t \mathcal{G}_{\uparrow,\mathbf{k},\omega_2}^> \mathcal{G}_{\downarrow,\mathbf{k},\omega_1+\omega_2}^< \\
&- G_{0,\omega_1}^< \mathcal{G}_{\uparrow,\mathbf{k},\omega_2}^< \mathcal{G}_{\downarrow,\mathbf{k},\omega_1+\omega_2}^> + G_{0,\omega_1}^< \mathcal{G}_{\uparrow,\mathbf{k},\omega_2}^{\bar{t}} \mathcal{G}_{\downarrow,\mathbf{k},\omega_1+\omega_2}^{\bar{t}}], \quad (4.9)
\end{aligned}$$

where we have used the relation; $\int dt'' H_1(t_0/2 - t'') H_1(t_0/2 + t'') = \int_{-t_0/2}^{t_0/2} dt''$. This calculation explicitly reflects our procedure (Fig. 4.1) that we begin to apply the pumping magnetic field at $t = -t_0/2$ and switch off at $t = t_0/2$.

Note that eq. (4.9) is very the macroscopic condensate order parameter after switching off the pumping magnetic field (Fig. 4.1); $\langle a(t; t > t_0/2) \rangle$. It is clear that only the zero mode of magnons is relevant to the order parameter $\langle a(t; t > t_0/2) \rangle$.

4.3 Quasi-equilibrium Magnon BEC

We continue to apply the pumping magnetic field for enough finite time t_0 (i.e. from $t = -t_0/2$ to $t = t_0/2$, Fig. 4.1) to satisfy the approximation; $\int_{-t_0/2}^{t_0/2} dt'' [e^{-i(-\omega_1 - \Omega)t''} + e^{-i(-\omega_1 + \Omega)t''}] \simeq 2\pi[\delta(\omega_1 + \Omega) + \delta(\omega_1 - \Omega)]$. From the viewpoint of experiments, compared with the period of the applied pumping magnetic field (i.e. $2\pi/\Omega \sim$ a few nanoseconds), it will be enough to apply Γ for a few milliseconds. Consequently, the macroscopic condensate order parameter after switching off the pumping magnetic field $\langle a(t; t > t_0/2) \rangle$ results in

$$\begin{aligned}
\langle a(t; t > t_0/2) \rangle &= ia_0^2 \sqrt{\frac{\tilde{S}}{2}} \frac{\Gamma_0}{8\pi^2} \int_0^\infty dk \int d\omega Jk \\
&\times \left\{ e^{i\Omega t} \left[G_{0,-\Omega}^t \mathcal{G}_{\uparrow,\mathbf{k},\omega+\Omega}^t \mathcal{G}_{\downarrow,\mathbf{k},\omega}^t - G_{0,-\Omega}^t \mathcal{G}_{\uparrow,\mathbf{k},\omega+\Omega}^> \mathcal{G}_{\downarrow,\mathbf{k},\omega}^< \right. \right. \\
&- G_{0,-\Omega}^< \mathcal{G}_{\uparrow,\mathbf{k},\omega+\Omega}^< \mathcal{G}_{\downarrow,\mathbf{k},\omega}^> + G_{0,-\Omega}^< \mathcal{G}_{\uparrow,\mathbf{k},\omega+\Omega}^{\bar{t}} \mathcal{G}_{\downarrow,\mathbf{k},\omega}^{\bar{t}} \left. \right] \\
&+ e^{-i\Omega t} \left[G_{0,\Omega}^t \mathcal{G}_{\uparrow,\mathbf{k},\omega-\Omega}^t \mathcal{G}_{\downarrow,\mathbf{k},\omega}^t - G_{0,\Omega}^t \mathcal{G}_{\uparrow,\mathbf{k},\omega-\Omega}^> \mathcal{G}_{\downarrow,\mathbf{k},\omega}^< \right. \\
&- G_{0,\Omega}^< \mathcal{G}_{\uparrow,\mathbf{k},\omega-\Omega}^< \mathcal{G}_{\downarrow,\mathbf{k},\omega}^> + G_{0,\Omega}^< \mathcal{G}_{\uparrow,\mathbf{k},\omega-\Omega}^{\bar{t}} \mathcal{G}_{\downarrow,\mathbf{k},\omega}^{\bar{t}} \left. \right] \left. \right\} \quad (4.10)
\end{aligned}$$

$$=:\langle a \rangle_{\mathbf{k}'=0}, \quad (4.11)$$

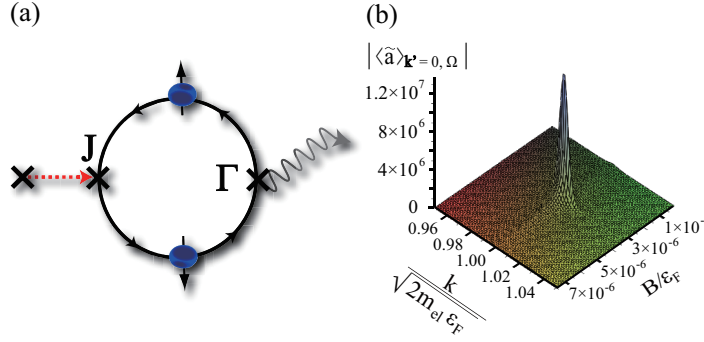


Figure 4.3: (Color online). (a) A Feynman diagram of $\langle a \rangle_{\mathbf{k}'=0}$. The solid, dotted, and wavy lines represent the conduction electrons, magnons, and pumping magnetic fields, respectively. (b) Plot of $|\langle \tilde{a} \rangle_{\mathbf{k}'=0, \Omega}|$ which has been generated by FMR ($\Omega = B$). As a typical case,[27] each parameter is set as follows;[12, 103, 65, 118, 27] $\epsilon_F = 5.6$ eV, $T = 300$ K, $(2m_{\text{el}})^{-1} = 4$ eV Å², $(2m)^{-1} = 0.3$ eV Å², $a_0 = 3$ Å, $S = 1/2$, $\Omega/\epsilon_F = 3.2 \times 10^{-6}$, and $J/\epsilon_F = 0.002$.

where the variable k' represents the wavenumber of magnons.

Here let us introduce the dimensionless macroscopic condensate order parameter, $\langle a \rangle_{\mathbf{k}'=0}/\mathcal{A}$, and the one in the wavenumber-space of the conduction electrons, $\langle \tilde{a} \rangle_{\mathbf{k}'=0, -n\Omega}$, as follows;

$$\langle a \rangle_{\mathbf{k}'=0} =: \mathcal{A} \int_0^\infty \frac{dk}{\sqrt{2m_{\text{el}}\epsilon_F}} \sum_{n=\pm 1} e^{in\Omega t} \langle \tilde{a} \rangle_{\mathbf{k}'=0, -n\Omega} \quad (4.12)$$

with

$$\mathcal{A} := \frac{im_{\text{el}}a_0^2\Gamma_0}{4\pi^2} \sqrt{\frac{\tilde{S}}{2}}, \quad (4.13)$$

and

$$\begin{aligned} \langle \tilde{a} \rangle_{\mathbf{k}'=0, -\Omega} &:= \int d\omega \sqrt{\frac{\epsilon_F}{2m_{\text{el}}}} Jk \\ &\times \left[G_{0, -\Omega}^t \mathcal{G}_{\uparrow, \mathbf{k}, \omega + \Omega}^t \mathcal{G}_{\downarrow, \mathbf{k}, \omega}^t - G_{0, -\Omega}^t \mathcal{G}_{\uparrow, \mathbf{k}, \omega + \Omega}^> \mathcal{G}_{\downarrow, \mathbf{k}, \omega}^< \right. \\ &\left. - G_{0, -\Omega}^< \mathcal{G}_{\uparrow, \mathbf{k}, \omega + \Omega}^< \mathcal{G}_{\downarrow, \mathbf{k}, \omega}^> + G_{0, -\Omega}^{\bar{t}} \mathcal{G}_{\uparrow, \mathbf{k}, \omega + \Omega}^{\bar{t}} \mathcal{G}_{\downarrow, \mathbf{k}, \omega}^{\bar{t}} \right] \end{aligned} \quad (4.14)$$

$$\begin{aligned} \langle \tilde{a} \rangle_{\mathbf{k}'=0, \Omega} &:= \int d\omega \sqrt{\frac{\epsilon_F}{2m_{\text{el}}}} Jk \\ &\times \left[G_{0, \Omega}^t \mathcal{G}_{\uparrow, \mathbf{k}, \omega - \Omega}^t \mathcal{G}_{\downarrow, \mathbf{k}, \omega}^t - G_{0, \Omega}^t \mathcal{G}_{\uparrow, \mathbf{k}, \omega - \Omega}^> \mathcal{G}_{\downarrow, \mathbf{k}, \omega}^< \right. \\ &\left. - G_{0, \Omega}^< \mathcal{G}_{\uparrow, \mathbf{k}, \omega - \Omega}^< \mathcal{G}_{\downarrow, \mathbf{k}, \omega}^> + G_{0, \Omega}^{\bar{t}} \mathcal{G}_{\uparrow, \mathbf{k}, \omega - \Omega}^{\bar{t}} \mathcal{G}_{\downarrow, \mathbf{k}, \omega}^{\bar{t}} \right], \end{aligned} \quad (4.15)$$

where the Fermi energy reads ϵ_F . The variable $k/\sqrt{2m_{\text{el}}\epsilon_F}$ represents the dimensionless wavenumber of the conduction electrons. Because the expansion coefficient of each angular frequency mode satisfies the relation; $|\langle \tilde{a} \rangle_{\mathbf{k}'=0, -\Omega}| \ll |\langle \tilde{a} \rangle_{\mathbf{k}'=0, \Omega}|$, eq. (4.12) can be approximated as

$$\langle a \rangle_{\mathbf{k}'=0} \simeq \mathcal{A} \int_0^\infty \frac{dk}{\sqrt{2m_{\text{el}}\epsilon_F}} e^{-i\Omega t} \langle \tilde{a} \rangle_{\mathbf{k}'=0, \Omega}. \quad (4.16)$$

It is clear from eq. (4.16) and Fig. 4.3 (b) that the absolute value of the macroscopic condensate order parameter¹ $|\langle a \rangle_{\mathbf{k}'=0}|$ takes a non-zero value being constant under the time-development after switching off the pumping magnetic field, in which the $U(1)$ -symmetry of the systems has been recovered. Therefore we can conclude that the zero-mode of magnons has undergone magnon BEC, which corresponds to the (uniform) coherent precession[45, 44] with the period $2\pi/\Omega$ in the language of original spins, after switching off. That is, by applying the pumping magnetic field (a few GHz) for a few milliseconds, magnon BEC can be realized also in quantum spin-pumping systems after switching off.

In conclusion, we have theoretically shown that the coherent magnon state generated by applying the pumping magnetic field (a few GHz) for a few milliseconds is stable even after switching off with keeping the macroscopic condensate parameter a non-zero value $\langle a(t; t > t_0/2) \rangle \neq 0$. Therefore, we theoretically predict that magnon BEC can be realized also in quantum spin-pumping systems after switching off by applying the pumping magnetic field (a few GHz) for a few milliseconds.

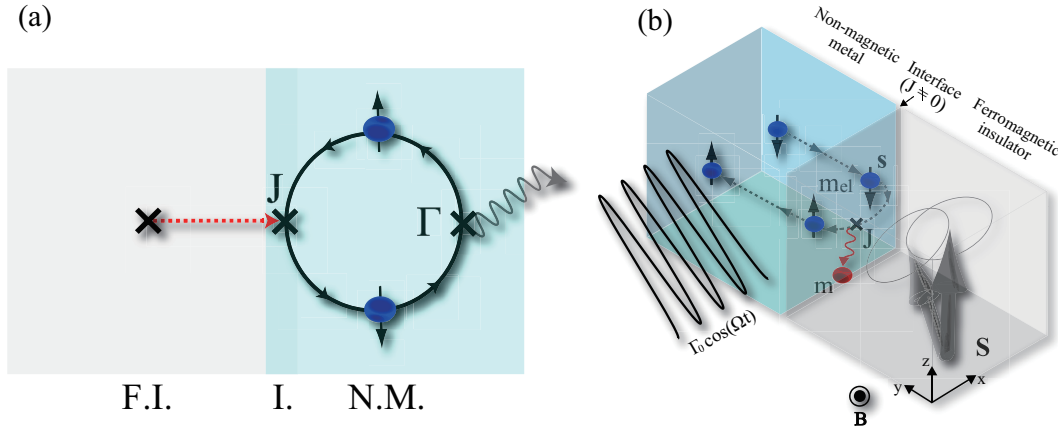


Figure 4.4: (Color online). (a) In our scenario, magnons are generated by the exchange interaction with the conduction electrons. Microwaves indirectly affect magnons via the conduction electrons. (b) Microwaves are directly applied to only the metal.

Features of Magnon BEC

Here let us remark that this magnon BEC is generated not directly by microwaves, but by the exchange interaction J between the conduction electrons and the magnons, which characterizes spin pumping. Then, this microscopic mechanism of magnon BEC is unique to this system.[119] As shown in Fig. 4.4 (a), microwaves indirectly affect magnons via the conduction electrons. Therefore we theoretically propose that also by applying microwaves into only the non-magnetic metal (Fig. 4.4 (b)), magnon BEC will occur in quantum spin-pumping systems.² This is our theoretical prediction.

¹Strictly speaking, the above $|\langle a \rangle_{\mathbf{k}'=0}|$ in eq. (4.16) does not represent the number of condensate magnons; it is that of pumped magnons and by enjoying thermalization processes, it becomes the number of condensate magnons. Regarding the detail, please see (Appendix) Chap. 10.

²According to the discussion with Prof. Bauer, to experimentally apply only one subsystem will be too difficult to achieve.

4.4 Dynamics after Switching off Microwaves

On the other hand, in real materials, there are variety kinds of additional degrees of freedoms. In particular, the effects of phonons is inevitable. Then, we now investigate the contribution of phonons to the dynamics (i.e. magnon BEC state) after switching off.

For the purpose, we here introduce dimensionless³ variables; J , η , $J_{\text{m.m.}}$, where each parameter represents the dimensionless coupling constant denoting the magnitude of the interaction between magnons-electrons, magnons-phonons, and magnons-magnons, respectively (Fig. 4.5).

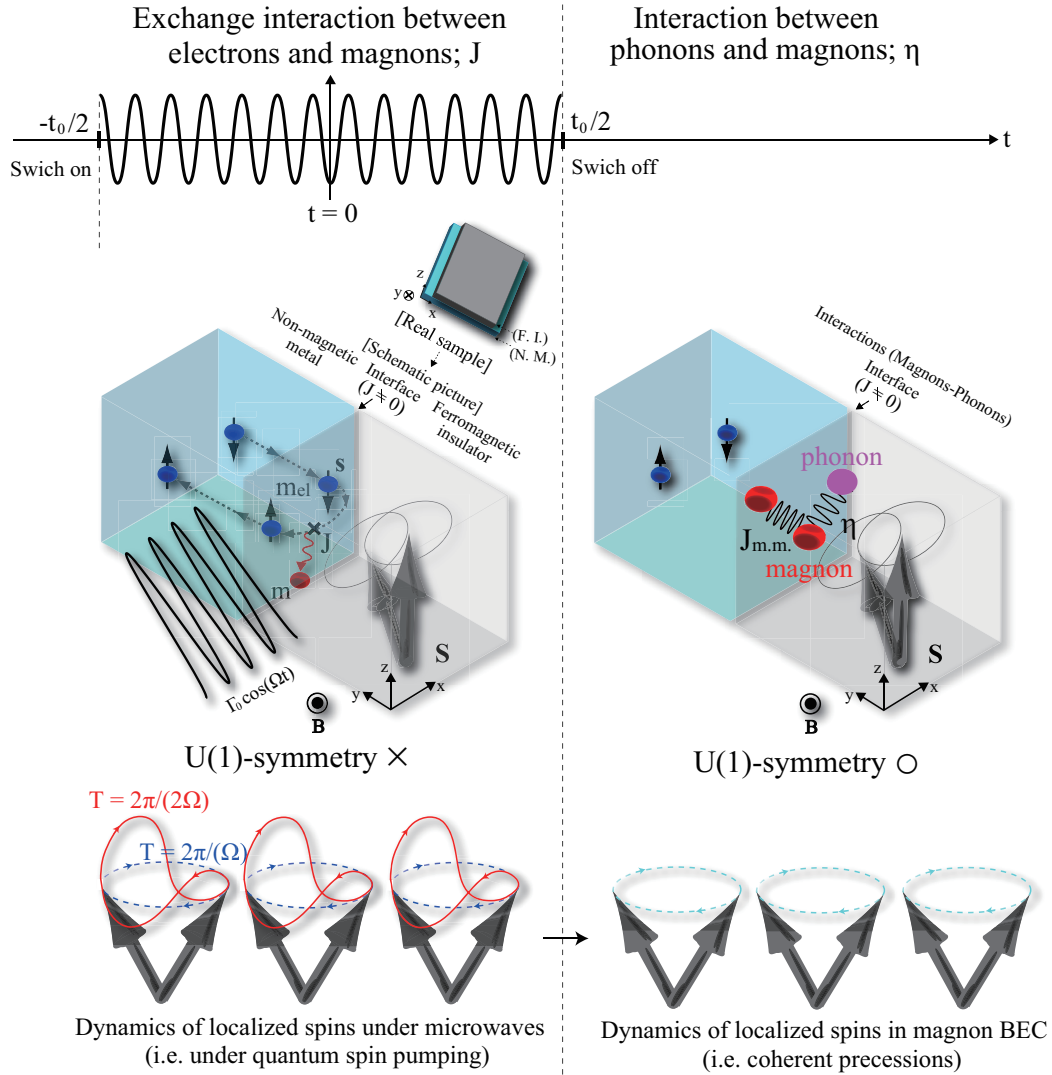


Figure 4.5: (Color online). Schematic pictures of the dynamics of ferromagnetic localized spins under switching on and off.

We then assume the relation; $\eta > J$. This means the situation where magnons mainly interact with phonons after switching off. Under this situation, we closely investigate the time-development of the order parameter $\langle a(t) \rangle$ after switching off by using Heisenberg's equation

³Although we often use the symbol J in various kinds of contexts, please do not confuse and please identify each other. In terms of the interaction among ferromagnets; $-\tilde{J}_{\text{ferro.}} \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j$. That is, we have notated as follows; $\tilde{\eta}/\tilde{J}_{\text{ferro.}} := \eta (< 1)$, $\tilde{J}/\tilde{J}_{\text{ferro.}} := J (< 1)$, and $\tilde{J}_{\text{m.m.}}/\tilde{J}_{\text{ferro.}} := J_{\text{m.m.}} (< 1)$. In other words, we have taken $\tilde{J}_{\text{ferro.}} = 1$. In general, the Fermi energy $\tilde{\epsilon}_F$ is far larger than $\tilde{J}_{\text{ferro.}}$.

of motion and explicitly show that magnon BEC state is stable also in spin-pumping systems. This is the main purpose of the present work.

Last, it would be helpful to remark that our present analysis, from the perturbative calculation based on Keldysh formalism to the analysis after switching off, is applicable to the system where the following condition is satisfied;

$$1 > \eta > J > \eta^2. \quad (4.17)$$

The reason reads as follows; in the previous work, we have shown that the order parameter $\langle a(t) \rangle$ becomes a non-zero value at the interface of quantum spin-pumping systems where magnons interact with electrons via J under applied microwaves Γ ; $\langle a(t) \rangle \propto J\Gamma$. Of course we have noted that magnon-phonon interactions gives some effects on the order parameter, but it gives the following contribution; $\langle a(t) \rangle \propto \eta^2\Gamma$ or $\langle a(t) \rangle \propto J\Gamma\eta^2$. Therefore, if the condition is satisfied;

$$J\Gamma > \eta^2\Gamma \text{ and } J\Gamma > J\Gamma\eta^2, \quad (4.18)$$

that is,

$$J > \eta^2 \text{ and } \eta^2 < 1, \quad (4.19)$$

we can conclude that the contribution arising from the exchange interaction between the conduction electrons and the magnons J at the interface under microwaves Γ , $\langle a(t) \rangle \propto J\Gamma$, is the main (largest) one in the present situation.⁴ On top of this, from now on, we consider the situation where magnons mainly interact with phonons after switching off. For the purpose, the following relation is required;

$$J < \eta. \quad (4.20)$$

Therefore, by combining the eqs. (4.19) and (4.20), we have reached the condition eq. (4.17) on which our present analysis is useful;

$$1 > \eta > J > \eta^2. \quad (4.21)$$

4.4.1 The Interaction Between Magnons and Phonons

We assume the interaction between magnons and phonons described by the Hamiltonian $V_{\text{mag.pho}}$;

$$V_{\text{mag.pho}} = \int d\mathbf{x} \left\{ \eta a^\dagger(\mathbf{x}) a(\mathbf{x}) [b^\dagger(\mathbf{x}) + b(\mathbf{x})] + A a^\dagger(\mathbf{x}) a(\mathbf{x}) + P b^\dagger(\mathbf{x}) b(\mathbf{x}) \right\} \quad (4.22)$$

$$= \eta a_{\mathbf{k}+\mathbf{q}}^\dagger a_{\mathbf{k}} b_{\mathbf{q}} + \eta a_{\mathbf{k}}^\dagger a_{\mathbf{k}+\mathbf{q}} b_{\mathbf{q}}^\dagger + A a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + P b_{\mathbf{q}}^\dagger b_{\mathbf{q}}, \quad (4.23)$$

where the variables $a/a^\dagger$, $b/b^\dagger$ denote magnon annihilation/creation and phonon annihilation/creation operators, and the variable η represents the magnitude of the interaction

⁴In addition, we have defined and introduced the interface as an effective area where the exchange interaction between the conduction electrons and the magnons J works, which is the key to spin pumping; without J , spin pumping cannot occur. That is, the interface of spin-pumping systems can be characterized only by J , which is particular to the interface of spin-pumping systems. On top of this, we now investigate the possibility for the occurrence of magnon BEC after switching off microwaves, that is, after spin pumping. Therefore it will be natural to clarify the contribution of the exchange interaction J to the macroscopic order parameter $\langle a \rangle$.

between magnons and phonons. Here let me emphasize that although the operators $a/a^\dagger$ describes the spin angular momentum, $b/b^\dagger$ does not and hence, the total spin angular momentum of the systems described by $V_{\text{mag.pho}}$ is conserved. That is, $V_{\text{mag.pho}}$ has the $U(1)$ -symmetry; $a \mapsto ae^{i\theta}$ where θ is constant. In other words, the number of magnons under $V_{\text{mag.pho}}$ is conserved; $n(\mathbf{x}) = [n(\mathbf{x}), V_{\text{mag.pho}}]/i = 0$ with $n(\mathbf{x}) = a^\dagger(\mathbf{x})a(\mathbf{x})$.

Now, we consider the situation where the magnitude of the magnon-phonon interaction is weaker than that of Fermi energy and the exchange interaction among ferromagnetic localized spins and hence, we here have assumed that the variable η denotes a dimensionless value; $\eta < 1$.

We have already seen that only the zero-mode of magnons takes a non-zero value under the applied microwaves;

$$\langle a_{\mathbf{k}=0} \rangle \sim J\Gamma \neq 0 \text{ and } \langle a_{\mathbf{k} \neq 0} \rangle = 0 \quad (4.24)$$

and hence, from now on, we focus on the time-development of the zero-modes of magnons after switching off microwaves, where the $U(1)$ -symmetry of the systems has been recovered.

The time-development of physical quantities can be described by the Heisenberg's equation of motion and it reads as follows in terms of the zero-mode of magnons under eq. (4.23) or $V_{\text{mag.pho}}$;

$$\dot{a}_0 = \frac{1}{i}[a_0, V_{\text{mag.pho}}] \quad (4.25)$$

$$= -i\eta a_{\mathbf{k}} b_{-\mathbf{k}} - i\eta a_{\mathbf{q}} b_{\mathbf{q}}^\dagger - iAa_0. \quad (4.26)$$

This differential equation represents the quantum dynamics of magnons after switching off microwaves. By solving this equation, we can obtain the explicit form of the time-development of magnons in general.

On the other hand, we can easily find a practical problem; eq. (4.26) is not closed in terms of the zero-mode of magnons a_0 . That is, eq. (4.26) includes the contribution arising from other modes $a_{\mathbf{k}(\neq 0)}$ as well as a_0 . Therefore, it is a extremely tough issue to directly solve eq. (4.26).

In order to overcome this situation, we employ the same procedure (i.e. approximation) with Bogoliubov et al.,[120]⁵ which is applicable to the system where the number of magnons belonging to $\mathbf{k} = \mathbf{0}$ is much larger than that of magnons in $\mathbf{k} \neq \mathbf{0}$, and derive an effective Hamiltonian. As mentioned, it is because we have applied microwaves and they give $\langle a_{\mathbf{k}=0} \rangle \neq 0$ and $\langle a_{\mathbf{k} \neq 0} \rangle = 0$, our system satisfies the condition for the approximation by Bogoliubov et al.

4.4.2 An Effective Hamiltonian via the Procedure by Bogoliubov et al.

We here focus on a term in eq. (4.23), $a_{\mathbf{k}+\mathbf{q}}^\dagger a_{\mathbf{k}}$, and it can be classified as follows;

$$a_{\mathbf{k}+\mathbf{q}}^\dagger a_{\mathbf{k}} = a_{\mathbf{q}(\neq 0)}^\dagger a_{\mathbf{0}} + a_{\mathbf{0}}^\dagger a_{\mathbf{0}} + a_{\mathbf{k}+\mathbf{q}(\neq 0)}^\dagger a_{\mathbf{k}(\neq 0)} + a_{\mathbf{0}}^\dagger a_{\mathbf{k}(\neq 0)}. \quad (4.27)$$

Following the procedure by Bogoliubov et al.,[120] it can be approximated as

$$a_{\mathbf{k}+\mathbf{q}}^\dagger a_{\mathbf{k}} = a_{\mathbf{q}(\neq 0)}^\dagger a_{\mathbf{0}} + a_{\mathbf{0}}^\dagger a_{\mathbf{0}} + a_{\mathbf{k}+\mathbf{q}(\neq 0)}^\dagger a_{\mathbf{k}(\neq 0)} + a_{\mathbf{0}}^\dagger a_{\mathbf{k}(\neq 0)} \quad (4.28)$$

$$\sim a_{\mathbf{0}}^\dagger a_{\mathbf{0}}, \quad (4.29)$$

⁵Regarding the detail, please see the famous textbook.[120]

because we had applied microwaves and it has given the situation where the number of magnons who belong to $\mathbf{k} = \mathbf{0}$ is much larger than that of magnons in $\mathbf{k} \neq \mathbf{0}$; $|\text{micro.}\rangle$. Therefore, we are allowed to assume the following relation with respect to transition amplitude;

$$\langle \text{micro.} | a_0^\dagger a_0 | \text{micro.} \rangle \gg \langle \text{micro.} | a_{\mathbf{q}(\neq \mathbf{0})}^\dagger a_0 | \text{micro.} \rangle \quad (4.30)$$

$$\sim \langle \text{micro.} | a_0^\dagger a_{\mathbf{k}(\neq \mathbf{0})} | \text{micro.} \rangle \quad (4.31)$$

$$\gg \langle \text{micro.} | a_{\mathbf{k}+\mathbf{q}(\neq \mathbf{0})}^\dagger a_{\mathbf{k}(\neq \mathbf{0})} | \text{micro.} \rangle \quad (4.32)$$

Hence, we have obtained a relation;

$$a_{\mathbf{k}+\mathbf{q}}^\dagger a_{\mathbf{k}} \sim a_0^\dagger a_0. \quad (4.33)$$

By using the same procedure, we reach a relation;

$$a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \sim a_0^\dagger a_0. \quad (4.34)$$

As the result, we obtain an effective Hamiltonian of eq. (4.23);

$$V_{\text{mag.pho}} \sim V_{\text{mag.pho}}^{\text{eff.}} = \eta a_0^\dagger a_0 b_0 + \eta a_0^\dagger a_0 b_0^\dagger + A a_0^\dagger a_0 + P b_{\mathbf{q}}^\dagger b_{\mathbf{q}}. \quad (4.35)$$

This is very the effective Hamiltonian we have wanted and from now on, we discuss on the basis of eq. (4.35), $V_{\text{mag.pho}}^{\text{eff.}}$.

Under the effective Hamiltonian eq. (4.35) or $V_{\text{mag.pho}}^{\text{eff.}}$, the time-development of the zero-mode of magnons reads as follows;

$$\dot{a}_0 = \frac{1}{i} [a_0, V_{\text{mag.pho}}^{\text{eff.}}] \quad (4.36)$$

$$= -i\eta a_0 b_0 - i\eta a_0 b_0^\dagger - iA a_0. \quad (4.37)$$

In addition, the time-development of the zero-mode of phonon in eq. (4.35) reads as follows;

$$\dot{b}_0 = \frac{1}{i} [b_0, V_{\text{mag.pho}}^{\text{eff.}}] \quad (4.38)$$

$$= -i\eta a_0^\dagger a_0 - iP b_0. \quad (4.39)$$

Eqs. (4.37) and (4.39) fully represent the quantum dynamics after switching off. Then, we here employ a classical approximation;

$$a_0 \mapsto \phi \in \mathbb{C} \text{ where } \phi \equiv \phi_1 + i\phi_2 \text{ with } \phi_1, \phi_2 \in \mathbb{R} \quad (4.40)$$

$$b_0 \mapsto \Phi \in \mathbb{C} \text{ where } \Phi \equiv \Phi_1 + i\Phi_2 \text{ with } \Phi_1, \Phi_2 \in \mathbb{R}, \quad (4.41)$$

in which ϕ and Φ represent complex scholar fields and $\phi_{1(2)}$ and $\Phi_{1(2)}$ denote real scholar fields.

As the result, eqs. (4.37) and (4.39) can be rewritten as

$$\dot{\phi}_1 = (2\eta\Phi_1 + A)\phi_2 \quad (4.42)$$

$$\dot{\phi}_2 = -(2\eta\Phi_1 + A)\phi_1 \quad (4.43)$$

$$\dot{\Phi}_1 = P\Phi_2 \quad (4.44)$$

$$\dot{\Phi}_2 = -\eta(\phi_1^2 + \phi_2^2) - P\Phi_1. \quad (4.45)$$

These relations, eqs. (4.42)-(4.45), govern the classical dynamics after switching off, which has been plotted in Fig. 4.6.

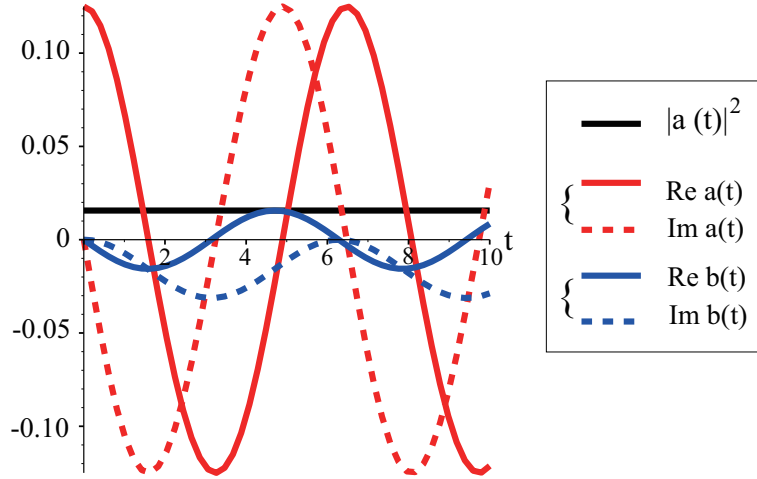


Figure 4.6: (Color online). Plots of eqs. (4.42)-(4.45). Each parameter reads as follows; $\eta = 1$, $A = 1$, $P = 1$. It is clear that the macroscopic condensate order parameter $|a(t)|^2$ takes a time-independent non-zero value, which means the occurrence of magnon BEC.

4.4.3 An Effective Hamiltonian Including Magnon-Magnon Interactions

By using the same procedure, we can take the effects of magnon-magnon interaction into account. The magnon-magnon interaction, $V_{\text{mag}} = J_{\text{m.m.}} a_{\mathbf{k}+\mathbf{q}}^\dagger a_{-\mathbf{q}}^\dagger a_{\mathbf{k}+\mathbf{p}} a_{-\mathbf{p}}$, can be approximated as follows;

$$V_{\text{mag}} = J_{\text{m.m.}} a_{\mathbf{k}+\mathbf{q}}^\dagger a_{-\mathbf{q}}^\dagger a_{\mathbf{k}+\mathbf{p}} a_{-\mathbf{p}} \quad (4.46)$$

$$\sim J_{\text{m.m.}} a_0^\dagger a_0^\dagger a_0 a_0 \quad (4.47)$$

$$=: V_{\text{mag}}^{\text{eff.}} \quad (4.48)$$

It is because

$$\langle \text{micro.} | a_0^\dagger a_0^\dagger a_0 a_0 | \text{micro.} \rangle \gg \langle \text{micro.} | a_{\mathbf{q}(\neq 0)}^\dagger a_{-\mathbf{q}(\neq 0)}^\dagger a_0 a_0 | \text{micro.} \rangle \quad (4.49)$$

$$\sim \langle \text{micro.} | a_0^\dagger a_0^\dagger a_{\mathbf{p}(\neq 0)} a_{-\mathbf{p}(\neq 0)} | \text{micro.} \rangle \quad (4.50)$$

$$\gg \langle \text{micro.} | a_{\mathbf{k}+\mathbf{q}(\neq 0)}^\dagger a_{-\mathbf{q}(\neq 0)}^\dagger a_{\mathbf{k}+\mathbf{p}(\neq 0)} a_{-\mathbf{p}(\neq 0)} | \text{micro.} \rangle. \quad (4.51)$$

Then, the time-development under $V_{\text{mag}}^{\text{eff.}}$ becomes

$$\dot{a}_0 = \frac{1}{i} [a_0, V_{\text{mag}}^{\text{eff.}}] \quad (4.52)$$

$$= -2i J_{\text{m.m.}} a_0^\dagger a_0 a_0. \quad (4.53)$$

By using the same approximation with the last one (eqs. (4.40) and (4.41)), eq. (4.53) gives the new contribution;

$$\dot{\phi}_1 = 2J_{\text{m.m.}} (\phi_1^2 + \phi_2^2) \phi_2 \quad (4.54)$$

$$\dot{\phi}_2 = -2J_{\text{m.m.}} (\phi_1^2 + \phi_2^2) \phi_1. \quad (4.55)$$

Consequently, magnons under magnon-magnon interaction as well as magnon-phonon interactions (i.e. eqs. (4.42)-(4.45)) obey the following differential equations;

$$\dot{\phi}_1 = (2\eta\Phi_1 + A)\phi_2 + 2J_{\text{m.m.}}(\phi_1^2 + \phi_2^2)\phi_2 \quad (4.56)$$

$$\dot{\phi}_2 = -(2\eta\Phi_1 + A)\phi_1 - 2J_{\text{m.m.}}(\phi_1^2 + \phi_2^2)\phi_1 \quad (4.57)$$

$$\dot{\Phi}_1 = P\Phi_2 \quad (4.58)$$

$$\dot{\Phi}_2 = -\eta(\phi_1^2 + \phi_2^2) - P\Phi_1. \quad (4.59)$$

We have confirmed that the behavior of each field is qualitatively the same with eqs. (4.42)-(4.45). That is, magnon BEC state is stable also in the presence of magnon-magnon interactions as well as in that of magnon-phonon interactions.⁶

Lastly, let us again remark that our present analysis is applicable to the system where the following condition is satisfied;⁷

$$1 > \eta > J > \eta^2. \quad (4.60)$$

4.5 Conclusion

On the basis of the Schwinger-Keldysh formalism, we have evaluated the macroscopic condensate order parameter in quantum spin-pumping systems under the influence of phonons and have shown that quasi-equilibrium magnon BEC occurs after switching off microwaves. Remarkably, the magnons are generated not by microwaves but by the exchange interaction with the conduction electrons characterizing spin-pumping systems. Therefore our magnon BEC in this work is peculiar to quantum spin-pumping systems.

4.6 Supplement; Protection of Information

We discuss the features of magnon BEC, the protection of information, which results from the macroscopic coherent state.

Magnon BEC

Magnon BEC, Ψ_{BEC} , is the macroscopic state in the sense that a macroscopic numbers (N -particles) of magnons have formed a coherent quantum state described by a common wavefunction ϕ ;[61]

$$\Psi_{\text{BEC}}(1, 2, \dots, N-1, N) = \Pi_{i=1}^N \phi_i. \quad (4.61)$$

Then the density matrix ρ_{BEC}^N , which captures the fundamental information of the systems,⁸ can be represented as

$$\rho_{\text{BEC}}^N = \Psi_{\text{BEC}}^*(1, 2, \dots, N-1, N) \Psi_{\text{BEC}}(1, 2, \dots, N-1, N). \quad (4.62)$$

⁶Until now, we have been treating the dissipation-less system. Regarding the effects of dissipations, please see (Appendix) Chap. 11.

