

On universality in penalisation problems with multiplicative weights

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Abstract

We give a general framework for the universality classes of σ -finite measures in penalisation problems with multiplicative weights. We discuss penalisation problems for Brownian motions, Lévy processes and Langevin processes in our framework.

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1 Introduction

For a measure μ and a non-negative measurable function f , we write $\mu[f]$ for the integral $\int f d\mu$.

For a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $(\mathcal{F}_s)_{s \geq 0}$, and for a non-negative process $\Gamma = (\Gamma_t)_{t \geq 0}$ called a *weight*, we mean by a *penalisation* a problem of finding a limit probability P^Γ on $(\Omega, \mathcal{F})$ called the *penalised probability* such that

$$\frac{P[F_s \Gamma_t]}{P[\Gamma_t]} \xrightarrow{t \rightarrow \infty} P^\Gamma[F_s] \quad (1.1)$$

is satisfied for all $s \geq 0$ and all bounded $\mathcal{F}_s$ -measurable functional F_s . Under the penalised probability P^Γ , the process $(\Gamma_t)_{t \geq 0}$ is prevented from taking small values; this is why Roynette–Vallois–Yor [14] (see also [15]) called this problem the penalisation. Conditioning a process to stay in a domain D may be regarded as a special case of the penalisation, as we take the weight $\Gamma_t = 1_{\{\tau_D > t\}}$ where τ_D denotes the exit time of D .

Although the penalised probability P^Γ depends upon the weight Γ , we can often find a σ -finite measure $\mathcal{P}$ on $(\Omega, \mathcal{F})$ independent of a particular weight such that

$$P^\Gamma(A) = \frac{\mathcal{P}[\Gamma_\infty; A]}{\mathcal{P}[\Gamma_\infty]}, \quad A \in \mathcal{F} \quad (1.2)$$

holds with a suitable limit Γ_∞ of Γ_t in a certain class of weights Γ . In this case we say that Γ belongs to the *universality class* of $\mathcal{P}$. The aim of this paper is to gain a clear

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insight into the universality classes in penalisation problems. For this purpose, we confine ourselves to multiplicative weights.

Let $\{B = (B_t)_{t \geq 0}, W_x\}$ denote the canonical representation of the one-dimensional Brownian motion with $W_x(B_0 = x) = 1$ and let $\mathcal{F}_t^B = \sigma(B_s : s \leq t)$ denote the natural filtration of the coordinate process B . Let $\tau_D = \inf\{t \geq 0 : B_t = 0\}$ denote the exit time of B from the non-zero real $D = \mathbb{R} \setminus \{0\}$. Let $x \in D$ be fixed. It is then well-known that

$$W_x[F_s | \tau_D > t] \xrightarrow[t \rightarrow \infty]{} W_x^{\pm 3B}[F_s] = \frac{1}{|x|} W_x[F_s | B_s | 1_{\{\tau_D > s\}}] \quad (1.3)$$

for all bounded $\mathcal{F}_s^B$ -measurable functional F_s , where $W_x^{\pm 3B}$ denotes the law of $\pm$ times 3-dimensional Bessel process starting from x . This conditioning to avoid zero may be regarded as a special case of the penalisation with the weight being given by $\Gamma_t = 1_{\{\tau_D > t\}}$. Note that $W_x^{\pm 3B}$ is locally absolutely continuous with respect to W_x , i.e. $W_x^{\pm 3B}|_{\mathcal{F}_s^B}$ is absolutely continuous with respect to $W_x|_{\mathcal{F}_s^B}$ for all $s \geq 0$. But $W_x^{\pm 3B}$ and W_x are mutually singular on $\mathcal{F}_\infty^B := \sigma(B)$, because $W_x^{\pm 3B}(\tau_D = \infty) = W_x(\tau_D < \infty) = 1$. While the original process $\{B, W_x\}$ is recurrent, the *penalised process* $\{B, W_x^{\pm 3B}\}$ is transient.

Roynette–Vallois–Yor ([13] and [12]) have studied the penalisation problems for the one-dimensional Brownian motion. They determined the penalised probabilities for $\Gamma_t = f(\overline{X}_t)$, a function of a supremum, $\Gamma_t = f(L_t)$, a function of a local time at 0, and $\Gamma_t = \exp(-\int_0^t v(B_s)ds)$, a Kac killing weight. For the special case $\Gamma_t = e^{-L_t}$, we have

$$\frac{W_0[F_s e^{-L_t}]}{W_0[e^{-L_t}]} \xrightarrow[t \rightarrow \infty]{} W_0^\Gamma[F_s] = \frac{1}{1 + |x|} W_0[F_s(1 + |B_s|)e^{-L_s}] \quad (1.4)$$

for all $s \geq 0$ and all bounded $\mathcal{F}_s^B$ -measurable functional F_s . Although W_0^Γ is locally absolutely continuous with respect to W_0 , the two measures W_0^Γ and W_0 are mutually singular on $\mathcal{F}_\infty^B$, because $W_0^\Gamma(L_\infty < \infty) = W_0(L_\infty = \infty) = 1$. While the original process $\{B, W_0\}$ is recurrent, the penalised process $\{B, W_0^\Gamma\}$ is transient.

