

§0 Intro.

Generalized van der Corput Sequences

and

Dynamical Systems

$x_0, x_1, \dots, x_N, \dots \in [0, 1)^k$

I : interval of $[0, 1)^k$

$A(I, N, \{x_n\}) := \#\{n \mid x_n \in I, n < N\}$

Def 1.

$$D_N = D_N(\{x_n\}) := \sup_{I \subset \mathbb{T}^k} \left| \frac{A(I, N, \{x_n\})}{N} - \lambda_k(I) \right|$$

is called discrepancy of $\{x_n\}_{n=0}^\infty$

star discrepancy

$$D_N^*(\{x_n\}) := \sup \left| \sum_{j=0}^{N-1} \chi_j(x_j) \right|$$

$$D_N^* \leq D_N \leq 4^k D_N^*$$

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2003/8/28 発表

Key words

- van der Corput seq.
- low discrepancy
- Orbit of dynamical systems

— 0 —

Th (Koksma-Hlawka's Ineq.)

f : be of bounded variation on $[0, 1]^k$
in the sense of Hardy & Krause.

then

$$\left| \frac{1}{N} \sum_{n=0}^{N-1} f(x_n) - \int_{[0, 1]^k} f(x) dx \right| \leq V(f) D_N^*(\{x_n\})$$

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Def 2. $\{x_n\} \in [0, 1]^R$ is called a low discrepancy sequence if

$$\exists c: DN \leq c \log^R N / N \quad \text{for all } N$$

§1 van der Corput sequence.

$$n = a_k 2^k + a_{k-1} 2^{k-1} + \dots + a_0 2^0, \quad a_k \neq 0$$

$$\varphi(n) := (a_k, a_{k-1}, \dots, a_0)$$

$$x_n := \frac{a_0}{2} + \frac{a_1}{2^2} + \dots + \frac{a_k}{2^{k+1}}, \quad n = 0, 1, 2, \dots$$

$\{x_n\}$ is called Van der Corput seq.

Prop 1 Van der Corput seq. is a low discrepancy seq.

List of

$$\varphi(n) = (a_k, \dots, a_0)$$

and

$$x_n = \frac{a_0}{2} + \frac{a_1}{2^2} + \dots + \frac{a_k}{2^{k+1}}$$

$\varphi(0) = (0)$	$x(0) = (0)$	
$\varphi(1) = (1)$	$x(1) = (0, 5)$	
$\varphi(2) = (1, 0)$	$x(2) = (0, 25)$	
$\varphi(3) = (1, 1)$	$x(3) = (0, 75)$	
$\varphi(4) = (1, 0, 0)$	$x(4) = (0, 125)$	
$\varphi(5) = (1, 0, 1)$	$x(5) = (0, 625)$	
$\varphi(6) = (1, 1, 0)$	$x(6) = (0, 375)$	
$\varphi(7) = (1, 1, 1)$	$x(7) = (0, 875)$	
$\varphi(8) = (1, 0, 0, 0)$	$x(8) = (0, 0625)$	
$\varphi(9) = (1, 0, 0, 1)$	$x(9) = (0, 5625)$	
$\varphi(10) = (1, 0, 1, 0)$	$x(10) = (0, 3125)$	
$\varphi(11) = (1, 0, 1, 1)$	$x(11) = (0, 8125)$	
$\varphi(12) = (1, 1, 0, 0)$	$x(12) = (0, 1875)$	
$\varphi(13) = (1, 1, 0, 1)$	$x(13) = (0, 6875)$	
$\varphi(14) = (1, 1, 1, 0)$	$x(14) = (0, 4375)$	
$\varphi(15) = (1, 1, 1, 1)$	$x(15) = (0, 9375)$	
$\varphi(16) = (1, 0, 0, 0, 0)$	$x(16) = (0, 03125)$	
$\varphi(17) = (1, 0, 0, 0, 1)$	$x(17) = (0, 33125)$	
$\varphi(18) = (1, 0, 0, 1, 0)$	$x(18) = (0, 28125)$	
$\varphi(19) = (1, 0, 0, 1, 1)$	$x(19) = (0, 78125)$	
$\varphi(20) = (1, 0, 1, 0, 0)$	$x(20) = (0, 15625)$	
$\varphi(21) = (1, 0, 1, 0, 1)$	$x(21) = (0, 65625)$	
$\varphi(22) = (1, 0, 1, 1, 0)$	$x(22) = (0, 40625)$	
$\varphi(23) = (1, 0, 1, 1, 1)$	$x(23) = (0, 90625)$	
$\varphi(24) = (1, 1, 0, 0, 0)$	$x(24) = (0, 09375)$	
$\varphi(25) = (1, 1, 0, 0, 1)$	$x(25) = (0, 59375)$	
$\varphi(26) = (1, 1, 0, 1, 0)$	$x(26) = (0, 34375)$	
$\varphi(27) = (1, 1, 0, 1, 1)$	$x(27) = (0, 84375)$	
$\varphi(28) = (1, 1, 1, 0, 0)$	$x(28) = (0, 21875)$	
$\varphi(29) = (1, 1, 1, 0, 1)$	$x(29) = (0, 71875)$	
$\varphi(30) = (1, 1, 1, 1, 0)$	$x(30) = (0, 46875)$	
$\varphi(31) = (1, 1, 1, 1, 1)$	$x(31) = (0, 96875)$	
$\varphi(32) = (1, 0, 0, 0, 0, 0)$	$x(32) = (0, 015625)$	
$\varphi(33) = (1, 0, 0, 0, 0, 1)$	$x(33) = (0, 515625)$	