⁷We have already confirmed that even in the case $J > \eta$, magnon BEC is stable (i.e. $|a(t)| = (\text{constant})$) if only the effective magnetic field of the conduction electrons is equal to that of magnons; otherwise, the quantity $|a(t)|$ begins to oscillate.

⁸In the sense that the expectation value of physical quantities $\langle \mathcal{O} \rangle$ can be characterized by the density matrix of the systems ρ ; $\langle \mathcal{O} \rangle = \text{tr}(\rho \mathcal{O})$.

Now recall that the loss of information corresponds to the procedure partial tracing out in terms of the degrees of freedoms in the language of physics. That is, the loss of information which the N -th particle acquires can be represented as follows;

$$\int d\mathbf{r}_N \rho_{\text{BEC}}^N. \quad (4.63)$$

Remarkably, in the case of BEC, the density matrix is qualitatively invariant even after partial tracing out;

$$\int d\mathbf{r}_N \rho_{\text{BEC}}^N = \rho_{\text{BEC}}^{N-1}. \quad (4.64)$$

In this sense, it can be concluded that (magnon) BEC state is robust against the loss of information.

Superposition State

On the other hand, in the general cases of quantum mechanics, the result dramatically changes; in the usual cases of quantum mechanics, the N -th particle state can be represented by the superposition of quantum states ϕ and φ ;⁹

$$\Psi_{\text{cat}}(1, 2, \dots, N-1, N) = \Pi_{i=1}^N \phi_i + \eta \Pi_{i=1}^N \varphi_i. \quad (4.65)$$

Then, the density matrix becomes

$$\rho_{\text{cat}}^N = \Psi_{\text{cat}}^*(1, 2, \dots, N-1, N) \Psi_{\text{cat}}(1, 2, \dots, N-1, N). \quad (4.66)$$

Now, it should be stressed that in sharp contrast to the case of BEC, the density matrix is dramatically changes after partial tracing out;

$$\int d\mathbf{r}_N \rho_{\text{cat}}^N \neq \rho_{\text{cat}}^{N-1}. \quad (4.67)$$

Then, it can be concluded that the general quantum (superposition) state is fragile against noises.

In conclusion, as shown in eq. (4.64), the macroscopic coherent state is robust against the loss of information,[61] which is the strong point of magnon BEC.

⁹The variable η denotes a coefficient.

Chapter 5

Magnon BEC Theory; SSB

We now investigate the possibility for the occurrence of magnon BEC resulting from SSB in spin-pumping systems. In order to go beyond the perturbative analysis, we employ a non-perturbative theory and clarify the condition for the stable magnon BEC in spin-pumping systems without using microwaves.

5.1 SSB and Magnon BEC

In the present study, we identify the expectation value of the bosonic annihilation operator $\langle \Psi \rangle$ with the macroscopic condensate order parameter[54] and adopt as the criterion for the occurrence of BEC.[45, 121] That is, a non-zero value of the order parameter $\langle \Psi \rangle \neq 0$ under the $U(1)$ -symmetric Hamiltonian does mean the occurrence of BEC, which is associated with $U(1)$ -SSB.¹ This definition of BEC has now been very commonly used in the literature.[45, 58, 121]

On the basis of this definition of BEC, we investigate the possibility for the occurrence of the $U(1)$ -SSB of the vacuum in spin-pumping systems, which is associated with a non-zero value of the order parameter under the $U(1)$ -symmetric Hamiltonian. In order to go beyond the perturbative analysis by the Schwinger-Keldysh formalism,[30] we employ a powerful theoretical technique non-perturbative theory,[39] which does not rely on the assumption called the adiabatic theorem (i.e. the well-known Gell-Mann and Low theorem).[38, 39, 122] Therefore we can analyze beyond a perturbative theory.[30]

5.2 Goldstone Model

Before going on to the main subject, let us briefly review[39, 115, 89] the idea of the SSB of the vacuum in classical field theory. As an example, we employ the (so called) Goldstone model whose potential term is given as (see also Fig. 5.2)

$$V_{\text{Goldstone}}(\varphi) := -\mu\varphi\varphi^* + \mathcal{J}(\varphi\varphi^*)^2 \quad (5.1)$$

$$= -\mu |\varphi|^2 + \mathcal{J} |\varphi|^4, \quad (5.2)$$

¹As remarked, regarding the issue on the criterion for BEC in interacting systems, we in part agree[41] with the argument by Leggett in his textbook[58] (see pp. 31-40); in addition to this commonly used macroscopic condensate order parameter,[45] he has introduced (i.e. presented) the discussion by Penrose-Onsager[59] and Yang[60] about the generalization of original (noninteracting) BEC to interacting cases. We have been tackling this issue.[41]

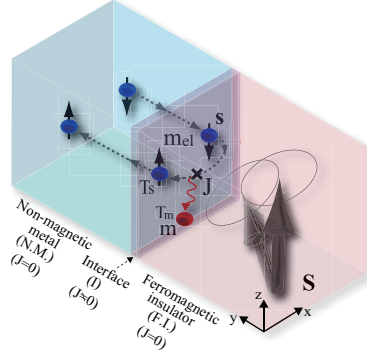


Figure 5.1: (Color online). A schematic picture of thermal spin pumping mediated by magnons where, in sharp contrast to quantum spin-pumping systems, microwaves are not applied.

in which the variable $\varphi(\in \mathbb{C})$ denotes a complex scalar field. The parameter $\mu(\in \mathbb{R})$ represents the dimensionless chemical potential and $\mathcal{J}(\in \mathbb{R})$ does a dimensionless coupling constant. It is clear that the Goldstone model possesses the global $U(1)$ -symmetry; $\varphi \mapsto e^{i\theta}\varphi$ with $\theta \equiv (\text{const.}) \in \mathbb{R}$. For the stability of the system or the vacuum (i.e. the ground state), there should be a lower bound on the energy level of the system.[123] Thus the condition is required (see also Fig. 5.5 (a));

$$0 < \mathcal{J}. \quad (5.3)$$

From here on, we will assume that the possible vacuum states are invariant under translations and they are time-independent.[39, 115] Thus, the candidate of the stable vacuum of the system is given as the stationary point of the effective potential $V_{\text{Goldstone}}^{\text{eff}}$:[39, 89]

$$\frac{\partial V_{\text{Goldstone}}^{\text{eff}}}{\partial \varphi} = 0. \quad (5.4)$$

In addition, within the classical theory in the sense that we omit the quantum effects (i.e. loop corrections) and discuss within the tree-level, the effective potential $V_{\text{Goldstone}}^{\text{eff}}$ is reduced to the usual one $V_{\text{Goldstone}}$:[39]

$$V_{\text{Goldstone}}^{\text{eff}}(|\varphi|) = -\mu |\varphi|^2 + \mathcal{J} |\varphi|^4 + \mathcal{O}(\hbar) \quad (5.5)$$

$$= V_{\text{Goldstone}}(|\varphi|) + \mathcal{O}(\hbar). \quad (5.6)$$

Thus the minimum-energy classical configuration is a uniform field $\varphi = \varphi_0$ with φ_0 chosen to minimize the potential $V_{\text{Goldstone}}$:[39, 115]

$$V_{\text{Goldstone}}(|\varphi|) = \mathcal{J} \left[|\varphi|^2 - \frac{\mu}{2\mathcal{J}} \right]^2 - \frac{\mu^2}{4\mathcal{J}}. \quad (5.7)$$

Consequently when the chemical potential is positive ($0 < \mu$), the vacuum expectation value of the field φ_0 reads

$$|\varphi_0| = \sqrt{\frac{\mu}{2\mathcal{J}}} \quad (\neq 0). \quad (5.8)$$

As the result, the $U(1)$ -SSB of the vacuum does occur when the chemical potential is positive ($0 < \mu$, Fig. 5.2 (b)); otherwise not (i.e. $\mu \leq 0$, Fig. 5.2 (a)).

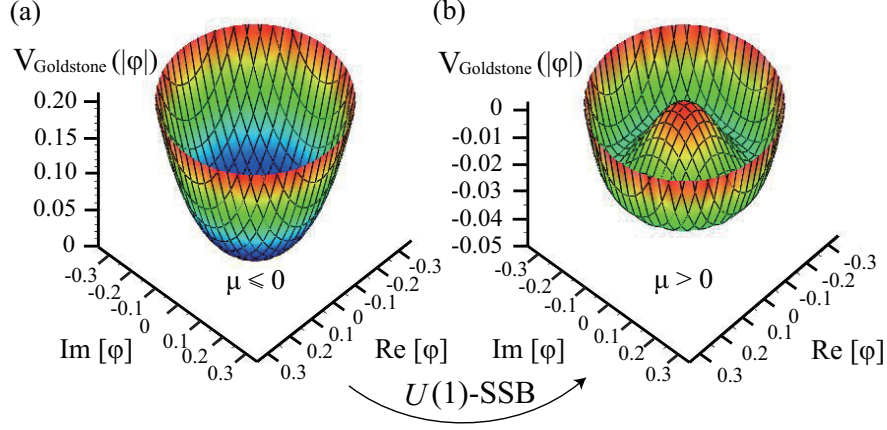


Figure 5.2: (Color online). Schematic pictures of the $U(1)$ -SSB of the vacuum in the Goldstone model $V_{\text{Goldstone}}(|\varphi|)$; eq. (5.7). When the chemical potential becomes positive ($0 < \mu$), the Goldstone model forms the Mexican-hat potential[124] (b) and the $U(1)$ -symmetry of the vacuum is spontaneously broken; eq. (5.8). As an example, each parameter is set as follows; (a) $\mathcal{J} = +10$, $\mu = -1$ and (b) $\mathcal{J} = +10$, $\mu = +1$. It will be useful to see also Fig. 5.5 (a).

5.2.1 Minimally Generalized Goldstone Model

We have seen that the $U(1)$ -symmetry of the vacuum in the Goldstone model where only one complex scalar field φ acts is spontaneously broken when the chemical potential μ is properly adjusted; this is the rigorous theoretical result based on the non-perturbative analysis. The Goldstone model has been used to describe a dilute Bose gas in the classical limit at $T = 0$ as a (phenomenological) standard model;[89, 124] e.g. magnons, regardless[121] of ferromagnets[125, 45] or antiferromagnets.[54, 126, 124, 127] On the other hand, in real materials and experiments, there does exist variety kinds of freedoms besides the one on which we focus, such as magnetic impurities, phonons,[62] and photons[46] et al. (see also sec. 5.3.1). Therefore it is desirable to extend the Goldstone model so as to include the effects of such degrees of freedoms by introducing a new complex scalar field $\psi(\in \mathbb{C})$ which couples with the usual field φ .

Now, our strategy of the generalization of the Goldstone model reads as follows; for clearness, we exclusively focus on when the usual chemical potential μ is negative ($\mu \leq 0$). In that case, the model reads

$$V_{\text{Goldstone}}(\varphi) = \mathcal{B} |\varphi|^2 + \mathcal{J} |\varphi|^4, \quad (5.9)$$

in which we have denoted as, $-\mu =: \mathcal{B} (\geq 0)$, for convenience (see also sec. 5.3.1).² In this case, it is apparent that the vacuum expectation value of the field becomes zero ($\varphi_0 = 0$), which is not associated with the $U(1)$ -SSB of the vacuum (Fig. 5.2 (a)). Now, we investigate the possibility for the occurrence of the $U(1)$ -SSB of the vacuum owing to the coupling with other degrees of freedom represented by a complex scalar field $\psi(\in \mathbb{C})$ such as

$$(\varphi\psi^* + \varphi^*\psi), \quad \psi\psi^*, \quad (5.10)$$

and

$$|\varphi|^2 |\psi|^2, \quad (5.11)$$

²Note that the sign of $\mathcal{B}$ is opposite from the one of the chemical potential μ .

which do not violate the $U(1)$ -symmetry of the system; $(\varphi, \psi) \mapsto e^{i\theta}(\varphi, \psi)$, with $\theta \equiv (\text{const.}) \in \mathbb{R}$. It is expected that these couplings bring effective chemical potential to φ (φ^*). As the result, the total chemical potential might become positive and the $U(1)$ -SSB of the vacuum might be generated.

5.2.2 Minimal Model

We introduce the minimally generalized Goldstone model $V_{U(1)\text{-mini.}}(\varphi, \psi)$ by adding couplings, $(\varphi\psi^* + \varphi^*\psi)$ and $\psi\psi^*$, into the Goldstone model;

$$V_{U(1)\text{-mini.}}(\varphi, \psi) := V_{\text{Goldstone}}(\varphi) - \gamma(\varphi\psi^* + \varphi^*\psi) + \kappa\psi\psi^* \quad (5.12)$$

$$= \mathcal{B} |\varphi|^2 + \mathcal{J} |\varphi|^4 - \gamma(\varphi\psi^* + \varphi^*\psi) + \kappa |\psi|^2, \quad (5.13)$$

where each variable, $\gamma(\in \mathbb{R}$ and $\gamma > 0$) and $\kappa(\in \mathbb{R})$, represents a dimensionless coupling constant. The minimally generalized Goldstone model $V_{U(1)\text{-mini.}}$ includes two kinds of fields, φ and ψ . Therefore for the stability of the vacuum, there should be a lower bound on the energy level of the system in respect to ψ as well as φ ; [121] in terms of φ , the minimally generalized Goldstone model can be expressed as

$$V_{U(1)\text{-mini.}}(\varphi) = \mathcal{J} |\varphi|^4 + \mathcal{B} |\varphi|^2 - \gamma(\varphi\psi^* + \varphi^*\psi) + \mathcal{O}((\varphi^{(*)})^0). \quad (5.14)$$

Therefore the condition is required; $0 < \mathcal{J}$. In addition, from the viewpoint of ψ , $V_{U(1)\text{-mini.}}$ can be regarded as

$$V_{U(1)\text{-mini.}}(\psi) = \kappa |\psi|^2 - \gamma(\varphi\psi^* + \varphi^*\psi) + \mathcal{O}((\psi^{(*)})^0). \quad (5.15)$$

Thus the condition should be satisfied;

$$0 < \kappa. \quad (5.16)$$

Otherwise, the saddle point (see Fig. 5.4 (a) as an example) cannot be eliminated from the condition for the stationary point represented by eq. (5.4); the saddle point gives an unstable state and the situation is out of the aim of the present study.

Under these conditions (i.e. inequalities (5.3) and (5.16)), through the same procedure with Sec. 5.2 and within the classical theory, we seek the true stable vacuum of the minimally generalized Goldstone model. The condition for the stationary point in respect to ψ gives

$$\frac{\partial V_{U(1)\text{-mini.}}}{\partial \psi} = 0 \Rightarrow \psi^* = \frac{\gamma}{\kappa} \varphi^*. \quad (5.17)$$

On the point, $V_{U(1)\text{-mini.}}$ can be rewritten as

$$V_{U(1)\text{-mini.}}(\varphi, \psi = \frac{\gamma}{\kappa} \varphi) = (\mathcal{B} - \frac{\gamma^2}{\kappa}) |\varphi|^2 + \mathcal{J} |\varphi|^4 \quad (5.18)$$

$$= \mathcal{J} \left[|\varphi|^2 - \frac{1}{2\mathcal{J}} (\frac{\gamma^2}{\kappa} - \mathcal{B}) \right]^2 - \frac{1}{4\mathcal{J}} (\frac{\gamma^2}{\kappa} - \mathcal{B})^2 \quad (5.19)$$

$$=: V_{U(1)\text{-mini.}}(|\varphi|). \quad (5.20)$$

It is clear that the minimally generalized Goldstone model $V_{U(1)\text{-mini.}}$ is reduced to the standard one $V_{\text{Goldstone}}$ with the effective potential $(\gamma^2/\kappa - \mathcal{B})$; as expected, the coupling with other degrees of freedoms ψ has brought the effective chemical potential γ^2/κ (see eq. (5.18)). As

the result, the total chemical potential can become positive and hence, the $U(1)$ -SSB of the vacuum occurs (see Fig. 5.3 (a)) when

$$(0 <) \mathcal{B} < \frac{\gamma^2}{\kappa}. \quad (5.21)$$

Under this condition, the vacuum expectation value φ_0 (see Fig. 5.3 (b)) reads

$$|\varphi_0| = \sqrt{\frac{1}{2\mathcal{J}}\left(\frac{\gamma^2}{\kappa} - \mathcal{B}\right)}. \quad (5.22)$$

Here let us emphasize that when $\gamma = 0$ or $\kappa = 0$ (see inequalities (5.16) and (5.21)), the $U(1)$ -SSB of the vacuum cannot occur. That is, the γ -term as well as the κ -term in eq. (5.13) is essential for the occurrence of the $U(1)$ -SSB of the vacuum. Therefore we have named $V_{U(1)\text{-mini.}}$ the minimally generalized Goldstone model.

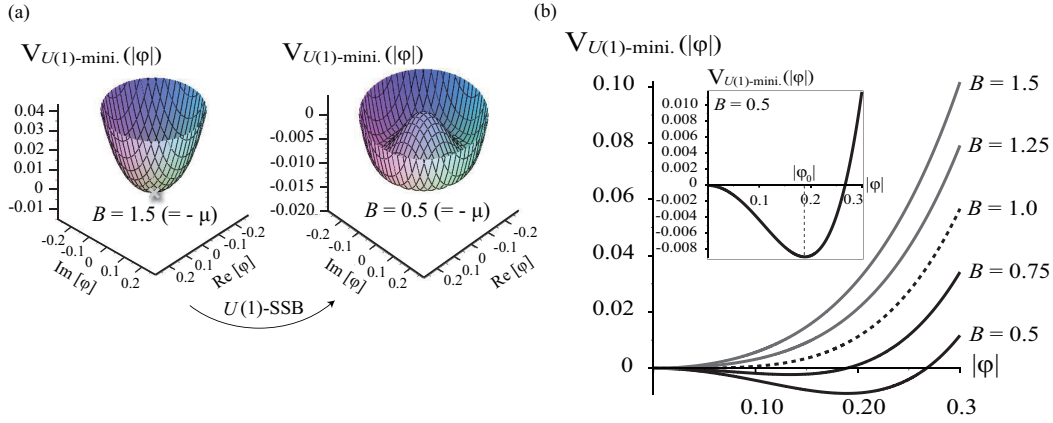


Figure 5.3: (Color online). Schematic pictures of the $U(1)$ -SSB of the vacuum in the minimally generalized Goldstone model $V_{U(1)\text{-mini.}}(|\varphi|)$; eq. (5.20). As an example, each dimensionless parameter is set as follows; $\mathcal{J} = 7$ and $\kappa = \gamma = 1$. Therefore for the occurrence of the $U(1)$ -SSB, the parameter $\mathcal{B}$ must satisfy the condition (inequality (5.21)); $(0 <) \mathcal{B} < 1$. (a) Even when parameter μ is negative ($0 < \mathcal{B} := -\mu$, see also Fig. 5.2 (a)), the $U(1)$ -SSB of the vacuum can occur in the minimally generalized Goldstone model because the effective potential is not μ , but $(\gamma^2/\kappa - \mathcal{B})$ (eq. (5.18)). (b) When $\mathcal{B} = 0.5$, the $U(1)$ -symmetry of the vacuum is spontaneously broken and the vacuum expectation value φ_0 (eq. (5.22)) becomes $|\varphi_0| \simeq 0.189$.

5.2.3 Stability of Vacuum

It would be useful to investigate the stability of the vacuum to confirm the importance of the repulsive interaction, [123] $\mathcal{J} |\varphi|^4$ with $0 < \mathcal{J}$, for the realization of the stable vacuum. For simplicity here, each coupling constant in eq. (5.13) is set as follows; $\mathcal{J} = \kappa = \mathcal{B} = 0$. On this condition, $V_{U(1)\text{-mini.}}$ becomes

$$V_{U(1)\text{-mini.}}^{\mathcal{J}=\kappa=\mathcal{B}=0}(\varphi, \psi) = -\gamma(\varphi\psi^* + \varphi^*\psi) \quad (5.23)$$

$$= -\gamma \begin{pmatrix} \varphi^* & \psi^* \end{pmatrix} \mathcal{A} \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, \quad (5.24)$$

with

$$\mathcal{A} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \mathcal{A}^\dagger. \quad (5.25)$$

It is clear that $V_{U(1)\text{-mini.}}^{\mathcal{J}=\kappa=\mathcal{B}=0}$ takes quadratic form and the matrix $\mathcal{A}$ is Hermitian. Therefore $V_{U(1)\text{-mini.}}^{\mathcal{J}=\kappa=\mathcal{B}=0}$ can be easily diagonalized[121] via an unitary matrix U as follows;

$$V_{U(1)\text{-mini.}}^{\mathcal{J}=\kappa=\mathcal{B}=0}(|\Phi_+|, |\Phi_-|) = -\gamma(|\Phi_+|^2 - |\Phi_-|^2), \quad (5.26)$$

with

$$\frac{1}{\sqrt{2}} \begin{pmatrix} \varphi + \psi \\ \varphi - \psi \end{pmatrix} =: \begin{pmatrix} \Phi_+ \\ \Phi_- \end{pmatrix}. \quad (5.27)$$

Fig. 5.4 (a) describes $V_{U(1)\text{-mini.}}^{\mathcal{J}=\kappa=\mathcal{B}=0}(|\Phi_+|, |\Phi_-|)$. It is apparent that the origin (i.e. $|\Phi_+|=|\Phi_-|=0$) has become a saddle point, which is not stable or the true vacuum; there are no true stable vacuum states in $V_{U(1)\text{-mini.}}^{\mathcal{J}=\kappa=\mathcal{B}=0}$.

Note that even when a non-zero value is given to each coupling constant (κ and $\mathcal{B}$), the situation does not change as long as $\mathcal{J}$ is zero; for simplicity here, we take $\gamma = 1$. On this condition, $V_{U(1)\text{-mini.}}$ becomes

$$V_{U(1)\text{-mini.}}^{\mathcal{J}=0, \gamma=1}(\varphi, \psi) = \mathcal{B}|\varphi|^2 + \kappa|\psi|^2 - (\varphi\psi^* + \varphi^*\psi) \quad (5.28)$$

$$= \begin{pmatrix} \varphi^* & \psi^* \end{pmatrix} \mathcal{A}' \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, \quad (5.29)$$

with

$$\mathcal{A}' := \begin{pmatrix} \mathcal{B} & -1 \\ -1 & \kappa \end{pmatrix} = (\mathcal{A}')^\dagger. \quad (5.30)$$

Also in this case, it is clear that $V_{U(1)\text{-mini.}}^{\mathcal{J}=0, \gamma=1}$ takes quadratic form and the matrix $\mathcal{A}'$ is Hermitian. Therefore $\mathcal{A}'$ can be diagonalized via an unitary matrix U' ;

$$U'^\dagger \mathcal{A}' U' = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}. \quad (5.31)$$

Each eigenvalue, $\lambda_\pm$, is determined by the following characteristic equation;

$$|\lambda E - \mathcal{A}'| = 0 \quad (5.32)$$

$$\Leftrightarrow \lambda = \frac{(\mathcal{B} + \kappa) \pm \sqrt{(\mathcal{B} + \kappa)^2 - 4(\mathcal{B}\kappa - 1)}}{2} \quad (5.33)$$

$$= \frac{(\mathcal{B} + \kappa) \pm \sqrt{(\mathcal{B} - \kappa)^2 + 4}}{2} \quad (5.34)$$

$$=: \lambda_\pm, \quad (5.35)$$

with $\lambda_- < \lambda_+$ by definition and note that $0 < \lambda_+$.³ According to eq. (5.33), λ_- becomes positive when $1 < \mathcal{B}\kappa (\neq 0)$; otherwise negative or zero ($\lambda_- \leq 0$).

By using these eigenvalues $\lambda_\pm$, $V_{U(1)\text{-mini.}}^{\mathcal{J}=0, \gamma=1}$ can be diagonalized as

$$V_{U(1)\text{-mini.}}^{\mathcal{J}=0, \gamma=1}(\Phi'_+, \Phi'_-) = \lambda_+ |\Phi'_+|^2 + \lambda_- |\Phi'_-|^2, \quad (5.36)$$

³Remember that $0 < \mathcal{B}$ and $0 < \kappa$.

in which the newly introduced complex scalar fields $\Phi'_\pm$ are represented by using an unitary matrix U' as $(\varphi^* \ \psi^*)U'^\dagger = ((\Phi'_+)^* \ (\Phi'_-)^*)$. Fig. 5.4 (b) describes $V_{U(1)\text{-mini.}}^{\mathcal{J}=0, \gamma=1}$ when $0 \leq \lambda_-$. In this case, though the origin (i.e. $|\Phi'_+| = |\Phi'_-| = 0$) is the stable vacuum state, it is not generated by the $U(1)$ -SSB; it is simply the original vacuum and it in fact gives $\varphi_0 = 0$. Let us remark that this can be easily confirmed also by the same procedure with sec. 5.2.2 (i.e. eq. (5.18)); $V_{U(1)\text{-mini.}}^{\mathcal{J}=0, \gamma=1}(\varphi, \psi = \varphi/\kappa) = (\mathcal{B} - 1/\kappa) |\varphi|^2$. On the other hand, when $\lambda_- < 0$, the situation is the same with $V_{U(1)\text{-mini.}}^{\mathcal{J}=\kappa=\mathcal{B}=0}$; Fig. 5.4 (a).

Therefore, we conclude that the true stable vacuum state associated with the $U(1)$ -SSB does not exist[121] without the repulsive interaction; $\mathcal{J} |\varphi|^4$ with $0 < \mathcal{J}$.

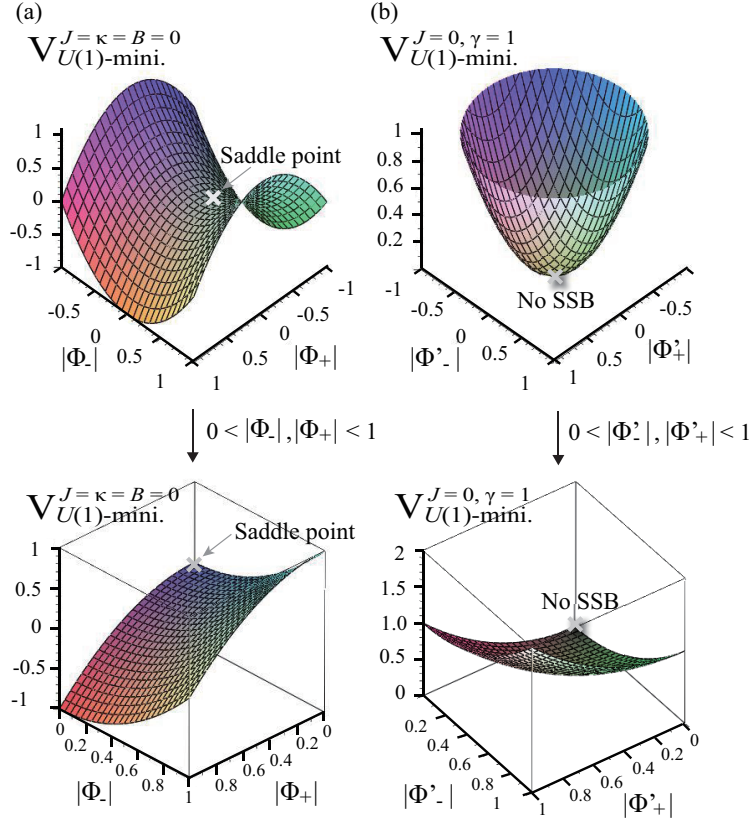


Figure 5.4: (Color online). (a) Plot of $V_{U(1)\text{-mini.}}^{\mathcal{J}=\kappa=\mathcal{B}=0}$ (eq. (5.26)) with $\gamma = 1$. The origin (i.e. $|\Phi_+| = |\Phi_-| = 0$) is a saddle point, which is not stable or the true vacuum; there are no true stable vacuum states in $V_{U(1)\text{-mini.}}^{\mathcal{J}=\kappa=\mathcal{B}=0}$. (b) Plot of $V_{U(1)\text{-mini.}}^{\mathcal{J}=0, \gamma=1}$ (eq. (5.36)) with $\lambda_+ = \lambda_- = +1$. The origin (i.e. $|\Phi'_+| = |\Phi'_-| = 0$) is the original vacuum; the state is not generated by the $U(1)$ -SSB and it in fact gives $\varphi_0 = 0$. (a) (b) Note that although the variables are restricted to $0 \leq |\Phi_{\pm}^{(\prime)}|$ by definition, we also have plotted the region; $-1 \leq |\Phi_{\pm}^{(\prime)}|$ for clearness.

5.3 Theoretical Proposal

We consider the application of the minimally generalized Goldstone model $V_{U(1)\text{-mini.}}$ to spin-pumping systems; the minimally generalized Goldstone model can be regarded to describe the dynamics of magnons interacting with spins carried by the conduction electrons at $T = 0$.

5.3.1 Anisotropic Exchange Interaction

In our previous work[30] based on the Schwinger-Keldysh formalism (i.e. a perturbative theory), we have studied thermal spin pumping[98] mediated by magnons in a ferromagnetic insulator and non-magnetic metal junction (Fig. 4.1), in which localized spins $\mathbf{S}$ isotropically interact with the conduction electrons $\mathbf{s}$ at the interface;

$$V_{\text{iso.}} := -2\gamma' \mathbf{S} \cdot \mathbf{s} = -2\gamma' (S^x s^x + S^y s^y + S^z s^z). \quad (5.37)$$

The variable $\gamma'(>0)$ represents the magnitude of the exchange interaction. As far as our perturbative analysis,[30] the macroscopic condensate order parameter becomes zero and magnon BEC cannot occur. Now, by considering the correspondence of the spin-pumping system (i.e. magnon-electron system) with the minimally generalized Goldstone model $V_{U(1)\text{-mini.}}$, we investigate the possibility for the occurrence of the stable magnon BEC state in spin-pumping systems on the basis of a non-perturbative theory.

For the purpose, first, let us consider the same situation[30] except the point that ferromagnetic localized spins $\mathbf{S}$ anisotropically interact with the conduction electrons $\mathbf{s}$;

$$V_{\text{aniso.}} := -2\gamma' (S^x s^x + S^y s^y + \Delta S^z s^z) \quad (5.38)$$

$$= -2\gamma' \left(\frac{S^+ s^- + S^- s^+}{2} + \Delta S^z s^z \right), \quad (5.39)$$

in which Δ represents the magnitude of the anisotropic; $0 \leq \Delta \leq 1$. Reflecting the fact that the minimally generalized Goldstone model (eq. (5.13)) has not included $|\varphi|^2 |\psi|^2$, we here focus on the strong anisotropic limit, $\Delta \rightarrow 0$;

$$V_{\text{aniso.}} \xrightarrow{\Delta \rightarrow 0} -\gamma' (S^+ s^- + S^- s^+) \quad (5.40)$$

$$=: V_{\text{aniso.}(\Delta=0)}. \quad (5.41)$$

Via the Holstein-Primakoff transformation; $S^+ = \sqrt{2S}a + \mathcal{O}(1/\sqrt{S})$, $S^- = \sqrt{2S}a^\dagger + \mathcal{O}(1/\sqrt{S})$, $S^z = S - a^\dagger a$, $V_{\text{aniso.}(\Delta=0)}$ can be expressed in terms of magnon creation/annihilation operators as follows;

$$V_{\text{aniso.}(\Delta=0)} = -\sqrt{2S}\gamma' (as^- + a^\dagger s^+). \quad (5.42)$$

5.3.2 Correspondence to Spin-pumping System

We take the classical limit[89] and each operator is replaced with a commutative complex scalar field (i.e. c-number); $a \xrightarrow{\text{classical}} \varphi \in \mathbb{C}$, $s^+ \xrightarrow{\text{classical}} \psi \in \mathbb{C}$. As the result, in the classical limit, the strong anisotropic exchange interaction between magnons (i.e. spin waves) and the conduction electrons $V_{\text{aniso.}(\Delta=0)}^{\text{cla.}}$ is rewritten as follows;

$$V_{\text{aniso.}(\Delta=0)}^{\text{cla.}} = -\sqrt{2S}\gamma' (\varphi\psi^* + \varphi^*\psi). \quad (5.43)$$

It is clear that this term corresponds to the γ -term in the minimally generalized Goldstone model $V_{U(1)\text{-mini.}} = \mathcal{B} |\varphi|^2 + \mathcal{J} |\varphi|^4 - \gamma(\varphi\psi^* + \varphi^*\psi) + \kappa |\psi|^2$; in the language of the spin-pumping system, the $\mathcal{B}$ -term describes the couplings with the effective magnetic field along the z-axis for magnons, the κ -term represents the interaction between up-spins and down-spins of the conduction electrons, and the $\mathcal{J}$ -term corresponds to the magnon-magnon interaction. Therefore, if these quantities satisfy the condition for the occurrence of the $U(1)$ -SSB shown in Sec. 5.2.2 (inequality (5.21)); $(0 <) \mathcal{B} < (\gamma^2/\kappa)$, the stable magnon BEC state can be realized even under the interaction with the conduction electrons. The vacuum expectation value φ_0 , which corresponds to the macroscopic condensate order parameter of magnons (i.e. spin waves), becomes a non-zero value (see Fig. 5.3); $|\varphi_0| = \sqrt{(\gamma^2/\kappa - \mathcal{B})/2\mathcal{J}}$.

5.3.3 Stable Magnon BEC

Here let us again remark that, as stressed in sec. 5.2.2, for the occurrence of the stable magnon BEC state in the spin-pumping system, the repulsive interaction between up-spins and down-spins of the conduction electrons (i.e. $0 < \kappa$) is essential as well as the repulsive magnon-magnon interaction (i.e. $0 < \mathcal{J}$), which can be realized, as an example,[121] owing to the dipolar interaction.[125] This is the rigorous theoretical result based on the non-perturbative theory beyond a perturbative one[30] and the one in sharp contrast to the far-seeing work by Bender et al.,[91] who has already pointed out the possibility for the occurrence of magnon BEC in spin-pumping systems before our present work by using the similar model with us except the point that they have omitted the magnon-magnon interaction. On top of this, we would like to remark that they have not included the effects of the exchange interaction between the conduction electrons and the magnons (eq. (5.42) or (5.43)), which characterize spin-pumping systems.

Lastly, it might be useful to mention that in Sec. 5.2.3, we have closely investigated the stability of the vacuum in the minimally generalized Goldstone model $V_{U(1)\text{-mini.}}$. There, we have concluded that the true stable vacuum state associated with the $U(1)$ -SSB does not exist without the repulsive interaction (see Fig. 5.4); $\mathcal{J} |\varphi|^4$ with $0 < \mathcal{J}$. This means, in the language of the above spin-pumping system, that the true stable magnon BEC state cannot exist without the repulsive magnon-magnon interaction; although the minimally generalized Goldstone model $V_{U(1)\text{-mini.}}^{\mathcal{J}=0, \gamma=1}$ can possess the stable vacuum state when $0 \leq \lambda_-$ (see Fig. 5.4 (b)), it is not generated by the $U(1)$ -SSB and it in fact gives $\varphi_0 = 0$. Therefore we suspect that magnon BEC cannot occur[121] in the spin-pumping system[91] described by the minimally generalized Goldstone model with $\mathcal{J} = 0$ (i.e. $V_{U(1)\text{-mini.}}^{\mathcal{J}=\kappa=\mathcal{B}=0}$ or $V_{U(1)\text{-mini.}}^{\mathcal{J}=0, \gamma=1}$). Here, let us point out that BEC should not be identified with superfluid[128, 89, 121] and hence there might exist a superfluid phase in that case (i.e. $\mathcal{J} = 0$).[121, 91]

5.3.4 Discussion; Field Theory and One-body Quantum Mechanics

We have discussed this issue at University of Basel during my stay on July 9 (2013) with Dr. Tserkovnyak, who is the coauthor of the paper by Bender. They have argued that from the viewpoint of one-body quantum mechanics, their free magnons actually undergo BEC because they are Bose-particles and the state (i.e. magnon BEC state) is stable. Of course we have agreed that their discussion has captured some aspects of their and our dynamics, but we still would like to remark that their stability arises from the definition of the theory they have employed. That is, it results from the fact that one-body quantum mechanics assume that there are no creations and annihilations of particles and their state is stable by definition. Then, we suspect that one-body quantum mechanics is not useful or is not applicable to the dynamics where the creations and annihilations of particles due to many-body interactions play the key role. Therefore, we consider that their discussion is applicable to only the ferromagnetic insulators at low temperature.

Let us stress that at the interface of spin-pumping systems (i.e. the interface between ferromagnetic insulators and non-magnetic metals), there does exist the exchange interaction between the conduction electrons and the magnons, which characterize the spin pumping effects; without the exchange interaction, spin pumping does not occur. Then, we consider that in order to investigate the possibility for the generation of spin current carried by condensate magnons (i.e. which is associated with magnon BEC), one should first take the effects of the exchange interaction into account. The, we have employed the present approach and have found that magnon-magnon interactions are essential to the realization of stable magnon

BEC.