Najnudel–Roynette–Yor ([8]) have introduced the σ -finite measure $\mathscr{W}_0$ defined by

$$\mathscr{W}_0 = \int_0^\infty \frac{du}{\sqrt{2\pi}u} \Pi^{(u)} \bullet W_0^{s3B}, \quad (1.5)$$

where $\Pi^{(u)}$ stands for the law of the Brownian bridge from 0 to 0 of length u , W_0^{s3B} for the law of the symmetrised Bessel process, and $\bullet$ for the law of the concatenated path of two independent paths. They proved that the penalised probability W_0^Γ for any weight Γ in the previous paragraph is absolutely continuous on $\mathcal{F}_\infty^B$ with respect to $\mathscr{W}_0$:

$$W_0^\Gamma[F] = \frac{\mathscr{W}_0[F\Gamma_\infty]}{\mathscr{W}_0[\Gamma_\infty]} \quad (1.6)$$

for all bounded $\mathcal{F}_\infty^B$ -measurable functional F . Moreover, if we define $\mathscr{W}_x(\cdot) = \mathscr{W}_0(x + B \in \cdot)$, we have

$$W_x^{\pm 3B}[F] = \frac{\mathscr{W}_x[F; \tau_D = \infty]}{\mathscr{W}_x(\tau_D = \infty)} \quad (1.7)$$

for all $x > 0$ and all bounded $\mathcal{F}_\infty^B$ -measurable functional F . In other words, all the weights belong to the universality class of $\mathcal{W}_x$.

K.Yano–Y.Yano–Yor [20, 21], Y.Yano [22] and recently Takeda–K.Yano [16] studied the penalisation problems for one-dimensional stable Lévy processes and found out that there are two different universality classes. In this paper, we would like to give a general framework to characterise universality classes, where we will give some new results.

Groeneboom–Jongbloed–Wellner [6] studied the conditioning to stay positive for the Langevin process. Profeta [10] studied penalisation problems with several kinds of weights. In this paper, we shall also discuss universality classes for those penalisation problems.

This paper is organized as follows. In Section 2 we develop a general study on penalised probabilities with multiplicative weights. In Section 3 we define the unweighted measures and discuss the subsequent Markov property of them. In Section 4 we state and prove our main theorems on universality classes. In Section 5 we give a general discussion on penalisation problems with multiplicative weights. In Sections 6, 7 and 8, we look at some known results of penalisation problems for Brownian motions, Lévy processes and Langevin processes in our framework. In Section 9 as an appendix, we discuss extension of the transformed probability measures given by local absolute continuity.

2 Penalised probability

For a measure μ and a non-negative measurable function f , we write $f \cdot \mu$ for the transformed measure defined by $(f \cdot \mu)(A) = \int_A f d\mu$ for all measurable set A . Let $(\mathcal{F}_s)_{s \geq 0}$ be a filtration. For two measures μ and ν , we say that μ is *locally absolutely continuous* with respect to ν if $\mu|_{\mathcal{F}_s}$ is absolutely continuous with respect to $\nu|_{\mathcal{F}_s}$ for all $s \geq 0$. We say the two measures are *locally equivalent* if they are locally absolutely continuous with respect to each other. For a parameterised family $(\mu_\lambda)_\lambda$ of finite measures and a finite measure μ , we say that

$$\lim_\lambda \mu_\lambda = \mu \text{ along } (\mathcal{F}_s)_{s \geq 0} \quad (2.1)$$

if

$$\lim_\lambda \mu_\lambda[F_s] = \mu[F_s] \quad (2.2)$$

holds for all $s \geq 0$ and all bounded measurable functional F_s .

Let S be a locally compact separable metric space and let $\mathbb{D}$ denote the space of càdlàg paths from $[0, \infty)$ to S . Let $X = (X_t)_{t \geq 0}$ denote the coordinate process: $X_t(\omega) = \omega(t)$ for $t \geq 0$ and $\omega \in \mathbb{D}$. Let $\mathcal{F}_t^X = \sigma(X_s : s \leq t)$ denote the natural filtration of X and set $\mathcal{F}_t = \bigcap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon}^X$ so that $(\mathcal{F}_t)_{t \geq 0}$ is a right-continuous filtration. We write $\mathcal{F}_\infty = \sigma(\bigcup_{t \geq 0} \mathcal{F}_t) = \sigma(X)$. For $t \geq 0$, let θ_t denote the shift operator of $\mathbb{D}$: $\theta_t \omega(s) = \omega(t+s)$ for $s \geq 0$.

Let $\{X, \mathcal{F}_\infty, (P_x)_{x \in S}\}$ denote the canonical representation of a strong Markov process taking values in S with respect to the augmented filtration $(\mathcal{G}_t)_{t \geq 0}$ of $(\mathcal{F}_t)_{t \geq 0}$. A process

$\Gamma = (\Gamma_t)_{t \geq 0}$ is called a *weight* if it is a non-negative càdlàg process. A weight Γ is called *multiplicative* if Γ is adapted to $(\mathcal{F}_t)_{t \geq 0}$ and

$$\Gamma_t = \Gamma_s \cdot (\Gamma_{t-s} \circ \theta_s), \text{ } P_x\text{-a.s. for all } 0 \leq s \leq t < \infty \text{ and all } x \in S. \quad (2.3)$$

Let Γ be a multiplicative weight. Since $\Gamma_0 = \Gamma_0 \cdot (\Gamma_0 \circ \theta_0) = \Gamma_0^2$, we note that

$$\text{for any } x \in S \text{ we have either } P_x(\Gamma_0 = 1) = 1 \text{ or } P_x(\Gamma_0 = 0) = 1. \quad (2.4)$$

We set

$$S^\Gamma = \{x \in S : P_x(\Gamma_0 = 1) = 1\}. \quad (2.5)$$

It is easy to see that

$$\tau^\Gamma := \inf\{t \geq 0 : X_t \notin S^\Gamma\} = \inf\{t \geq 0 : \Gamma_t = 0\} \text{ } P_x\text{-a.s. for all } x \in S, \quad (2.6)$$

since $[\Gamma_{t_0} = 0 \text{ implies } \Gamma_t = 0 \text{ for all } t \geq t_0]$ because of the multiplicativity.