sketch of proof.

$$P_k := \{x_0, x_1, \dots, x_{2^k-1}\}$$

Lemma 1

$$(1) \quad N = 2^{k_1} + 2^{k_2} + \dots + 2^{k_s}$$

$$\begin{aligned} & \{x_0, \dots, x_{2^{k_1}-1}, x_{2^{k_1}}, \dots, x_{2^{k_1}+2^{k_2}-1}, \dots, x_{2^{k_1}+2^{k_2}+\dots+2^{k_{s-1}}-1}, \dots, x_{2^{k_1}+2^{k_2}+\dots+2^{k_s}-1}\} \\ &= P_{k_1} \cup (P_{k_2} \cup \frac{1}{2^{k_1}}) \cup \dots \cup (P_{k_s} \cup \frac{1}{2^{k_1}} + \dots + \frac{1}{2^{k_{s-1}}}) \end{aligned}$$

$$(2) \quad P_1 = \{0, 1/2\}$$

$$P_2 = P_1 \cup (P_1 + 1/2)$$

$$P_k = P_{k-1} \cup (P_{k-1} + 1/2^k)$$

Self-similar

Lemma 2

$$| \#(P_k \cap [0, a]) - a \cdot 2^k | \leq 1$$

Bounded Remainder

$$\frac{a}{2^k} \leq \frac{a-1}{2^k} < a \leq \frac{a+1}{2^k}$$

in 2.79 Lemma 11.3

$$| \{x_0, \dots, x_{N-1} \cap [0, a] \} - a \cdot N |$$

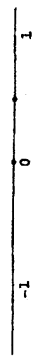
$$= | \sum_{i=1}^s \#(P_{k_i} \cup \frac{1}{2^{k_1}} + \dots + \frac{1}{2^{k_{i-1}}}) \cap [0, a] - a \cdot 2^{k_i} |$$

$$\leq k_1 \neq \log N$$

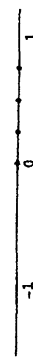
$$(\because) 2^{k_1} \leq N < 2^{k_1+1}$$

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P[1]



P[2]



P[3]



P[4]



P[5]



figure of P_k , (Def. of P_k found in page 5)

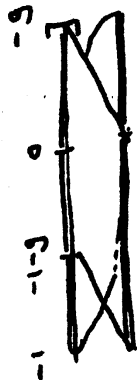
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§2 rotations

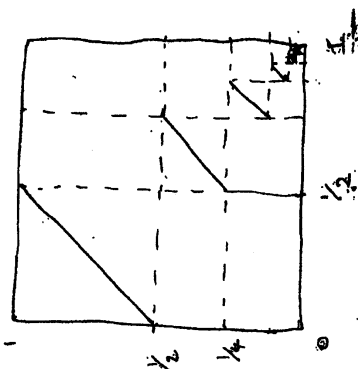
$$g := \frac{1-\sqrt{5}}{2}$$

$$T_g: (-1, -g] \rightarrow (-1, -g]$$

$$T_g(x) = \begin{cases} x+g & \text{if } x \in (-1, -g] \\ x+1 & \text{if } x \in (-1, -1+g] \end{cases}$$



Plot of (x_n, x_{n+1}) , $n=0,1,2,\dots$



Then we have the graph of
Kakutani transf. T_K
 (adding machines)

Prop $T_K: [0,1] \rightarrow [0,1]$ Kakutani
 then $x_n = T_K^n(0)$

Proposition $x_n := T_g^n(0)$, $n=0,1,2,\dots$
 is a low discrepancy seq.