Lastly, let us again emphasize that one should understand (or take more seriously into account) the fact that although the total spin angular momentum of spin-pumping systems is conserved, each spin conservation law of the conduction electrons and the magnons is broken due to the exchange interaction. As the result, torque term arises and it plays the central role on spin pumping

5.3.5 Generalized Goldstone Model

The next focus lies on whether the stable magnon BEC state could exist under a finite (i.e. non-zero) Δ regime in the above spin-pumping system.[30] For the purpose, we include the term $|\varphi|^2 |\psi|^2$, which arises from the Δ -term in $V_{\text{aniso.}}$ (eq. (5.39));

$$V_{U(1)}(\varphi, \psi) := V_{U(1)\text{-mini.}}(\varphi, \psi) - \alpha |\varphi|^2 |\psi|^2 \quad (5.44)$$

$$\begin{aligned} &= \mathcal{B} |\varphi|^2 + \mathcal{J} |\varphi|^4 - \alpha |\varphi|^2 |\psi|^2 \\ &- \gamma(\varphi\psi^* + \varphi^*\psi) + \kappa |\psi|^2, \end{aligned} \quad (5.45)$$

where $\alpha(\in \mathbb{R})$ is the corresponding dimensionless coupling constant. From the viewpoint of the correspondence with the above spin-pumping system described by $V_{\text{aniso.}}$, we restrict α to a positive value; $0 < \alpha$.

In the language of the spin-pumping system (i.e. the Holstein-Primakoff transformation), the variable $|\varphi|^2$ represents the number of magnons obeying the parastatistics[129] and hence, the relation; $|\varphi|^2 < S = \mathcal{O}(1)$, is required by definition. Moreover because we here treat the extremely low temperature regime (i.e. $T = 0$), the variable $|\varphi|^2$ is supposed to be very small enough to satisfy the relation; $|\varphi|^2 \ll \mathcal{O}(1)$. Therefore when we choose variables, κ and α , to satisfy the condition; $\kappa/\alpha = \mathcal{O}(1)$, we are allowed to assume the relation; $|\varphi|^2 \ll \kappa/\alpha \Leftrightarrow \alpha |\varphi|^2 \ll \kappa$. Also from the viewpoint of the stability of the system in respect to $\psi(\psi^*)$, $V_{U(1)}(\psi) = (\kappa - \alpha |\varphi|^2) |\psi|^2 + \mathcal{O}(\psi^{(*)}) + \mathcal{O}(\psi^{(*)0})$, the relation is strongly required. Thus from now on, we discuss on the basis of the assumption; $\alpha |\varphi|^2 \ll \kappa$. In other words, the following our analysis is adequate in the region.

Through the same procedure with the minimally generalized Goldstone model $V_{U(1)\text{-mini.}}$ and the approximation; $(\kappa - \alpha |\varphi|^2)^{-1} \simeq (1 + \alpha |\varphi|^2 / \kappa) / \kappa$, the generalized Goldstone model $V_{U(1)}$ on the point, $\psi = \gamma\varphi / (\kappa - \alpha |\varphi|^2) \Leftarrow \partial V_{U(1)} / (\partial \psi) = 0$, reads

$$V_{U(1)}(\varphi, \psi = \frac{\gamma}{\kappa - \alpha |\varphi|^2} \varphi) = (\mathcal{B} - \frac{\gamma^2}{\kappa})\chi + (\mathcal{J} - \frac{\alpha\gamma^2}{\kappa^2})\chi^2 - \frac{2\alpha^2\gamma^2}{\kappa^3}\chi^3 \quad (5.46)$$

$$=: V_{U(1)}(\chi), \quad (5.47)$$

with $\chi := |\varphi|^2 (> 0)$.

Here let us denotes the solution of the equation, $dV_{U(1)}(\chi)/(d\chi) = 0$, as $\chi_{\pm}$ with $\chi_- < \chi_+$ by definition (see Fig. 5.5 (b)). The coefficient of χ^3 in $V_{U(1)}(\chi)$, $2\alpha^2\gamma^2/(\kappa^3)$, takes a positive value. Therefore for the occurrence of the $U(1)$ -SSB associated with the stable magnon BEC state, the condition is required (Fig. 5.5 (b));

$$0 < \chi_- \quad (5.48)$$

and

$$S < \chi_0. \quad (5.49)$$

That is, when

$$\frac{\alpha\gamma^2}{\kappa^2} < \mathcal{J}, \quad (5.50)$$

$$\frac{\gamma^2}{\kappa} - \frac{\kappa^3}{6\alpha^2\gamma^2} \left(\mathcal{J} - \frac{\alpha\gamma^2}{\kappa^2} \right)^2 < \mathcal{B} < \frac{\gamma^2}{\kappa}, \quad (5.51)$$

and

$$S < \chi_0 \quad (5.52)$$

with

$$\chi_0 = \frac{\kappa^3}{6\alpha^2\gamma^2} \left[\left(\mathcal{J} - \frac{\alpha\gamma^2}{\kappa^2} \right) + 2\sqrt{\left(\mathcal{J} - \frac{\alpha\gamma^2}{\kappa^2} \right)^2 + \frac{6\alpha^2\gamma^2}{\kappa^3} \left(\mathcal{B} - \frac{\gamma^2}{\kappa} \right)} \right], \quad (5.53)$$

the $U(1)$ -SSB associated with the stable magnon BEC state (χ_-) occurs;

$$\chi_- = \frac{\kappa^3}{6\alpha^2\gamma^2} \left[\left(\mathcal{J} - \frac{\alpha\gamma^2}{\kappa^2} \right) - \sqrt{\left(\mathcal{J} - \frac{\alpha\gamma^2}{\kappa^2} \right)^2 + \frac{6\alpha^2\gamma^2}{\kappa^3} \left(\mathcal{B} - \frac{\gamma^2}{\kappa} \right)} \right]. \quad (5.54)$$

Let us remark that magnons obey the parastatistics and hence when $S > \chi_0$, the state χ_- becomes the classically metastable state[39, 115, 130] and it does not give the absolute minimum. That is, the state χ_- is not the true stable vacuum and it can decay to the true vacuum by the quantum-mechanical tunneling effect[39] (see Fig. 5.5 (a) as an example). Of course we have noted that we have been theoretically discussing within the classical theory, but quantum effects are inevitable in real materials (i.e. experiments). Thus, the condition $S < \chi_0$ (eq. (5.52)) is required for the experimental realization of stable magnon BEC in spin-pumping systems with the non-zero Δ -term (i.e. α -term).

5.4 Summary and Outlook

In order to investigate the possibility for the stable magnon BEC state associated with SSB in spin-pumping systems, we have employed a non-perturbative theory to go beyond the perturbative analysis and have extended the standard Goldstone model. For the realization of the stable magnon BEC state, the repulsive interaction between up-spins and down-spins of the conduction electrons is essential as well as the repulsive magnon-magnon interaction. By realizing the condition we have clarified in this work, the true stable magnon BEC state can be experimentally observed also in spin-pumping systems without using microwaves.

On the other hand, to extend the system at finite temperature with the influence of quantum effects is left as a future work. On top of this, we consider that to clarify the effects of the unusual energy dispersion of the lowest magnon mode in YIG, which is a relevant material to the experiment of magnon BEC[46, 47, 56] and spin pumping,[65, 86] is a significant theoretical issue.

In addition, as stressed by Hick et al.[128] BEC of quasiparticles is not necessarily associated with superfluidity.[89] In other words, they should not be identified.[121] Of course it is roughly expected, owing to Bogoliubov theory,[89] that superfluid of magnons is associated with magnon BEC in spin-pumping systems, but to reveal the detailed relationship between magnon BEC and superfluid[44] of magnons in spin-pumping systems is left as a important future work.

BEC state (i.e. coherent state) is the robust macroscopic quantum state against the loss of information (see also Sec. 4.6). Therefore we hope this work becomes a bridge between the research on spintronics and magnon BEC to lead to the information technologies.

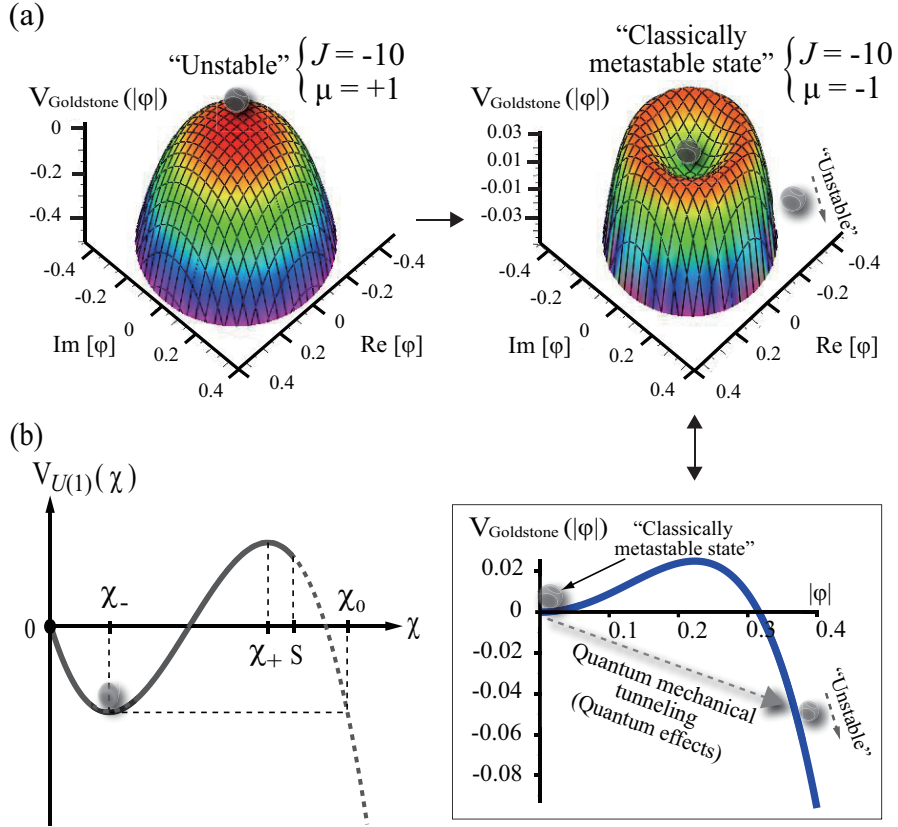


Figure 5.5: (Color online). (a) Plot of the Goldstone model $V_{\text{Goldstone}}$ (eq. (5.7)) with $\mathcal{J} < 0$. The $U(1)$ -SSB of the vacuum does not occur. The origin (i.e. $|\varphi| = 0$) is unstable ($0 \leq \mu$) or the classically metastable state ($\mu < 0$). (b) A schematic picture of the generalized Goldstone model $V_{U(1)}(\chi)$ (eq. (5.47)). When $\chi_- \leq 0$, the situation is the same with (a). The state χ_0 is defined as $V_{U(1)}(\chi = \chi_0) = V_{U(1)}(\chi = \chi_-)$.

Chapter 6

Dissipation Theory

Until now, we have been treating dissipation-less systems. On the other hand, in real materials, dissipation effects are inevitable. Then, we now investigate the dissipation effects described by the Caldeira-Leggett model as well as the LLG equation and clarify the distinction between them from the viewpoint of the time-development of the macroscopic condensate order parameter (i.e. magnon BEC).

The famous Caldeira-Leggett model,[63, 64] which is a prototype of a system-reservoir model for the description of dissipation phenomena (Fig. 6.1), was originally introduced to describe a Brownian particle of mass M with coordinate x which moves in a potential $V(x)$. Then, by employing the Caldeira-Leggett model[63, 64] with the help of the Holstein-Primakoff transformation,[39] we investigate the dissipation effects on magnon BEC[88] and clarify the distinction from those described by the LLG equation.¹

6.1 Caldeira-Leggett Model

The master equation² which the Caldeira-Leggett model gives reads as follows:[64]

$$\langle \dot{x} \rangle = \frac{1}{M} \langle p \rangle \quad (6.1)$$

$$\langle \dot{p} \rangle = -\langle V'(x) \rangle - 2\gamma \langle p \rangle. \quad (6.2)$$

It should be noted that, the variables x and p with $[x, p] = i$ can be represented in terms of magnon annihilation and creation operators, a and $a^\dagger$ with $[a, a^\dagger] = 1$, and vice versa, which reads as follows.

The Hamiltonian of the zero-mode of magnons, which has been produced by microwaves, under the static magnetic field³ B becomes

$$\mathcal{H}_B = B a_0^\dagger a_0, \quad (6.3)$$

with $[a_0, a_0^\dagger] = 1$.

In order to reduce[121] the dynamics of magnons to the Caldeira-Leggett model, we use

¹Regarding the detail, please see (Appendix) Chap. 11.

²In the original paper by Caldeira and Leggett,[63] they have discussed on the basis of the path-integral method.

³It includes the g -factor and Bohr magneton.

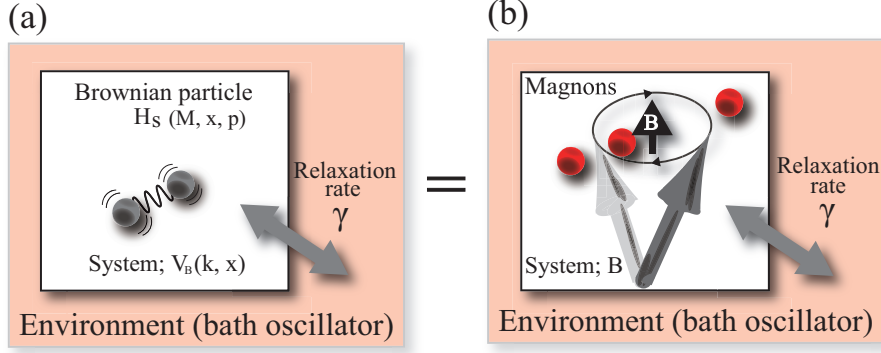


Figure 6.1: (Color online) Schematic pictures of the Caldeira-Leggett model; the correspondence between Brownian particles and magnons under dissipations due to phonons.

the following well-known relations[39] (i.e. harmonic oscillator variables);⁴

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right) \quad (6.4)$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{ip}{m\omega}\right). \quad (6.5)$$

In other words,

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger) \quad (6.6)$$

$$p = \frac{1}{i}\sqrt{\frac{\hbar m\omega}{2}}(a - a^\dagger). \quad (6.7)$$

This satisfies the relations; $[x, p] = i$ and $[a, a^\dagger] = 1$. We then replace the representation; $a \rightarrow a_0$, $a^\dagger \rightarrow a_0^\dagger$. Consequently, the Hamiltonian $\mathcal{H}_B$ can be rewritten as follows;

$$\mathcal{H}_B = Ba_0^\dagger a_0 \quad (6.8)$$

$$= \frac{1}{2M}p^2 + V_B(x) + (\text{const.}) \quad (6.9)$$

$$= \mathcal{H}_s, \quad (6.10)$$

with

$$V_B(x) = \frac{1}{2}kx^2, \quad M := \frac{\hbar m\omega}{B}, \quad k := \frac{Bm\omega}{\hbar}. \quad (6.11)$$

This means that the dynamics of the zero-mode of magnons after switching off under the static magnetic field B corresponds to that of a Brownian particle with the mass M trapped to the (harmonic) potential $V(x) = kx^2/2$; see Fig. 6.1. Then by using the Caldeira-Leggett model, eqs. (6.1) and (6.2), we can study the dynamics of the zero-mode of magnons after switching off with dissipation effects due to phonons. This is the most important point of the present work.

⁴The variable m denotes the mass of oscillators and ω represents the angular frequency.

We here replace the expectation value, $\langle a_0 \rangle$ and $\langle a_0^\dagger \rangle$, with the real scholar field ϕ_1, ϕ_2 (i.e. classical approximations);⁵

$$\langle a_0 \rangle = \sqrt{\frac{m\omega}{2\hbar}}(\langle x \rangle + \frac{i\langle p \rangle}{m\omega}) \rightarrow \phi_1 + i\phi_2 \quad (6.12)$$

$$\langle a_0^\dagger \rangle = \sqrt{\frac{m\omega}{2\hbar}}(\langle x \rangle - \frac{i\langle p \rangle}{m\omega}) \rightarrow \phi_1 - i\phi_2. \quad (6.13)$$

That is,

$$\langle x(t) \rangle = \sqrt{\frac{2\hbar}{m\omega}}\phi_1 \quad (6.14)$$

$$\langle p(t) \rangle = \sqrt{2\hbar m\omega}\phi_2. \quad (6.15)$$

Therefore by using the notation, $\langle a_0(t) \rangle =: \phi_1(t) + i\phi_2(t)$ with $\phi_{1(2)} \in \mathbf{R}$, the above master equation (i.e. eqs. (6.1) and (6.2)) of the Caldeira-Leggett model under the condition $\gamma \ll B$ can be rewritten as follows;

$$\dot{\phi}_1(t) = \omega_B \phi_2(t) \quad (6.16)$$

$$\dot{\phi}_2(t) = -\omega_B \phi_1(t) - 2\gamma \phi_2(t), \quad (6.17)$$

with $\omega_B := B/\hbar$.

Figs. 6.2 (a) and (b) shows the time-development of the macroscopic condensate order parameter $|\langle a(t) \rangle|$ under dissipations described by Caldeira-Leggett model. It is clear that there are several plateau regions; $|\langle a(t) \rangle| = (\text{const.})$ as the function of time t .

6.2 LLG Equation

We also investigate the dissipation effect described by LLG equation on magnon BEC. Then, we clarify the distinction between dissipation effects by the Caldeira-Leggett model and the LLG equation from the viewpoint of magnon BEC.

The LLG equation,[22, 23, 24] which is often used for the study of spintronics, reads ($\mathbf{H} = (\Gamma(t), 0, B)$)

$$\dot{\mathbf{S}} = \mathbf{S} \times \mathbf{H} - \alpha \mathbf{S} \times \dot{\mathbf{S}}, \quad (6.18)$$

where spins are treated as classical variables and the effects of dissipation are phenomenologically taken into account as the Gilbert damping term (i.e. α -term).[26] Each component of LLG equation becomes

$$\dot{S}^x = BS^y - \alpha(S^y \dot{S}^z - S^z \dot{S}^y) + \mathcal{O}(\alpha^2), \quad (6.19)$$

$$\dot{S}^y = -BS^x + \Gamma S^z - \alpha(S^z \dot{S}^x - S^x \dot{S}^z) + \mathcal{O}(\alpha^2), \quad (6.20)$$

$$\dot{S}^z = -\Gamma S^y - \alpha(S^x \dot{S}^y - S^y \dot{S}^x) + \mathcal{O}(\alpha^2). \quad (6.21)$$

When microwaves are applied for a certain time, they give a non-zero value to S^x and S^y . After that, one switches off microwaves, which corresponds to the procedure $\Gamma(t) \rightarrow 0$. As the result, the condensate order parameter $|a(t)| \sim |S^x(t) + iS^y(t)| = \sqrt{(S^x(t))^2 + (S^y(t))^2}$ varies as shown in Fig. 6.2 (c).

⁵That is, we neglect quantum fluctuations.

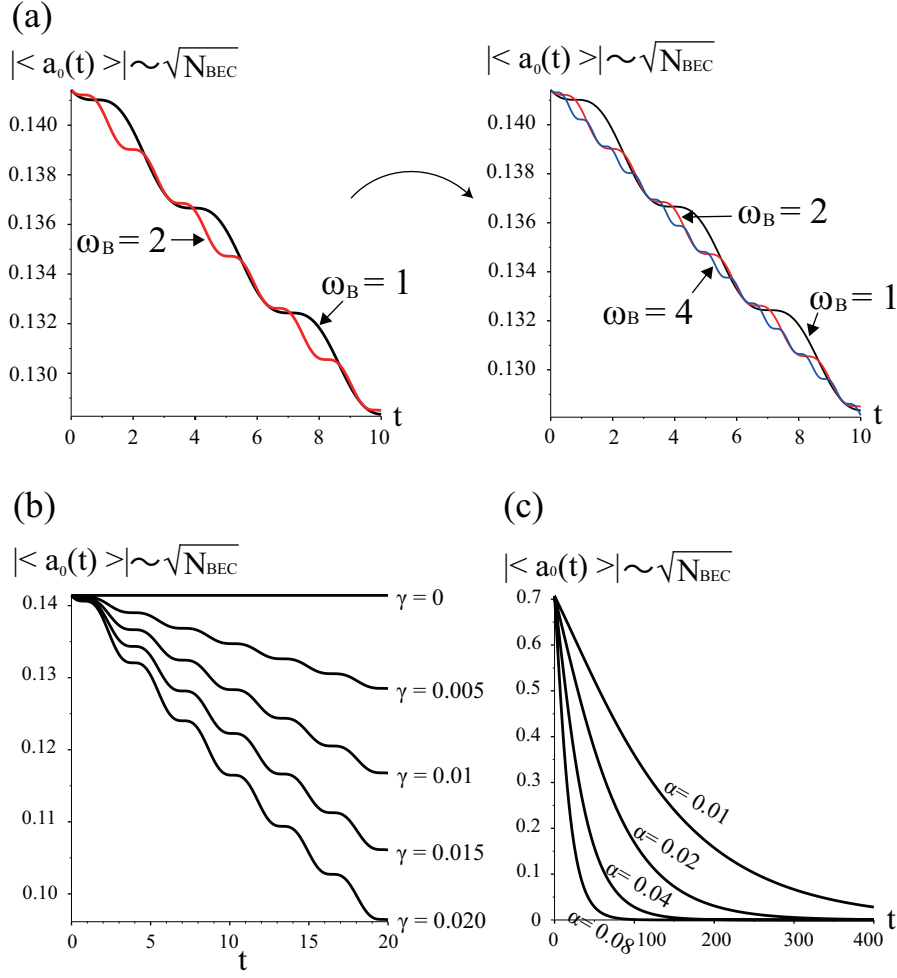


Figure 6.2: (Color online) (a) (b) The time-development of the macroscopic condensate order parameter $|\langle a_0(t) \rangle| = \sqrt{\phi_1(t)^2 + \phi_2(t)^2}$ under dissipations described by Caldeira-Leggett model with (a) $\gamma = 0.01$ and (b) $\omega_B = 1$. There are several plateau regions. (c) That by LLG equation with $B = 1$. There are no plateau regions.

Distinction

It is clear that there are no plateaus in Fig. 6.2 (c) and this result is in sharp contrast to the case of the analysis based on Caldeira-Leggett model; Figs. 6.2 (a) and (b). Then, we conclude that the emergence of plateaus is the unique property of the system described by Caldeira-Leggett model.

6.3 Conclusion

The emergence of the plateau region in the macroscopic condensate order parameter of magnon BEC is peculiar to the system described by Caldeira-Leggett model (see also (Appendix) Chap. 11), which is in sharp contrast to the system described by LLG equation

To clarify the microscopic mechanism for the emergence of the plateaus might become an important issue.⁶[134, 135, 136]

⁶At this stage, I do not think our plateaus emerging in Caldeira-Leggett model in terms of magnon BEC

have some relations with the recently attractive novel phenomenon called pre-thermalization.[131, 132, 133, 43] Our plateaus have emerged simply due to the combination of the relaxation (i.e. decay) effect and the oscillation effect. For the clear and deep understanding, we need further detailed analysis.

Chapter 7

Theory on Josephson Effects in Magnon BEC through Aharonov-Casher Phase

As the synthesis of our works, we[137] present the microscopic theory on the Josephson effect[73, 70] in magnon BEC[78] through quantum-mechanical phases called A-C phases.[66, 67] As mentioned in Sec. 1.1.5, the A-C phase[66, 67] is dual to the Aharonov-Bohm phase[69, 4, 5] and each quantum-mechanical phase is just a special case of geometric phase called Berry's phase, which causes phase shifts; as a charged particle feels a magnetic flux (i.e. a magnetic vector potential) and obtain the Aharonov-Bohm phase, a moving magnetic dipole is affected by electric fields and acquires the Aharonov-Casher phase (see Fig. 1.5).

We then microscopically construct the theory on the Josephson effect[70, 72, 73] of magnon BEC through A-C phases, which are under our control via electric fields and work as driving forces. This is the phenomenon analogous to the magnon Hall effect[138, 139] in the sense that transverse Josephson spin currents of the magnon condensate are generated by electric fields via A-C phases.

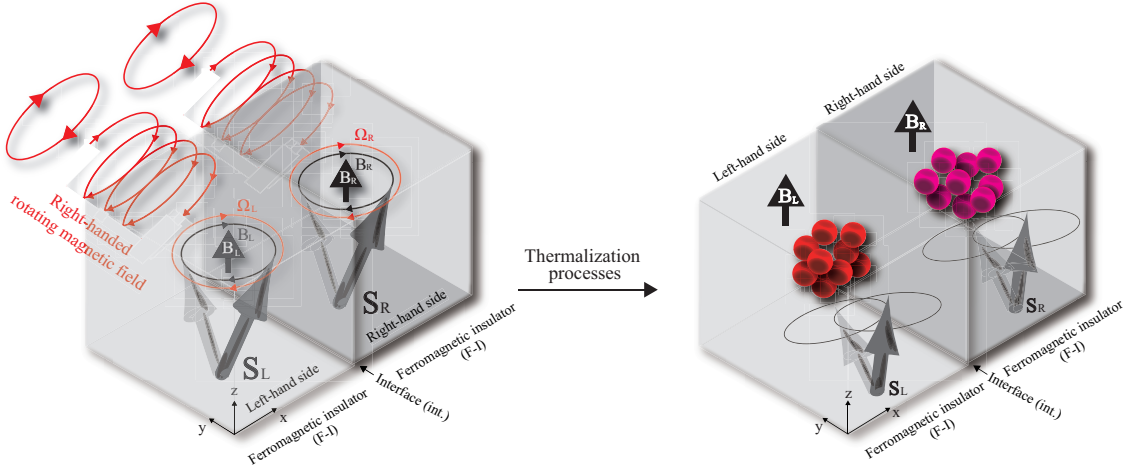


Figure 7.1: (Color online) Schematic pictures of Josephson effects by the magnon condensate, namely the magnon BEC Josephson junction. Each ferromagnetic insulator consisting of the magnon condensate is weakly connected to each other via the tunneling Hamiltonian V_{ex} . Regarding thermalization processes, please see (Appendix) Chap. 10.

7.1 System and Hamiltonian

We consider the junction shown in Fig. 7.1 in which two ferromagnetic insulators are weakly connected to each other via the exchange interaction J_{ex} ;

$$V_{\text{ex}} := -J_{\text{ex}} l_0^3 \int_{\mathbf{x} \in (\text{int.})} d\mathbf{x} \mathbf{S}_L \cdot \mathbf{S}_R, \quad (7.1)$$

where $\mathbf{S}_{L(R)}$ denotes the spin variable of the left(right)-hand side ferromagnetic insulator whose lattice constant reads l_0 . Via the Holstein-Primakoff transformation, $S^+ = S^x + iS^y = \sqrt{2\tilde{S}}a + \mathcal{O}((\tilde{S})^{-1/2}) = (S^-)^\dagger$ and $S^z = \tilde{S} - a^\dagger a$ with $\tilde{S} := S/(l_0)^3$, eq. (7.1) is reduced to the following tunneling Hamiltonian[130] consisting of magnon creation/annihilation operators which satisfy the commutation relation $[a(\mathbf{x}), a^\dagger(\mathbf{x}')] = \delta(\mathbf{x} - \mathbf{x}')$;

$$V_{\text{ex}} = -J_{\text{ex}} S \int_{\mathbf{x} \in (\text{int.})} d\mathbf{x} (a_L a_R^\dagger + a_L^\dagger a_R), \quad (7.2)$$

in which the interface is defined as an effective area where magnons coming from both sides coexist to be interchanged via the above tunneling Hamiltonian V_{ex} (i.e. the tunneling amplitude $J_{\text{ex}} \neq 0$) originating from proximity effects. Here let us remark that in eq. (7.2), we have omitted the terms arising from the z-component of spin variables in eq. (7.1), such as $a^\dagger a$ and $a^\dagger a^\dagger a a$, because they commute with the magnon number density $a^\dagger a$ and hence they are not relevant to tunneling effects on which we focus in the present work.

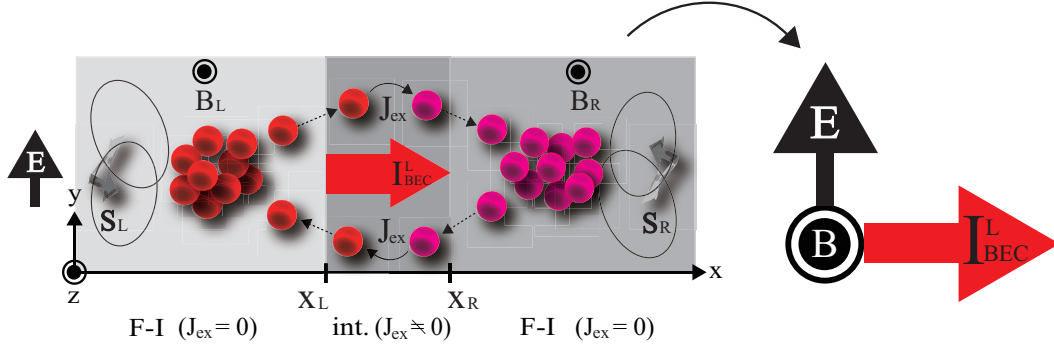


Figure 7.2: (Color online) Enlarged view of the magnon BEC Josephson junction. The electric field $\mathbf{E} = (0, E, 0)$, which produces A-C phases and induces transverse spin currents along the x -axis, is applied to the whole system.

Under the tunneling Hamiltonian V_{ex} , we clarify the transport properties of the quasi-equilibrium magnon condensate, which can be produced by using the method called microwave pumping (see also Chap. 4 and (Appendix) Chap. 10);[46] applied microwaves excite the zero-mode of magnons (i.e. the uniform state) and drive the system out of equilibrium to reach a certain non-equilibrium steady state. After that, microwaves are switched off and the system experiences thermalization processes. As the result, magnons dynamically condense and we obtain the quasi-equilibrium magnon condensate which has acquired the following macroscopic condensate order parameter,[45, 61, 46]

$$\langle a(t) \rangle = \sqrt{n_{\text{BEC}}(t)} e^{-iBt}, \quad (7.3)$$

where the variable $n_{\text{BEC}}(t)$ denotes the number density of the magnon condensate and B represents the applied magnetic field including the g -factor and Bohr magneton along the

quantization axis (i.e. z-axis). It should be noted that only the zero-mode of magnons is relevant to this dynamical condensation and they work as the macroscopic quantum state, namely the macroscopic coherent precession with the angular frequency B .

Dephasing Effects

Let us remark that we now treat the weakly connected junction consisting of magnons in BEC (Fig. 7.1). Therefore we assume dephasing effects significant enough that coherence between the left and right subsystems is destroyed and the density matrix for the entire system ρ is always in the form,[91]

$$\rho = \rho_L \otimes \rho_R, \quad (7.4)$$

which gives the following evaluation;

$$\langle a_L(t) a_R^\dagger(t) \rangle := \text{tr}[\rho a_L(t) a_R^\dagger(t)] \quad (7.5)$$

$$= \langle a_L(t) \rangle_L \langle a_R^\dagger(t) \rangle_R, \quad (7.6)$$

where¹

$$\text{tr}[\rho_L a_L(t)] =: \langle a_L(t) \rangle_L \quad (7.7)$$

$$= \sqrt{n_{\text{BEC}}^L(t)} e^{-iB_L t} \quad (7.8)$$

and

$$\text{tr}[\rho_R a_R(t)] =: \langle a_R(t) \rangle_R = \sqrt{n_{\text{BEC}}^R(t)} e^{-iB_R t}. \quad (7.9)$$

From now on, on the basis of the above condition, we investigate the tunneling effects of the magnon condensate under quantum-mechanical phases. This is the main purpose of the present work.

7.2 Aharonov-Casher Phase

A moving dipole moving in an electric field acquires a quantum-mechanical phase called the A-C phase,[66, 140, 141] which is dual to the Aharonov-Bohm phase;[69, 4, 5] each quantum-mechanical phase is just a special case of geometric phase called Berry's phase. As the result, under the influence of the applied electric field $\mathbf{E} = (0, E, 0)$ shown in Fig. 7.1, the tunneling Hamiltonian V_{ex} is reduced to the form $V_{\text{ex}}^{\text{A-C}}$;^[68]

$$V_{\text{ex}}^{\text{A-C}} = -J_{\text{ex}} S \int_{\mathbf{x} \in (\text{int.})} d\mathbf{x} (a_L a_R^\dagger e^{-i\theta_{\text{A-C}}} + a_L^\dagger a_R e^{i\theta_{\text{A-C}}}) \quad (7.10)$$

with the A-C phase given by

$$\theta_{\text{A-C}} := \frac{g\mu_B}{\hbar c^2} \int_{\mathbf{x}_L}^{\mathbf{x}_R} d\mathbf{x} \cdot (\mathbf{E} \times \mathbf{e}_z). \quad (7.11)$$

It is clear that this Hamiltonian $V_{\text{ex}}^{\text{A-C}}$ has the $U(1)$ -symmetry (i.e. invariant under the following global gauge transformation; $a \mapsto a e^{i\vartheta}$ where the variable ϑ denotes a constant real value). Therefore the total number of the magnon condensate in the whole system is

¹The variable $B_{\text{L(R)}}$ represents the applied magnetic field to the left (right)-hand side.

conserved. Then, the (tunneling) spin current density $I_{\text{BEC}}^{\text{L}}$ passing through the interface from the left-hand side to the right one along the x -axis (Fig. 7.2) can be defined as follows;

$$I_{\text{BEC}}^{\text{L}} := -\dot{n}_{\text{BEC}}^{\text{L}}, \quad (7.12)$$

where $n_{\text{BEC}}^{\text{L}}(t) := a_{\text{L}}^{\dagger} a_{\text{L}}$ represents the number density of the magnon condensate in the left-hand side.

7.3 Josephson Spin Current

By using the Heisenberg's equation of motion,

$$\dot{n}_{\text{BEC}}^{\text{L}} = [n_{\text{BEC}}^{\text{L}}, V_{\text{ex}}^{\text{A-C}}]/i, \quad (7.13)$$

we obtain the expectation value of the spin current density $I_{\text{BEC}}^{\text{L}}$ carried by magnons in BEC;

$$\begin{aligned} \langle I_{\text{BEC}}^{\text{L}} \rangle &= 2J_{\text{ex}}S \sqrt{n_{\text{BEC}}^{\text{L}}(t)n_{\text{BEC}}^{\text{R}}(t)} \\ &\times \sin[(B_{\text{L}} - B_{\text{R}})t + \theta_{\text{A-C}}] + \mathcal{O}((J_{\text{ex}})^2), \end{aligned} \quad (7.14)$$

where

$$\theta_{\text{A-C}} = (g\mu_{\text{B}}/\hbar c^2)E \int_{\mathbf{x}_{\text{L}}}^{\mathbf{x}_{\text{R}}} d\mathbf{x} \cdot \mathbf{e}_x \quad (7.15)$$

$$= (g\mu_{\text{B}}/\hbar c^2)E\Delta x \quad (7.16)$$

with $\Delta x := x_{\text{R}} - x_{\text{L}} (> 0)$ and the variable $n_{\text{BEC}}^{\text{R}}(t) := a_{\text{R}}^{\dagger} a_{\text{R}}$ denotes the number density of the magnon condensate in the right-hand side. Let us mention that we here have taken dephasing effects shown in Sec. 7.1 into account to evaluate $\langle I_{\text{BEC}}^{\text{L}} \rangle$.

Detail of Evaluation

Taking dephasing effects into account (eqs. (7.4), (7.6), (7.8) and (7.9)), we have evaluated the spin current density as follows;

$$-\langle I_{\text{BEC}}^{\text{L}} \rangle := \langle \dot{n}_{\text{BEC}}^{\text{L}} \rangle \quad (7.17)$$

$$= \left\langle \frac{1}{i} [n_{\text{BEC}}^{\text{L}}, V_{\text{ex}}^{\text{A-C}}] \right\rangle \quad (7.18)$$

$$= \left\langle \frac{1}{i} [a_{\text{L}}^{\dagger} a_{\text{L}}, -J_{\text{ex}}S \int_{\mathbf{x} \in (\text{interface})} d\mathbf{x} (a_{\text{L}} a_{\text{R}}^{\dagger} e^{-i\theta_{\text{A-C}}} + a_{\text{L}}^{\dagger} a_{\text{R}} e^{i\theta_{\text{A-C}}})] \right\rangle \quad (7.19)$$

$$= \langle iJ_{\text{ex}}S (-a_{\text{L}} a_{\text{R}}^{\dagger} e^{-i\theta_{\text{A-C}}} + a_{\text{L}}^{\dagger} a_{\text{R}} e^{i\theta_{\text{A-C}}}) \rangle \quad (7.20)$$

$$= iJ_{\text{ex}}S (\langle a_{\text{L}}^{\dagger} \rangle_{\text{L}} \langle a_{\text{R}} \rangle_{\text{R}} e^{i\theta_{\text{A-C}}} - \langle a_{\text{L}} \rangle_{\text{L}} \langle a_{\text{R}}^{\dagger} \rangle_{\text{R}} e^{-i\theta_{\text{A-C}}}) \quad (7.21)$$

$$= iJ_{\text{ex}}S \frac{\sqrt{N_{\text{BEC}}^{\text{L}}(t)N_{\text{BEC}}^{\text{R}}(t)}}{V} (e^{i(B_{\text{L}} - B_{\text{R}})t} e^{i\theta_{\text{A-C}}} - e^{-i(B_{\text{L}} - B_{\text{R}})t} e^{-i\theta_{\text{A-C}}}) \quad (7.22)$$

$$= iJ_{\text{ex}}S \frac{\sqrt{N_{\text{BEC}}^{\text{L}}(t)N_{\text{BEC}}^{\text{R}}(t)}}{V} (e^{i[(B_{\text{L}} - B_{\text{R}})t + \theta_{\text{A-C}}]} - e^{-i[(B_{\text{L}} - B_{\text{R}})t + \theta_{\text{A-C}}]}) \quad (7.23)$$

$$= -\frac{2J_{\text{ex}}S}{V} \sqrt{N_{\text{BEC}}^{\text{L}}(t)N_{\text{BEC}}^{\text{R}}(t)} \sin[(B_{\text{L}} - B_{\text{R}})t + \theta_{\text{A-C}}] \quad (7.24)$$

$$= -2J_{\text{ex}}S \sqrt{n_{\text{BEC}}^{\text{L}}(t)n_{\text{BEC}}^{\text{R}}(t)} \sin[(B_{\text{L}} - B_{\text{R}})t + \theta_{\text{A-C}}]. \quad (7.25)$$

Thus, we have obtained eq. (7.14).