We introduce the following assumptions:

(A1) There is a Borel function φ^Γ on S such that $\varphi^\Gamma > 0$ on S^Γ and

$$P_x[\Gamma_t \varphi^\Gamma(X_t)] = \varphi^\Gamma(x) \text{ for all } x \in S \text{ and } t \geq 0. \quad (2.7)$$

(A2) It holds that

$$P_x[\Gamma_{\mathbf{e}(q)}] \rightarrow 0 \text{ as } q \downarrow 0 \text{ for all } x \in S^\Gamma, \quad (2.8)$$

where we abuse P_x for the extended probability measure of P_x supporting a standard exponential variable $\mathbf{e}$ independent of $\mathcal{F}_\infty$ and we set $\mathbf{e}(q) = \mathbf{e}/q$ for $q > 0$.

Note that, by the dominated convergence theorem, the condition **(A2)** follows from the following condition:

(A2') It holds that

$$P_x[\Gamma_t] \rightarrow 0 \text{ as } t \rightarrow \infty \text{ for all } x \in S^\Gamma. \quad (2.9)$$

By the multiplicativity, the condition (2.7) is equivalent to the condition that

$$(\Gamma_t \varphi^\Gamma(X_t))_{t \geq 0} \text{ is a right-continuous } ((\mathcal{G}_t)_{t \geq 0}, P_x)\text{-martingale for all } x \in S \quad (2.10)$$

(for right-continuity, see, e.g., [5, Theorem 5.8]). Under **(A1)**, for $x \in S^\Gamma$, we may define a probability measure P_x^Γ on $(\mathbb{D}, \mathcal{F}_\infty)$, which we call the *penalised probability* of P_x for Γ , by the following (see Section 9):

$$P_x^\Gamma|_{\mathcal{F}_t} = \frac{\Gamma_t \varphi^\Gamma(X_t)}{\varphi^\Gamma(x)} \cdot P_x|_{\mathcal{F}_t} \text{ for all } t \geq 0. \quad (2.11)$$

It is then immediate that the *penalised process* $\{X, \mathcal{F}_\infty, (P_x^\Gamma)_{x \in S}\}$ is a Markov process with respect to $(\mathcal{F}_t)_{t \geq 0}$.

We write $\xrightarrow{P}$ for convergence in probability. In addition to **(A1)** and **(A2)**, we also introduce the following assumptions:

(A3) There is a non-negative finite $\mathcal{F}_\infty$ -measurable functional Γ_∞ such that

$$P_x^\Gamma \left(\Gamma_t \xrightarrow[t \rightarrow \infty]{} \Gamma_\infty > 0 \right) = 1 \text{ for all } x \in S^\Gamma. \quad (2.12)$$

Note that in many examples we have **(A3)** and $P_x(\liminf_{t \rightarrow \infty} \Gamma_t = 0) = 1$, which implies that the two measures P_x^Γ and P_x are mutually singular on $\mathcal{F}_\infty$.

The following is a routine argument.

Proposition 2.1. *Let Γ be a multiplicative weight. Then the following hold.*

(i) Under **(A1)**, it holds that

$$P_x^\Gamma(\tau^\Gamma = \infty) = 1 \text{ for all } x \in S^\Gamma. \quad (2.13)$$

(ii) Under **(A1)**, **(A2)** and **(A3)**, it holds that

$$P_x^\Gamma \left(\varphi^\Gamma(X_t) \xrightarrow[t \rightarrow \infty]{} \infty \right) = 1 \text{ for all } x \in S^\Gamma. \quad (2.14)$$

Proof. (i) We apply the optional stopping theorem to the $((\mathcal{G}_t)_{t \geq 0}, P_x)$ -martingale $M_t := \Gamma_t \varphi^\Gamma(X_t) / \varphi^\Gamma(x)$ (by **(A1)**) to see that

$$P_x^\Gamma(\tau^\Gamma > t) = P_x[M_t; \tau^\Gamma > t] \quad (2.15)$$

$$= P_x[M_{t \wedge \tau^\Gamma}] - P_x[M_{t \wedge \tau^\Gamma}; \tau^\Gamma \leq t] \quad (2.16)$$

$$= P_x[M_0] - P_x[M_{\tau^\Gamma}; \tau^\Gamma \leq t] = 1, \quad (2.17)$$

which implies that $P_x^\Gamma(\tau^\Gamma = \infty) = 1$.

(ii) Let $0 \leq s \leq t < \infty$ and $A_s \in \mathcal{F}_s$. We then have

$$\begin{aligned} P_x^\Gamma \left[\frac{1}{\Gamma_t \varphi^\Gamma(X_t)}; A_s \right] &= \frac{1}{\varphi^\Gamma(x)} P_x(A_s, \tau^\Gamma > t) \\ &\leq \frac{1}{\varphi^\Gamma(x)} P_x(A_s, \tau^\Gamma > s) = P_x^\Gamma \left[\frac{1}{\Gamma_s \varphi^\Gamma(X_s)}; A_s \right]. \end{aligned} \quad (2.18)$$