Sketch of the proof (new?)

Fibonacci substitution

$$\sigma: 1 \rightarrow 12$$

$$2 \rightarrow 1$$

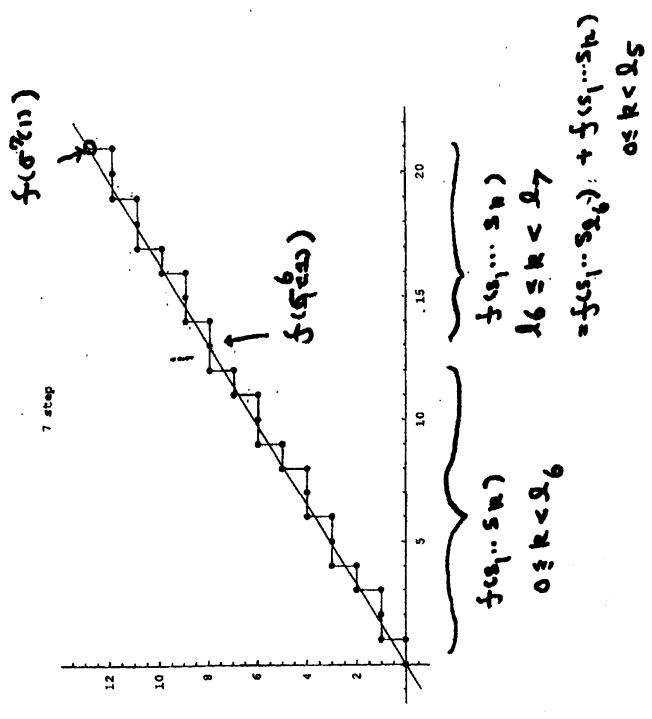
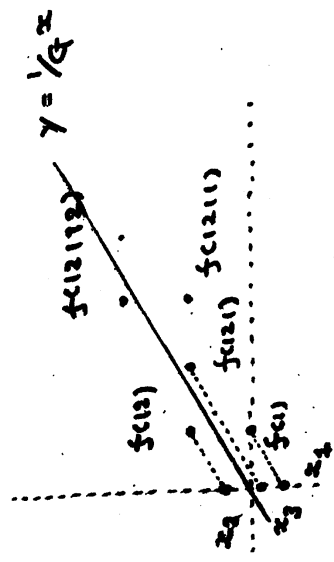
$$L\sigma = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad L\sigma^n = \begin{bmatrix} f_n & f_{n-1} \\ f_{n-1} & f_{n-2} \end{bmatrix}$$

$$\begin{aligned} \sigma^0 c_1 &= 12 & (\sigma^0 c_1) &= 1 \\ \sigma^1 c_1 &= 121 & (\sigma^1 c_1) &= 2 \\ \sigma^2 c_1 &= 12112 & (\sigma^2 c_1) &= 3 \\ & \vdots & & \\ \sigma^n c_1 &= s_1 s_2 \dots s_n & (\sigma^n c_1) &= 2n \\ & \boxed{l_n = 2n-1 + 2n-2} & , l_0 &= 1, l_1 = 2 \end{aligned}$$

$$\begin{aligned} f: \cup \{1, 2\}^n &\rightarrow \mathbb{Z}^2 \\ \begin{matrix} 1 \\ 2 \end{matrix} &\mapsto \begin{matrix} f(1) = e_1 \\ f(2) = e_2 \end{matrix} \\ f(s_1 \dots s_k) &:= f(s_1) + \dots + f(s_k) \end{aligned}$$

$$\wedge: \mathbb{R}^2 \rightarrow \gamma\text{-axis along } \left[\frac{1}{\sqrt{q}} \right]$$

$$x_n := \wedge f(s_1 \dots s_n) \quad (x_0 = 0)$$



$$\begin{aligned} f(s_1 \dots s_n) &= f(s_1 \dots s_{n-1}) + f(s_1 \dots s_n) \\ 0 \leq k &< 2n \\ 0 \leq k &< 2n \end{aligned}$$

figure of \$f(s_1, \dots, s_n)\$ as \$k < 2n\$
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property of $P_k = \{x_0, x_1, \dots, x_{k-1}\}$