Canonically Conjugate Relations

Here it will be useful to derive eq. (7.14) by using other kinds of methods, namely the canonically conjugate relations between the number of particles N and the phase θ ;² $[\theta, N] = i \Rightarrow N = -\partial\mathcal{H}/(\partial\theta)$ with Hamiltonian $\mathcal{H}$.

By using the same procedure with eq. (7.25), namely dephasing effects, the tunneling Hamiltonian under the A-C phase reads as follows;

$$\langle V_{\text{ex}}^{\text{A-C}} \rangle = -J_{\text{ex}}S \int_{\mathbf{x} \in (\text{int.})} d\mathbf{x} (\langle a_{\text{L}} \rangle_{\text{L}} \langle a_{\text{R}}^{\dagger} \rangle_{\text{R}} e^{-i\theta_{\text{A-C}}} + \langle a_{\text{L}}^{\dagger} \rangle_{\text{L}} \langle a_{\text{R}} \rangle_{\text{R}} e^{i\theta_{\text{A-C}}}) \quad (7.26)$$

$$= -2J_{\text{ex}}S \sqrt{n_{\text{BEC}}^{\text{L}}(t)n_{\text{BEC}}^{\text{R}}(t)} \cos[(B_{\text{L}} - B_{\text{R}})t + \theta_{\text{A-C}}] \cdot V_{\text{int.}} \quad (7.27)$$

with $V_{\text{int.}} := \int_{\mathbf{x} \in (\text{int.})} d\mathbf{x}$.

For convenience, we consider the situation, $B_{\text{L}} = B_{\text{R}}$, and it becomes

$$\langle V_{\text{ex}}^{\text{A-C}} \rangle = -2J_{\text{ex}}S \sqrt{n_{\text{BEC}}^{\text{L}}(t)n_{\text{BEC}}^{\text{R}}(t)} \cos\theta_{\text{A-C}} \cdot V_{\text{int.}} \quad (7.28)$$

We here use the canonically conjugate relations; $[\theta_{\text{A-C}}, (V_{\text{int.}} \cdot n_{\text{BEC}}^{\text{L}})] = i \Rightarrow (V_{\text{int.}} \cdot \dot{n}_{\text{BEC}}^{\text{L}}) = -\partial\mathcal{H}/(\partial\theta_{\text{A-C}})$. Then, we obtain the following results;³

$$V_{\text{int.}} \cdot \dot{n}_{\text{BEC}}^{\text{L}} = -\frac{\partial\mathcal{H}}{\partial\theta_{\text{A-C}}} \quad (7.29)$$

$$= -\frac{\partial V_{\text{ex}}^{\text{A-C}}}{\partial\theta_{\text{A-C}}} \quad (7.30)$$

$$= -\frac{\partial \left(-2J_{\text{ex}}S \sqrt{n_{\text{BEC}}^{\text{L}}(t)n_{\text{BEC}}^{\text{R}}(t)} \cos\theta_{\text{A-C}} \cdot V_{\text{int.}} \right)}{\partial\theta_{\text{A-C}}} \quad (7.31)$$

$$= -2J_{\text{ex}}S \sqrt{n_{\text{BEC}}^{\text{L}}(t)n_{\text{BEC}}^{\text{R}}(t)} \sin\theta_{\text{A-C}} \cdot V_{\text{int.}} \quad (7.32)$$

$$=: -V_{\text{int.}} \cdot I_{\text{BEC}}^{\text{L}} \quad (7.33)$$

$$\Leftrightarrow I_{\text{BEC}}^{\text{L}} = 2J_{\text{ex}}S \sqrt{n_{\text{BEC}}^{\text{L}}(t)n_{\text{BEC}}^{\text{R}}(t)} \sin\theta_{\text{A-C}}. \quad (7.34)$$

This is very the eq. (7.14). That is, as remarked, we have derived eq. (7.14) by using the canonically conjugate relations between the number of particles and the phase.

7.4 Features

Eq. (7.14) shows that the spin current density $I_{\text{BEC}}^{\text{L}}$ is proportional to the number density of the magnon condensate, which means that the spin current can flow as long as magnons are in condensation. That is, the condensation of magnons arising from macroscopic quantum effects plays the essential role on the generation of tunneling spin currents. Therefore we can conclude that this phenomenon producing transverse spin currents (Fig. 7.1) can be recognized as the Josephson effects of the magnon condensate.

Remarkably, according to eq. (7.14), the Josephson effect of magnon BEC arises in first order[72] in the tunneling amplitude J_{ex} , which is in sharp contrast to the Josephson effect of superconductors.[142, 143] Therefore this macroscopic phenomenon is dominant in the weakly coupled junction.

²We have hit upon this procedure owing to the discussion with Prof. Pascal (France).[119]

³The Hamiltonian $\mathcal{H}$ corresponds to $V_{\text{ex}}^{\text{A-C}}$ (i.e. eq. (7.27)) in our case.

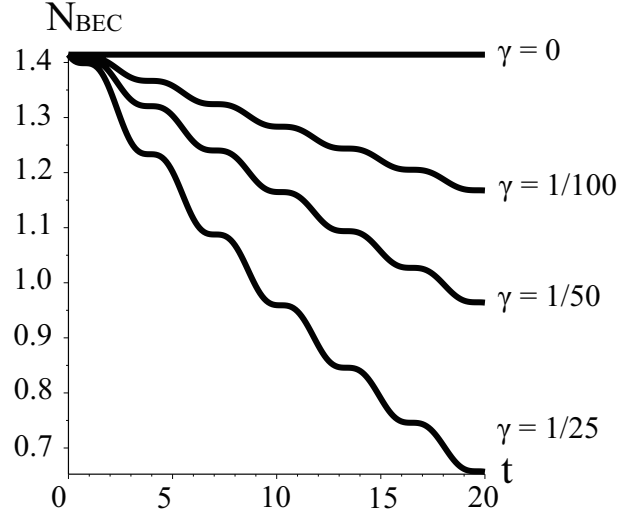


Figure 7.3: The plot of the decay process of the magnon condensate due to dissipation effects described by the Caldeira-Leggett model (see Sec. 11), where the variable N_{BEC} represents the total number of magnons in BEC. Regarding the detail, please see (Appendix) Chap. 11.

7.4.1 Dissipation Effects

Although we have remarked that Josephson spin currents flow as long as magnons are in condensation, there does exist other kinds of degrees of freedoms in real materials such as impurities and phonons et al., which cause dissipation.

Therefore in order to present an analysis which is relevant to experiments, we take dissipation effects on magnon condensates into account; we have investigated the dissipation effects due to phonons on magnon condensates by using the Caldeira-Leggett model, which has been plotted in Fig. 7.3.

Regarding the detail, please see (Appendix) Chap. 11.

7.4.2 Josephson Effects

We now consider the case; $B_L \neq B_R$. Then, we obtain the Josephson effect of magnon BEC, in which the Josephson spin current density reads

$$\langle I_{\text{BEC}}^L \rangle = 2J_{\text{ex}}S\sqrt{n_{\text{BEC}}^L(t)n_{\text{BEC}}^R(t)}\sin[(B_L - B_R)t + \theta_{\text{A-C}}]. \quad (7.35)$$

In this situation, the A-C phase $\theta_{\text{A-C}}$ simply gives the initial phase. That is, even in the absence of the A-C phase or electric field, $\theta_{\text{A-C}} \propto E = 0$, the Josephson spin current does exist because it is induced by the difference of the applied magnetic field (i.e. $B_L - B_R \neq 0$) characterizing the period of the Josephson effect. This is the qualitatively natural result reflecting the well-known fact[144] that the role of the magnetic field gradient on spin currents is the same with that of the electric field on charge currents.

Let us remark that the applied magnetic field B , which corresponds to the chemical potential in the language of the Josephson effects in Bose atoms,[77] is under our control. Therefore one can easily tune the period of the Josephson effect of the magnon condensate. This point will be useful to the observation of this phenomenon.

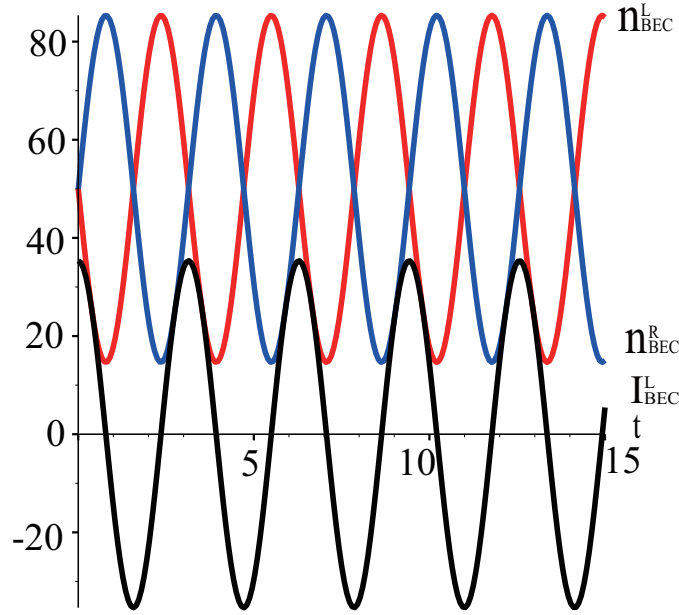


Figure 7.4: (Color online) The Josephson effect generated purely by A-C phases. We obtain the Josephson effect by solving the differential equation, eq. (7.14), in terms of n_{BEC} and $\dot{n}_{\text{BEC}}$ with the help of the following relations; $I_{\text{BEC}}^{\text{L}} = -\dot{n}_{\text{BEC}}^{\text{L}}$, $I_{\text{BEC}}^{\text{R}} = -\dot{n}_{\text{BEC}}^{\text{R}}$ and $\dot{n}_{\text{BEC}}^{\text{L}} + \dot{n}_{\text{BEC}}^{\text{R}} = 0$ (i.e. the total number of the magnon condensate in the whole system is conserved). As an example, we have employed the following values; $J_{\text{ex}} = 1/10$, $S = 1/2$, $B_{\text{L}} - B_{\text{R}} = 0$, $\theta_{\text{A-C}} = \pi/4$, $n_{\text{BEC}}^{\text{L}}(0) = n_{\text{BEC}}^{\text{R}}(0) = 50$.

Our next focus lies on the case; $B_{\text{L}} = B_{\text{R}}$. Then owing to the A-C phase, we obtain the Josephson effect of magnon BEC shown in Fig. 7.4, in which the Josephson spin current density reads

$$\langle I_{\text{BEC}}^{\text{L}} \rangle = 2J_{\text{ex}}S\sqrt{n_{\text{BEC}}^{\text{L}}(t)n_{\text{BEC}}^{\text{R}}(t)}\sin\theta_{\text{A-C}}. \quad (7.36)$$

This means this Josephson effect is generated purely by A-C phases with the help of the magnon condensate. In fact, when the A-C phase is tuned to satisfy the condition,

$$\theta_{\text{A-C}} \equiv 0 \pmod{\pi}, \quad (7.37)$$

the Josephson spin current is not generated; $\langle I_{\text{BEC}}^{\text{L}} \rangle = 0$. This is the features of the Josephson effect characterized purely by A-C phases. This condition is useful to the observation of the Josephson effect in magnon BEC because the A-C phase is under our control via the applied electric field.

Here it should be stressed that this Josephson effect is generated purely by the applied electric field via A-C phases, which induces transverse Josephson spin currents. That is, the direction of the Josephson spin current (i.e. along the x -axis) is perpendicular to that of the applied electric field (Fig. 7.1). Therefore this Josephson effect can be viewed as the phenomenon analogous to the magnon Hall effect.[138, 139]

7.4.3 Distinction

Let us mention that a similar mechanism for the emergence of persistent spin currents has been discussed in the articles,[140, 141] in which the spin current is carried by thermally

excited magnons. Therefore in the zero-temperature limit ($T = 0$), their Josephson effects cease to work.

On the other hand, our spin current is by magnon condensates resulting fully from quantum effects. Then, our Josephson effects continue to work even in the zero-temperature case.

7.5 Summary

Conclusions

We have presented a microscopic theory on the Josephson effects[70, 71, 72, 73, 74, 75, 76, 77] of magnon BEC[46, 47, 48, 49, 50, 51, 52, 53] through the A-C phases.[66, 141, 79, 80, 68] Our formalism gives the clear understanding that Josephson effects result fully from quantum effects, the quantum-mechanical phase called the A-C phase and magnon BEC, and Josephson spin currents are produced as long as magnons are in condensation.

To the best of our knowledge, the required techniques for the experiments (i.e. the microwave pumping method),[65, 46, 47, 48, 49, 50] the Josephson effect of magnon BEC, have already been established.[46, 65] That is, the first observation of the Josephson effects in magnon BEC is now possible. Moreover, the observation also deserves the indirect detection of the A-C phase.[79, 80] Thus, our results open a new door to experimentally exploring ferromagnetic insulators[65] by means of magnon BEC, which is expected to lead to applications to spintronics devices.

Discussions

Magnons in condensation form a macroscopic quantum (coherent) state, which is robust against the loss of information.[61] Therefore, the method we have newly proposed in this work to generate spin current in ferromagnetic insulators,[65] namely Josephson effects of magnon BEC, is significant also from the viewpoint of potential applications to information and communication technologies.

On top of this, the quasi-equilibrium magnon BEC, which plays the essential role in our method, can be produced at room temperature by using the microwave pumping method,[46] which has been already experimentally established.[46, 47, 48, 49, 50, 51, 52, 53] That is, we have no need to cool the samples, which is in sharp contrast to the original Josephson effects of superconductors. Consequently, the energy cost becomes far smaller than the one of superconductors. This is the virtue of our method.

On the other hand, to theoretically propose a way to extend the condensation time of magnons at room temperature will be an important issue,⁴ which will be addressed in the near future.

Outlooks

On preparing this thesis, we[119, 142] have found that our present analysis is based on the unconscious assumption that the relative phase between magnon BECs takes a constant value all the time.

Now, in order to overcome this issue, we have been studying on the basis of the two-state model[74, 75] consisting of the canonically conjugate variables, the relative phase and the order parameter of the population imbalance;[76] we relate the microscopic spin model of

⁴Regarding this issue, Dr. I. Danshita has kindly advised us that it might be helpful to use antiferromagnetic insulators (TiCuCl_3) and realize magnon BEC as the ground state, which will lead to the stable magnon BEC. This is the experimentally established method by Nikuni et al.[54]

single magnon BEC to the the Gross-Pitaevskii Hamiltonian[74, 75, 71] and microscopically derive the two-state model. Then, on the basis of the two-state model, we have been analyzing the features of the Josephson effects in magnon BEC through the A-C phase. The result will be addressed soon later.[137]

Let us stress that we have confirmed that the conclusion of this chapter does not change at all; the Josephson effect in magnon BEC is generated by the A-C phase and the phenomenon is analogous to the magnon Hall effect[138] in the sense that it generates the transverse Josephson spin current. We have been clarifying the condition of the A-C phase for the DC Josephson effect.

7.6 Supplements; Berry's Phase

As mentioned, the quantum-mechanical phase called the A-C phase[66] is dual to the Aharonov-Bohm phase[69, 4, 5] and each quantum-mechanical phase is just a special case of geometric phase called Berry's phase.[67]

Then, we show that Berry's phase naturally arises from the standard quantum mechanics on the condition called adiabatic assumption.[145]⁵

Theorem

Under the adiabatic assumption, quantum states $|\psi(t)\rangle$ acquire geometric phases $\gamma(t)$ called Berry's phase;

$$|\psi(t)\rangle = \exp\left[i\gamma_n(t) - i \int_0^t ds E_n(\mathbf{R}(s))\right] |n, \mathbf{R}(t)\rangle \quad (7.38)$$

$$\gamma_n(t) = i \int_0^t \langle n, \mathbf{R}(s) | \frac{d}{ds} | n, \mathbf{R}(s) \rangle ds. \quad (7.39)$$

Preparation

A quantum state $|\psi(t)\rangle$ obeys the Schrödinger eq.,

$$\mathcal{H}(\mathbf{R}(t)) |\psi(t)\rangle = i \frac{d}{dt} |\psi(t)\rangle. \quad (7.40)$$

Then let us suppose that it belongs to the n -th energy eigenstate $|n, \mathbf{R}(0)\rangle$ at $t = 0$;

$$|\psi(0)\rangle = |n, \mathbf{R}(0)\rangle \quad (7.41)$$

$$\mathcal{H}(\mathbf{R}(0)) |n, \mathbf{R}(0)\rangle = E_n(\mathbf{R}(0)) |n, \mathbf{R}(0)\rangle, \quad (7.42)$$

where the variable $\mathbf{R}(t)$ represents a set of parameters as a function of time t , which develops adiabatically.

Then, we assume that the quantum state also develops adiabatically. That is, the state remains being belonging to the n -th energy eigenstate $|n, \mathbf{R}(t)\rangle$ all times;

$$\mathcal{H}(\mathbf{R}(t)) |n, \mathbf{R}(t)\rangle = E_n(\mathbf{R}(t)) |n, \mathbf{R}(t)\rangle. \quad (7.43)$$

Proof

⁵This section has been constructed on the basis of the nice textbook by Prof. M. Nakahara.[145]

On the basis of the above condition, we can easily obtain Berry's phase shown in eq. (7.39). We show it on the basis of Schrödinger eq., eq. (7.40), as follows;

$$(\text{LHS of eq. (7.40)}) = \mathcal{H}(\mathbf{R}(t)) | \psi(t) \rangle \quad (7.44)$$

$$\begin{aligned} &= \mathcal{H}(\mathbf{R}(t)) \exp \left[i\gamma_n(t) - i \int_0^t ds E_n(\mathbf{R}(s)) \right] | n, \mathbf{R}(t) \rangle \\ &= \exp \left[i\gamma_n(t) - i \int_0^t ds E_n(\mathbf{R}(s)) \right] \mathcal{H}(\mathbf{R}(t)) | n, \mathbf{R}(t) \rangle \\ &= \exp \left[i\gamma_n(t) - i \int_0^t ds E_n(\mathbf{R}(s)) \right] E_n(\mathbf{R}(t)) | n, \mathbf{R}(t) \rangle, \end{aligned} \quad (7.45)$$

where we have used the fact that $\exp \left[i\gamma_n(t) - i \int_0^t ds E_n(\mathbf{R}(s)) \right]$ is a c-number and eq. (7.43).

On the other hand, the RHS of eq. (7.40) can be rewritten as follows;

$$(\text{RHS of eq. (7.40)}) = i \frac{d}{dt} | \psi(t) \rangle \quad (7.46)$$

$$= i \frac{d}{dt} \left\{ \exp \left[i\gamma_n(t) - i \int_0^t ds E_n(\mathbf{R}(s)) \right] | n, \mathbf{R}(t) \rangle \right\} \quad (7.47)$$

$$\begin{aligned} &= \left[-\frac{d\gamma_n(t)}{dt} + E_n(\mathbf{R}(t)) \right] \exp \left[i\gamma_n(t) - i \int_0^t ds E_n(\mathbf{R}(s)) \right] | n, \mathbf{R}(t) \rangle \\ &+ \exp \left[i\gamma_n(t) - i \int_0^t ds E_n(\mathbf{R}(s)) \right] i \frac{d}{dt} | n, \mathbf{R}(t) \rangle. \end{aligned} \quad (7.48)$$

Now, recall that RHS of eq. (7.45) and RHS of eq. (7.48) originate from eq. (7.40). Therefore (RHS of eq. (7.45)) = (RHS of eq. (7.48)).

Consequently,

$$\begin{aligned} \frac{d\gamma_n(t)}{dt} \exp \left[i\gamma_n(t) - i \int_0^t ds E_n(\mathbf{R}(s)) \right] | n, \mathbf{R}(t) \rangle &= \exp \left[i\gamma_n(t) - i \int_0^t ds E_n(\mathbf{R}(s)) \right] \\ &\times i \frac{d}{dt} | n, \mathbf{R}(t) \rangle \end{aligned} \quad (7.49)$$

$$\Rightarrow \frac{d\gamma_n(t)}{dt} | n, \mathbf{R}(t) \rangle = i \frac{d}{dt} | n, \mathbf{R}(t) \rangle \quad (7.50)$$

$$\Rightarrow \frac{d\gamma_n(t)}{dt} = i \langle n, \mathbf{R}(t) | \frac{d}{dt} | n, \mathbf{R}(t) \rangle. \quad (7.51)$$

By integrating out, we obtain Berry's phase;⁶

$$\gamma_n(t) = i \int_0^t \langle n, \mathbf{R}(s) | \frac{d}{ds} | n, \mathbf{R}(s) \rangle ds. \blacksquare \quad (7.52)$$

⁶Note that $\gamma_n(t=0) = 0$ by definition, which is shown in eqs. (7.38) and (7.41).

Part III

Summary and Appendices

Chapter 8

Summary; Concluding Remarks

Summary

In this thesis, we have microscopically investigated the non-equilibrium quantum transport phenomena of magnetization on the basis of the Schwinger-Keldysh formalism. The central physical quantity is the spin current, which has been attracting special attentions from the viewpoint of fundamental science and the potential application to quantum information and communication technologies (i.e. spintronics).

Firstly, we have reformulated spin pumping theory by focusing on the non-equilibrium spin-flip process arising from quantum effects at the interface between metals and insulators, and have found that spin pumping is characterized by SRT breaking the spin conservation law of the conduction electrons. Consequently, the net pumped spin current is represented in terms of only SRT and it is recognized as the non-linear response to microwaves working as quantum fluctuations. Then, we have clarified the distinction from the classical phenomenological theory and have confirmed that our formalism corresponds to the quantum version of the preceding classical theory. Our quantum spin pumping theory predicts that spin pumping is generated also by ESR as well as the usual method via FMR. This theoretical prediction is the important milestone for the validity of our quantum theory.

After that, stimulated by the recent development of experimental techniques by Demokritov et al., we have shown that quasi-equilibrium (dynamical) magnon BEC occurs also in spin-pumping systems as well as magnon BEC associated with SSB. In addition, we have constructed a phenomenological theory on quasi-equilibrium (dynamical) magnon BEC where thermalization processes, which works as a bridge between magnon pumping and quasi-equilibrium magnon BEC, are phenomenologically included.

Then, as the synthesis of the above works, we utilize magnon condensates and theoretically propose a new kind of methods for the generation of spin currents arising from macroscopic quantum effects; Josephson effects in magnon BEC through the A-C phase. In contrast to the usual spin pumping, this is a qualitatively new method in the sense that transverse Josephson spin currents are carried by magnons in BEC in insulators through the influence of the A-C phase. The magnon BEC is a macroscopic state with quantum coherence and it is robust against the loss of information. Therefore, the method we have newly proposed in this work is significant also from the viewpoint of potential applications to information and communication technologies. Remarkably, the quasi-equilibrium magnon BEC can be produced at room temperature by using the microwave pumping method, which has been already experimentally established. That is, we have no need to cool the samples, which is in sharp contrast to the original Josephson effects of superconductors. Consequently, the energy cost becomes far smaller than the one of superconductors. This is the virtue of our method. Thus, we have theoretically opened a new door to experimentally exploring spin currents in

insulators.

In addition, the A-C phase, which plays the key role on our method, is under our control via the applied electric field. Then, we consider that the required experimental techniques have already been ensured and by using the method we propose in this work with the help of the microwave pumping method, the first observation of the Josephson effects in magnon BEC and the resultant indirect detection of the A-C phase are now possible.

We hope that our works become a bridge between spintronics and magnon BEC to stimulate each other and realize the first observation of the Josephson effect in magnon BEC.

Outlooks

On the other hand, we realize that there remain important and interesting issues to be tackled.

Although we have constructed a phenomenological theory on quasi-equilibrium (dynamical) magnon BEC, establishing the microscopic description of thermalization processes resulting from the peculiar dispersion relations of the material YIG which is relevant to spin pumping and magnon BEC (Fig. 11.12 (i)), is an urgent issue. According to the experiments, quasi-equilibrium magnon BEC (and magnon BEC) can be realized only in YIG. Therefore, we can easily imagine that thermalization processes, which are peculiar to YIG, play an essential role in quasi-equilibrium magnon BEC and work as the bridge between magnon pumping and resultant quasi-equilibrium magnon BEC. Remember that applied microwaves can in principle excite only the zero-mode of magnons and through the thermalization process, quasi-equilibrium magnon BEC at the lowest energy mode having a non-zero wave-number is realized. We expect that the microscopic description of thermalization processes might lead to the longer condensate life-time. Hence, establishing quasi-equilibrium magnon BEC theory based on the microscopic description of thermalization processes will be significant from the viewpoint of fundamental science as well as the potential application to quantum spintronics devices.

On top of this, to theoretically propose a method to experimentally realize the direct measurement of spin currents carried by the conduction electrons or magnons will also open a new door.

We believe to theoretically propose theories which stimulate experiments will be *one of the main tasks*¹ of us theorists and we would like to continue to contribute to the aspect of theoretical physics.

¹I believe to theoretically stimulate theorists will be also one of the main tasks (or the best) of us theorists. Anyway, I mean I do not want to do any kinds of the parody (i.e. *sarumane* in Japanese).

Chapter 9

Appendix; Review of Schwinger-Keldysh Formalism

We quickly review the Schwinger-Keldysh formalism,[36, 37] which is the main method we have employed in this thesis. We achieve this aim on the basis of the following sophisticated articles,[31, 32, 33, 22, 34, 35] from which we have learned the essence. Then, regarding the detail, please refer to the original explanation.

Let us recall the points of Keldysh formalism. Thanks to the Schwinger-Keldysh closed time path, the Schwinger-Keldysh formalism (i.e. closed time path formalism or the real-time formalism) is free from the assumption of the adiabaticity and the corresponding theorem of Gell-Mann and Low.[38, 39] Therefore, within the perturbative theory via the Schwinger-Keldysh (or contour-ordered) Green's functions, the formalism can deal with an arbitrary time-dependent Hamiltonian[40, 41] and treat the system mechanically out of the equilibrium.¹ On top of this, this formalism is applicable to systems at finite temperature; the well-known Matsubara formalism (i.e. the imaginary-time formalism), which also can deal with thermodynamic average values, can be regarded as a simple corollary[35] of the Schwinger-Keldysh formalism. That is, the Schwinger-Keldysh formalism includes the Matsubara formalism and information about finite temperature is contained in the greater and the lesser Green's functions. Consequently, we can treat non-equilibrium phenomena at finite temperature thanks to the Schwinger-Keldysh formalism. These are the strong points of the formalism.² This will be the strong point of the formalism.³

9.1 Fundamental of Schwinger-Keldysh Formalism

9.1.1 Gell-Mann and Low Theorem

The concept of the adiabatic switching on interactions shown in Fig. 9.1 (a) represents a mathematical tool to obtain exact eigenstates of the interacting system by employing those of noninteracting systems, which reads as follows.

¹I have employed the terminology mechanically so as to mean that the non-equilibrium situations are induced by a time-dependent Hamiltonian. An attempt to generalize the Schwinger-Keldysh formalism so as to treat systems thermally out of equilibrium has been very recently reported by Dr. A. Shitade,[42] whose guiding principle is the gauge covariance.

²Regarding the Floquet's theorem, which is another powerful tool to analyze (periodic) non-equilibrium dynamics, please see pp. 29 in the review article by Aoki et al.[43]

³This formalism has been attracting special attentions also in the study of cosmology.[41, 40] In the context, this formalism is often called in-in formalism,[146, 147, 148] which reflects the Schwinger-Keldysh closed-time path.

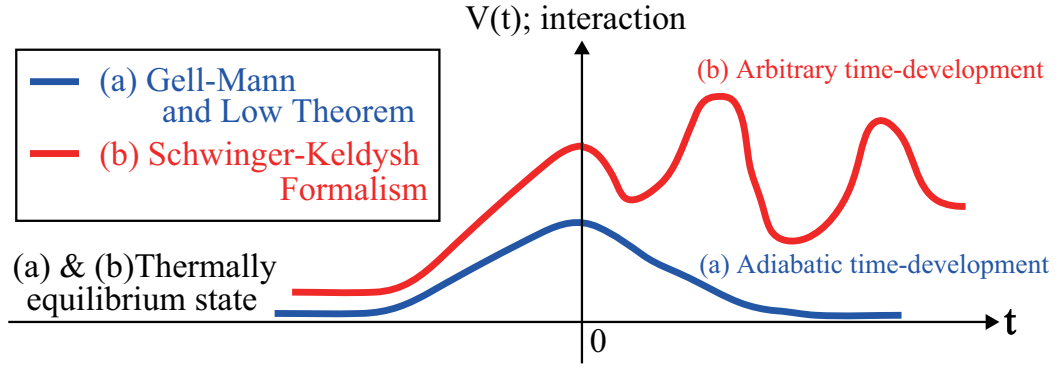


Figure 9.1: (Color online) Schematic pictures of the time-development of interactions in each formalism. The procedure, adiabatic time-development of the interactions, is not essential to the Schwinger-Keldysh formalism.

We consider the systems shown in Fig. 9.1 (a), which can be described by the following Hamiltonian $\mathcal{H}(t)$;

$$\mathcal{H}(t) = \mathcal{H}_0 + \mathcal{H}_1 e^{-\epsilon|t|} \quad (9.1)$$

$$\mathcal{H}_0 \Phi = E_0 \Phi, \quad (9.2)$$

where ϵ is a small positive quantity. The perturbation (i.e. interaction) $\mathcal{H}_1$ is turned on and off infinitely slowly (i.e. adiabatically). It is clear that at very large times, both in the past and in the future, the Hamiltonian $\mathcal{H}(t)$ is reduced to $\mathcal{H}_0$, which is solvable (i.e. can be diagonalized).

Gell-Mann Low theorem[122, 38]⁴ states that if the following quantity exists,

$$\lim_{\epsilon \rightarrow 0+} \frac{U_\epsilon(0, -\infty)|\Phi\rangle}{\langle\Phi|U_\epsilon(0, -\infty)|\Phi\rangle} =: \frac{|\Psi\rangle}{\langle\Phi|\Psi\rangle} \quad (9.3)$$

$$=: |\mathcal{X}\rangle, \quad (9.4)$$

with the full adiabatic evolution operator U_ϵ ,

$$U(t, t_0) := \text{T exp} \left[-i \int_{t_0}^t dt' \mathcal{H}_1(t') \right] \quad (9.5)$$

$$U_\epsilon(0, -\infty) := \text{T exp} \left[-i \int_{-\infty}^0 dt' e^{\epsilon t'} \mathcal{H}_1(t') \right], \quad (9.6)$$

then we can obtain an eigenstate of $\mathcal{H}$;

$$\mathcal{H}|\mathcal{X}\rangle = E|\mathcal{X}\rangle. \quad (9.7)$$

Here it should be noted that this theorem use a trick[31] that the true interacting state in the system under adiabatic time-development (i.e. adiabatic switching on and off) at time $t = 0$ can be obtained modulo a phase factor $e^{i\phi}$;

$$U_\epsilon(0, -\infty)|\Phi\rangle =: |\Psi\rangle \quad (9.8)$$

$$\Rightarrow U_\epsilon(\infty, -\infty)|\Phi\rangle = e^{i\phi}|\Phi\rangle \quad (9.9)$$

$$\Leftrightarrow \langle\Phi|U_\epsilon(\infty, -\infty)|\Phi\rangle = e^{i\phi}. \quad (9.10)$$

⁴In the original paper,[122] this theorem written in the Appendix.

Then, it should be remembered that this theorem is applicable to only the system under adiabatic time-development (i.e. adiabatic switching on and off) as shown in Fig. 9.1 (a) ; as has been intensively stressed by Kamenev in his textbook,[31] the crucial assumption is that the state is unique, which is independent of the details of the switching procedure and always keeps the eigenstate, up to a possible phase factor.

It is clear that this is not true to the case out of equilibrium; let us consider the situation when the system starts from some arbitrary non-equilibrium state and interactions are switched on and off. Then one can easily imagine that the system enjoys some unpredictable state, which depends on the switching procedure, and we have no knowledge of the final state.

9.1.2 Emergence of Two-branch Contour

Therefore in order to treat such non-equilibrium situations, we have to construct a theory that avoids references to the state at the final state. It has been achieved by using the closed time path shown in Fig. 9.2 and generalizing time-ordering to contour-ordering, which has been first theoretically proposed by Schwinger;[36] his suggestion is to take the final state to be exactly the same as the (well-known) initial one by introducing the time evolution along the two-branch contour c shown in Fig. 9.2. In this formalism, the above artificial procedure on which Gell-Mann and Low theorem is based (i.e. adiabatic switching on and off of interactions) is not needed owing to the closed time path (i.e. the two-branch contour c).

Lastly, we would like to remark that, except the newly introduced the closed time path (i.e. the two-branch contour c), the following the Schwinger-Keldysh formalism (i.e. closed time path formalism or the real-time formalism)⁵ is based simply on the standard quantum-statistical mechanics; the well-known (traditional) quantum field theory called Matsubara formalism (i.e. the imaginary-time formalism), which is applicable to only equilibrium cases, can be regarded as a simple corollary[35] of the Schwinger-Keldysh formalism.

One will be happy to know that this non-equilibrium perturbation theory in fact has a simpler structure than the standard equilibrium theory and includes the traditional theories such as Kubo-formula and Matsubara formalism.⁶

Standard Quantum-statistical Mechanics

Here, on the basis of the standard quantum-statistical mechanics, we show that the closed time path shown in Fig. 9.2 (i.e. the two-branch contour $c = c_{\rightarrow} + c_{\leftarrow}$) naturally emerges and we demonstrate how to evaluate the expectation value of physical quantities by using it.

We consider the systems described by the following Hamiltonian $\mathcal{H}(t)$;

$$\mathcal{H}(t) = \mathcal{H}_0 + V(t), \quad (9.11)$$

where $\mathcal{H}_0$ represents the solvable (i.e. can be diagonalized) Hamiltonian, which plays the role as a non-perturbative Hamiltonian, and $V(t)$ denotes the perturbative term whose effects on the systems is assumed to be small enough that we are allowed to treat it as a perturbative term.⁷ Let us stress that in sharp contrast to the case of Gell-Mann and Low theorem, the term $V(t)$ can arbitrarily work under the time-development; Fig. 9.1 (b). That is, $V(t)$ is an arbitrary time-dependent Hamiltonian.

⁵The Schwinger-Keldysh formalism is the real-time description of non-equilibrium quantum-statistical mechanics.

⁶We shall take advantage of the absence of (so-called) denominator-problems.

⁷Needless to say, an exact solution of a quantum field theory is a mission impossible in general. At

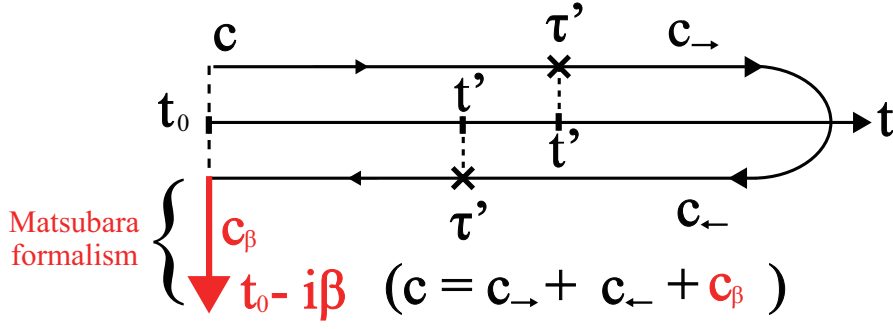


Figure 9.2: (Color online) The closed time path contour c ; $c = c_{\rightarrow} + c_{\leftarrow} + c_{\beta}$, in which τ' represents the contour variable defined on the closed time path c . The path c_{β} corresponds to the Matsubara formalism (i.e. imaginary time formalism), which includes the information about temperature.