This shows that $N_t := 1/\{\Gamma_t \varphi^\Gamma(X_t)\}$ is a non-negative P_x^Γ -supermartingale with respect to the completed filtration $(\overline{\mathcal{F}}_t^{P_x^\Gamma})_{t \geq 0}$ of $(\mathcal{F}_t)_{t \geq 0}$, and consequently it converges P_x^Γ -a.s. as $t \rightarrow \infty$ to some random variable N_∞ . By **(A3)**, we see that

$$\frac{1}{\varphi^\Gamma(X_t)} = \Gamma_t N_t \xrightarrow[t \rightarrow \infty]{} \Gamma_\infty N_\infty \quad P_x^\Gamma\text{-a.s.}, \quad (2.19)$$

which implies $1/\varphi^\Gamma(X_{e(q)}) \xrightarrow[q \downarrow 0]{P_x^\Gamma} \Gamma_\infty N_\infty$. Using Fatou's lemma, we obtain

$$P_x^\Gamma[\Gamma_\infty N_\infty] \leq \liminf_{q \downarrow 0} P_x^\Gamma \left[\frac{1}{\varphi^\Gamma(X_{e(q)})} \right] = \frac{1}{\varphi^\Gamma(x)} \lim_{q \downarrow 0} P_x[\Gamma_{e(q)}] = 0 \quad (2.20)$$

by **(A2)**. Hence we obtain (2.14). $\square$

3 Subsequent Markov property

Let Γ be a multiplicative weight satisfying **(A1)**, **(A2)** and **(A3)**. For $x \in S^\Gamma$, we may define a measure $\mathcal{P}_x^\Gamma$ on $(\mathbb{D}, \mathcal{F}_\infty)$, which we call the *unweighted measure* of P_x^Γ , by

$$\mathcal{P}_x^\Gamma = \varphi^\Gamma(x) \Gamma_\infty^{-1} \cdot P_x^\Gamma \quad \text{on } \mathcal{F}_\infty. \quad (3.1)$$

Note that $\mathcal{P}_x^\Gamma$ is σ -finite on $\mathcal{F}_\infty$, because $\mathbb{D} = \bigcup_{n \in \mathbb{N}} \{\Gamma_\infty > 1/n\}$, $\mathcal{P}_x^\Gamma$ -a.e. and

$$\mathcal{P}_x^\Gamma(\Gamma_\infty > 1/n) \leq n\varphi^\Gamma(x) < \infty \quad \text{for all } n \in \mathbb{N}. \quad (3.2)$$

The family of the unweighted measures satisfies the following property.

Theorem 3.1. *Let Γ be a multiplicative weight satisfying **(A1)**-(**A3**). Then, for any $x \in S^\Gamma$, any non-negative $\mathcal{F}_t$ -measurable functional F_t and any non-negative $\mathcal{F}_\infty$ -measurable functional G , it holds that*

$$\mathcal{P}_x^\Gamma[F_t(G \circ \theta_t)] = P_x[F_t \mathcal{P}_{X_t}^\Gamma[G]; \tau^\Gamma > t]. \quad (3.3)$$

Proof. By definition of $\mathcal{P}_x^\Gamma$, we have

$$\mathcal{P}_x^\Gamma[(F_t \Gamma_t)((G \Gamma_\infty) \circ \theta_t)] = \mathcal{P}_x^\Gamma[F_t(G \circ \theta_t) \Gamma_\infty] \quad (3.4)$$

$$= \varphi^\Gamma(x) P_x^\Gamma[F_t(G \circ \theta_t)]. \quad (3.5)$$

By the Markov property for X under P_x^Γ , by the local equivalence between P_x^Γ and P_x , and by the global equivalence between P_x^Γ and $\mathcal{P}_x^\Gamma$, we obtain

$$(3.5) = \varphi^\Gamma(x) P_x^\Gamma[F_t P_{X_t}^\Gamma[G]] \quad (3.6)$$

$$= P_x[F_t \varphi^\Gamma(X_t) \Gamma_t P_{X_t}^\Gamma[G]] \quad (3.7)$$

$$= P_x[F_t \Gamma_t \mathcal{P}_{X_t}^\Gamma[G \Gamma_\infty]], \quad (3.8)$$

where we used the fact obtained from Proposition 2.1 that $X_t \in S^\Gamma$, P_x -a.s. on $\{\Gamma_t > 0\}$. Thus we obtain

$$\mathcal{P}_x^\Gamma[F_t \Gamma_t (G \Gamma_\infty) \circ \theta_t] = P_x[F_t \Gamma_t \mathcal{P}_{X_t}^\Gamma[G \Gamma_\infty]]. \quad (3.9)$$

Replacing F_t by $F_t \Gamma_t^{-1} 1_{\{\tau^\Gamma > t\}}$ and G by $G \Gamma_\infty^{-1} 1_{\{\Gamma_\infty > 0\}}$, we obtain the desired identity, since $\tau^\Gamma = \infty$ and $\Gamma_\infty > 0$, $\mathcal{P}_x^\Gamma$ -a.e. The proof is now complete. $\square$

Theorem 3.1 asserts that, the process under $\mathcal{P}_x^\Gamma$ behaves until a fixed time t as the process under P_x killed upon leaving S^Γ , and it starts afresh at time t to behave as the process under $\mathcal{P}_{X_t}^\Gamma$. In this sense, we may call this property (3.3) the *subsequent Markov property*.

4 Universality class

Let $\mathcal{E}$ be a particular multiplicative weight satisfying **(A1)**-**(A3)**. We would like to give a sufficient condition for existence of a positive function $c(x)$ such that

$$S^\Gamma \subset S^\mathcal{E} \quad \text{and} \quad \mathcal{P}_x^\Gamma = c(x)1_{\{\Gamma_\infty > 0\}} \cdot \mathcal{P}_x^\mathcal{E} \quad \text{for all } x \in S^\Gamma. \quad (4.1)$$

We note that $[\mathcal{P}_x^\Gamma = c(x)1_{\{\Gamma_\infty > 0\}} \cdot \mathcal{P}_x^\mathcal{E}]$ yields $[\Gamma \text{ belongs to the universality class of } \mathcal{P}_x^\mathcal{E}]$ in the sense we mentioned in Introduction.