(1) $P_{k+1} = P_k \cup (P_{k-1} + \frac{1}{g_k})$ self-similar

(2) $|\pi(P_{k+1} \setminus P_k) - \lambda(P_k)| \leq 1$

Boundary
Remainder

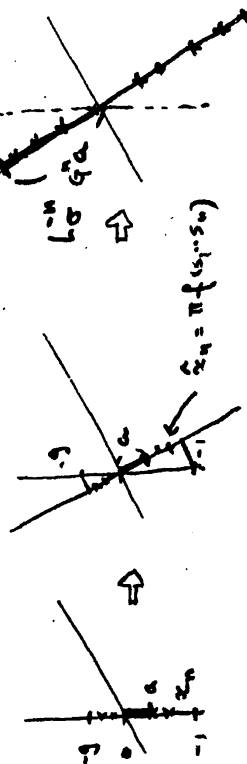
How can we get (2)?

From $\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ G \end{bmatrix} = -G \begin{bmatrix} -1 \\ G \end{bmatrix}$

$x_n = \hat{\pi}(f(s_1, \dots, s_n))$

$(-G)^n x_n = \hat{\pi}(\hat{L}_G^n f(s_1, \dots, s_n))$

$\hat{L}_G^n f(s_1, \dots, s_n) \in$



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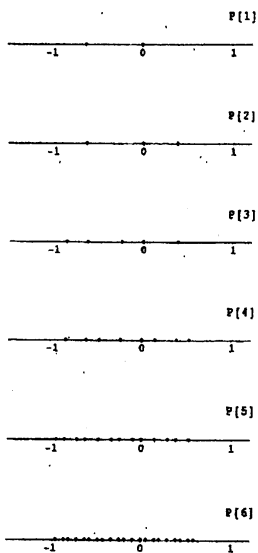


figure of $P_k = \{x_0, x_1, \dots, x_{k-1}\}$

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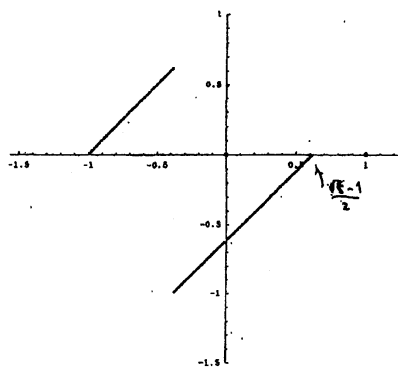


figure of (x_n, x_{n+1})
= graph of T_g

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Remark 1 $R_d(z) = z + d \pmod{1}$

directional $x_n = R_d^n(x)$ is a low disc.

$\Leftrightarrow \frac{1}{n} \sum_{j=1}^n a_j$ is bounded

where $d = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots}}$

(See Springer Lecture notes 1651)

§3 $\frac{1}{b} \sum_{j=1}^b a_j \in \mathbb{C} \pmod{1}$ — (I)

$$X := \left\{ \sum_{j=1}^{\infty} a_j (-1+i)^{j+1} \mid a_j \in \{0,1\} \right\}$$

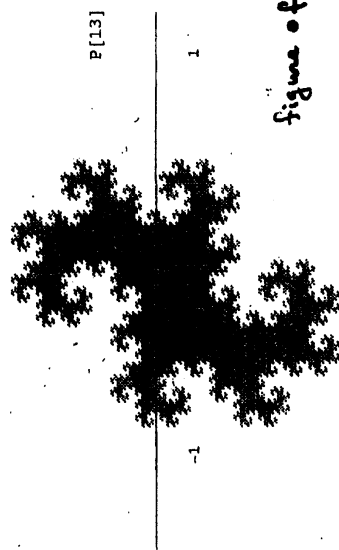


figure of X

For $n = a_n 2^n + a_{n-1} 2^{n-1} + \dots + a_0 2^0$

$$x_n := \sum_{j=0}^n \frac{a_j}{(i-1)^{j+1}} \in \mathbb{C}$$

we call generalized von der Corput seq.

or

$$\hat{x}_n \equiv x_n \pmod{1} \quad \hat{x}_n \in \square$$

and $\exists T_K : X \rightarrow X$ (Kakutani adding 1).

s.t. $x_n = T_K^n(x_0)$

$$P_k := \{x_0, x_1, \dots, x_{2^k-1}\}$$

$$\hat{P}_k := \{\hat{x}_0, \dots, \hat{x}_{2^k-1}\}$$

then property

$$(i) P_k = P_{k-1} \cup (P_{k-1} - \frac{1}{(i-1)2^k})$$

In particular self similar

$$\hat{P}_{2^m} = \left\{ \frac{p}{2^m} + \frac{q}{2^m} : 0 \leq q < 2^m \right\}$$