We assume that the initial state belongs to a thermally equilibrium state (Fig. 9.1) and according to the standard quantum-statistical mechanics, the expectation value of a physical quantity $\mathcal{O}$ reads $\langle \mathcal{O} \rangle = \text{tr}(\rho \mathcal{O})$. Therefore by employing the canonical ensemble (i.e. $\rho = e^{-\beta \mathcal{H}}$ and $Z_0 := \text{tr}[e^{-\beta \mathcal{H}}]$), it can be rewritten as follows;

$$\langle \mathcal{O}(t) \rangle = \frac{1}{Z_0} \text{tr}[e^{-\beta \mathcal{H}} \mathcal{O}] \quad (9.12)$$

$$= \frac{1}{Z_0} \sum_{\alpha} \langle \alpha(t) | e^{-\beta \mathcal{H}} \mathcal{O} | \alpha(t) \rangle \quad (9.13)$$

$$= \frac{1}{Z_0} \sum_{\alpha} \langle \alpha | U^{\dagger}(t, t_0) e^{-\beta \mathcal{H}} \mathcal{O} U(t, t_0) | \alpha \rangle \quad (9.14)$$

$$= \frac{1}{Z_0} \sum_{\alpha} \langle \alpha | e^{-\beta \mathcal{H}} U^{\dagger}(t, t_0) \mathcal{O} U(t, t_0) | \alpha \rangle \quad (9.15)$$

$$= \frac{1}{Z_0} \sum_{\alpha} \langle \alpha | U(-i\beta + t_0, t_0) U^{\dagger}(t, t_0) \mathcal{O} U(t, t_0) | \alpha \rangle, \quad (9.16)$$

where we have employed the relations;

$$i\hbar \frac{\partial}{\partial t} |\alpha(t)\rangle = \mathcal{H}(t) |\alpha(t)\rangle \quad (9.17)$$

$$|\alpha(t)\rangle = U(t, t_0) |\alpha\rangle \quad (9.18)$$

$$\text{with } U(t, t_0) = \text{T exp} \left[-i \int_{t_0}^t dt' \mathcal{H}(t') \right]. \quad (9.19)$$

On top of this, it should be noted that the Boltzmann factor $e^{-\beta \mathcal{H}}$ can be represented in terms of the time-development operator $U(t, t_0)$ by using the complex time plane c_{β} (Fig. 9.2);

$$e^{-\beta \mathcal{H}} = U(-i\beta + t_0, t_0). \quad (9.20)$$

This explicitly shows the fact that the Schwinger-Keldysh formalism includes Matsubara formalism.

present, the only general method available for obtaining knowledge from the fundamental principle about the dynamics of a system is the perturbative study.

As the result, eq. (9.16) can be rewritten as follows;

$$\langle \mathcal{O}(t) \rangle = \frac{1}{Z_0} \text{tr}[U(-i\beta + t_0, t_0) U^\dagger(t, t_0) \mathcal{O} U(t, t_0)]. \quad (9.21)$$

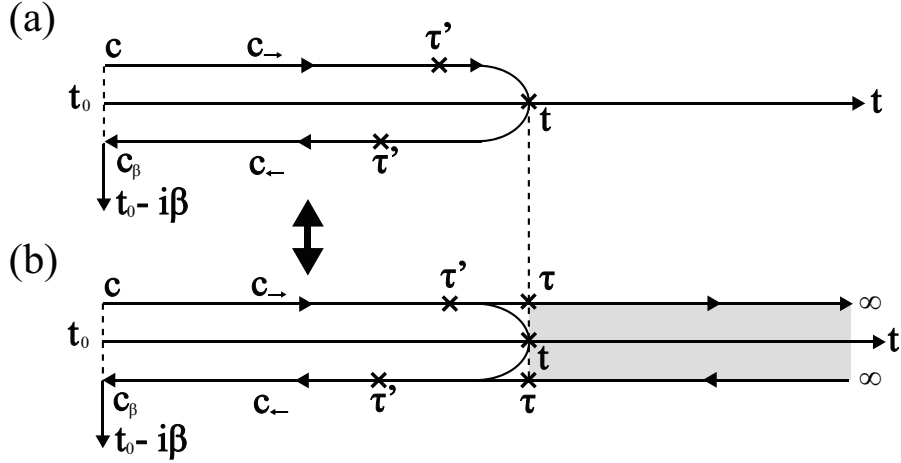


Figure 9.3: (Color online) The enlarged closed time path contour c by using the identity; $U_V^\dagger(\infty, t) U_V(\infty, t) = 1$. The contributions arising from the hatched parts in the path (b) cancel each other. Therefore the path (b) is reduced to the one (a).

9.1.3 Schwinger-Keldysh Closed-time Path

We then introduce the following representations;

$$U(t, t_0) =: U_0(t, t_0) U_V(t, t_0) \quad (9.22)$$

$$V_{\mathcal{H}_0}(t) := U_0^\dagger(t, t_0) V(t) U_0(t, t_0) \quad (9.23)$$

$$U_V(t, t_0) = \text{T exp}[-i \int_{t_0}^t dt' V_{\mathcal{H}_0}(t')]. \quad (9.24)$$

Consequently, eq. (9.21) reads as follows;

$$\langle \mathcal{O}(t) \rangle = \frac{1}{Z_0} \text{tr}[U(-i\beta + t_0, t_0) U^\dagger(t, t_0) \mathcal{O} U(t, t_0)] \quad (9.25)$$

$$\begin{aligned} &= \frac{1}{Z_0} \text{tr}[(U_0(-i\beta + t_0, t_0) U_V(-i\beta + t_0, t_0)) \\ &\times (U_V^\dagger(t, t_0) U_0^\dagger(t, t_0)) \mathcal{O} (U_0(t, t_0) U_V(t, t_0))] \end{aligned} \quad (9.26)$$

$$\begin{aligned} &= \frac{1}{Z_0} \text{tr}[U_0(-i\beta + t_0, t_0) \{U_V(-i\beta + t_0, t_0) U_V^\dagger(t, t_0)\} \\ &\times (U_0^\dagger(t, t_0) \mathcal{O} U_0(t, t_0)) \{U_V(t, t_0)\}], \end{aligned} \quad (9.27)$$

in which we have employed the following identity; $U_V^\dagger(\infty, t)U_V(\infty, t) = 1$. As the result, $\langle \mathcal{O}(t) \rangle$ can be rewritten in two ways shown in eqs. (9.28) and (9.29);

$$\begin{aligned} \langle \mathcal{O}(t) \rangle &= \frac{1}{Z_0} \text{tr}[U_0(-i\beta + t_0, t_0) \{U_V(-i\beta + t_0, t_0) U_V^\dagger(t, t_0)\} \\ &\times \{U_V^\dagger(\infty, t) U_V(\infty, t)\} \left(\mathcal{O}_{\mathcal{H}_0}(t) \right) \{U_V(t, t_0)\}] \end{aligned} \quad (9.28)$$

$$\begin{aligned} &= \frac{1}{Z_0} \text{tr}[U_0(-i\beta + t_0, t_0) \{U_V(-i\beta + t_0, t_0) U_V^\dagger(t, t_0)\} \left(\mathcal{O}_{\mathcal{H}_0}(t) \right) \\ &\times \{U_V^\dagger(\infty, t) U_V(\infty, t)\} \{U_V(t, t_0)\}]. \end{aligned} \quad (9.29)$$

Now, recall the relations; $e^{-\beta \mathcal{H}} = U(-i\beta + t_0, t_0)$ and $c = c_{\rightarrow} + c_{\leftarrow} + c_{\beta}$. Then, $\langle \mathcal{O}(t) \rangle$ can be represented by the following simple ways;

$$\langle \mathcal{O}(t) \rangle = \frac{1}{Z_0} \text{tr}[e^{-\beta \mathcal{H}_0} \left\{ T_c \exp[-i \int_c d\tau' V_{\mathcal{H}_0}(\tau')] \right\} \left(\mathcal{O}_{\mathcal{H}_0}(\tau) \right)] \quad (9.30)$$

$$= \left\langle T_c \exp[-i \int_c d\tau' V_{\mathcal{H}_0}(\tau') \mathcal{O}_{\mathcal{H}_0}(\tau)] \right\rangle_0, \quad (9.31)$$

with

$$\int_c d\tau' = \left(\int_{c_{\rightarrow}} + \int_{c_{\leftarrow}} + \int_{c_{\beta}} \right) d\tau', \quad (9.32)$$

where τ and τ' represent the contour variables defined on the contour c (Fig. 9.2) and T_c is the contour ordering symbol which orders the products of operators located on the contour c according to the position of their contour time argument on the closed contour; earlier contour time places an operator to the right.

Here it should be noted that $\langle \mathcal{O} \rangle_0$ represents $\langle \mathcal{O} \rangle_0 = \text{tr}[e^{-\beta \mathcal{H}_0} \mathcal{O}]$. Therefore we are allowed to use the Wick's theorem on the practical perturbative calculations.

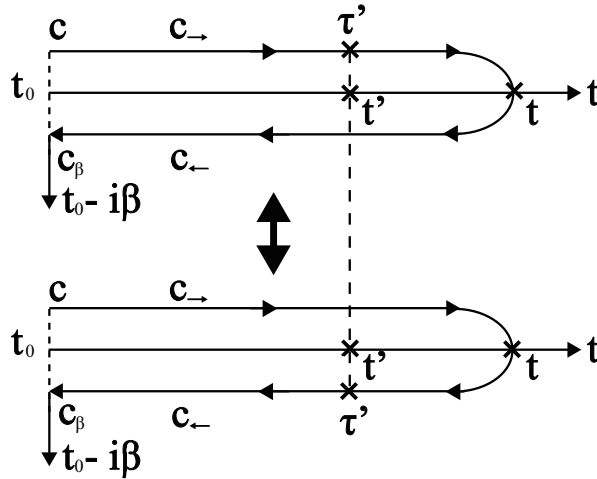


Figure 9.4: The closed time path contour c . It does not matter whichever the contour variable τ is located, i.e. forward path $c_{\rightarrow}$ or backward path $c_{\leftarrow}$.

In addition, let us remark that in the above calculations eqs. (9.27)-(9.31), we have positively used the identity;

$$U_V^\dagger(\infty, t)U_V(\infty, t) = 1. \quad (9.33)$$

This means that the resultant closed contour gives the unit operator, $U_V^\dagger(\infty, t)U_V(\infty, t) = 1$, and then, the contributions coming from the hatched parts cancel each other (Fig. 9.3). Now, let us focus on the resultant relations;

$$\begin{aligned} \langle \mathcal{O}(t) \rangle &= \frac{1}{Z_0} \text{tr}[U_0(-i\beta + t_0, t_0) \{U_V(-i\beta + t_0, t_0) U_V^\dagger(t, t_0)\} \\ &\times \{U_V^\dagger(\infty, t) U_V(\infty, t)\} \left(\mathcal{O}_{\mathcal{H}_0}(t) \right) \{U_V(t, t_0)\}] \end{aligned} \quad (9.34)$$

$$\begin{aligned} &= \frac{1}{Z_0} \text{tr}[U_0(-i\beta + t_0, t_0) \{U_V(-i\beta + t_0, t_0) U_V^\dagger(t, t_0)\} \left(\mathcal{O}_{\mathcal{H}_0}(t) \right) \\ &\times \{U_V^\dagger(\infty, t) U_V(\infty, t)\} \{U_V(t, t_0)\}] \end{aligned} \quad (9.35)$$

$$= \frac{1}{Z_0} \text{tr}[e^{-\beta \mathcal{H}_0} \left\{ T_c \exp[-i \int_c d\tau' V_{\mathcal{H}_0}(\tau')] \right\} \left(\mathcal{O}_{\mathcal{H}_0}(\tau) \right)]. \quad (9.36)$$

It is clear that two branches, forward path c_+ or backward path c_- , naturally emerges in the Schwinger-Keldysh formalism as shown in Fig. 9.3; eq. (9.34) describes the procedure on forward path c_+ in Fig. 9.3 (b) and eq. (9.35) represents the one on backward path c_- in Fig. 9.3 (b).

Consequently in the non-equilibrium quantum field theory (i.e. the Schwinger-Keldysh formalism or the real-time formalism or the closed-time path formalism), as shown in eqs. (9.76) and (9.90), four kinds of Green's functions, lesser and greater Green's functions as well as retarded and advanced ones, are introduced on the two branches in order to describe the dynamics out of equilibrium. This is in sharp contrast to the case of the usual quantum field theory (i.e. imaginary-time formalism).⁸

Eqs. (9.34) and (9.35) mean that it does not matter whichever the contour variable τ is located, i.e. forward path c_+ or backward path c_- (Figs. 9.3 and 9.4). On top of this, let us remark that if we are not interested in transient phenomena in a system or physics on short-time scales, we are allowed to use the Schwinger-Keldysh closed-time path shown in Fig. 9.5; $\int_c d\tau \rightarrow \int_{c_+} d\tau_+ + \int_{c_-} d\tau_- = \int_{-\infty}^{\infty} d\tau_+ + \int_{\infty}^{-\infty} d\tau_- = \int_{-\infty}^{\infty} d\tau_+ - \int_{-\infty}^{\infty} d\tau_-$, where the contributions arising from the imaginary part vanishes.

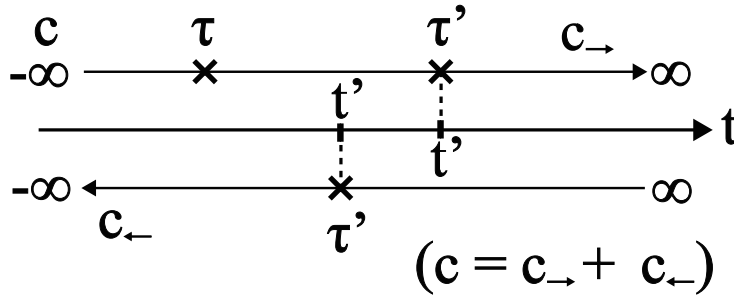


Figure 9.5: The Schwinger-Keldysh closed-time path or the Schwinger-Keldysh (i.e. real-time) contour.

9.1.4 Langreth Method

Let us introduce the (so-called) Langreth method, which is the useful theoretical tools for the practical perturbative calculations.

⁸As you know, the usual quantum field theory (i.e. imaginary-time formalism) can be constructed by only retarded and advanced Green's functions.

In the practical perturbative calculations, we often encounter contour matrix-multiplications which take the following form;^[33]⁹

$$\mathcal{C}(t_1, t'_1) := \int_{c_t} d\tau [\mathcal{A}(t_1, \tau) \mathcal{B}(\tau, t'_1)] \quad (9.37)$$

$$=: (\mathcal{A} \square \mathcal{B})(t_1, t'_1), \quad (9.38)$$

where $\mathcal{A}$ and $\mathcal{B}$ are functions of the contour variables.¹⁰ We have adopted this convenient expression because only the contour-time variables defined on contour c_t shown in Fig. 9.6 play a role;

$$\int_{c_t} d\tau = \int_{c_{t_1}} d\tau + \int_{c_{t'_1}} d\tau. \quad (9.39)$$

As a concrete example, let us demonstrate the analytical procedure for the lesser component (the case $\mathcal{C}^<$) shown in Fig. 9.6, which can be evaluated as follows;^[22]

$$\mathcal{C}^<(t_1, t'_1) := \int_{c_t} d\tau [\mathcal{A}(t_1, \tau) \mathcal{B}(\tau, t'_1)]^< \quad (9.40)$$

$$= \left(\int_{c_{t_1}} + \int_{c_{t'_1}} \right) d\tau [\mathcal{A}(t_1, \tau) \mathcal{B}(\tau, t'_1)]^< \quad (9.41)$$

$$= \int_{c_{t_1}} d\tau [\mathcal{A}(t_1, \tau) \mathcal{B}(\tau, t'_1)]^< + \int_{c_{t'_1}} d\tau [\mathcal{A}(t_1, \tau) \mathcal{B}(\tau, t'_1)]^< \quad (9.42)$$

$$= \int_{c_{t_1}} d\tau \mathcal{A}(t_1, \tau) \mathcal{B}^<(\tau, t'_1) + \int_{c_{t'_1}} d\tau \mathcal{A}^<(t_1, \tau) \mathcal{B}(\tau, t'_1). \quad (9.43)$$

We here focus on the first term $\int_{c_{t_1}} d\tau \mathcal{A}(t_1, \tau) \mathcal{B}^<(\tau, t'_1)$, which reads as follows (Fig. 9.6);

$$\int_{c_{t_1}} d\tau \mathcal{A}(t_1, \tau) \mathcal{B}^<(\tau, t'_1) = \left(\int_{c_{t_1} \rightarrow} + \int_{c_{t_1} \leftarrow} \right) d\tau \mathcal{A}(t_1, \tau) \mathcal{B}^<(\tau, t'_1) \quad (9.44)$$

$$= \int_{c_{t_1} \rightarrow} d\tau \mathcal{A}(t_1, \tau) \mathcal{B}^<(\tau, t'_1) + \int_{c_{t_1} \leftarrow} d\tau \mathcal{A}(t_1, \tau) \mathcal{B}^<(\tau, t'_1) \quad (9.45)$$

$$= \int_{-\infty}^{t_1} d\tau \mathcal{A}^>(t_1, \tau) \mathcal{B}^<(\tau, t'_1) + \int_{t_1}^{-\infty} d\tau \mathcal{A}^<(t_1, \tau) \mathcal{B}^<(\tau, t'_1). \quad (9.46)$$

We here have ordered the contour times, τ and t_1 , on the basis of the position they have been located on each contour (i.e. each contour time argument); $\int_{c_{t_1} \rightarrow} d\tau$ and $\int_{c_{t_1} \leftarrow} d\tau$. The time-relation between τ and t_1 depends on which path τ is located on; forward path $\int_{c_{t_1} \rightarrow} d\tau$ or backward path $\int_{c_{t_1} \leftarrow} d\tau$.

⁹It should be stressed that in this section, we follow the notation of the textbook.^[33, 35]

¹⁰Although we often use the symbol $\mathcal{A}$ in this thesis, please do not confuse and identify each other.

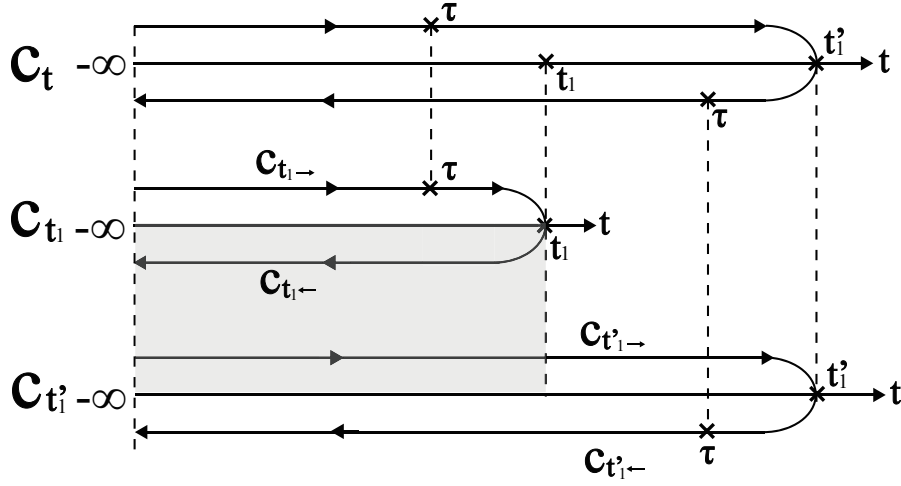


Figure 9.6: (Color online) The procedure for the Langreth method; $\int_{C_t} d\tau = \int_{C_{t_1}} d\tau + \int_{C_{t'_1}} d\tau$.

Consequently, it can be rewritten as follows;

$$\begin{aligned} \int_{C_{t_1}} d\tau \mathcal{A}(t_1, \tau) \mathcal{B}^<(\tau, t'_1) &= \int_{-\infty}^{\infty} d\tau \theta(t_1 - \tau) \mathcal{A}^>(t_1, \tau) \mathcal{B}^<(\tau, t'_1) \\ &+ \int_{-\infty}^{\infty} d\tau \theta(t_1 - \tau) \mathcal{A}^<(t_1, \tau) \mathcal{B}^<(\tau, t'_1) \end{aligned} \quad (9.47)$$

$$= \int_{-\infty}^{\infty} d\tau \theta(t_1 - \tau) [\mathcal{A}^>(t_1, \tau) - \mathcal{A}^<(t_1, \tau)] \mathcal{B}^<(\tau, t'_1) \quad (9.48)$$

$$= \int_{-\infty}^{\infty} d\tau \mathcal{A}^r(t_1, \tau) \mathcal{B}^<(\tau, t'_1). \quad (9.49)$$

That is,

$$\int_{C_{t_1}} d\tau \mathcal{A}(t_1, \tau) \mathcal{B}^<(\tau, t'_1) = \int_{-\infty}^{\infty} d\tau \mathcal{A}^r(t_1, \tau) \mathcal{B}^<(\tau, t'_1). \quad (9.50)$$

By using the same procedure, the latter term in eq. (9.43) $\int_{C_{t'_1}} d\tau \mathcal{A}^<(t_1, \tau) \mathcal{B}(\tau, t'_1)$ can be evaluated as follows;

$$\int_{C_{t'_1}} d\tau \mathcal{A}^<(t_1, \tau) \mathcal{B}(\tau, t'_1) = \int_{-\infty}^{\infty} d\tau \mathcal{A}^<(t_1, \tau) \mathcal{B}^a(\tau, t'_1). \quad (9.51)$$

As the result, we obtain the following relations;

$$\mathcal{C}^<(t_1, t'_1) = \int_{-\infty}^{\infty} d\tau [\mathcal{A}^r(t_1, \tau) \mathcal{B}^<(\tau, t'_1) + \mathcal{A}^<(t_1, \tau) \mathcal{B}^a(\tau, t'_1)] \quad (9.52)$$

$$=: (\mathcal{A}^r \circ \mathcal{B}^< + \mathcal{A}^< \circ \mathcal{B}^a), \quad (9.53)$$

where $\circ$ symbolizes matrix multiplication in real time, integrating over the internal real-time variable from minus infinity to plus infinity of times.[35]

List of Formula

By using the same approach, we obtain the following relations;

$$\mathcal{C}^r(t_1, t'_1) = \int_{-\infty}^{\infty} d\tau \mathcal{A}^r(t_1, \tau) \mathcal{B}^r(\tau, t'_1) \quad (9.54)$$

$$= \mathcal{A}^r \circ \mathcal{B}^r, \quad (9.55)$$

$$\mathcal{C}^a(t_1, t'_1) = \int_{-\infty}^{\infty} d\tau \mathcal{A}^a(t_1, \tau) \mathcal{B}^a(\tau, t'_1) \quad (9.56)$$

$$= \mathcal{A}^a \circ \mathcal{B}^a. \quad (9.57)$$

In summary, we have obtained the following relations;[22, 33]

$$(\mathcal{AB})^< = \mathcal{A}^r \mathcal{B}^< + \mathcal{A}^< \mathcal{B}^a \quad (9.58)$$

$$\Leftrightarrow (\mathcal{A} \square \mathcal{B})^< = \mathcal{A}^r \circ \mathcal{B}^< + \mathcal{A}^< \circ \mathcal{B}^a \quad (9.59)$$

$$(\mathcal{A} \square \mathcal{B})^r = \mathcal{A}^r \circ \mathcal{B}^r \quad (9.60)$$

$$(\mathcal{A} \square \mathcal{B})^a = \mathcal{A}^a \circ \mathcal{B}^a, \quad (9.61)$$

which will be useful on your practical perturbative calculations. By using these relations, we can evaluate also $\mathcal{A} \square \mathcal{B} \square \mathcal{D}$ as well as $\mathcal{A} \square \mathcal{B}$;

$$(\mathcal{A} \square \mathcal{B} \square \mathcal{D})^< = [\mathcal{A} \square (\mathcal{B} \square \mathcal{D})]^< \quad (9.62)$$

$$= \mathcal{A}^r \circ (\mathcal{B} \square \mathcal{D})^< + \mathcal{A}^< \circ (\mathcal{B} \square \mathcal{D})^a \quad (9.63)$$

$$= \mathcal{A}^r \circ \mathcal{B}^r \circ \mathcal{D}^< + \mathcal{A}^r \circ \mathcal{B}^< \circ \mathcal{D}^a + \mathcal{A}^< \circ \mathcal{B}^a \circ \mathcal{D}^a \quad (9.64)$$

$$\Leftrightarrow (\mathcal{A} \square \mathcal{B} \square \mathcal{D})^< = \mathcal{A}^r \circ \mathcal{B}^r \circ \mathcal{D}^< + \mathcal{A}^r \circ \mathcal{B}^< \circ \mathcal{D}^a + \mathcal{A}^< \circ \mathcal{B}^a \circ \mathcal{D}^a. \quad (9.65)$$

9.2 Concrete Example

For simplicity here, we consider the perturbative term, V' ,

$$V' = \Gamma(t) \int d\mathbf{x} [a(\mathbf{x}, t) + a^\dagger(\mathbf{x}, t)], \quad (9.66)$$

and evaluate the expectation value of the bosonic annihilate operator, a , as an example;[84]

$$\langle a(\tau) \rangle = \left\langle T_c a(\tau) \exp[-i \int_c d\tau' V(\tau')] \right\rangle \quad (9.67)$$

$$= -i \int d\mathbf{x}' \int_c d\tau' \Gamma(\tau') \langle T_c a(\mathbf{x}, \tau) a^\dagger(\mathbf{x}', \tau') \rangle + \mathcal{O}(\Gamma^2) \quad (9.68)$$

$$\equiv -i \int d\mathbf{x}' \mathcal{I}. \quad (9.69)$$

Here T_c is the path-ordering operator defined on the Schwinger-Keldysh closed time path,[35] c (see Fig. 9.7).[32, 22, 33] We express the Schwinger-Keldysh closed time path as a sum of the forward path, $c_{\rightarrow}$, and the backward path, $c_{\leftarrow}$; $c = c_{\rightarrow} + c_{\leftarrow}$. [35, 32, 22, 33] We take τ which denotes the contour variable defined on the Schwinger-Keldysh closed time path on forward path, $c_{\rightarrow}$. Even when τ is located on backward path, $c_{\leftarrow}$, the result of this calculation is invariant because each Green's function, G^r , G^a , $G^<$, $G^>$, is not independent;[32, 31, 34, 22] they obey,¹¹

$$G^r - G^a = G^> - G^<. \quad (9.70)$$

¹¹The spectral function $\mathcal{A}_{\text{sp}}$ is defined as follows; $\mathcal{A}_{\text{sp}} := i(G^r - G^a) = i(G^> - G^<)$.

Note that this relation comes into effect also for the fermionic case;[32, 31, 34, 22]

$$\mathcal{G}^r - \mathcal{G}^a = \mathcal{G}^> - \mathcal{G}^<. \quad (9.71)$$

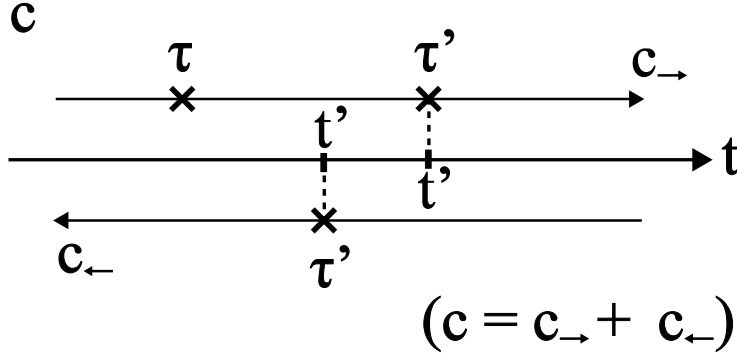


Figure 9.7: The Schwinger-Keldysh closed time path, c . Both the forward and backward paths are actually on the real axis but shifted slightly upwards and downwards, respectively, to distinguish them clearly.

The integral on the Schwinger-Keldysh closed time path of eq. (9.69), $\mathcal{I}$, is executed by taking an identity[32, 34] into account

$$\int_c d\tau' = \int_{c_{\rightarrow}} d\tau' + \int_{c_{\leftarrow}} d\tau', \quad (9.72)$$

as

$$\mathcal{I} = i \int_{-\infty}^{\infty} dt' \Gamma(t') [G^t(t, t') - G^<(t, t')]. \quad (9.73)$$

By using the relation, $G^r(t, t') = G^t(t, t') - G^i(t, t')$, we obtain

$$\mathcal{I} = i \int_{-\infty}^{\infty} dt' \Gamma(t') G^r(t, t'). \quad (9.74)$$

9.2.1 Bosonic Keldysh Green's Function

In this section, we show the point of the bosonic Keldysh Green's functions.

The bosonic Keldysh Green's function, $G(\tau, \tau')$, is defined as[31, 32, 34]

$$G(\tau, \tau') := -i \langle T_c a(\tau) a^\dagger(\tau') \rangle. \quad (9.75)$$

Depending on the points where τ and τ' are located on the Schwinger-Keldysh closed time path (i.e. $c = c_{\rightarrow} + c_{\leftarrow}$), the bosonic Keldysh Green's function is expressed as[31, 32, 34]

$$G(\tau, \tau') = \begin{cases} G^<(t, t') = -i \langle a^\dagger(t') a(t) \rangle, & \text{when } \tau \in c_{\rightarrow}, \tau' \in c_{\leftarrow}. \\ G^>(t, t') = -i \langle a(t) a^\dagger(t') \rangle, & \text{when } \tau \in c_{\leftarrow}, \tau' \in c_{\rightarrow}. \\ G^t(t, t'), & \text{when } \tau \in c_{\rightarrow}, \tau' \in c_{\rightarrow}. \\ G^{\bar{t}}(t, t'), & \text{when } \tau \in c_{\leftarrow}, \tau' \in c_{\leftarrow}. \end{cases} \quad (9.76)$$

It should be noted that each Green's function is not independent;[31, 32, 34]

$$G^r - G^a = G^> - G^< \Leftrightarrow G^t + G^{\bar{t}} = G^> + G^<. \quad (9.77)$$

In addition, these relations[34, 35] would be useful on calculation;

$$G^t = G^a + G^>, \quad (9.78)$$

$$G^{\bar{t}} = G^< - G^a, \quad (9.79)$$

$$G^r = G^t - G^< \quad (9.80)$$

$$= G^> - G^{\bar{t}}, \quad (9.81)$$

$$\langle a^\dagger(t)a(t) \rangle = iG^<(t, t). \quad (9.82)$$

By executing the Fourier transformation, the lesser and greater Green's functions for free bosons become[31, 34]

$$G_{\mathbf{k}', \omega'}^< = -f_B(\omega')(G_{\mathbf{k}', \omega'}^a - G_{\mathbf{k}', \omega'}^r) \quad (9.83)$$

$$= -2\pi i f_B(\omega')\delta(\omega' - \omega_{\mathbf{k}'}), \quad (9.84)$$

$$G_{\mathbf{k}', \omega'}^> = -2\pi i [1 + f_B(\omega')]\delta(\omega' - \omega_{\mathbf{k}'}), \quad (9.85)$$

$$G_{\mathbf{k}', \omega'}^K := G_{\mathbf{k}', \omega'}^< + G_{\mathbf{k}', \omega'}^> \quad (9.86)$$

$$= -2\pi i [1 + 2f_B(\omega')]\delta(\omega' - \omega_{\mathbf{k}'}), \quad (9.87)$$

$$= 2i \text{Im} G_{\mathbf{k}', \omega'}^r \coth(\beta\omega'/2). \quad (9.88)$$

The last one, G^K , represents the Keldysh Green's function [31, 32] and the relation is called the bosonic fluctuation-dissipation theorem.[35]

9.2.2 Fermionic Keldysh Green's Function

It would be useful to compare with the (spinless) Fermionic Keldysh Green's function, $\mathcal{G}(\tau, \tau')$, which is defined as[31, 32, 34, 22, 33]

$$\mathcal{G}(\tau, \tau') := -i\langle T_c c(\tau) c^\dagger(\tau') \rangle. \quad (9.89)$$

Depending on the points where τ and τ' are located on the Schwinger-Keldysh closed time path (i.e. $c = c_{\rightarrow} + c_{\leftarrow}$), the Fermionic Keldysh Green's function is expressed as[31, 32, 34, 22, 33]

$$\mathcal{G}(\tau, \tau') = \begin{cases} \mathcal{G}^<(t, t') = i\langle c^\dagger(t')c(t) \rangle, & \text{when } \tau \in c_{\rightarrow}, \tau' \in c_{\leftarrow}. \\ \mathcal{G}^>(t, t') = -i\langle c(t)c^\dagger(t') \rangle, & \text{when } \tau \in c_{\leftarrow}, \tau' \in c_{\rightarrow}. \\ \mathcal{G}^t(t, t'), & \text{when } \tau \in c_{\rightarrow}, \tau' \in c_{\rightarrow}. \\ \mathcal{G}^{\bar{t}}(t, t'), & \text{when } \tau \in c_{\leftarrow}, \tau' \in c_{\leftarrow}. \end{cases} \quad (9.90)$$

Note that with reflecting the statistical properties, the sign of the lesser Green's function is opposite from the bosonic case. In addition, they satisfy the relation;[31, 32, 34, 22, 33]

$$\mathcal{G}^r - \mathcal{G}^a = \mathcal{G}^> - \mathcal{G}^< \Leftrightarrow \mathcal{G}^t + \mathcal{G}^{\bar{t}} = \mathcal{G}^> + \mathcal{G}^<, \quad (9.91)$$

and

$$\mathcal{G}^t = \mathcal{G}^r + \mathcal{G}^< \quad (9.92)$$

$$= \mathcal{G}^a + \mathcal{G}^>, \quad (9.93)$$

$$\mathcal{G}^{\bar{t}} = -\mathcal{G}^r + \mathcal{G}^> \quad (9.94)$$

$$= -\mathcal{G}^a + \mathcal{G}^<. \quad (9.95)$$

By executing the Fourier transformation, the lesser and greater Green's functions for free Fermions become[31, 34, 22, 33]

$$\mathcal{G}_{\mathbf{k},\omega}^< = f_F(\omega)(\mathcal{G}_{\mathbf{k},\omega}^a - \mathcal{G}_{\mathbf{k},\omega}^r) \quad (9.96)$$

$$= 2\pi i f_F(\omega) \delta(\omega - \omega_{\mathbf{k}}), \quad (9.97)$$

$$\mathcal{G}_{\mathbf{k},\omega}^> = -2\pi i [1 - f_F(\omega)] \delta(\omega - \omega_{\mathbf{k}}), \quad (9.98)$$

$$\mathcal{G}_{\mathbf{k},\omega}^K := \mathcal{G}_{\mathbf{k},\omega}^< + \mathcal{G}_{\mathbf{k},\omega}^> \quad (9.99)$$

$$= -2\pi i [1 - 2f_F(\omega)] \delta(\omega - \omega_{\mathbf{k}}) \quad (9.100)$$

$$= 2i \text{Im} \mathcal{G}_{\mathbf{k},\omega}^r \tanh(\beta\omega/2). \quad (9.101)$$

The last one, $\mathcal{G}^K$, represents the Keldysh Green's function [31, 32] and the relation is called the fermionic fluctuation-dissipation theorem.[35, 33]

9.3 Detail of Calculation

Here we show the detailed procedure of the calculation in Sec. 2.6.[27]

The interface is, in general, a weak coupling regime;[12] the exchange interaction, J , is supposed to be smaller than the Fermi energy and the exchange interaction among ferromagnets. Thus $\mathcal{H}'_{\text{ex}}$ can be treated as a perturbative term. In addition, we apply weak transverse magnetic fields. Then we can treat $\mathcal{H}'_{\text{ex}}$, V_{el}^Γ , and V_{mag}^Γ as perturbative terms to evaluate the SRT, eq. (2.18).

9.3.1 Explicit Form of SRT

Through the standard procedure of the Schwinger-Keldysh (or contour-ordered) Green's function,[100, 31, 32] the Langreth method,[33, 22, 34, 35] each term of the SRT can be evaluated as follows; the first term of eq. (2.18) reads

$$\begin{aligned} & \left\langle iJ a_0^3 \sqrt{\frac{\tilde{S}}{2}} a^\dagger(\mathbf{x}, t) \left[1 - \frac{a^\dagger(\mathbf{x}, t) a(\mathbf{x}, t)}{4\tilde{S}} \right] c^\dagger(\mathbf{x}, t) \sigma^+ c(\mathbf{x}, t) \right\rangle \\ &= \frac{JS}{2} \left(\frac{\Gamma_0}{2} \right)^2 \int \frac{d\mathbf{k}}{(2\pi)^3} \int \frac{d\omega}{2\pi} \left[1 - \frac{i}{\tilde{S}} \int \frac{d\mathbf{k}'}{(2\pi)^3} \int \frac{d\omega'}{2\pi} \mathcal{G}_{\mathbf{k}',\omega'}^< \right] \\ & \times \left\{ \left[e^{2i\Omega t} \mathcal{G}_{0,\Omega}^a + \mathcal{G}_{0,-\Omega}^a \right] \left[\mathcal{G}_{\downarrow,\mathbf{k},\omega-\Omega}^t \mathcal{G}_{\uparrow,\mathbf{k},\omega}^t - \mathcal{G}_{\downarrow,\mathbf{k},\omega-\Omega}^< \mathcal{G}_{\uparrow,\mathbf{k},\omega}^> \right] \right. \\ & \left. + \left[e^{-2i\Omega t} \mathcal{G}_{0,-\Omega}^a + \mathcal{G}_{0,\Omega}^a \right] \left[\mathcal{G}_{\downarrow,\mathbf{k},\omega+\Omega}^t \mathcal{G}_{\uparrow,\mathbf{k},\omega}^t - \mathcal{G}_{\downarrow,\mathbf{k},\omega+\Omega}^< \mathcal{G}_{\uparrow,\mathbf{k},\omega}^> \right] \right\} \\ & + \mathcal{O}(J^2) + \mathcal{O}(\Gamma^4) + \mathcal{O}(JS^{-1}). \end{aligned} \quad (9.102)$$

The variable $\mathcal{G}^{t(<, >)}$ is the fermionic time-ordered (lesser, greater) Green's function, and $\mathcal{G}^{<(a)}$ is the bosonic lesser (advanced) one.