Theorem 4.1 (Universality theorem). *Let $\mathcal{E}$ and Γ be two multiplicative weights satisfying **(A1)**-**(A3)**. Suppose there exists a positive function $c(x)$ such that*

$$P_x^\mathcal{E} \left(\Gamma_t \xrightarrow[t \rightarrow \infty]{} \Gamma_\infty \right) = 1, \quad \frac{\varphi^\Gamma(X_t)}{\varphi^\mathcal{E}(X_t)} \xrightarrow[t \rightarrow \infty]{P_x^\mathcal{E}} c(x) \quad \text{for all } x \in S^\Gamma \quad (4.2)$$

and

$$P_x^\Gamma \left(\mathcal{E}_t \xrightarrow[t \rightarrow \infty]{} \mathcal{E}_\infty > 0 \right) = 1, \quad \frac{\varphi^\Gamma(X_t)}{\varphi^\mathcal{E}(X_t)} \xrightarrow[t \rightarrow \infty]{P_x^\Gamma} c(x) \quad \text{for all } x \in S^\Gamma. \quad (4.3)$$

(Notice that these assumptions do not follow from **(A3)**.) Then (4.1) holds.

Proof. Let $x \in S^\Gamma$ be fixed. Since $P_x = P_x^\Gamma$ on $\mathcal{F}_0$, we have

$$P_x(\mathcal{E}_0 = 1) = P_x^\Gamma(\mathcal{E}_0 = 1) \geq P_x^\Gamma(\mathcal{E}_\infty > 0) = 1, \quad (4.4)$$

which shows $x \in S^\mathcal{E}$. By the assumptions, we have

$$R_t \xrightarrow[t \rightarrow \infty]{P_x^\mathcal{E}} R_\infty \quad \text{and} \quad R_t \xrightarrow[t \rightarrow \infty]{P_x^\Gamma} R_\infty \quad \text{with} \quad R_t = \frac{\Gamma_t \varphi^\Gamma(X_t)}{\varphi^\mathcal{E}(X_t)} \quad \text{and} \quad R_\infty = c(x)\Gamma_\infty. \quad (4.5)$$

Let $s > 0$ and let F_s be a non-negative $\mathcal{F}_s$ -measurable functional. For $t > s$, we have

$$P_x^\mathcal{E} \left[F_s \cdot \frac{R_t}{1 + R_t + \mathcal{E}_t} \right] = \frac{1}{\varphi^\mathcal{E}(x)} P_x \left[F_s \cdot \frac{R_t}{1 + R_t + \mathcal{E}_t} \cdot \mathcal{E}_t \varphi^\mathcal{E}(X_t) \right] \quad (4.6)$$

$$= \frac{\varphi^\Gamma(x)}{\varphi^\mathcal{E}(x)} P_x^\Gamma \left[F_s \cdot \frac{R_t}{1 + R_t + \mathcal{E}_t} \cdot \frac{\mathcal{E}_t \varphi^\mathcal{E}(X_t)}{\Gamma_t \varphi^\Gamma(X_t)} \right] \quad (4.7)$$

$$= \frac{\varphi^\Gamma(x)}{\varphi^\mathcal{E}(x)} P_x^\Gamma \left[F_s \cdot \frac{\mathcal{E}_t}{1 + R_t + \mathcal{E}_t} \right]. \quad (4.8)$$

Letting $t \rightarrow \infty$ and applying the dominated convergence theorem, we obtain

$$P_x^\mathcal{E} \left[F_s \cdot \frac{R_\infty}{1 + R_\infty + \mathcal{E}_\infty} \right] = \frac{\varphi^\Gamma(x)}{\varphi^\mathcal{E}(x)} P_x^\Gamma \left[F_s \cdot \frac{\mathcal{E}_\infty}{1 + R_\infty + \mathcal{E}_\infty} \right]. \quad (4.9)$$

Since $s > 0$ and F_s are arbitrary, we obtain

$$c(x)\varphi^\mathcal{E}(x)\Gamma_\infty \cdot P_x^\mathcal{E} = \varphi^\Gamma(x)\mathcal{E}_\infty \cdot P_x^\Gamma, \quad (4.10)$$

which yields

$$c(x)1_{\{\Gamma_\infty > 0\}} \cdot \mathcal{P}_x^\mathcal{E} = 1_{\{\mathcal{E}_\infty > 0\}} \cdot \mathcal{P}_x^\Gamma = \mathcal{P}_x^\Gamma, \quad (4.11)$$

since $P_x^\Gamma(\mathcal{E}_\infty > 0) = 1$. We thus obtain the desired result. $\square$

5 Penalisation problems

We give two systematic methods of ensuring the conditions **(A1)** and **(A2)** in penalisation problems.

5.1 Constant clock

We give a general framework for penalisation problems with constant clock.

Proposition 5.1. *Let Γ be a multiplicative weight. Let $\rho(t)$ be a function such that*

$$\rho(t) \xrightarrow[t \rightarrow \infty]{} \infty \quad \text{and} \quad \frac{\rho(t)}{\rho(t-s)} \xrightarrow[t \rightarrow \infty]{} 1 \quad \text{for all } s > 0, \quad (5.1)$$

or in other words, $\rho(\log t)$ is divergent and slowly varying at $t = \infty$. Suppose there exists a process $(M_s)_{s \geq 0}$ such that $P_x(M_0 > 0) = P_x(\Gamma_0 = 1)$ for all $x \in S$ and

$$\rho(t)P_x[\Gamma_t|\mathcal{F}_s] \xrightarrow[t \rightarrow \infty]{} M_s \quad \text{in } L^1(P_x) \text{ for all } x \in S \text{ and all } s \geq 0. \quad (5.2)$$