See figure

But P_k has not Bounded Remainder.

$$\# \left| \hat{P}_{2^m} \cap \left[\frac{a}{b}, \frac{a}{b} + \frac{1}{b} \right] \right| < 2^m$$

$$\text{Therefore } N \neq 2^m \quad 2^m \neq \sqrt{N}$$

$$DN \leq \sqrt{N} \log N$$

$$X(x_1, \dots, x_k) := \left\{ \sum_{j=1}^{\infty} a_j / (c(i))^{j+1} : (a_1, \dots, a_k) = (x_1, \dots, x_k) \right\}$$

is a bounded remainder set w.r.t. $\{P_k\}$

$\Rightarrow$ Fujita-Niwomija-Ito.

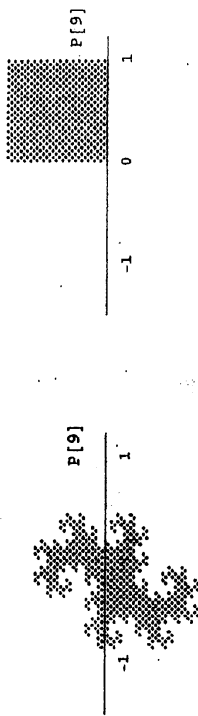
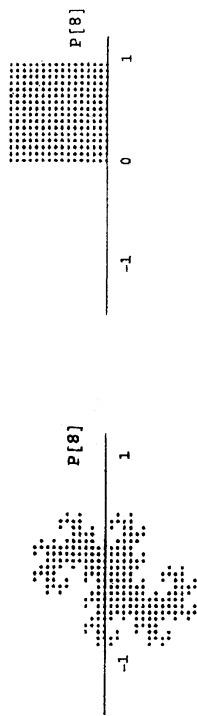
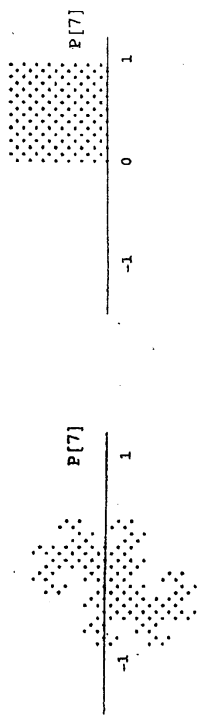


Figure of P_k and $\hat{P}_k$, $k=7,8,9$

Halton Sequence

$$n = a_k 2^k + a_{k-1} 2^{k-1} + \dots + a_0$$

$$n = b_j 3^j + b_{j-1} 3^{j-1} + \dots + b_0$$

$$x_n := \left(\sum_{k=0}^{\infty} a_k 2^{-(k+1)}, \sum_{j=0}^{\infty} b_j 3^{-(j+1)} \right)$$

is called Halton seq.

Theorem On Halton sequence

$$D_N \sim \log^2 N \cdot N \quad (\text{low discrepancy})$$

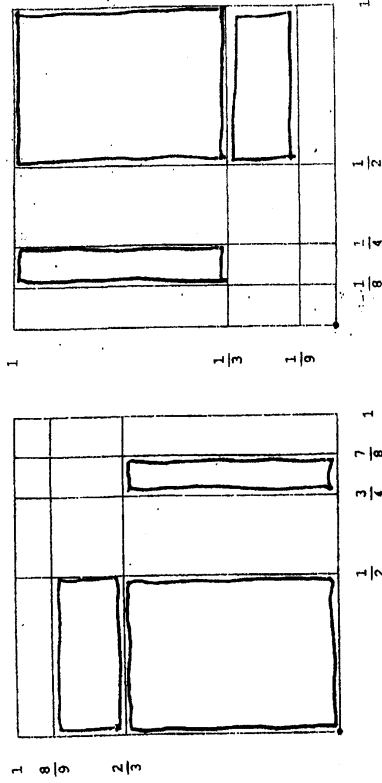
Remark M. Mori construct the 2-dim Dynamical System T_n on $[0,1]^2$

such that

$$x_n = T_n^{\infty}(x_0)$$

and $\{x_n\}$ is low discrepancy.