We here have taken the extended time (i.e. the contour variable) defined on the Schwinger-Keldysh closed time path,[35, 32, 22, 33, 34] c , on the forward path $c_{\rightarrow}$ (see also Fig. 9.7); $c = c_{\rightarrow} + c_{\leftarrow}$. Even when the time is located on the backward path $c_{\leftarrow}$, the result of the calculation does not change because each Green's function is not independent; $\mathcal{G}^r - \mathcal{G}^a = \mathcal{G}^< - \mathcal{G}^>$, where $\mathcal{G}^{r(a)}$ represents the retarded (advanced) Green's function. This relation comes into effect also for the bosonic case (see eq. (9.77)).

Here it would be useful to mention that under the thermal equilibrium condition where temperature difference does not exist between ferromagnet and non-magnetic metal, the $\mathcal{O}(J^2\Gamma^0)$ term including no quantum fluctuations cannot contribute to spin pumping because of the balance between thermal fluctuations in ferromagnet and those in non-magnetic metal.[96]

The second term of eq. (2.18) reads

$$\begin{aligned}
& \left\langle -iJa_0^3 \sqrt{\frac{\tilde{S}}{2}} \left[1 - \frac{a^\dagger(\mathbf{x}, t)a(\mathbf{x}, t)}{4\tilde{S}} \right] a(\mathbf{x}, t)c^\dagger(\mathbf{x}, t)\sigma^- c(\mathbf{x}, t) \right\rangle \\
&= -\frac{JS}{2} \left(\frac{\Gamma_0}{2}\right)^2 \int \frac{d\mathbf{k}}{(2\pi)^3} \int \frac{d\omega}{2\pi} \left[1 - \frac{i}{\tilde{S}} \int \frac{d\mathbf{k}'}{(2\pi)^3} \int \frac{d\omega'}{2\pi} G_{\mathbf{k}', \omega'}^< \right] \\
&\times \left\{ \left[e^{2i\Omega t} G_{0, \Omega}^r + G_{0, -\Omega}^r \right] \left[\mathcal{G}_{\uparrow, \mathbf{k}, \omega - \Omega}^t \mathcal{G}_{\downarrow, \mathbf{k}, \omega}^t - \mathcal{G}_{\uparrow, \mathbf{k}, \omega - \Omega}^< \mathcal{G}_{\downarrow, \mathbf{k}, \omega}^> \right] \right. \\
&\quad \left. + \left[e^{-2i\Omega t} G_{0, -\Omega}^r + G_{0, \Omega}^r \right] \left[\mathcal{G}_{\uparrow, \mathbf{k}, \omega + \Omega}^t \mathcal{G}_{\downarrow, \mathbf{k}, \omega}^t - \mathcal{G}_{\uparrow, \mathbf{k}, \omega + \Omega}^< \mathcal{G}_{\downarrow, \mathbf{k}, \omega}^> \right] \right\} \\
&\quad + \mathcal{O}(J^2) + \mathcal{O}(\Gamma^4) + \mathcal{O}(JS^{-1}),
\end{aligned} \tag{9.103}$$

where the variable G^r is the bosonic retarded Green's function. The third term of eq. (2.18) reads

$$\begin{aligned}
& \left\langle \frac{\Gamma(t)c^\dagger(\mathbf{x}, t)\sigma^+ c(\mathbf{x}, t)}{4i} \right\rangle \\
&= \frac{JS}{2} \left(\frac{\Gamma_0}{2}\right)^2 \int \frac{d\mathbf{k}}{(2\pi)^3} \int \frac{d\omega}{2\pi} \left[1 - \frac{i}{\tilde{S}} \int \frac{d\mathbf{k}'}{(2\pi)^3} \int \frac{d\omega'}{2\pi} G_{\mathbf{k}', \omega'}^< \right] \\
&\times \left\{ (e^{2i\Omega t} + 1) G_{0, -\Omega}^r \left[\mathcal{G}_{\downarrow, \mathbf{k}, \omega - \Omega}^t \mathcal{G}_{\uparrow, \mathbf{k}, \omega}^t - \mathcal{G}_{\downarrow, \mathbf{k}, \omega - \Omega}^< \mathcal{G}_{\uparrow, \mathbf{k}, \omega}^> \right] \right. \\
&\quad \left. + (e^{-2i\Omega t} + 1) G_{0, \Omega}^r \left[\mathcal{G}_{\downarrow, \mathbf{k}, \omega + \Omega}^t \mathcal{G}_{\uparrow, \mathbf{k}, \omega}^t - \mathcal{G}_{\downarrow, \mathbf{k}, \omega + \Omega}^< \mathcal{G}_{\uparrow, \mathbf{k}, \omega}^> \right] \right\} \\
&\quad + \mathcal{O}(J^0) + \mathcal{O}(J^2) + \mathcal{O}(\Gamma^4) + \mathcal{O}(JS^{-1}).
\end{aligned} \tag{9.104}$$

We omit the $\mathcal{O}(J^0)$ terms because they contain no contributions of magnons via the exchange interaction and are not relevant to the spin pumping effect; the terms are out of the purpose of the present study.

The last term of eq. (2.18) reads

$$\begin{aligned}
& \left\langle -\frac{\Gamma(t)c^\dagger(\mathbf{x}, t)\sigma^- c(\mathbf{x}, t)}{4i} \right\rangle \\
&= -\frac{JS}{2} \left(\frac{\Gamma_0}{2}\right)^2 \int \frac{d\mathbf{k}}{(2\pi)^3} \int \frac{d\omega}{2\pi} \left[1 - \frac{i}{\tilde{S}} \int \frac{d\mathbf{k}'}{(2\pi)^3} \int \frac{d\omega'}{2\pi} G_{\mathbf{k}', \omega'}^< \right] \\
&\times \left\{ (e^{2i\Omega t} + 1) G_{0, -\Omega}^a \left[\mathcal{G}_{\uparrow, \mathbf{k}, \omega - \Omega}^t \mathcal{G}_{\downarrow, \mathbf{k}, \omega}^t - \mathcal{G}_{\uparrow, \mathbf{k}, \omega - \Omega}^< \mathcal{G}_{\downarrow, \mathbf{k}, \omega}^> \right] \right. \\
&\quad \left. + (e^{-2i\Omega t} + 1) G_{0, \Omega}^a \left[\mathcal{G}_{\uparrow, \mathbf{k}, \omega + \Omega}^t \mathcal{G}_{\downarrow, \mathbf{k}, \omega}^t - \mathcal{G}_{\uparrow, \mathbf{k}, \omega + \Omega}^< \mathcal{G}_{\downarrow, \mathbf{k}, \omega}^> \right] \right\} \\
&\quad + \mathcal{O}(J^0) + \mathcal{O}(J^2) + \mathcal{O}(\Gamma^4) + \mathcal{O}(JS^{-1}).
\end{aligned} \tag{9.105}$$

Finally, the SRT can be rearranged as

$$\langle \mathcal{T}_s^z \rangle = \sum_{n=0, \pm 1} \langle \mathcal{T}_s^z(n) \rangle e^{2in\Omega t}, \quad (9.106)$$

where

$$\begin{aligned} \langle \mathcal{T}_s^z(1) \rangle &= \frac{JS}{2} \left(\frac{\Gamma_0}{2} \right)^2 \int \frac{d\mathbf{k}}{(2\pi)^3} \int \frac{d\omega}{2\pi} \left[1 - \frac{i}{\tilde{S}} \int \frac{d\mathbf{k}'}{(2\pi)^3} \int \frac{d\omega'}{2\pi} G_{\mathbf{k}', \omega'}^< \right] \\ &\times \left[(G_{0, \Omega}^a + G_{0, -\Omega}^r) (\mathcal{G}_{\downarrow, \mathbf{k}, \omega - \Omega}^t \mathcal{G}_{\uparrow, \mathbf{k}, \omega}^t - \mathcal{G}_{\downarrow, \mathbf{k}, \omega - \Omega}^< \mathcal{G}_{\uparrow, \mathbf{k}, \omega}^>) \right. \\ &- (G_{0, \Omega}^r + G_{0, -\Omega}^a) (\mathcal{G}_{\uparrow, \mathbf{k}, \omega - \Omega}^t \mathcal{G}_{\downarrow, \mathbf{k}, \omega}^t - \mathcal{G}_{\uparrow, \mathbf{k}, \omega - \Omega}^< \mathcal{G}_{\downarrow, \mathbf{k}, \omega}^>) \left. \right], \end{aligned} \quad (9.107)$$

$$\begin{aligned} \langle \mathcal{T}_s^z(-1) \rangle &= \frac{JS}{2} \left(\frac{\Gamma_0}{2} \right)^2 \int \frac{d\mathbf{k}}{(2\pi)^3} \int \frac{d\omega}{2\pi} \left[1 - \frac{i}{\tilde{S}} \int \frac{d\mathbf{k}'}{(2\pi)^3} \int \frac{d\omega'}{2\pi} G_{\mathbf{k}', \omega'}^< \right] \\ &\times \left[(G_{0, -\Omega}^a + G_{0, \Omega}^r) (\mathcal{G}_{\downarrow, \mathbf{k}, \omega + \Omega}^t \mathcal{G}_{\uparrow, \mathbf{k}, \omega}^t - \mathcal{G}_{\downarrow, \mathbf{k}, \omega + \Omega}^< \mathcal{G}_{\uparrow, \mathbf{k}, \omega}^>) \right. \\ &- (G_{0, -\Omega}^r + G_{0, \Omega}^a) (\mathcal{G}_{\uparrow, \mathbf{k}, \omega + \Omega}^t \mathcal{G}_{\downarrow, \mathbf{k}, \omega}^t - \mathcal{G}_{\uparrow, \mathbf{k}, \omega + \Omega}^< \mathcal{G}_{\downarrow, \mathbf{k}, \omega}^>) \left. \right], \end{aligned} \quad (9.108)$$

$$\begin{aligned} \langle \mathcal{T}_s^z(0) \rangle &= \frac{JS}{2} \left(\frac{\Gamma_0}{2} \right)^2 \int \frac{d\mathbf{k}}{(2\pi)^3} \int \frac{d\omega}{2\pi} \left[1 - \frac{i}{\tilde{S}} \int \frac{d\mathbf{k}'}{(2\pi)^3} \int \frac{d\omega'}{2\pi} G_{\mathbf{k}', \omega'}^< \right] \\ &\times \left[(G_{0, -\Omega}^a + G_{0, \Omega}^r) (\mathcal{G}_{\downarrow, \mathbf{k}, \omega - \Omega}^t \mathcal{G}_{\uparrow, \mathbf{k}, \omega}^t - \mathcal{G}_{\downarrow, \mathbf{k}, \omega - \Omega}^< \mathcal{G}_{\uparrow, \mathbf{k}, \omega}^>) \right. \\ &+ (G_{0, \Omega}^a + G_{0, -\Omega}^r) (\mathcal{G}_{\downarrow, \mathbf{k}, \omega + \Omega}^t \mathcal{G}_{\uparrow, \mathbf{k}, \omega}^t - \mathcal{G}_{\downarrow, \mathbf{k}, \omega + \Omega}^< \mathcal{G}_{\uparrow, \mathbf{k}, \omega}^>) \\ &- (G_{0, -\Omega}^r + G_{0, \Omega}^a) (\mathcal{G}_{\uparrow, \mathbf{k}, \omega - \Omega}^t \mathcal{G}_{\downarrow, \mathbf{k}, \omega}^t - \mathcal{G}_{\uparrow, \mathbf{k}, \omega - \Omega}^< \mathcal{G}_{\downarrow, \mathbf{k}, \omega}^>) \\ &- (G_{0, \Omega}^r + G_{0, -\Omega}^a) (\mathcal{G}_{\uparrow, \mathbf{k}, \omega + \Omega}^t \mathcal{G}_{\downarrow, \mathbf{k}, \omega}^t - \mathcal{G}_{\uparrow, \mathbf{k}, \omega + \Omega}^< \mathcal{G}_{\downarrow, \mathbf{k}, \omega}^>) \left. \right]. \end{aligned} \quad (9.109)$$

9.3.2 Detail of the Procedure

From now on, we show how to calculate eqs. (9.102)-(9.105) in detail. We omit the label $\mathbf{x}$ when it is not relevant.

By adopting the Wick's theorem,[32, 35] the left-hand side (LHS) of eq. (9.102) reads

$$\begin{aligned} (\text{The LHS of eq. (9.102)}) &= \frac{iJa_0^3 \tilde{S}}{2} \int d\mathbf{x}' \int d\mathbf{x}'' \int_c d\tau' \int_c d\tau'' \Gamma(\tau') \Gamma(\tau'') \\ &\times \left(1 - \frac{\langle T_c a^\dagger(\mathbf{x}', \tau') a(\mathbf{x}', \tau') \rangle}{\tilde{S}} \right) \\ &\times \langle T_c a(\mathbf{x}'', \tau'') a^\dagger(\mathbf{x}, \tau) \rangle \langle T_c c_\downarrow(\mathbf{x}, \tau) c_\downarrow^\dagger(\mathbf{x}', \tau') \rangle \\ &\times \langle T_c c_\uparrow(\mathbf{x}', \tau') c_\uparrow^\dagger(\mathbf{x}, \tau) \rangle. \end{aligned} \quad (9.110)$$

By employing the relation;

$$\int_c d\tau' \int_c d\tau'' = \left(\int_{c \rightarrow} d\tau' + \int_{c \leftarrow} d\tau' \right) \left(\int_{c \rightarrow} d\tau'' + \int_{c \leftarrow} d\tau'' \right) \quad (9.111)$$

$$\begin{aligned} &= \int_{c \rightarrow} d\tau' \int_{c \rightarrow} d\tau'' + \int_{c \rightarrow} d\tau' \int_{c \leftarrow} d\tau'' \\ &+ \int_{c \leftarrow} d\tau' \int_{c \rightarrow} d\tau'' + \int_{c \leftarrow} d\tau' \int_{c \leftarrow} d\tau'', \end{aligned} \quad (9.112)$$

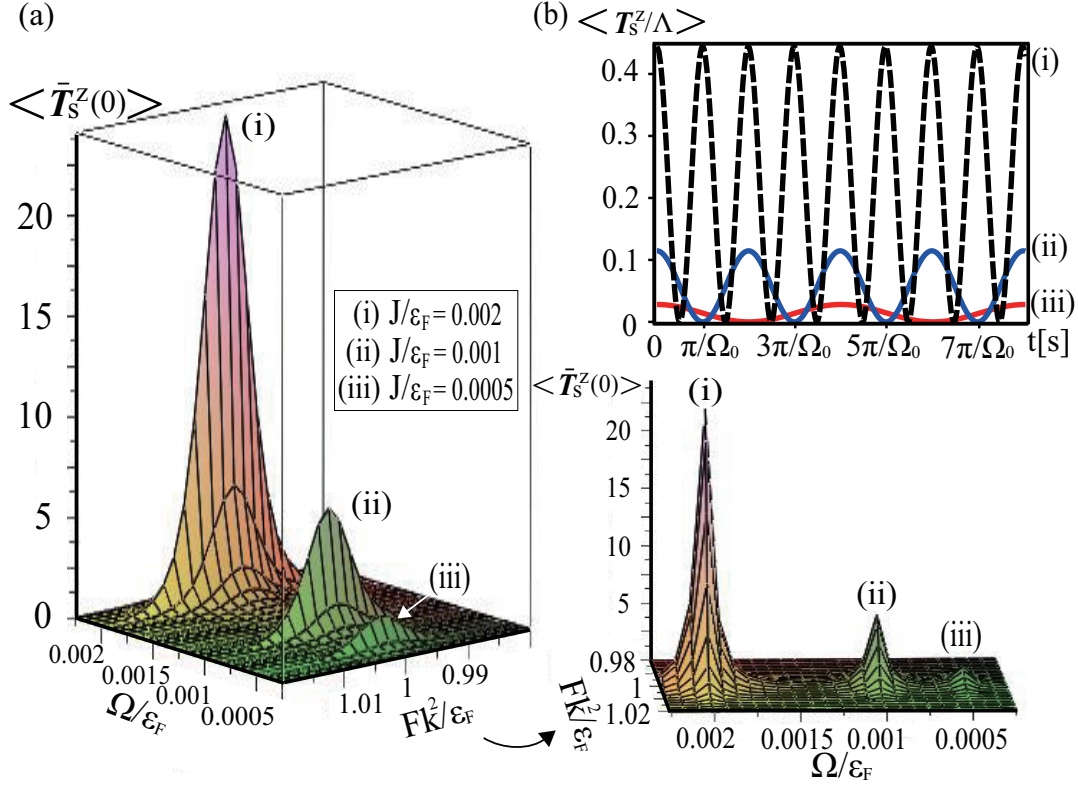


Figure 9.8: (Color online) (a) The angular frequency dependence of the SRT; $\langle \bar{T}_s^z(0) \rangle$. A sharp peak exists on the point where the resonance condition, $\Omega = J$, is satisfied. Each quantity, $\langle \bar{T}_s^z(\pm 1) \rangle$, has the same structure with $\langle \bar{T}_s^z(0) \rangle$. (b) The time evolution of the SRT, $\langle T_s^z/\Lambda \rangle$, at the resonance point ($\Omega = J$); (i) $0.446 \times \cos^2(\Omega_0 t)$, (ii) $0.112 \times \cos^2(\Omega_0 t/2)$, (iii) $0.0281 \times \cos^2(\Omega_0 t/4)$, where $\Omega_0 \equiv 1.70 \times 10^{13} \text{ s}^{-1}$.

and the Langreth method[33, 22, 34] with eqs. (9.76) and (9.90), the right-hand side (RHS) of eq. (9.110) can be expressed as

$$\begin{aligned}
 (\text{The RHS of eq.(9.110)}) &= \frac{Ja_0^3 \tilde{S}}{2} \int d\mathbf{x}' \int d\mathbf{x}'' \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dt'' \Gamma(t') \Gamma(t'') \\
 &\times \left(1 - \frac{i}{\tilde{S}} G^<(t', t') \right) [G^t(t'', t) - G^>(t'', t)] \\
 &\times [\mathcal{G}_{\downarrow}^t(t, t') \mathcal{G}_{\uparrow}^t(t', t) - \mathcal{G}_{\downarrow}^<(t, t') \mathcal{G}_{\uparrow}^>(t', t)]. \quad (9.113)
 \end{aligned}$$

We also here have taken τ on forward path, $c_{\rightarrow}$. As discussed in the last subsection, even when τ is located on backward path, $c_{\leftarrow}$, the result of this calculation is invariant.

Here it should be noted that

$$G^t(t'', t) - G^>(t'', t) = G^a(t'', t). \quad (9.114)$$

Then, eq. (9.113) can be rewritten as

$$\begin{aligned}
(\text{The RHS of eq.(9.113)}) &= \frac{Ja_0^3 \tilde{S}}{2} \int d\mathbf{x}' \int d\mathbf{x}'' \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dt'' \Gamma(t') \Gamma(t'') \\
&\times \left(1 - \frac{i}{\tilde{S}} G^<(t', t') \right) G^a(t'', t) \\
&\times [\mathcal{G}_{\downarrow}^{\dagger}(t, t') \mathcal{G}_{\uparrow}^{\dagger}(t', t) - \mathcal{G}_{\downarrow}^<(t, t') \mathcal{G}_{\uparrow}^>(t', t)]. \tag{9.115}
\end{aligned}$$

By executing Fourier transformation, we obtain the RHS of eq. (9.102).

Through the same procedure, remained terms are evaluated as follows;

$$\begin{aligned}
(\text{The LHS of eq.(9.103)}) &= -\frac{iJa_0^3 \tilde{S}}{2} \int d\mathbf{x}' \int d\mathbf{x}'' \int_c d\tau' \int_c d\tau'' \Gamma(\tau') \Gamma(\tau'') \\
&\times \left(1 - \frac{\langle T_c a^{\dagger}(\mathbf{x}', \tau') a(\mathbf{x}', \tau') \rangle}{\tilde{S}} \right) \\
&\times \langle T_c a(\mathbf{x}, \tau) a^{\dagger}(\mathbf{x}'', \tau'') \rangle \langle T_c c_{\uparrow}(\mathbf{x}, \tau) c_{\uparrow}^{\dagger}(\mathbf{x}', \tau') \rangle \\
&\times \langle T_c c_{\downarrow}(\mathbf{x}', \tau') c_{\downarrow}^{\dagger}(\mathbf{x}, \tau) \rangle \tag{9.116}
\end{aligned}$$

$$\begin{aligned}
&= -\frac{Ja_0^3 \tilde{S}}{2} \int d\mathbf{x}' \int d\mathbf{x}'' \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dt'' \Gamma(t') \Gamma(t'') \\
&\times \left(1 - \frac{i}{\tilde{S}} G^<(t', t') \right) G^r(t, t'') \\
&\times [\mathcal{G}_{\uparrow}^{\dagger}(t, t') \mathcal{G}_{\downarrow}^{\dagger}(t', t) - \mathcal{G}_{\uparrow}^<(t, t') \mathcal{G}_{\downarrow}^>(t', t)]. \tag{9.117}
\end{aligned}$$

$$\begin{aligned}
(\text{The LHS of eq.(9.104)}) &= \frac{iJa_0^3 \tilde{S}}{2} \int d\mathbf{x}' \int d\mathbf{x}'' \int_c d\tau' \int_c d\tau'' \Gamma(\tau) \Gamma(\tau'') \\
&\times \left(1 - \frac{\langle T_c a^{\dagger}(\mathbf{x}', \tau') a(\mathbf{x}', \tau') \rangle}{\tilde{S}} \right) \\
&\times \langle T_c a(\mathbf{x}', \tau') a^{\dagger}(\mathbf{x}'', \tau'') \rangle \langle T_c c_{\downarrow}(\mathbf{x}, \tau) c_{\downarrow}^{\dagger}(\mathbf{x}', \tau') \rangle \\
&\times \langle T_c c_{\uparrow}(\mathbf{x}', \tau') c_{\uparrow}^{\dagger}(\mathbf{x}, \tau) \rangle \tag{9.118}
\end{aligned}$$

$$\begin{aligned}
&= \frac{Ja_0^3 \tilde{S}}{2} \int d\mathbf{x}' \int d\mathbf{x}'' \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dt'' \Gamma(t) \Gamma(t'') \\
&\times \left(1 - \frac{i}{\tilde{S}} G^<(t', t') \right) G^r(t', t'') \\
&\times [\mathcal{G}_{\downarrow}^{\dagger}(t, t') \mathcal{G}_{\uparrow}^{\dagger}(t', t) - \mathcal{G}_{\downarrow}^<(t, t') \mathcal{G}_{\uparrow}^>(t', t)]. \tag{9.119}
\end{aligned}$$

Note that we have adopted the relation; $G^t - G^< = G^r = G^> - G^{\bar{t}}$.

$$\begin{aligned}
(\text{The LHS of eq.(9.105)}) &= -\frac{iJa_0^3\tilde{S}}{2} \int d\mathbf{x}' \int d\mathbf{x}'' \int_c d\tau' \int_c d\tau'' \Gamma(\tau) \Gamma(\tau'') \\
&\times \left(1 - \frac{\langle T_c a^\dagger(\mathbf{x}', \tau') a(\mathbf{x}', \tau') \rangle}{\tilde{S}}\right) \\
&\times \langle T_c a(\mathbf{x}'', \tau'') a^\dagger(\mathbf{x}', \tau') \rangle \langle T_c c_\uparrow(\mathbf{x}, \tau) c_\uparrow^\dagger(\mathbf{x}', \tau') \rangle \\
&\times \langle T_c c_\downarrow(\mathbf{x}', \tau') c_\downarrow^\dagger(\mathbf{x}, \tau) \rangle \quad (9.120)
\end{aligned}$$

$$\begin{aligned}
&= -\frac{Ja_0^3\tilde{S}}{2} \int d\mathbf{x}' \int d\mathbf{x}'' \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dt'' \Gamma(t) \Gamma(t'') \\
&\times \left(1 - \frac{i}{\tilde{S}} G^<(t', t'')\right) G^a(t'', t') \\
&\times [\mathcal{G}_\uparrow^t(t, t') \mathcal{G}_\downarrow^t(t', t) - \mathcal{G}_\uparrow^<(t, t') \mathcal{G}_\downarrow^>(t', t)]. \quad (9.121)
\end{aligned}$$

Note that we have employed the relation; $G^< - G^{\bar{t}} = G^a = G^t - G^>$.

By using Fourier transformation, we obtain the RHS of eqs. (9.103)-(9.105). For the purpose, the following relations, each Green's function including the the effects of impurities as the lifetime, will be useful;¹²

$$G_{\mathbf{k}', \omega'}^a = [\omega' - \omega_{\mathbf{k}'} - i/(2\tau_m)]^{-1} \quad (9.122)$$

$$= (G_{\mathbf{k}', \omega'}^r)^*, \quad (9.123)$$

$$\mathcal{G}_{\sigma, \mathbf{k}, \omega}^a = [\omega - \omega_{\sigma, \mathbf{k}} - i/(2\tau)]^{-1} \quad (9.124)$$

$$= (\mathcal{G}_{\sigma, \mathbf{k}, \omega}^r)^*, \quad (9.125)$$

$$G_{\mathbf{k}', \omega'}^< = -f_B(\omega') (G_{\mathbf{k}', \omega'}^a - G_{\mathbf{k}', \omega'}^r), \quad (9.126)$$

$$\mathcal{G}_{\sigma, \mathbf{k}, \omega}^< = f_F(\omega) (\mathcal{G}_{\sigma, \mathbf{k}, \omega}^a - \mathcal{G}_{\sigma, \mathbf{k}, \omega}^r), \quad (9.127)$$

$$\mathcal{G}_{\sigma, \mathbf{k}, \omega}^t = \mathcal{G}_{\sigma, \mathbf{k}, \omega}^r + \mathcal{G}_{\sigma, \mathbf{k}, \omega}^< \quad (9.128)$$

$$= \mathcal{G}_{\sigma, \mathbf{k}, \omega}^> + \mathcal{G}_{\sigma, \mathbf{k}, \omega}^a, \quad (9.129)$$

¹²Here let us mention that, in real materials, there does exist impurity scattering. We assume that this is the main cause for the finite lifetime of the magnons and the conduction electrons. Moreover the rate of impurities such as lattice defects and nonmagnetic impurities is, in general, far larger than that of magnetic impurities. Therefore we phenomenologically introduce the lifetime and regard it as a constant parameter. Then we adopt Green's functions including the the effects of impurities as the lifetime, and calculate the SRT by using them; eqs. (9.122)-(9.129). This is the procedure called dressed perturbation theory. Though, as the result, it might be better to execute the accompanying vertex corrections from viewpoints of theoretical aspects, we have not done for the aim now explained; to put it briefly, in order to clarify that pumped spin currents are generated purely by quantum fluctuations, we have not executed vertex corrections. It should be noted that before our present study, Takeuchi et al.[104] have already studied spin pumping on the basis of the Schwinger-Keldysh formalism under the same condition with ours except two points; (i) they have treated localized spins as not magnons but classical variables, and (ii) they have not applied any transverse magnetic fields. On their condition, they have clarified that under the uniform magnetization spin currents cannot be generated without vertex corrections (i.e. multiple scatterings of impurities). In other words, they have already revealed that spin currents can be generated by the effects of multiple scatterings of impurities, i.e. vertex corrections. Thus, the main purpose of the present study is to propose an alternative way for the generation of spin currents without using vertex corrections, i.e. multiple scatterings of impurities; we propose a method for the generation of spin currents by using time-dependent transverse magnetic fields, which are under our control and act as quantum fluctuations. Therefore we call this method quantum spin pumping. In order to clarify that pumped spin currents are induced purely by quantum fluctuations, we have not included the effects of multiple scatterings of impurities (i.e. we have not executed vertex corrections). We now recognize that to include vertex corrections into our quantum spin pumping theory might be a significant task.

where the variables τ_m , τ , $f_B(\omega')$, and $f_F(\omega)$ are the lifetime of magnons, that of the conduction electrons, the Bose distribution function, and the Fermi one, respectively. The energy dispersion relation reads $\omega_{\mathbf{k}'} \equiv Dk'^2 + B$ and $\omega_{\sigma,\mathbf{k}} \equiv Fk^2 - (JS + B/2)\sigma - \mu$, where $D \equiv 1/(2m)$, $F \equiv 1/(2m_{\text{el}})$, $\sigma = +1, -1 (= \uparrow, \downarrow)$, and μ denotes the chemical potential.

We consider a weak magnetic field regime and omit the $\mathcal{O}([(JS + B)/\epsilon_F]^2)$ terms, where ϵ_F represents the Fermi energy. Through the Sommerfeld expansion, the chemical potential is determined as, $\mu(T) = \epsilon_F - (\pi k_B T)^2/(12\epsilon_F) + \mathcal{O}(T^4)$.

Chapter 10

Appendix; Magnon Pumping and Resultant Magnon BEC

Although we have calculated the macroscopic condensate order parameter $\langle a(t) \rangle$ of quantum spin-pumping systems in Chap. 4, strictly speaking, it does not represent the number of condensate magnons; it is that of pumped magnons. We should recognize that magnons pumped by applied microwaves undergo magnon BEC after thermalization processes.[62, 52, 53] Here, we show how to effectively or phenomenologically include the above thermalization processes into calculations[137] by employing ferromagnetic insulators (for convenience) as an example.¹

10.1 Macroscopic Condensate Order Parameter

We apply a right-handed rotating magnetic field whose frequency reads $\Omega(>0)$ to ferromagnetic insulators at low-temperature (a few K) for convenience (Fig. 10.1);

$$V_R := \Gamma_0 \int d^3x (e^{i\Omega t} S^+ + e^{-i\Omega t} S^-) \quad (10.1)$$

$$= 2\Gamma_0 \int d^3x [S^x \cos(-\Omega t) + S^y \sin(-\Omega t)]. \quad (10.2)$$

¹Let us stress that this method has been developed owing to discussion with Dr. Kevin of University of Basel during my stay (from July 2 to July 19 in 2013) and this procedure is mainly based on his creative idea.

By using the standard procedure of the Schwinger-Keldysh formalism, the condensate order parameter can be evaluated as follows;[84]

$$\langle a(t) \rangle = -i\sqrt{2\tilde{S}\Gamma_0} \int d^3x' \int_c d\tau' \langle T_c a(\tau) a^\dagger(\tau') \rangle e^{-i\Omega\tau'} + \mathcal{O}((\Gamma_0)^3) \quad (10.3)$$

$$= -i\sqrt{2\tilde{S}\Gamma_0} \int d^3x' \{i \int_{-\infty}^{\infty} dt' [g^t(t, t') - g^<(t, t')]\} e^{-i\Omega t'} \quad (10.4)$$

$$= -i\sqrt{2\tilde{S}\Gamma_0} \int d^3x' \{i \int_{-\infty}^{\infty} dt' g^r(t, t')\} e^{-i\Omega t'} \quad (10.5)$$

$$= \sqrt{2\tilde{S}\Gamma_0} \int d^3x' \int_{-\infty}^{\infty} dt' [g^r(t, t')] e^{-i\Omega t'} \quad (10.6)$$

$$= \sqrt{2\tilde{S}\Gamma_0} \int d^3x' \int_{-\infty}^{\infty} dt' \left[\frac{1}{V} \sum_{\mathbf{k}} \int \frac{d\omega}{2\pi} e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}') - i\omega(t - t')} g_{\mathbf{k}, \omega}^r \right] e^{-i\Omega t'} \quad (10.7)$$

$$= \sqrt{2\tilde{S}\Gamma_0} \int_{-\infty}^{\infty} dt' e^{-i\Omega t'} \int \frac{d\omega}{2\pi} e^{-i\omega(t - t')} g_{0, \omega}^r \quad (10.8)$$

$$=: \langle a_0 \rangle, \quad (10.9)$$

where the variable $g^r/g^t/g^<$ represents the retarded/time-ordered/lesser green's function. It is clear that the applied right-handed rotating magnetic field produces only the zero-mode of magnons.

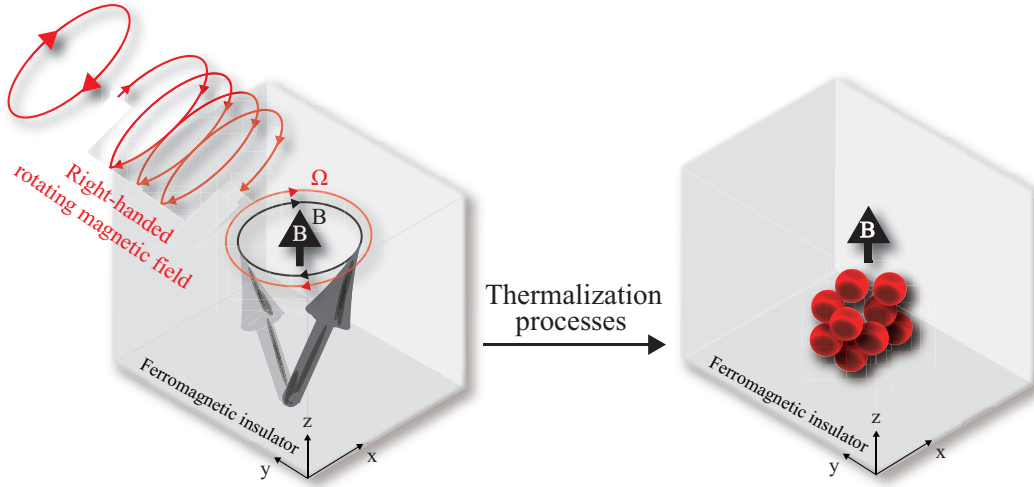


Figure 10.1: (Color online) A schematic picture of our system whose volume reads V ; we apply right-handed rotating magnetic fields whose frequency reads $\Omega(> 0)$ to ferromagnetic insulators at low-temperature (a few K) under the static magnetic fields B .

The retarded Green's function $g_{\mathbf{k}, \omega}^r$ reads

$$g_{\mathbf{k}, \omega}^r = \frac{1}{\omega - \omega_{\mathbf{k}} + i0} \quad \text{with} \quad \omega_{\mathbf{k}} = \frac{\hbar^2 \mathbf{k}^2}{2m} - \mu, \quad (10.10)$$

in which the variable μ denotes the chemical potential. By using above relations, we can obtain the exact value of the order parameter $\langle a_0 \rangle$.

10.2 Pumping Process

It should be noted that under pumping, the chemical potential μ reads

$$\mu = -B =: \mu_B \quad (\text{under pumping}). \quad (10.11)$$

Then, $\langle a_0 \rangle$ in eq. (10.8) becomes

$$\langle a_0 \rangle_{\mu=-B} = -\pi i \Gamma_0 \sqrt{2\tilde{S}} e^{-iBt} \delta(B - \Omega), \quad (10.12)$$

and it is clear that this describes the coherent precession with the frequency B . On top of this, it should be noted that when we apply left-handed rotating magnetic fields (i.e. $\Omega \mapsto -\Omega$ with $\Omega > 0$), one cannot realize $\langle a_0 \rangle \neq 0$ and therefore, we cannot obtain magnon BEC by using left-handed rotating magnetic fields.