*Then the weight Γ satisfies **(A1)** and **(A2')** with*

$$\varphi^\Gamma(x) = \lim_{t \rightarrow \infty} \rho(t)P_x[\Gamma_t], \quad (5.3)$$

and the following penalisation limit with constant clock holds:

$$\frac{\Gamma_t \cdot P_x}{P_x[\Gamma_t]} \xrightarrow[t \rightarrow \infty]{} P_x^\Gamma \quad \text{along } (\mathcal{F}_s)_{s \geq 0} \text{ for all } x \in S^\Gamma. \quad (5.4)$$

Proof. The convergence (5.2) for $s = 0$ becomes (5.3). By the multiplicativity $\Gamma_t = \Gamma_s \cdot (\Gamma_{t-s} \circ \theta_s)$ and by the Markov property, we have

$$\rho(t)P_x[\Gamma_t|\mathcal{F}_s] = \frac{\rho(t)}{\rho(t-s)} \Gamma_s \cdot \rho(t-s)P_{X_s}[\Gamma_{t-s}] \xrightarrow[t \rightarrow \infty]{} \Gamma_s \varphi^\Gamma(X_s) \quad \text{in } P_x\text{-a.s.} \quad (5.5)$$

which yields $M_s = \Gamma_s \varphi^\Gamma(X_s)$. Hence we have

$$P_x[\Gamma_t \varphi^\Gamma(X_t)] = \lim_{u \rightarrow \infty} \rho(u)P_x[P_x[\Gamma_u|\mathcal{F}_t]] = \lim_{u \rightarrow \infty} \rho(u)P_x[\Gamma_u] = \varphi^\Gamma(x), \quad (5.6)$$

which shows that **(A1)** is satisfied. As $\rho(t) \rightarrow \infty$, we obtain **(A2')**. For $s > 0$ and for a bounded $\mathcal{F}_s$ -measurable functional F_s , we obtain

$$\rho(t)P_x[F_s \Gamma_t] = P_x[F_s \rho(t)P_x[\Gamma_t|\mathcal{F}_s]] \xrightarrow[t \rightarrow \infty]{} P_x[F_s M_s] = \varphi^\Gamma(x)P_x^\Gamma[F_s]. \quad (5.7)$$

This shows (5.4). □

5.2 Exponential clock

Conditioning and penalisation problems with exponential clock have been widely studied; see [3], [4], [9], [19] and [11]. We give a general framework for them.

Proposition 5.2. *Let $r(q)$ be a function defined for small $q > 0$ such that $r(q) \rightarrow \infty$ as $q \downarrow 0$. We abuse P_x for the extended probability measure of P_x supporting a standard exponential variable $\mathbf{e}$ independent of $(\mathcal{F}_t)_{t \geq 0}$ and set $\mathbf{e}(q) = \mathbf{e}/q$ for $q > 0$. Suppose there exists a process $(M_s)_{s \geq 0}$ such that $P_x(M_0 > 0) = P_x(\Gamma_0 = 1)$ for all $x \in S$ and*

$$\lim_{q \downarrow 0} r(q) P_x[\Gamma_{\mathbf{e}(q)} | \mathcal{F}_s] = \lim_{q \downarrow 0} r(q) P_x[\Gamma_{\mathbf{e}(q)} 1_{\{\mathbf{e}(q) > s\}} | \mathcal{F}_s] = M_s \quad \text{in } L^1(P_x) \quad (5.8)$$

for all $x \in S$ and all $s \geq 0$.

Then the weight Γ satisfies **(A1)** and **(A2)** with

$$\varphi^\Gamma(x) = \lim_{q \downarrow 0} r(q) P_x[\Gamma_{\mathbf{e}(q)}], \quad (5.9)$$

and the following penalisation limit with exponential clock holds:

$$\lim_{q \downarrow 0} \frac{\Gamma_{\mathbf{e}(q)} \cdot P_x}{P_x[\Gamma_{\mathbf{e}(q)}]} = \lim_{q \downarrow 0} \frac{\Gamma_{\mathbf{e}(q)} 1_{\{\mathbf{e}(q) > s\}} \cdot P_x}{P_x[\Gamma_{\mathbf{e}(q)}; \mathbf{e}(q) > s]} = P_x^\Gamma \quad \text{along } (\mathcal{F}_s)_{s \geq 0} \text{ for all } x \in S^\Gamma. \quad (5.10)$$

Proof. The convergence (5.8) for $s = 0$ becomes (5.9). By the multiplicativity $\Gamma_t = \Gamma_s \cdot (\Gamma_{t-s} \circ \theta_s)$, by the Markov property and by the memoryless property

$$\mathbf{e}(q) - s \text{ given } \{\mathbf{e}(q) > s\} \stackrel{\text{law}}{=} \mathbf{e}(q), \quad (5.11)$$

we have

$$r(q) P_x[\Gamma_{\mathbf{e}(q)} 1_{\{\mathbf{e}(q) > s\}} | \mathcal{F}_s] = e^{-qs} r(q) P_x[\Gamma_{\mathbf{e}(q)+s} | \mathcal{F}_s] \quad (5.12)$$

$$= e^{-qs} \Gamma_s r(q) P_{X_s}[\Gamma_{\mathbf{e}(q)}] \xrightarrow{q \downarrow 0} \Gamma_s \varphi^\Gamma(X_s) \quad P_x\text{-a.s.}, \quad (5.13)$$

which yields $M_s = \Gamma_s \varphi^\Gamma(X_s)$. Hence we obtain

$$P_x[\Gamma_t \varphi^\Gamma(X_t)] = P_x[M_t] = \lim_{q \downarrow 0} r(q) P_x[P_x[\Gamma_{\mathbf{e}(q)} | \mathcal{F}_t]] = \lim_{q \downarrow 0} r(q) P_x[\Gamma_{\mathbf{e}(q)}] = \varphi^\Gamma(x), \quad (5.14)$$

which shows that **(A1)** is satisfied. As $r(q) \rightarrow \infty$, we obtain **(A2)**. For $s > 0$ and for a bounded $\mathcal{F}_s$ -measurable functional F_s , we obtain

$$r(q) P_x[F_s \Gamma_{\mathbf{e}(q)}] = P_x[F_s r(q) P_x[\Gamma_{\mathbf{e}(q)} | \mathcal{F}_s]] \xrightarrow{q \downarrow 0} P_x[F_s M_s] = \varphi^\Gamma(x) P_x^\Gamma[F_s]. \quad (5.15)$$