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$T \rightarrow$
(Kakutani)

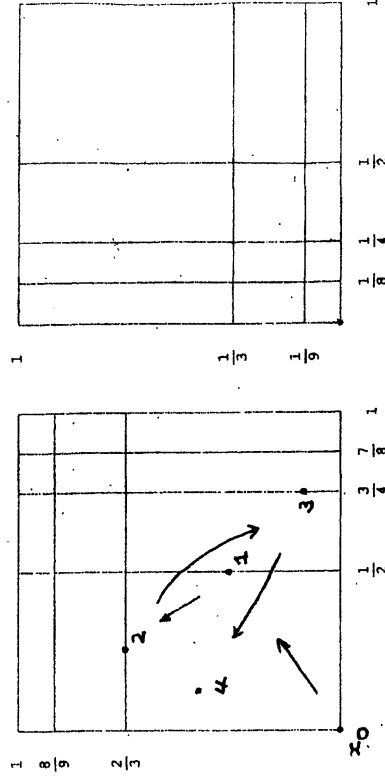
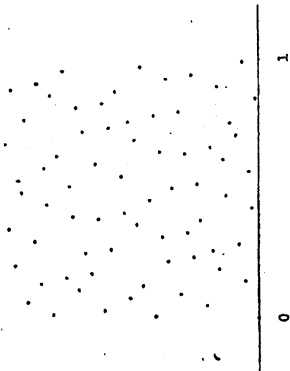


Figure of orbit

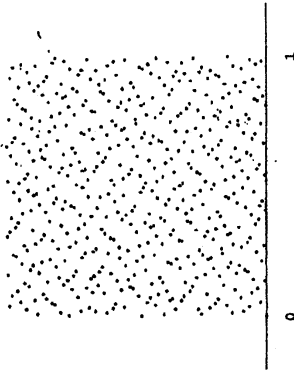
$$21 + \frac{1}{2}$$

vander2.nb

n=72

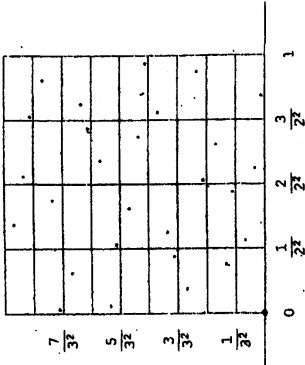


n=500

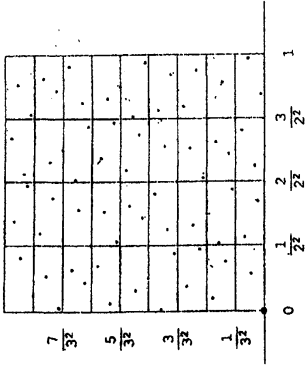


Where is
self similar?

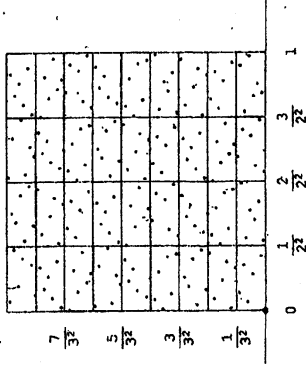
n=36



n=72



n=216



n=432

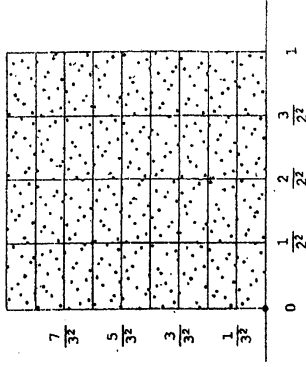


figure of $\{x_n | n=0, \dots, n-1\}$
of Halton Sequence.

Please attention the cardinality of points
in Box.

高次定数式 (II)

$$\sigma: \begin{matrix} 1 \rightarrow 12 \\ 2 \rightarrow 13 \\ 3 \rightarrow 1 \end{matrix} \quad L\sigma = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\lambda > 1, |\lambda| = 1, \lambda' = 1$$

$$|\sigma^n(1)| = l_n$$

$$l_n = l_{n-1} + l_{n-2} + l_{n-3}, \quad l_0 = 1, l_1 = 2, l_2 = 4$$

$$\sigma^n(1) = s_1 s_2 \dots s_n$$

$$\hat{\pi}: \mathbb{R}^3 \rightarrow \mathbb{R}^2 \text{ along } \begin{bmatrix} 1 \\ \lambda_1 \\ \lambda_2 \end{bmatrix}$$

$$x_n := \prod_{i=1}^n f(s_i, \dots, s_n)$$

$$\text{by } n = a_n l_n + a_{n-1} l_{n-1} + a_{n-2} l_{n-2} + \dots + a_0 l_0$$

$$x_n = \sum_{i=0}^n a_i \prod_{j=1}^n \begin{pmatrix} \delta_{ij} \\ r_{ij} \end{pmatrix} = \sum_{i=0}^n a_i (\delta_{ii} - p_i)$$