Here let us remark that in this case, $|\langle a_0 \rangle|^2$ represents the number density of magnons, n_{pump} , pumped by microwaves;

$$|\langle a_0 \rangle_{\mu=-B}|^2 = n_{\text{pump}}. \quad (10.13)$$

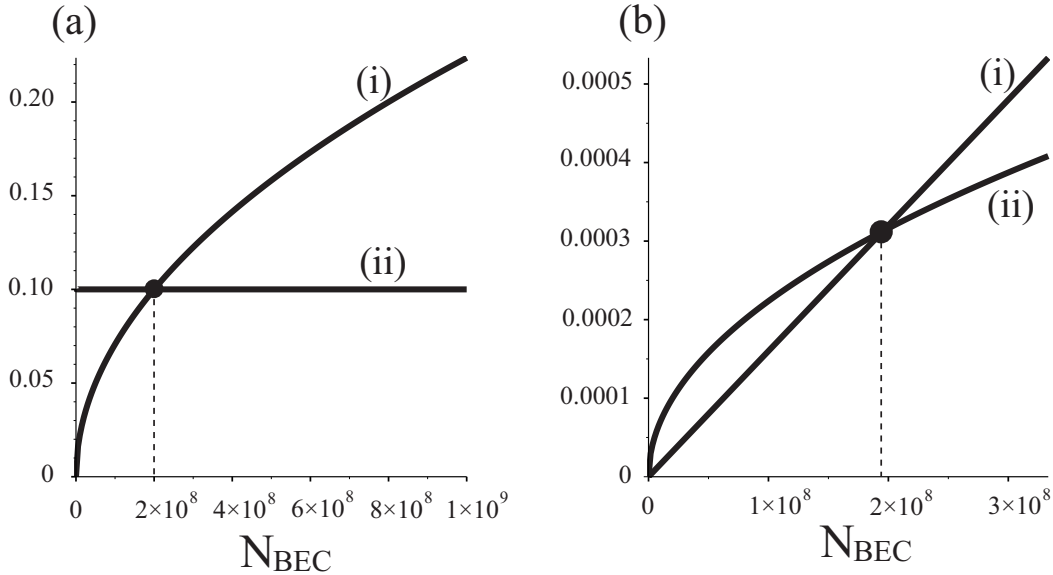


Figure 10.2: (a) The plot of each function as the function of N_{BEC} (i.e. the number of condensate magnons); (i) $\sqrt{\frac{N_{\text{BEC}}}{2\tilde{S}V}}$. (ii) $\frac{\Gamma_0}{k_B T} \frac{1}{|\frac{\hbar\Omega}{k_B T} - \frac{1}{N_{\text{BEC}}}|}$. As an example, I have set each values as follows; $\Omega = 1$ GHz (i.e. $\hbar\Omega = 1.054 \times 10^{-25}$ J), $T = 3$ K (i.e. $k_B T = 4.14 \times 10^{-23}$ J, $\hbar\Omega/(k_B T) = 2.5 \times 10^{-3}$), $\Gamma_0 = 1.054 \times 10^{-26}$ J (i.e. $\Gamma_0/(k_B T) = 2.5 \times 10^{-4}$), and $\tilde{S}V = 10^{10}$. (b) The plot of each function as the function of N_{BEC} (i.e. the number of condensate magnons); (i) $8 \frac{\Gamma_0 J}{(k_B T)^2} \frac{(l_0)^3}{V} \frac{N_{\text{BEC}}}{(\frac{\hbar\Omega}{k_B T} - \frac{1}{N_{\text{BEC}}})^2}$ (ii) $\sqrt{\frac{N_{\text{BEC}}}{2\tilde{S}V}}$.

10.3 Quasi-equilibrium Magnon BEC through Thermalization Processes

It should be noted that eq. (10.8) can be rewritten also as

$$|\langle a_0 \rangle| = \sqrt{2\tilde{S}\Gamma_0} |g_{0,\Omega}^r|. \quad (10.14)$$

Through thermalization processes, the system goes into a quasi-equilibrium state and consequently, a part of the pumped magnons condensates and the chemical potential acquires[118]²

$$\mu = \mu_B \text{ (under pumping)} \rightarrow \mu = \mu_{\text{BEC}} := -\frac{k_B T}{N_{\text{BEC}}} \text{ (under BEC)}, \quad (10.15)$$

where the variable N_{BEC} denotes the number of condensate magnons (i.e. the number density of condensate magnons n_{BEC} reads $N_{\text{BEC}} = V n_{\text{BEC}}$). Then by using eq. (10.14), we can obtain the number of condensate magnons because in this case, $|\langle a_0 \rangle|^2$ represents the number density of condensate magnons n_{BEC} ;

$$|\langle a_0 \rangle_{\mu=\mu_{\text{BEC}}}|^2 = n_{\text{BEC}} \quad (10.16)$$

$$\Rightarrow |\langle a_0 \rangle_{\mu=\mu_{\text{BEC}}}| = \sqrt{\frac{N_{\text{BEC}}}{V}}. \quad (10.17)$$

That is, by solving the following self-consistent equation, we can obtain N_{BEC} ;

$$\sqrt{\frac{N_{\text{BEC}}}{V}} = \sqrt{2\tilde{S}\Gamma_0} \frac{1}{|\hbar\Omega - \frac{k_B T}{N_{\text{BEC}}}|} \quad (10.18)$$

$$\Leftrightarrow \sqrt{\frac{N_{\text{BEC}}}{2\tilde{S}V}} = \frac{\Gamma_0}{k_B T} \frac{1}{|\frac{\hbar\Omega}{k_B T} - \frac{1}{N_{\text{BEC}}}|}. \quad (10.19)$$

Fig. 10.2 (a) represents each function; (i) $\sqrt{\frac{N_{\text{BEC}}}{2\tilde{S}V}}$ and (ii) $\frac{\Gamma_0}{k_B T} \frac{1}{|\frac{\hbar\Omega}{k_B T} - \frac{1}{N_{\text{BEC}}}|}$. It is clear that the number N_{BEC} of the intersection represents the number of condensate magnons.

In conclusion, the macroscopic condensate order parameter becomes

$$\langle a(t) \rangle = \sqrt{\frac{N_{\text{BEC}}(t)}{V}} e^{-iBt} \quad (10.20)$$

$$= \sqrt{n_{\text{BEC}}(t)} e^{-iBt} \quad (10.21)$$

under quasi-equilibrium condensate state. This expression agrees with the one proposed by Bunkov and Volovik.[45]

10.4 Effects of Magnon-magnon Interactions

We now take the contributions arising from magnon-magnon interactions into;

$$V_{\text{mag}} := J_{\text{mag}}(l_0)^3 \int d^3x a^\dagger a^\dagger a a, \quad (10.22)$$

where the variable J_{mag} is the strength of the magnon-magnon interaction and l_0 denotes the lattice constant.

We consider the situation where the system is at low-temperature (a few K) and then, magnons are thin. In addition, we apply a weak rotating magnetic field Γ_0 (i.e. $\Gamma_0 \ll B$) and hence, we are allowed to use V_{mag} and V_R as perturbative terms.

²We have assumed that our system is at low-temperature (a few K).

By using the standard procedure of the Schwinger-Keldysh formalism, the condensate order parameter can be evaluated as follows;³

$$\begin{aligned} \langle a(t) \rangle &= -2 \langle T_c a(\tau) \int_c d\tau' \sqrt{2\tilde{S}} \Gamma_0 \int d^3x' e^{-i\Omega\tau'} a^\dagger(\tau') \int_c d\tau'' J(l_0)^3 \\ &\times \int d^3x'' a^\dagger(\tau'') a^\dagger(\tau'') a(\tau'') a(\tau'') \rangle + \mathcal{O}(\Gamma^2) + \mathcal{O}(J^2) \end{aligned} \quad (10.23)$$

$$\begin{aligned} &= -2\sqrt{2\tilde{S}}\Gamma_0 J(l_0)^3 \int d^3x' \int d^3x'' \int_c d\tau' \int_c d\tau'' e^{-i\Omega\tau'} \\ &\times \langle T_c a(\tau) a^\dagger(\tau') a^\dagger(\tau'') a(\tau'') a(\tau'') \rangle \end{aligned} \quad (10.24)$$

$$\begin{aligned} &= -8\sqrt{2\tilde{S}}\Gamma_0 J(l_0)^3 \int d^3x' \int d^3x'' \int_c d\tau' \int_c d\tau'' e^{-i\Omega\tau'} \\ &\times \langle T_c a(\tau) a^\dagger(\tau'') \rangle \langle T_c a(\tau'') a^\dagger(\tau') \rangle \langle T_c a(\tau'') a^\dagger(\tau'') \rangle. \end{aligned} \quad (10.25)$$

From now on, we focus on the term $\mathcal{A}$ defined as

$$\mathcal{A} := \int_c d\tau' \int_c d\tau'' \langle T_c a(\tau) a^\dagger(\tau'') \rangle \langle T_c a(\tau'') a^\dagger(\tau') \rangle \langle T_c a(\tau'') a^\dagger(\tau'') \rangle, \quad (10.26)$$

and evaluate it by using the relation;

$$\int_c d\tau' = \left(\int_{c \rightarrow} + \int_{c \leftarrow} \right) d\tau' \quad (10.27)$$

and consequently

$$\int_c d\tau' \int_c d\tau'' = \left(\int_{c \rightarrow} + \int_{c \leftarrow} \right) d\tau' \left(\int_{c \rightarrow} + \int_{c \leftarrow} \right) d\tau'' \quad (10.28)$$

$$\begin{aligned} &= \int_{c \rightarrow} d\tau' \int_{c \rightarrow} d\tau'' \int_{c \rightarrow} d\tau' \int_{c \leftarrow} d\tau'' \\ &+ \int_{c \leftarrow} d\tau' \int_{c \rightarrow} d\tau'' \int_{c \leftarrow} d\tau' \int_{c \leftarrow} d\tau''. \end{aligned} \quad (10.29)$$

Then, eq. (10.26) can be rewritten as

$$\begin{aligned} \mathcal{A} &:= -i \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dt'' \{ g^t(t, t'') [g^t(t'', t') - g^<(t'', t')] \\ &+ g^<(t, t'') [g^{\bar{t}}(t'', t') - g^>(t'', t')] \} g^<(t'', t''), \end{aligned} \quad (10.30)$$

where the variable $g^r/g^t/g^</g^{\bar{t}}/g^>$ represents the retarded/time-ordered/lesser/anti-time ordered/greater green's function. We then use the relations; $g^t(t'', t') - g^<(t'', t') = g^r(t'', t')$ and $g^{\bar{t}}(t'', t') - g^>(t'', t') = -g^r(t'', t')$. As the result, eq. (10.30) reads

$$\mathcal{A} = -i \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dt'' g^r(t'', t') g^r(t, t'') g^<(t'', t''). \quad (10.31)$$

Therefore, the order parameter eq. (10.25) reads

$$\begin{aligned} \langle a(t) \rangle &= 8i\sqrt{2\tilde{S}}\Gamma_0 J(l_0)^3 \int d^3x' \int d^3x'' \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dt'' e^{-i\Omega t'} \\ &\times g^r(t'', t') g^r(t, t'') g^<(t'', t''). \end{aligned} \quad (10.32)$$

³ $g^t(t, t') - g^<(t, t') = g^r(t, t')$.

Then, we employ Fourier transform and eq. (10.32) becomes

$$\langle a(t) \rangle = \frac{8}{V} \sqrt{2\tilde{S}\Gamma_0} J(l_0)^3 e^{-i\Omega t} g_{0,\Omega}^r g_{0,\Omega}^r \sum_{\mathbf{k}} f_B(\omega_{\mathbf{k}}), \quad (10.33)$$

where $f_B(\omega_{\mathbf{k}})$ denotes the Bose-distribution function defined as

$$f_B(\omega_{\mathbf{k}}) = \frac{1}{e^{\beta\omega_{\mathbf{k}}} - 1}, \quad \text{with } \omega_{\mathbf{k}} = \frac{\hbar^2 \mathbf{k}^2}{2m} - \mu, \quad (10.34)$$

and

$$g_{\mathbf{k},\omega}^r = \frac{1}{\omega - \omega_{\mathbf{k}} + i0} \quad (10.35)$$

in which the variable μ denotes the chemical potential.

By using these relations, we can obtain the exact value of the order parameter $\langle a \rangle$. It is apparent that the following relation is satisfied;

$$|\langle a(t) \rangle| = \frac{8}{V} \sqrt{2\tilde{S}\Gamma_0} J(l_0)^3 g_{0,\Omega}^r g_{0,\Omega}^r \sum_{\mathbf{k}} f_B(\omega_{\mathbf{k}}). \quad (10.36)$$

10.4.1 Pumping Processes

It should be noted that under pumping (i.e. non-equilibrium state), the chemical potential μ reads

$$\mu = -B =: \mu_B \quad (\text{under pumping}). \quad (10.37)$$

In this case, $|\langle a \rangle|^2$ represents the number density of magnons, n_{pump} , pumped by microwaves;

$$|\langle a \rangle_{\mu=-B}|^2 = n_{\text{pump}}, \quad (10.38)$$

and consequently the number of pumped magnons N_{pump} reads $N_{\text{pump}} = V n_{\text{pump}}$.

10.4.2 Thermalization Processes and Resultant Quasi-equilibrium Magnon BEC

On the other hand, as soon as one switches off microwaves, the system instantaneous experiences thermalization processes[62, 52, 53] to go into a quasi-equilibrium state and as the result, a part of the pumped magnons condensates. In this situation, the chemical potential (we have assumed that our system is at low-temperature (a few K)) becomes

$$\mu = \mu_B \quad (\text{under pumping}) \rightarrow \mu = \mu_{\text{BEC}} := -\frac{k_B T}{N_{\text{BEC}}} \quad (\text{under BEC}), \quad (10.39)$$

where the variable N_{BEC} denotes the number of magnons which is in condensate (i.e. the number density of condensate magnons n_{BEC} reads $N_{\text{BEC}} = V n_{\text{BEC}}$). In this case, $|\langle a \rangle|^2$ represents the number density of condensate magnons n_{BEC} ;

$$|\langle a \rangle_{\mu=\mu_{\text{BEC}}}|^2 = n_{\text{BEC}} \quad (10.40)$$

$$\Rightarrow |\langle a \rangle_{\mu=\mu_{\text{BEC}}}| = \sqrt{\frac{N_{\text{BEC}}}{V}}. \quad (10.41)$$

On top of this, when the system undergoes BEC, the Bose-distribution function, eq.(10.34), becomes

$$\sum_{\mathbf{k}} f_{\text{B}}(\omega_{\mathbf{k}}) \sim N_{\text{BEC}}. \quad (10.42)$$

As the result, eq. (10.36) becomes

$$|\langle a \rangle_{\mu=\mu_{\text{BEC}}}| = 8\sqrt{2\tilde{S}}\Gamma_0 J \frac{(l_0)^3}{V} \frac{1}{(\hbar\Omega - \frac{k_{\text{B}}T}{N_{\text{BEC}}})^2} N_{\text{BEC}} \quad (10.43)$$

$$= 8\sqrt{2\tilde{S}} \frac{\Gamma_0 J}{(k_{\text{B}}T)^2} \frac{(l_0)^3}{V} \frac{N_{\text{BEC}}}{(\frac{\hbar\Omega}{k_{\text{B}}T} - \frac{1}{N_{\text{BEC}}})^2}. \quad (10.44)$$

Then by using eq. (10.41), we obtain the relation for the occurrence of quasi-equilibrium magnon BEC;

$$|\langle a \rangle_{\mu=\mu_{\text{BEC}}}| = \sqrt{\frac{N_{\text{BEC}}}{V}} \quad (10.45)$$

$$= 8\sqrt{2\tilde{S}} \frac{\Gamma_0 J}{(k_{\text{B}}T)^2} \frac{(l_0)^3}{V} \frac{N_{\text{BEC}}}{(\frac{\hbar\Omega}{k_{\text{B}}T} - \frac{1}{N_{\text{BEC}}})^2} \quad (10.46)$$

$$\Leftrightarrow \sqrt{\frac{N_{\text{BEC}}}{2\tilde{S}V}} = 8 \frac{\Gamma_0 J}{(k_{\text{B}}T)^2} \frac{(l_0)^3}{V} \frac{N_{\text{BEC}}}{(\frac{\hbar\Omega}{k_{\text{B}}T} - \frac{1}{N_{\text{BEC}}})^2}. \quad (10.47)$$

Fig. 10.2 (b) represents, as an example, each function; (i) $8 \frac{\Gamma_0 J}{(k_{\text{B}}T)^2} \frac{(l_0)^3}{V} \frac{N_{\text{BEC}}}{(\frac{\hbar\Omega}{k_{\text{B}}T} - \frac{1}{N_{\text{BEC}}})^2}$ (ii) $\sqrt{\frac{N_{\text{BEC}}}{2\tilde{S}V}}$. It is clear that the number N_{BEC} of the intersection represents the number of condensate magnons.

10.5 Outlook

In this chapter, we have developed a method to phenomenologically include the thermalization effects. To (more) microscopically describe the thermalization processes[62, 52, 53] on the basis of the intrinsic features of YIG (i.e. the double degeneracy and the resultant magnon BEC)[46, 47, 48, 51, 49, 50, 52, 53] will be an urgent issue. On that case, the similar mechanism called disoriented chiral condensation,[149, 150] which has been investigated in the context of Hadron physics, might be useful.

Chapter 11

Appendix; Dissipation Theory

We quickly review the dissipation theory on the basis of the sophisticated textbook.[64] In particular, we employ the Caldeira-Leggett model[63] and closely investigate the dissipation effects due to phonons on magnon BEC.[88]

11.1 Open and Closed Systems

Generally, an open system is a quantum system which is coupled to another quantum system called environment (Fig. 11.1 (a)). The state of the system on which we focus will change as a consequence of the interaction with surroundings as well as its internal dynamics. Then, the resultant change of states of our system cannot be described by the unitary operators (i.e. the usual Hamiltonian). That is, in sharp contrast to the case of closed systems,¹ the quantum dynamics of an open system cannot be represented only in terms of a unitary time evolution.

Here it should be remembered that in the closed systems, the time-development of our system can be described by the (well-known) von-Neumann or Liouville-von Neumann equation;

$$\frac{d}{dt}\rho(t) = -i[\mathcal{H}(t), \rho(t)], \quad (11.1)$$

where $\rho(t)$ represents the density matrix and $\mathcal{H}(t)$ denotes the Hamiltonian of our (closed) system. Therefore we can easily see that in open systems, the time-development of the density matrix cannot be represented as in the above von-Neumann equation. That is, it is associated with some additional terms which break the unitary time evolution of our (open) system;

$$\frac{d}{dt}\rho(t) = -i[\mathcal{H}(t), \rho(t)] + \dots \quad (11.2)$$

This results from the fact that the dynamics of our system is induced by the reduced Hamiltonian of the total system, which is the Hamiltonian one has obtained by tracing out some macroscopic degrees of freedoms of total system (i.e. often called environment),² and such our

¹Let us confirm the meaning of the terminology closed systems. We encounter the situation (our quantum spin-pumping system will be a good example.) where the system under consideration is driven by external forces, an external electromagnetic field for example. In such cases, if the dynamics of the system can still be described in terms of a possibly time-dependent Hamiltonian, the system still will be said to be closed.

²Here let us remark that the terminology reservoir corresponds to an environment with an infinite number of degrees of freedoms, which we cannot microscopically analyze, and consequently leads to an irreversible behavior of the open systems.

systems are often called reduced systems. In terms of density matrix, it can be represented as follows (Fig. 11.1 (a));

$$\rho_s(t) = \text{tr}_B \rho(t), \quad (11.3)$$

where $\rho_s(t)$ is the reduced density matrix of open systems, which can be obtained by partially tracing out over the degrees of freedom of the environment (i.e. bath) and plays the key role on the dynamics of open systems.

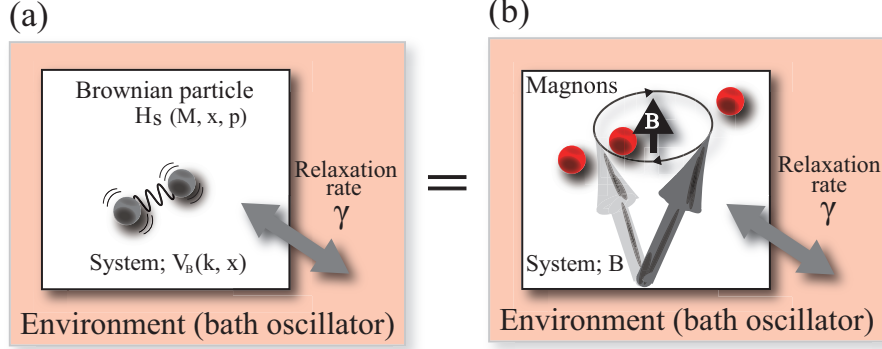


Figure 11.1: (Color online) Schematic pictures of the correspondence between the Caldeira-Leggett model (i.e. the dynamics of Brownian's particles) and our magnon systems. We have again listed the same picture for reader's convenience.

11.2 Caldeira-Leggett Model; Quantum Brownian Motion

Now let us introduce a prototype of a system-reservoir model for the description of dissipation phenomena in solid state physics called Caldeira-Leggett model,[63] which describes a Brownian particle of mass M with coordinate x moving in a potential $V(x)$; Fig. 11.1 (a). Then, the free Hamiltonian $\mathcal{H}_s$ of the particle is thus taken to be

$$\mathcal{H}_s = \frac{1}{2M}p^2 + V(x), \quad (11.4)$$

with the particle momentum p . This particle is assumed to be weakly³ and linearly coupled to a bath consisting of a large number of harmonic oscillators.

Caldeira-Leggett Master Equation

After the procedure based on Markovian quantum master equation, one reaches the following master equation called the Caldeira-Leggett master equation;

$$\frac{d}{dt}\rho_s(t) = -\frac{i}{\hbar}[\mathcal{H}_s, \rho_s(t)] - \frac{i\gamma}{\hbar}[x, \{p, \rho_s(t)\}] - \frac{2m\gamma k_B T}{\hbar^2}[x, [x, \rho_s(t)]], \quad (11.5)$$

where γ represents the relaxation rate. The first term of the right-hand side of the above master equation denotes the free coherent dynamics of the systems, the second one is the

³Enough that we are allowed to employ Born-Markov approximation.

dissipative term, and the last one describes thermal fluctuations plays the key role on decoherence phenomena.

With the help of the above master equation, one can easily obtain the following equation for the first moments⁴ of the coordinate and the momentum of Brownian particle;

$$\frac{d}{dt}\langle x \rangle = \frac{1}{M}\langle p \rangle, \quad (11.6)$$

$$\frac{d}{dt}\langle p \rangle = -\langle V'(x) \rangle - 2\gamma\langle p \rangle. \quad (11.7)$$

Preparation

One can easily show[41] them by simply using the following relations with $[x, p] = i\hbar$;

$$\langle \mathcal{O}(t) \rangle = \text{tr}(\rho_s(t)\mathcal{O}) \quad (11.8)$$

$$\frac{d}{dt}\langle \mathcal{O}(t) \rangle = \text{tr}(\dot{\rho}_s(t)\mathcal{O}) \quad (11.9)$$

$$\text{tr}[\mathcal{A}\mathcal{B}] = \text{tr}[\mathcal{B}\mathcal{A}]. \quad (11.10)$$

Proof

$$\frac{d}{dt}\langle x \rangle = \text{tr}(\dot{\rho}_s(t)x) \quad (11.11)$$

$$= \text{tr}\left(-\frac{i}{\hbar}[\mathcal{H}_s, \rho_s(t)]x\right) \quad (11.12)$$

$$= -\frac{i}{\hbar}\text{tr}\left(\left[\frac{1}{2M}p^2 + V(x), \rho_s(t)\right]x\right) \quad (11.13)$$

$$= -\frac{i}{\hbar}\text{tr}\left(\frac{p^2}{2M}\rho_s(t)x + V(x)\rho_s(t)x - \rho_s(t)\frac{p^2}{2M}x - \rho_s(t)V(x)x\right) \quad (11.14)$$

$$= -\frac{i}{\hbar}\text{tr}\left(\rho_s(t)x\frac{p^2}{2M} + \rho_s(t)xV(x) - \rho_s(t)\frac{p^2}{2M}x - \rho_s(t)xV(x)\right) \quad (11.15)$$

$$= -\frac{i}{2\hbar M}\text{tr}\left(\rho_s(t)(xp)p - \rho_s(t)p^2x\right), \quad (11.16)$$

in which we have used the relation; $\text{tr}[\mathcal{A}\mathcal{B}] = \text{tr}[\mathcal{B}\mathcal{A}]$. Then, it can be rewritten as

$$\frac{d}{dt}\langle x \rangle = -\frac{i}{2\hbar M}\text{tr}\left(\rho_s(t)(px + i\hbar)p - \rho_s(t)p^2x\right) \quad (11.17)$$

$$= -\frac{i}{2\hbar M}\text{tr}\left(\rho_s(t)[p(xp) + i\hbar p] - \rho_s(t)p^2x\right), \quad (11.18)$$

where we have employed the commutation relation; $[x, p] = i\hbar$. As the result,

$$\frac{d}{dt}\langle x \rangle = -\frac{i}{2\hbar M}\text{tr}\left(\rho_s(t)[p(px + i\hbar) + i\hbar p] - \rho_s(t)p^2x\right) \quad (11.19)$$

$$= -\frac{i}{2\hbar M}\text{tr}\left(\rho_s(t)2i\hbar p\right) \quad (11.20)$$

$$= \frac{1}{M}\text{tr}\left(\rho_s(t)p\right) \quad (11.21)$$

$$= \frac{\langle p \rangle}{M} \quad (11.22)$$

$$\Leftrightarrow \frac{d}{dt}\langle x \rangle = \frac{\langle p \rangle}{M}. \blacksquare \quad (11.23)$$

⁴Of course, we can also easily obtain the second moments.

By using the same procedure, the second equation, eq. (11.7), can be easily derived as follows;

$$\frac{d}{dt}\langle p \rangle = \text{tr}(\dot{\rho}_s(t)p) \quad (11.24)$$

$$= \text{tr}\left(\left\{-\frac{i}{\hbar}[\mathcal{H}_s, \rho_s(t)] - \frac{i\gamma}{\hbar}[x, \{p, \rho_s(t)\}]\right\}p\right). \quad (11.25)$$

Here one should note the relation;

$$[p, V(x)] = -i\hbar V'(x). \quad (11.26)$$

Then by employing this relation, we encounter the following relations;

$$\text{tr}\left(\left[\frac{p^2}{2M} + V(x), \rho_s(t)\right]p\right) = -i\hbar\langle V'(x) \rangle \quad (11.27)$$

$$\text{tr}\left([x, \{p, \rho_s\}]p\right) = -2\gamma\langle p \rangle. \quad (11.28)$$

As the result, we reach the equation;

$$\frac{d}{dt}\langle p \rangle = -\langle V'(x) \rangle - 2\gamma\langle p \rangle. \blacksquare \quad (11.29)$$

11.3 Caldeira-Leggett Model and Magnon

Let us again remark that the Caldeira-Leggett model,[63, 64] which is a prototype of a system-reservoir model for the description of dissipation phenomena, describes a Brownian particle of mass M with coordinate x which moves in a potential $V(x)$; Fig. 11.1. The free Hamiltonian $\mathcal{H}_s$ of the particle reads

$$\mathcal{H}_s = \frac{1}{2M}p^2 + V(x), \quad (11.30)$$

where p is the particle momentum. The particle is assumed to be coupled to a bath consisting of a large number of harmonic oscillator and in this model, the coordinate x of the Brownian particle is assumed to be coupled linearly to the coordinates of the bath oscillators.

Now, we adopt the semi-classical approximation ($\hbar \rightarrow 0$) and omit the decoherence effects (or decoherence term). Finally, the Heisenberg's equations of motion take the form;⁵

$$\dot{x}(t) = \frac{1}{M}p(t), \quad (11.31)$$

$$\dot{p}(t) = -V'(x(t)) - 2M\gamma\dot{x}(t), \quad (11.32)$$

where the friction force is equal to $-2M\gamma\dot{x}(t)$ and the variable γ represents the relaxation rate; Fig. 11.1.

⁵It should be noted that there are two expression of the Caldeira-Leggett model; one is represented via quantum master equation, the other is via Heisenberg's equation of motion. We have confirmed that they reach the same result in our study.

11.3.1 Mapping

Here, we restrict our discussion to ferromagnetic insulators and apply microwaves $\Gamma(t) = \Gamma_0 \cos(\Omega t)$ along the x-axis. This applied microwave Γ creates only the zero-mode of magnons $a_0 \neq 0$; that is, $a_{k \neq 0} = 0$. Therefore, we are allowed to focus on only the zero-mode of magnons after switching off microwaves.

The Hamiltonian of the zero-mode of magnons, which has been produced by microwaves, under the static magnetic field ⁶ B along the quantization axis reads as follows;

$$\mathcal{H}_B = B a_0^\dagger a_0, \quad (11.33)$$

with $[a_0, a_0^\dagger] = 1$.

In order to reduce [121, 52, 53] the dynamics of magnons to the Caldeira-Leggett model, we use the following well-known relations [39] (i.e. harmonic oscillator variables);

$$a = \sqrt{\frac{m\omega}{2\hbar}} \left(x + \frac{ip}{m\omega} \right) \quad (11.34)$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left(x - \frac{ip}{m\omega} \right). \quad (11.35)$$

In other words,

$$x = \sqrt{\frac{\hbar}{2m\omega}} (a + a^\dagger) \quad (11.36)$$

$$p = \frac{1}{i} \sqrt{\frac{\hbar m\omega}{2}} (a - a^\dagger). \quad (11.37)$$

This satisfies the relations; $[x, p] = i$ and $[a, a^\dagger] = 1$. We then replace the representation; $a \rightarrow a_0$, $a^\dagger \rightarrow a_0^\dagger$. Consequently, the Hamiltonian $\mathcal{H}_B$ can be rewritten as follows;

$$\mathcal{H}_B = B a_0^\dagger a_0 \quad (11.38)$$

$$= \frac{1}{2M} p^2 + V_B(x) + (\text{const.}) \quad (11.39)$$

$$= \mathcal{H}_s, \quad (11.40)$$

with

$$V_B(x) = \frac{1}{2} k x^2, \quad M := \frac{\hbar m\omega}{B}, \quad k := \frac{B m\omega}{\hbar}. \quad (11.41)$$

This means that the dynamics of the zero-mode of magnons after switching off under the static magnetic field B corresponds to that of a Brownian particle with the mass M trapped to the (harmonic) potential $V(x) = kx^2/2$; see Fig. 11.1. Then by using the equation of motion Caldeira-Leggett model gives, eqs. (11.31) and (11.32), we can study the dynamics of the zero-mode of magnons after switching off with dissipation effects due to phonons. This is the main point of the present analysis.

⁶It includes the g -factor and Bohr magneton.

11.3.2 The Time-development of the Zero-mode of Magnons

The Heisenberg's equations of motion the Caldeira-Leggett model gives reads as follows;

$$\dot{x}(t) = \frac{1}{M}p(t), \quad (11.42)$$

$$\dot{p}(t) = -V'(x(t)) - 2M\gamma\dot{x}(t). \quad (11.43)$$

We here replace the variables $a_0, a_0^\dagger$ with the real scholar field ϕ_1, ϕ_2 (i.e. classical approximations);⁷

$$a_0 = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right) \rightarrow \phi_1 + i\phi_2 \quad (11.44)$$

$$a_0^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{ip}{m\omega}\right) \rightarrow \phi_1 - i\phi_2. \quad (11.45)$$

That is,

$$x(t) = \sqrt{\frac{2\hbar}{m\omega}}\phi_1 \quad (11.46)$$

$$p(t) = \sqrt{2\hbar m\omega}\phi_2. \quad (11.47)$$

By substituting the relations into eqs. (11.42) and (11.43), the time-development of the zero-mode of magnons, $a_0 = \phi_1 + i\phi_2$, after switching off (i.e. $V(x) = V_B(x) = kx^2/2$) is governed by the following differential equations;

$$\dot{\phi}_1(t) = \omega_B\phi_2(t) \quad (11.48)$$

$$\dot{\phi}_2(t) = -\omega_B\phi_1(t) - \frac{2\gamma}{\omega_B}\dot{\phi}_1(t), \quad (11.49)$$

with $\omega_B := B/\hbar$. From now on, we discuss on the basis of these differential equations.

11.3.3 The Time-development under Microwaves

Before going to the main subject, the time-development after switching off, we investigate the dynamics under applied microwaves $V_\Gamma = \Gamma(t)(a_0 + a_0^\dagger)$ with $\Gamma(t) = \Gamma_0\cos(\Omega t)$; $V = V_B + V_\Gamma$. The microwaves give the new contributions, $\dot{\phi}_1^\Gamma(t)$ and $\dot{\phi}_2^\Gamma(t)$, into the differential equations (11.48) and (11.49);

$$\dot{\phi}_1^\Gamma(t) = 0 \quad (11.50)$$

$$\dot{\phi}_2^\Gamma(t) = \Gamma_0\cos(\Omega t). \quad (11.51)$$

Then, the time-development of the zero-modes magnons under applied microwaves reads as follows;

$$\dot{\phi}_1(t) = \omega_B\phi_2(t) \quad (11.52)$$

$$\dot{\phi}_2(t) = -\omega_B\phi_1(t) - \frac{2\gamma}{\omega_B}\dot{\phi}_1(t) + \Gamma_0\cos(\Omega t), \quad (11.53)$$

which has been plotted in Fig. 11.2.

It is clear that when the resonance (FMR) condition $\hbar\Omega = B$ is satisfied, the macroscopic condensate order parameter $|a(t)| = \sqrt{(\phi_1(t))^2 + (\phi_2(t))^2}$ takes a time-independent constant value after a certain period, which becomes the initial condition of the differential equations governing the dynamics after switching off (see Figs. 11.3 and 11.4, (Appendix) Sec. 11.3.4).

⁷That is, we neglect quantum fluctuations.

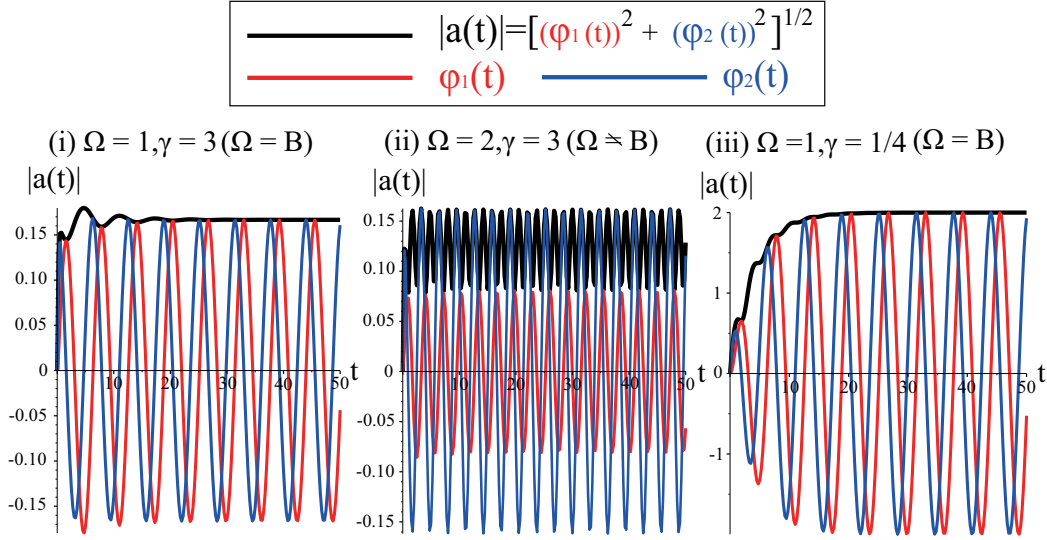


Figure 11.2: (Color online) Plots of the macroscopic condensate order parameter $|a(t)| = \sqrt{(\phi_1(t))^2 + (\phi_2(t))^2}$ governed by the differential equations (11.52) and (11.53) with the initial condition $\phi_1(0) = \phi_2(0) = 0$. For convenience, each parameter reads as follows; $\hbar = \Gamma_0 = \omega_B = B = 1$.

11.3.4 The Time-development after Switching off Microwaves

After switching off microwaves, the time-development is governed by the differential equations (11.48) and (11.49);⁸

$$\dot{\phi}_1(t) = \omega_B \phi_2(t) \quad (11.54)$$

$$\dot{\phi}_2(t) = -\omega_B \phi_1(t) - \frac{2\gamma}{\omega_B} \dot{\phi}_1(t), \quad (11.55)$$

with $\omega_B := B/\hbar$. These differential equation can be easily solved by considering $\ddot{\phi}_1$ in eq. (11.54) to use eq. (11.55) as follows;

$$(\text{eq. (11.54)}) \Rightarrow \ddot{\phi}_1 + 2\gamma \dot{\phi}_1 + \omega_B^2 \phi_1 = 0. \quad (11.56)$$

Then, it reads

$$\phi_1(t) = C_+ e^{\lambda_+ t} + C_- e^{\lambda_- t}, \quad (11.57)$$

with⁹

$$\lambda_+ = -\gamma + \sqrt{\gamma^2 - \omega_B^2}, \quad \lambda_- = -\gamma - \sqrt{\gamma^2 - \omega_B^2} \quad (11.58)$$

and the appropriate coefficients $C_{\pm}$ which satisfy the initial condition. In addition,

$$(\text{eq. (11.54)}) \Rightarrow \phi_2(t) = \frac{1}{\omega_B} [C_+ \lambda_+ e^{\lambda_+ t} + C_- \lambda_- e^{\lambda_- t}]. \quad (11.59)$$

⁸The initial condition can be obtained from Fig. 11.2.

⁹ $\lambda^2 + 2\gamma\lambda + \omega_B^2 = 0$.

Finally, the solution of the differential equations (11.54) and (11.55) becomes (see also Fig. 11.3)

$$\phi_1(t) = C_+ e^{\lambda_+ t} + C_- e^{\lambda_- t}, \quad (11.60)$$

$$\phi_2(t) = \frac{1}{\omega_B} [C_+ \lambda_+ e^{\lambda_+ t} + C_- \lambda_- e^{\lambda_- t}] \quad (11.61)$$

with

$$\lambda_+ = -\gamma + \sqrt{\gamma^2 - \omega_B^2}, \quad \lambda_- = -\gamma - \sqrt{\gamma^2 - \omega_B^2}. \quad (11.62)$$

Here let us remark that when the condition is satisfied, $\gamma = \omega_B$ (see Fig. 11.3 (B)), each solution takes the form;

$$\phi_1(t) = C_3 e^{-\gamma t} (1 + C_4 t), \quad (11.63)$$

$$\phi_2(t) = \frac{1}{\omega_B} C_3 e^{-\gamma t} [(C_4 - \gamma) - \gamma C_4 t], \quad (11.64)$$

with the appropriate coefficients C_3 and C_4 satisfying the initial condition.