This shows (5.10). □

6 Brownian penalisation revisited

Let us look at some results of Roynette–Vallois–Yor [13, 12] and Najnudel–Roynette–Yor [8] in our framework.

Let $\{B = (B_t)_{t \geq 0}, (W_x)_{x \in \mathbb{R}}\}$ denote the canonical representation of the one-dimensional Brownian motion with $W_x(B_0 = x) = 1$. Set $\overline{B}_t = \sup_{s \leq t} B_s$ and let L_t denote the local time of B at 0. For the shift operator on the path space, we have

$$B_{t+s} = B_t \circ \theta_s, \quad \overline{B}_{t+s} = \overline{B}_s \vee (\overline{B}_t \circ \theta_s), \quad L_{t+s} = L_s + (L_t \circ \theta_s). \quad (6.1)$$

For a technical reason, we set

$$S = \{(x, y, l) \in \mathbb{R}^3 : y \geq x, l \geq 0\} \quad (6.2)$$

as the state space and consider the coordinate process $X = (X_t)_{t \geq 0} = (X_t^B, X_t^{\sup}, X_t^{\text{lt}})_{t \geq 0}$ on the space of càdlàg paths from $[0, \infty)$ to S . Writing $a \vee b = \max\{a, b\}$, we define $P_{(x,y,l)}$ by the law on $\mathbb{D}$ of $(B, y \vee \overline{B}, l + L)$ under W_x , and adopt the notation of Section 2. By the identities (6.1), we see that the process $\{X, \mathcal{F}_\infty, (P_{(x,y,l)})_{(x,y,l) \in S}\}$ is a strong Markov process with respect to the augmented filtration.

(1) Supremum penalisation. For an integrable function $f : \mathbb{R} \rightarrow [0, \infty)$ such that for some $-\infty < y_0 \leq \infty$ we have $f(y) > 0$ for $y \leq y_0$ and $f(y) = 0$ for $y > y_0$, we set

$$\Gamma_t^{\sup, f} = \frac{f(X_t^{\sup})}{f(X_0^{\sup})} 1_{\{X_t^{\sup} \leq y_0\}}, \quad S^{\sup, f} = \{(x, y, l) \in S : y \leq y_0\}. \quad (6.3)$$

Then we see that $\Gamma^{\sup, f}$ is a multiplicative weight with $S^{\Gamma^{\sup, f}} = S^{\sup, f}$ (in what follows we will omit similar remarks). By Roynette–Vallois–Yor [12, Theorem 3.6], we see that all the assumptions of Proposition 5.1 are satisfied with $\rho(t) = \sqrt{\pi t/2}$ and

$$\varphi^{\sup, f}(x, y, l) = y - x + \frac{1}{f(y)} \int_y^{y_0} f(u) du, \quad (x, y, l) \in S^{\sup, f}, \quad (6.4)$$

so that **(A1)** and **(A2')** are satisfied. By the discussion of Roynette–Vallois–Yor [12, Subsection 1.4], we can derive that

$$P_{(x,y,l)}^{\sup, f}(X_\infty^{\sup} > a) = \frac{\int_a^{y_0} f(u) du}{(y - x)f(y) + \int_y^{y_0} f(u) du}, \quad y \leq a < \infty, \quad (6.5)$$

and hence that $[X_t^{\sup} = X_\infty^{\sup} \text{ for large } t]$ and $[\Gamma_t^{\sup, f} \rightarrow \Gamma_\infty^{\sup, f} > 0]$ $P_{(x,y,l)}^{\sup, f}$ -a.s., which shows **(A3)**. By (ii) of Proposition 2.1, we obtain the following known results:

$$P_{(x,y,l)}^{\sup, f} \left(X_t^B \rightarrow -\infty, \frac{\varphi^{\sup, f}(X_t)}{|X_t^B|} \rightarrow 1 \right) = 1. \quad (6.6)$$

(2) Local time penalisation. For an integrable function $f : [0, \infty) \rightarrow [0, \infty)$ such that for some $0 \leq l_0 \leq \infty$ we have $f(l) > 0$ for $l \leq l_0$ and $f(l) = 0$ for $l > l_0$, we set

$$\Gamma_t^{\text{lt},f} = \frac{f(X_t^{\text{lt}})}{f(X_0^{\text{lt}})} 1_{\{X_t^{\text{lt}} \leq l_0\}}, \quad S^{\text{lt},f} = \{(x, y, l) \in S : l \leq l_0\}. \quad (6.7)$$