Where is self-similar structure?
bounded remainder property?

$$P^{(p,q)} := \{x_n \mid 0 \leq n < 2^p 3^q\}$$

$$E_{c,d}^{(p,q)} := \left[\frac{c}{2^p}, \frac{c+1}{2^p} \right) \times \left[\frac{d}{3^q}, \frac{d+1}{3^q} \right)$$

property

$$(a) \quad \bar{x}_n \rightarrow (c, d), \quad \bar{x}_n \in E_{c,d}^{(p,q)}$$

is bijective

$$(0 \leq n < 2^p 3^q \text{ a.e.})$$

By Chinese remainder th.

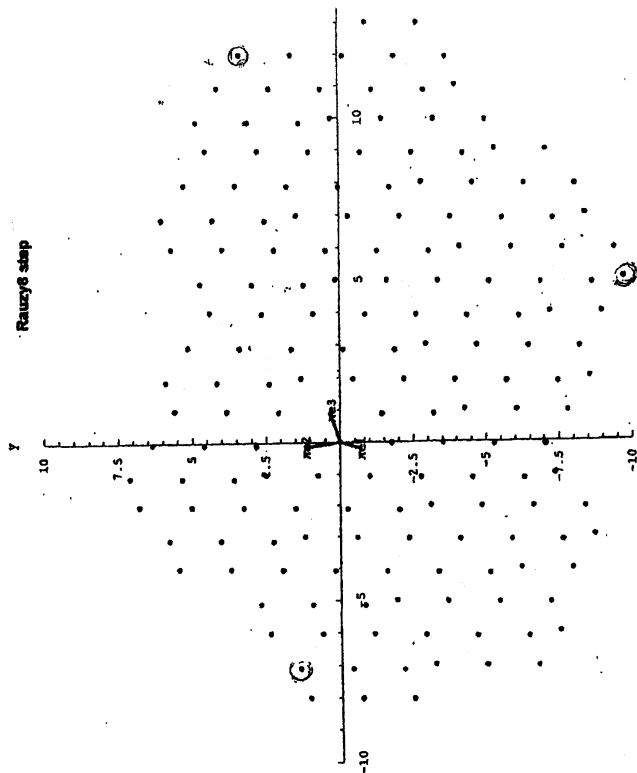
$$\left(\left\{ \bar{x}_n \in E_{c,d}^{(p,q)} \right\} = k \right)$$



Property $E_{c,d}^{(p,q)}$ is bounded remainder set

("self-similar")

Rauzy's step



$$\sigma: \begin{matrix} 1 \rightarrow 1,2 \\ 2 \rightarrow 1,3 \\ 3 \rightarrow 1,4 \end{matrix}$$

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Signe of $P_k, k=8$

$$P_k = P_{k-1} + \bigcirc + \bigcirc$$

n 2, 3, 4, ...

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Theorem (Arnoux - I)

 $X := \text{closure of } \{x_n\}$

then $\bigcup (X + \hat{n}z) = \mathbb{R}^2$
 $\hat{n}z \in L = \inf(e_2 - \hat{n}e_1) + m(\hat{n}e_3 - \hat{n}e_1) \mid m \in \mathbb{Z}$
 , that is,

$$\mathbb{R}^2 / L = X \cong \mathbb{T}^2$$

(2) $T: X \rightarrow X$
 $x \mapsto x - \hat{n}e_3 \pmod{L}$
 that is, T is a Rotation on $\mathbb{T}^2$

$$(3) \quad x_n = T^n(x)$$

$$P_k := \{x_0, x_1, \dots, x_{q_k-1}\}$$

(q_k-p_k)
(p_k-p_{k-1})