As an example, each solution of differential equations (11.54) and (11.55) has been plotted in Fig. 11.3; $\phi_1(t)$, $\phi_2(t)$, and $|a(t)| = \sqrt{\phi_1(t)^2 + \phi_2(t)^2}$.

11.3.5 Relaxation Processes

On top of this,¹⁰ it should be remembered that $\text{Re}(\lambda_{\pm})$ in eq. (11.58) characterize the time-scale (i.e. decay time) of each relaxation processes and each value $\text{Re}(\lambda_{\pm})$ has the following properties; when $\gamma > \omega_B$, $\text{Re}(\lambda_+) \neq \text{Re}(\lambda_-)$ and there are two relaxation processes¹¹ (see Fig. 11.3 (A) and Fig. 11.4 (3-II)), which are characterized by $\text{Re}(\lambda_+)$ and $\text{Re}(\lambda_-)$.

On the other hand when $\gamma < \omega_B$, $\text{Re}(\lambda_+) = \text{Re}(\lambda_-)$ and hence, there is only one relaxation process or time-scale (i.e. decay time) characterized by $\text{Re}(\lambda_+) = \text{Re}(\lambda_-)$; Figs. 11.3 (C) and (D) with Fig. 11.4 (3-I).¹²

11.3.6 Relaxation Processes (I)

As mentioned above, when $\gamma < \omega_B$, $\text{Re}(\lambda_+) = \text{Re}(\lambda_-)$ and hence, there is only one relaxation process or time-scale (i.e. decay time) characterized by $\text{Re}(\lambda_+) = \text{Re}(\lambda_-)$; see Fig. 11.4 (3-I). Some of them are plotted by Figs. 11.3 (C) and (D).

We here define and introduce a quantity τ_0 as

$$\tau_0 := \frac{1}{\text{Re}|\lambda_+|} = \frac{1}{\text{Re}|\lambda_-|} \quad (11.65)$$

$$= \frac{1}{\gamma} \quad (11.66)$$

and it has the features (see Fig. 11.4 (2) and Fig. 11.5 (I-a))

$$\frac{d\tau_0}{d\gamma} < 0. \quad (11.67)$$

¹⁰It will be useful to see Fig. 11.10, which is an analysis based on the original harmonic oscillator.

¹¹It should be noted that we now have been employing the Heisenberg's equation of motion the Caldeira-Leggett model gives; although the equation of motion, eqs. (6.1) and (6.2), we have obtained by using quantum master equations is applicable to only the weak dissipation case (i.e. $\gamma < \omega_B$) by definition, the present differential equations, eqs. (11.54) and (11.55), can be derived without quantum master equations and hence, they are applicable to also the strong case (i.e. $\gamma > \omega_B$).

¹²The imaginary part of $\lambda_{\pm}$ can be viewed as the parameter which characterizes the oscillation.

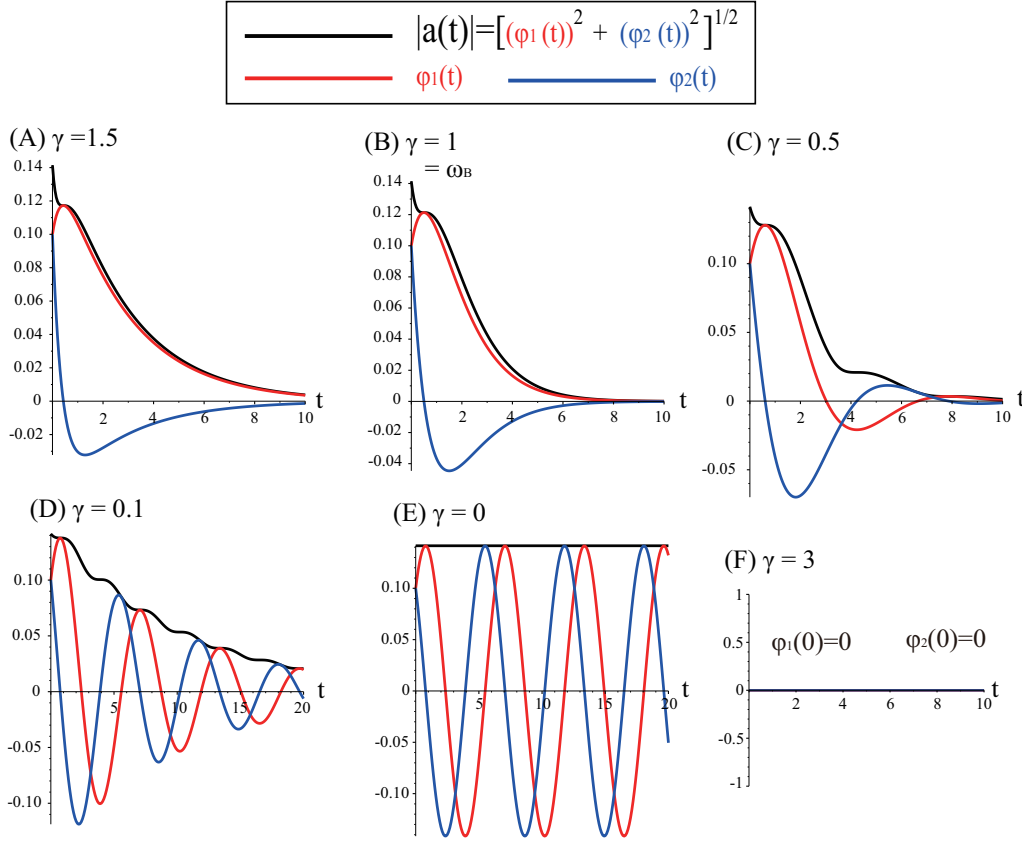


Figure 11.3: (Color online) (A)-(E) Plots of the macroscopic condensate order parameter $|a(t)| = \sqrt{(\phi_1(t))^2 + (\phi_2(t))^2}$ governed by the differential equations (11.54) and (11.55) with the initial condition $\phi_1(0) = \phi_2(0) = 1/10$. For convenience, each parameter reads as follows; $\hbar = \Gamma_0 = \omega_B = B = 1$. (F) This means that if $\phi_1(t=0) = \phi_2(t=0) = 0$, $|a(t)| = 0$ all the time. That is, in order to give the order parameter $a(t)$ a certain non-zero value, microwaves are essential.

11.3.7 Relaxation Processes (II)

As mentioned above, when $\gamma > \omega_B$, the following condition is satisfied $\text{Re}(\lambda_+) \neq \text{Re}(\lambda_-)$ and hence there are two relaxation processes, which are characterized by $\text{Re}(\lambda_+)$ and $\text{Re}(\lambda_-)$; see Fig. 11.4 (3-II).

We here introduce $\tau_{\pm}$, which characterizes the time-scale of the dynamics;

$$\tau_+ := \frac{1}{|\lambda_+|} = \frac{1}{\gamma - \sqrt{\gamma^2 - \omega_B^2}} \quad (11.68)$$

$$\rightarrow \frac{2\gamma}{\omega_B^2} \propto \gamma \quad (\text{when } \gamma \gg \omega_B), \quad (11.69)$$

$$\tau_- := \frac{1}{|\lambda_-|} = \frac{1}{\gamma + \sqrt{\gamma^2 - \omega_B^2}} \quad (11.70)$$

$$\rightarrow \frac{1}{2\gamma} \left(1 + \frac{\omega_B^2}{4\gamma^2}\right) \sim \frac{1}{2\gamma} \propto \frac{1}{\gamma} \quad (\text{when } \gamma \gg \omega_B). \quad (11.71)$$

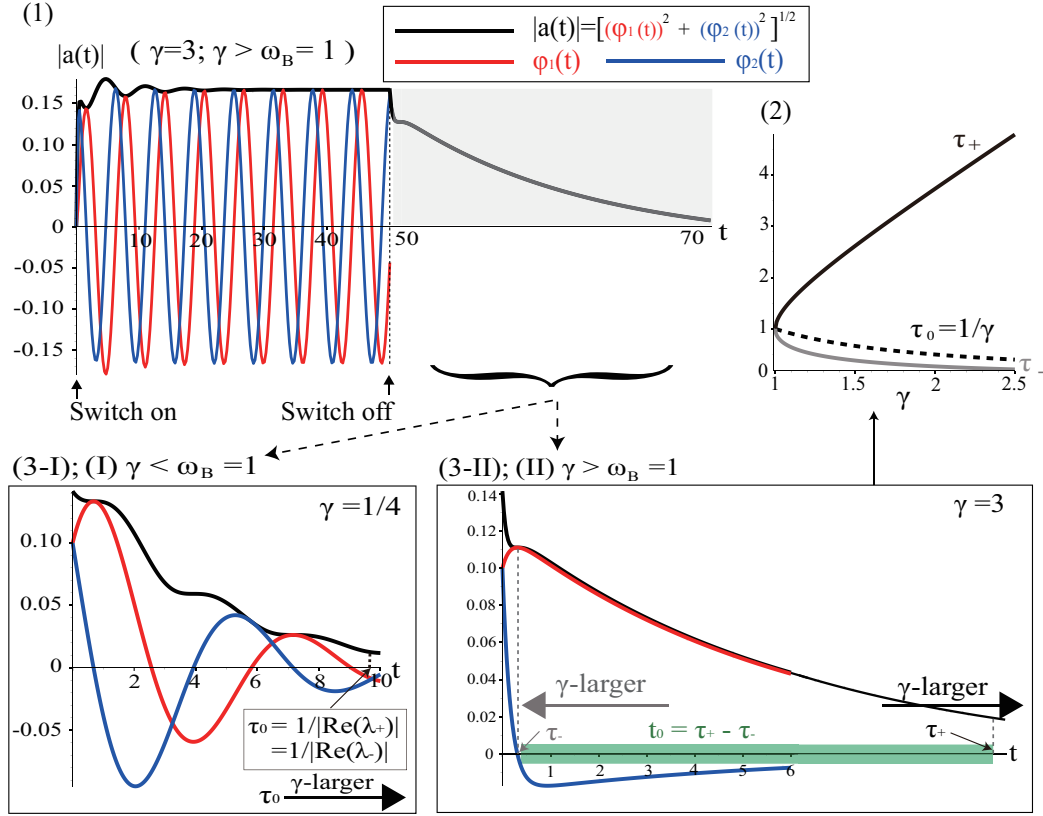


Figure 11.4: (Color online) (1) A series of the time-development of our system from under microwaves to after switching off. The dynamics after switching off can be classified into two ways; (I) $\gamma < \omega_B$ and (II) $\gamma > \omega_B$. (2) Plots of each time scale (i.e. decay time) of relaxation processes as a function of relaxation rate γ ; $\tau_{\pm}$ and $\tau_0 := 1/\gamma$ with $\tau_+ = 1/(\gamma - \sqrt{\gamma^2 - \omega_B^2})$, $\tau_- = 1/(\gamma + \sqrt{\gamma^2 - \omega_B^2})$. For simplicity, we have set $\omega_B = 1$. (3-I) $\gamma < \omega_B$; the relaxation process can be characterized only by τ_0 . (3-II) $\gamma > \omega_B$.

Then by using $\tau_{\pm}$, each variable, $\phi_1(t)$ and $\phi_2(t)$, can be rewritten as

$$\phi_1(t) = C_+ e^{-t/\tau_+} + C_- e^{-t/\tau_-}, \quad (11.72)$$

$$\phi_2(t) = \frac{1}{\omega_B} [C_+ \lambda_+ e^{-t/\tau_+} + C_- \lambda_- e^{-t/\tau_-}]. \quad (11.73)$$

It is clear that each variable $\tau_{\pm}$ characterizes the time-scale (i.e. decay time) of each relaxation processes.

On top of this, it should be remembered that (see Fig. 11.4 (2)) the behavior of each $\tau_{\pm}$ in respect to γ is different from each other;

$$\frac{d\tau_+}{d\gamma} > 0, \quad (11.74)$$

$$\frac{d\tau_-}{d\gamma} < 0. \quad (11.75)$$

Or (see Fig. 11.4 (3-II) green region and Fig. 11.6 (II-a) and (II-b))

$$\frac{dt_0}{d\gamma} > 0 \quad \text{with } t_0 := \tau_+ - \tau_-. \quad (11.76)$$

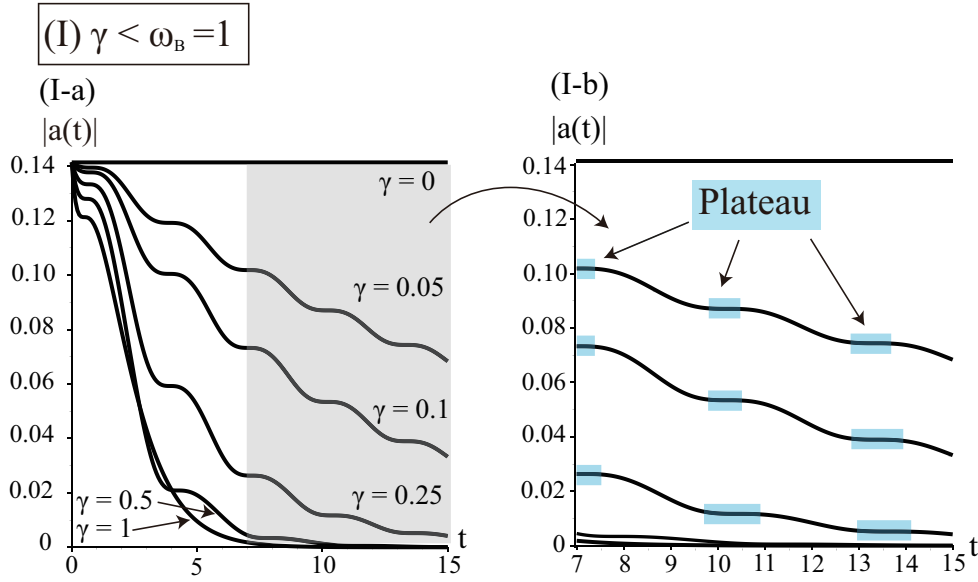


Figure 11.5: (Color online) The time-development of the order parameter $|a(t)|$ et al. under the relaxation processes (I); $\gamma < \omega_B$. (I-a) It is clear that when γ becomes larger, the decay-time becomes shorter. (I-b) There are a few plateau regions in $|a(t)|$ as a function of time t . On each region, the order parameter can be regarded to take the form; $a(t) \sim e^{i\mu t - i\alpha}$, then, $|a(t)| \sim (\text{constant})$ on each plateau region. This result can be understood that the order parameter $|a(t)|$ is stable under the time-evaluation.

Then, it is clear that the relaxation process characterized by τ_+ or t_0 (Fig. 11.4 (3-II)), that is, the process when $\tau_- < t < \tau_+$ is qualitatively fully different from the one when $t < \tau_-$ and the case (I) $\gamma < \omega_B$.

On top of this (see Fig. 11.6 (II-c')), it should be remembered that the time-development of the order parameter $|a(t)|$ of this relaxation process $\gamma > \omega_B$ characterized by $\tau_{\pm}$ can be divided into three regions (A), (B), and (C);

- (A) Faster relaxation process;
this region can be characterized by τ_- .
- (B) Slower relaxation process;
this region can be characterized by τ_+ or t_0 .
- (C) Plateau region;
in this region, the order parameter becomes time-independent. In this case $\gamma > \omega_B$, there is only one plateau region.

11.3.8 Emergence of Plateaus

Here it should be emphasized that in both cases (I) $\gamma < \omega_B$ and (II) $\gamma > \omega_B$, there are some region which should be regarded as plateau;¹³[134, 135, 136] see Figs. 11.4-11.9. On each

¹³At this stage, I do not think our plateaus emerging in Caldeira-Leggett model in terms of magnon BEC have some relations with the recently attractive novel phenomenon called pre-thermalization.[131, 132, 133, 43] Our plateaus have emerged simply due to the combination of the relaxation (i.e. decay) effect and the oscillation effect. For the clear and deep understanding, we need further detailed analysis.

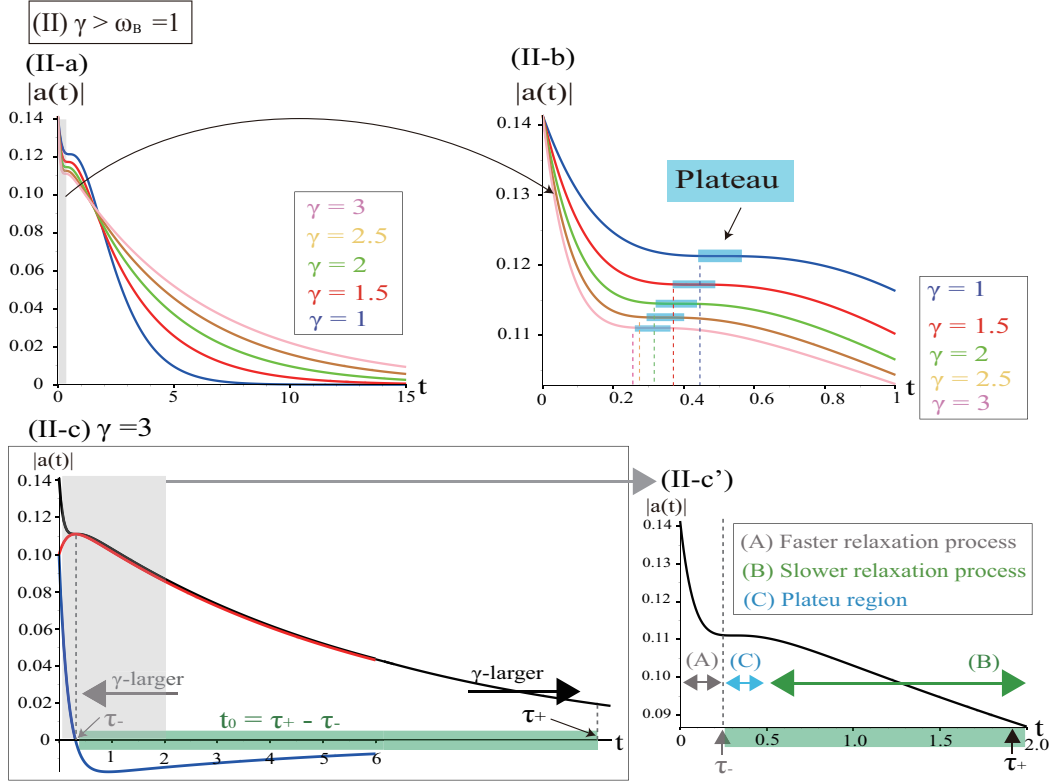


Figure 11.6: (Color online) The time-development of the order parameter $|a(t)|$ et al. under the relaxation processes (II); $\gamma > \omega_B$. (II-b) The position of each plateau depends on the value γ . (II-c) and (II-c'); the relaxation process can be characterized by $\tau_{\pm}$ and the time-development of the order parameter $|a(t)|$ can be divided into three regions; (A), (B), and (C). (A) Faster relaxation process; this region can be characterized by τ_- . (B) Slower relaxation process; this region can be characterized by τ_+ or t_0 . (C) Plateau region; in this region, the order parameter becomes time-independent. In this case $\gamma > \omega_B$, there is only one plateau region.

plateau, the order parameter takes the form (μ ; the non-equilibrium chemical potential[45])¹⁴

$$a(t) \sim e^{i\mu t - i\alpha} \quad (11.77)$$

and hence, it becomes

$$|a(t)| = \sqrt{N_0} \quad (11.78)$$

$$\sim (\text{constant}). \quad (11.79)$$

This agrees with the definition of magnon BEC proposed by Bunkov and Volovik[45] and this state might be regarded as the stable state under time-development. On each plateau, the number of condensate parameter does not change.

¹⁴Here it should be noted that we now consider non-equilibrium situations and hence, no one knows the exact form of the Bose-distribution function in non-equilibrium cases. We know the form of the Bose-distribution function only in equilibrium cases.

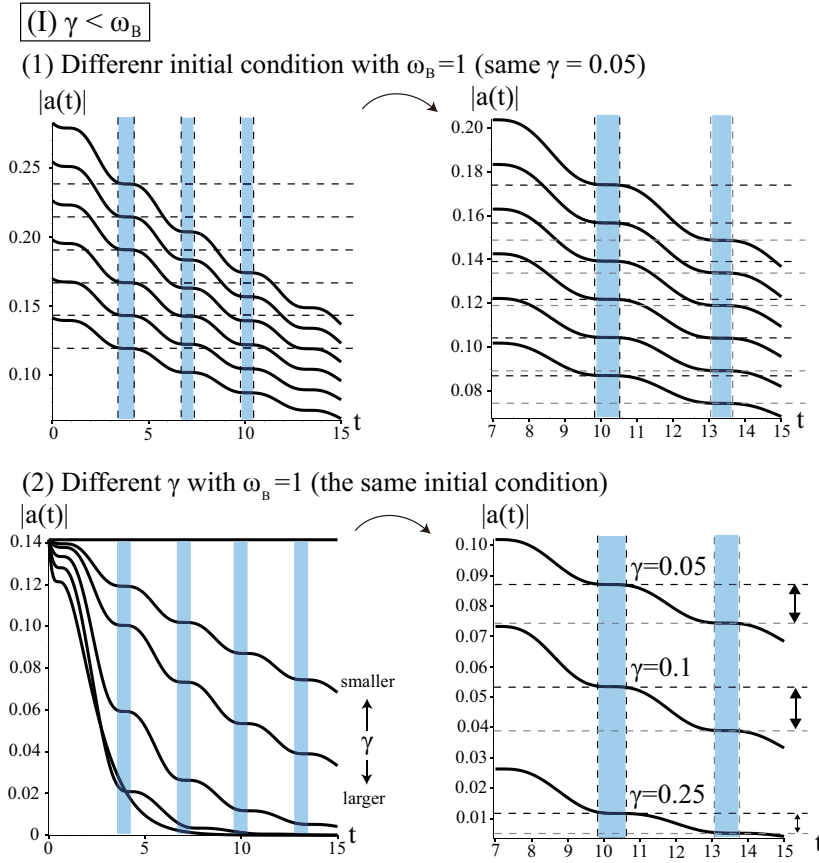


Figure 11.7: (Color online) On each plateau region, the order parameter $a(t)$ can be approximately regarded to have the form; $a(t) \sim e^{i\mu t - i\alpha}$ with the non-equilibrium chemical potential μ .^[45] This fact qualitatively agrees with the condition for (non-equilibrium) magnon BEC introduced by Bunkov and Volovik.^[45] (1) The shape of the function $|a(t)|$ is characterized not by the initial condition but by ω_B or B . (2) The position of each plateau does not depend on the value of γ . It depends on the value ω_B ; see also Fig. 11.8.

11.4 LLG Equation

Here, it will be useful to analyze relaxation processes from the view point of the LLG equation,^[22, 23, 24] which describes the time-evolution of ferromagnetic localized spins¹⁵ with taking the effects of dissipation into account as the phenomenologically introduced Gilbert damping term called α -term.^[26] On top of this, LLG equation is often used in the research area of spintronics. Then, it will be significant to evaluate the condensate order parameter

$$|a(t)| \sim |S^x(t) + iS^y(t)| = \sqrt{(S^x(t))^2 + (S^y(t))^2} \quad (11.80)$$

on the basis of LLG equation and compare the dynamics with the one based on Caldeira-Leggett model.

¹⁵They have treated spins as classical variables. One should remember that under microwaves, localized spins precess coherently.

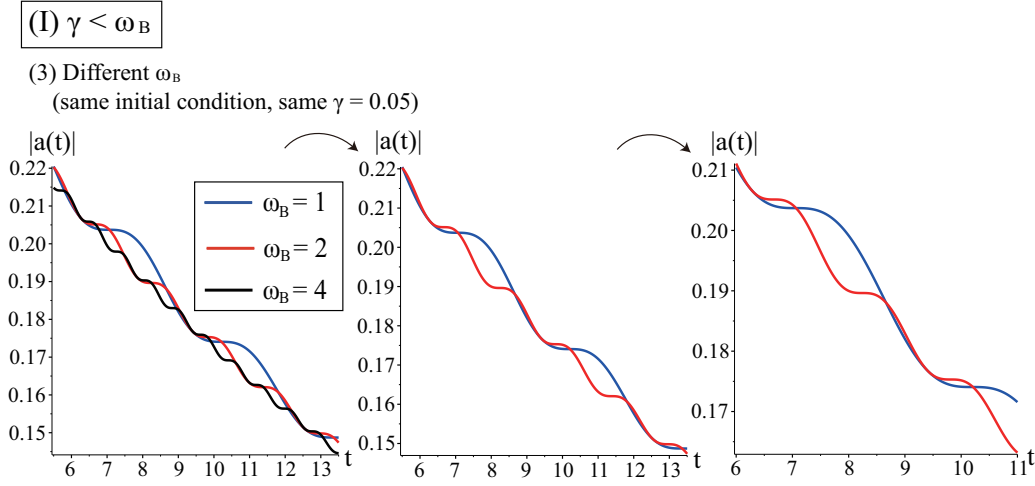


Figure 11.8: (Color online) It is clear that when (I) $\gamma < \omega_B$, the behavior of the plateau has been characterized by the applied static magnetic field B or ω_B .

Briefly speaking, the LLG equation we consider reads as follows;¹⁶

$$\dot{\mathbf{S}} = \mathbf{S} \times \mathbf{H} - \alpha \mathbf{S} \times \dot{\mathbf{S}} \quad (11.81)$$

$$\dot{S}_i = (\mathbf{S} \times \mathbf{H})_i - \alpha (\mathbf{S} \times \dot{\mathbf{S}})_i, \quad (11.82)$$

where the external magnetic field reads $\mathbf{H} = (\Gamma(t), 0, B)$ with index $i = x, y, z$. They have phenomenologically introduced the isotropic damping constant α , which represents the isotropic dissipation effects, to guarantee the conservation of the spin length; $|\mathbf{S}| = (\text{const.})$. By using eq. (11.82), we can easily show this fact; [41] $dS_i^2/(dt) = 0$. If we introduce an anisotropic damping constant, $\alpha \rightarrow \alpha(i)$,¹⁷ the spin length cannot be conserved in general; $dS_i^2/(dt) \neq 0$. They have treated spins as classical variables and the procedure will be appropriate under microwaves because localized spins precess coherently.¹⁸

Each component of LLG equation reads as follows;

$$\dot{S}^x = BS^y - \alpha(S^y \dot{S}^z - S^z \dot{S}^y) + \mathcal{O}(\alpha^2), \quad (11.83)$$

$$\dot{S}^y = -BS^x + \Gamma S^z - \alpha(S^z \dot{S}^x - S^x \dot{S}^z) + \mathcal{O}(\alpha^2), \quad (11.84)$$

$$\dot{S}^z = -\Gamma S^y - \alpha(S^x \dot{S}^y - S^y \dot{S}^x) + \mathcal{O}(\alpha^2), \quad (11.85)$$

which are plotted in Fig. 11.11.

As we know, microwave ($\Gamma(t)$) give some value to S^x and S^y . After switching off, the condensate order parameter $|a(t)| \sim |S^x(t) + iS^y(t)| = \sqrt{(S^x(t))^2 + (S^y(t))^2}$ and each component of spin variables $S^{x,y,z}$ varies as shown in Fig. 11.11.

Distinction

It is clear that there are no plateaus in Fig. 11.11 and this result is in sharp contrast to the case of the analysis based on Caldeira-Leggett model. In Fig. 11.11, there are no plateaus in $|a(t)| \sim |S^x(t) + iS^y(t)| = \sqrt{(S^x(t))^2 + (S^y(t))^2}$ and it has the form; $|a(t)| \sim |S^x(t) + iS^y(t)| \sim e^{-t/\tau}$. That is, the emergence of plateaus is the unique property of the system described by Caldeira-Leggett model.

¹⁶Regarding the details, please see [22].

¹⁷This means that the value of the damping constant depends on each component; x, y, z .

¹⁸We switch off microwaves, which corresponds to the procedure; $\Gamma(t) \rightarrow 0$.

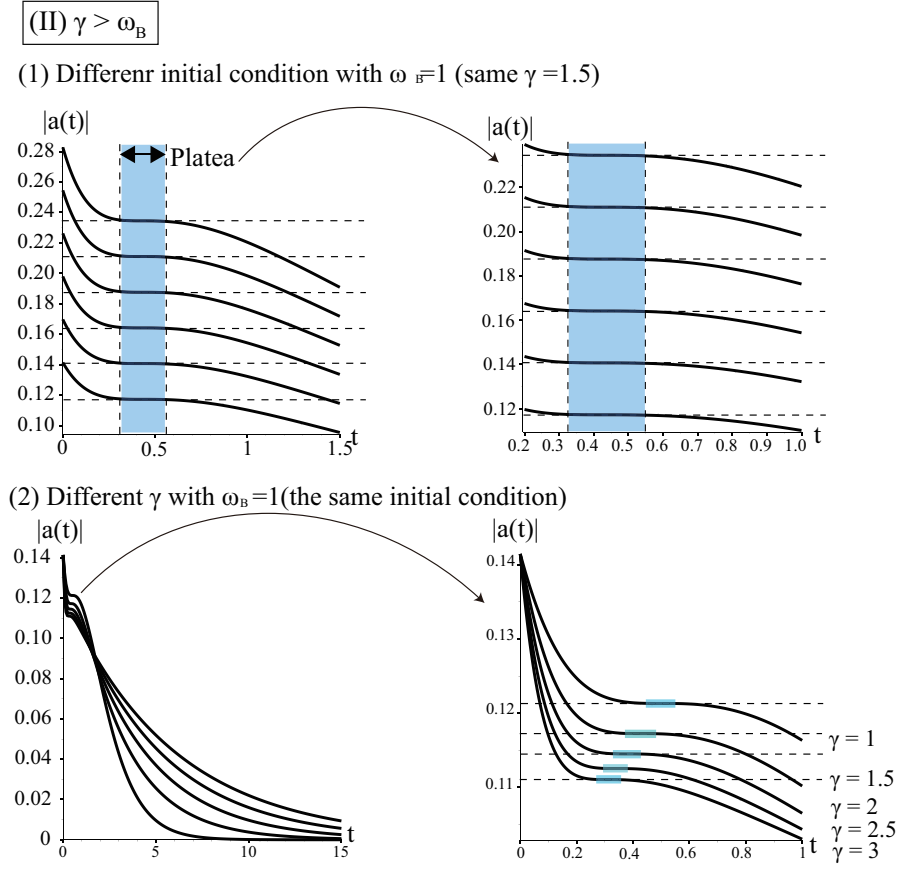


Figure 11.9: (Color online) In this case $\gamma > \omega_B$, there is only one plateau region. (1) The position of each plateau does not depend on the initial condition. (2) The position of each plateau does depend on the value γ .

11.5 Outlook; Thermalization Processes

Lastly, we remark on our outlook to obtain further deep understanding of thermalization processes.[51] On preparing for this manuscript, we have asked a few questions to Prof. Volovik[45] about the experiment by Demokritov et al.[46] Then, he has kindly replied it.¹⁹ Then, I have understood the comment of Volovik as shown in Fig. 11.12. That is, in the experimental observation of magnon BEC by Demokritov et al., the peculiar dispersion relation to YIG plays the key role; applied microwaves excite the non-zero mode of magnon in YIG (i.e. $k = 0$), which is not the lowest of magnon in YIG due to the peculiar dispersion relations to YIG (i.e. the double degeneracy of the ground state).[46, 47, 48, 49, 50, 52, 53] After that, due to the thermalization effects they call, the magnons start to move to the lowest energy states (i.e. $k = k_{\min.} \neq 0$). Consequently, quasi-equilibrium magnon BEC is formed in the lowest energy mode.

Prospect

In order to clarify the microscopic mechanism of the above thermalization process,[78,

¹⁹Let us cite a part of his reply. *Condensate is formed due to thermalization. Microwaves excite the higher modes of magnons. Due to thermalization the energy of created magnons decreases, and magnons start to populate the lowest energy states forming the condensate. If the thermalization time is shorter than the life time of magnons, the magnon BEC is formed.*

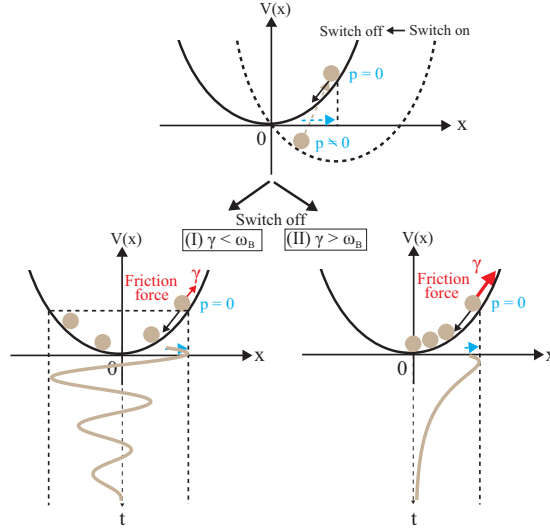


Figure 11.10: (Color online) An analysis of relaxation processes from the viewpoint of the original harmonic oscillator with friction forces. When the value of the friction force γ gets over a certain point, it ceases to oscillate (II).

78, 62] it will be helpful to use the analogy with the standard Goldstone model (Fig. 11.12); to extend the theory into non-equilibrium case is an urgent theoretical issue. On top of this, the similar mechanism called disoriented chiral condensation,[149, 150] which has been investigated in the context of Hadron physics, might be useful.[151]

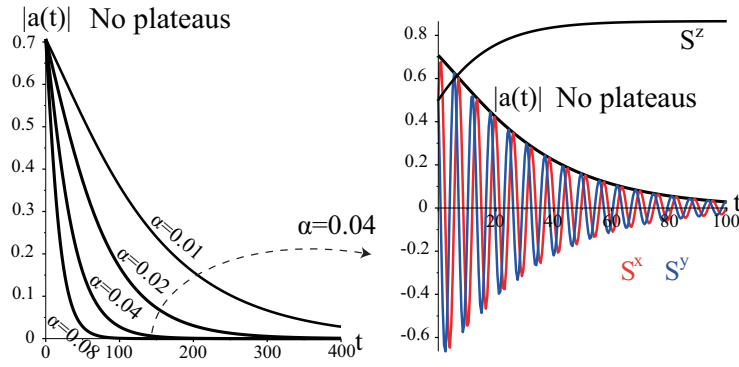


Figure 11.11: (Color online) The time-development by LLG equation after switching off. We have set $B = 1$. It is clear that there are no plateaus, which is in sharp contrast to the case of the analysis based on Caldeira-Leggett model.

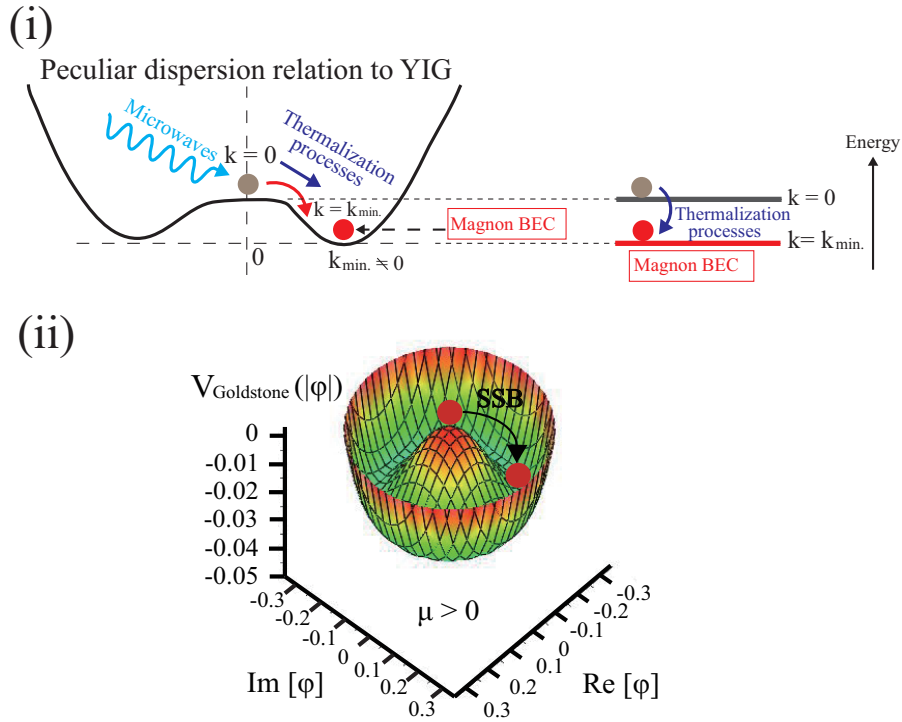


Figure 11.12: (Color online) (i) A schematic pictures of the microscopic mechanism of the experiment by Demokritov et al.[46] (ii) A schematic pictures of SSB generated in wine-bottle potential, which is often used to explain the emergence of magnon BEC in equilibrium situation. To extend the theory into non-equilibrium case is an urgent theoretical issue.

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