then

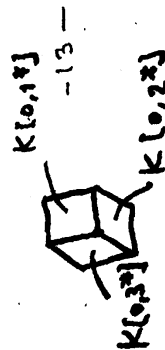
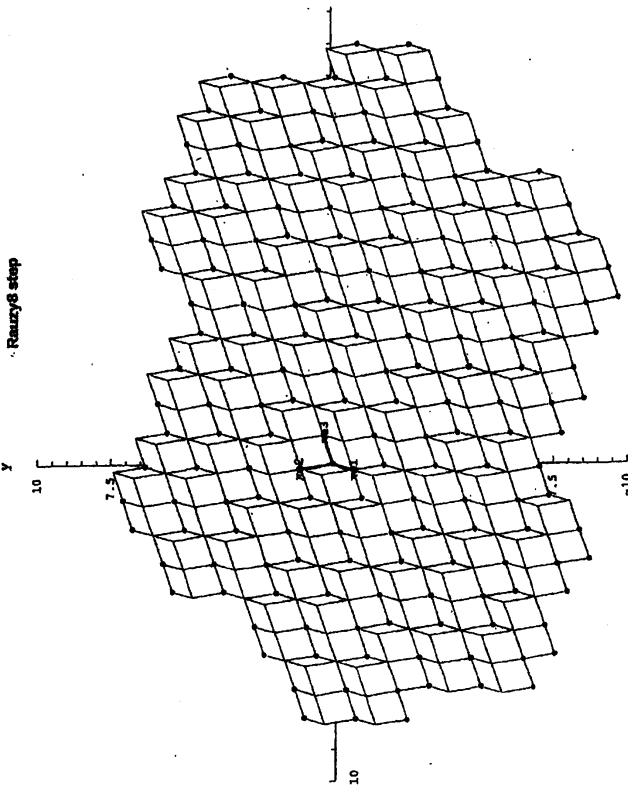
$$(1) \quad P_k = P_{k-1} \cup (P_{k-2} + \hat{\pi} \begin{pmatrix} p_k \\ p_k \end{pmatrix}) \cup (P_{k-3} + \hat{\pi} \begin{pmatrix} p_{k-1} \\ p_{k-1} \end{pmatrix})$$

self-similarity

For any $n = q_{k_1} + q_{k_2} + \dots + q_{k_r}$

$$P_n = P_{k_1} \cup (P_{k_2} + \hat{\pi} \begin{pmatrix} p_{k_1} \\ p_{k_1} \end{pmatrix}) \cup \dots \cup (P_{k_r} + \hat{\pi} \begin{pmatrix} p_{k_{r-1}} \\ p_{k_{r-1}} \end{pmatrix})$$

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- 26 - 1/2 -

Bounded Remainder π_2

$$\text{let } \square(a,b) := a \sum_b \pi_2$$

$$A_R^{(a,b)} = |\#(P_R \cap \square(a,b)) - |\square(a,b)| \cdot \ell_R|$$

if we find c :

$$A_R(a,b) < cR$$

$$\text{then } |\# \{x_n | n=0, \dots, N-1\} \cap \square(a,b) - |\square(a,b)| \cdot N| \leq c' R^2 \div c \log^2 N$$

Lemma

$$\#(P_R \cap \square(a,b)) = \#(\pi S \cap L_\sigma^{-k} \square(a,b))$$

$$\text{where } \pi S = \{ \pi x \mid x \in \mathbb{Z}^3, (x, u) \geq 0, (x - e_1) < 0 \}$$

Question

$$\# \pi S \cap L_\sigma^{-k} \square(a,b) = ?$$

Remark

$$\chi_{(\varepsilon_1, \dots, \varepsilon_k)} = \left\{ \sum_{i=1}^k a_i \chi^i \mid (a_1, a_2, \dots, a_k) = (\varepsilon_1, \dots, \varepsilon_k) \right\}$$

$\Rightarrow$ Fjitz - It

(is a bounded remainder set w.r.t. $\{P_R\}$. - 26 -

$d = (d_1, d_2, \dots, d_s)$
 $1, d_1, d_2, \dots, d_s$: linearly ind over $\mathbb{Q}$
 $z_n := (\{nd_1\}, \dots, \{nd_s\})$

Remark

• $D_N(z_n) = O(N^{-1}(1 + \log N)^{s+1})$

• d : algebraic

$D_N(z_n) = O(N^{-1+\epsilon})$ for every $\epsilon > 0$
 (Niederreiter)

• Assume $\exists \nu \geq 1, \epsilon > 0$:

$$\frac{\sum^s (h_1, \dots, h_s) < h_1 d_1 + \dots + h_s d_s > \epsilon}{\text{for all } (h_1, \dots, h_s) \neq (0, \dots, 0) \in \mathbb{Z}^s}$$

where $r(h) := \prod_{j=1}^s \max(1, |h_j|)$

then

$O D_N(z_n) = O(N^{-1} \log^{s+1} N)$ if $\nu = 1$

$D_N(z_n) = O(N^{-1/(s+1-\nu)} \log N)$
 if $\nu > 1$

proof is obtained from
 Erdős-Turán-Koksma's ineq.