By Roynette–Vallois–Yor [12, Theorem 3.13 and Lemma 3.15], we see that all the assumptions of Proposition 5.1 are satisfied with $\rho(t) = \sqrt{\pi t/2}$ and

$$\varphi^{\text{lt},f}(x, y, l) = |x| + \frac{1}{f(l)} \int_l^{l_0} f(u) du, \quad (x, y, l) \in S^{\text{lt},f}, \quad (6.8)$$

so that **(A1)** and **(A2')** are satisfied. Moreover, **(A3)** is also satisfied and

$$P_{(x,y,l)}^{\text{lt},f}(X_t^B \rightarrow \pm\infty) = \frac{x^\pm f(l) + \frac{1}{2} \int_l^{l_0} f(u) du}{|x| f(l) + \int_l^{l_0} f(u) du} \quad (6.9)$$

with $x^\pm = \max\{\pm x, 0\}$. It is then obvious that

$$P_{(x,y,l)}^{\text{lt},f}\left(|X_t^B| \rightarrow \infty, \frac{\varphi^{\text{lt},f}(X_t)}{|X_t^B|} \rightarrow 1\right) = 1. \quad (6.10)$$

Note that the conditioning to avoid zero, which we have mentioned in Introduction, can be regarded as a special case of the local time penalisation with the weight $1_{\{X_t^{\text{lt}}=0\}} = \Gamma_t^{\text{lt},f}$ for $f(l) = 1_{\{l=0\}}$.

(3) Kac killing penalisation with integrable potential. For an integrable function $v : \mathbb{R} \rightarrow [0, \infty)$ satisfying

$$0 < \int_{\mathbb{R}} (1 + |x|) v(x) dx < \infty, \quad (6.11)$$

we set

$$\Gamma_t^{\text{Kac},v} = \exp\left(-\int_0^t v(X_s^B) ds\right), \quad S^{\text{Kac},v} = S. \quad (6.12)$$

By Roynette–Vallois–Yor [13, Theorem 4.1], we see that all the assumptions of Proposition 5.1 are satisfied with $\rho(t) = \sqrt{\pi t/2}$ and $\varphi^{\text{Kac},v}(x, y, l) = \varphi_v(x)$ where φ_v is the unique solution to the Sturm–Liouville equation

$$\frac{1}{2} \frac{d^2 \varphi_v}{dx^2}(x) = v(x) \varphi_v(x), \quad \lim_{x \rightarrow \pm\infty} \frac{d\varphi_v}{dx}(x) = \pm 1. \quad (6.13)$$

so that **(A1)** and **(A2')** are satisfied. Moreover, **(A3)** is also satisfied and

$$P_{(x,y,l)}^{\text{Kac},v}(X_t^B \rightarrow -\infty) = \frac{1}{C_v} \int_x^\infty \frac{dy}{\varphi_v(y)^2}, \quad P_{(x,y,l)}^{\text{Kac},v}(X_t^B \rightarrow \infty) = \frac{1}{C_v} \int_{-\infty}^x \frac{dy}{\varphi_v(y)^2} \quad (6.14)$$

with $C_v = \int_{\mathbb{R}} \frac{dy}{\varphi_v(y)^2}$. By (6.13) it is obvious that

$$P_{(x,y,l)}^{\text{Kac},v} \left(|X_t^B| \rightarrow \infty, \frac{\varphi^{\text{Kac},v}(X_t)}{|X_t^B|} \rightarrow 1 \right) = 1. \quad (6.15)$$

(4) Kac killing penalisation with Heviside potential. For $\lambda > 0$, we set

$$\Gamma_t^{\text{Hev},\lambda} = \exp \left(-\lambda \int_0^t 1_{\{X_s^B > 0\}} ds \right), \quad S^{\text{Kac},v} = S. \quad (6.16)$$

By Roynette–Vallois–Yor [13, Theorem 5.1 and Example 5.4], we see that all the assumptions of Proposition 5.1 are satisfied with $\rho(t) = \sqrt{\pi t}/2$ and

$$\varphi^{\text{Hev},\lambda}(x, y, l) = \begin{cases} \frac{1}{\sqrt{2\lambda}} e^{-\sqrt{2\lambda}x} & (x \geq 0), \\ \frac{1}{\sqrt{2\lambda}} - x & (x < 0), \end{cases} \quad (6.17)$$

so that **(A1)** and **(A2')** are satisfied. Moreover, **(A3)** is also satisfied and

$$P_{(x,y,l)}^{\text{Hev},\lambda} \left(X_t^B \rightarrow -\infty, \frac{\varphi^{\text{Hev},\lambda}(X_t)}{|X_t^B|} \rightarrow 1 \right) = 1. \quad (6.18)$$

(*) The universality class of Brownian penalisation. Take $\mathcal{E}_t = \exp(-X_t^{\text{lt}})$ as a special case of (2) with $f(l) = e^{-l}$. (Note that, by Najnudel–Roynette–Yor [8, Theorem 1.1.2], the corresponding unweighted measure $\mathcal{P}_x^{\mathcal{E}}$ coincides with $\mathcal{W}_x$ given in Introduction.) By the above argument, we see that all the assumptions of Theorem 4.1 are satisfied with $\mathcal{E}$ and $\Gamma = \Gamma^{\text{sup},f}, \Gamma^{\text{lt},f}, \Gamma^{\text{Kac},v}$ or $\Gamma^{\text{Hev},\lambda}$, so that we obtain the following known result:

$$\mathcal{P}_{(x,y,l)}^{\Gamma} = 1_{\{\Gamma_{\infty} > 0\}} \cdot \mathcal{P}_{(x,y,l)}^{\mathcal{E}} \quad \text{for all } (x, y, l) \in S^{\Gamma}. \quad (6.19)$$

We remark the following obvious facts: It holds up to $\mathcal{P}_{(x,y,l)}^{\mathcal{E}}$ -null sets that

$$\mathbb{D} = \{X_t^B \rightarrow \infty \text{ or } X_t^B \rightarrow -\infty\}, \quad (6.20)$$

and that the event $\{\Gamma_{\infty} > 0\}$ becomes

$$\{\Gamma_{\infty}^{\text{sup},f} > 0\} = \{X_t^B \rightarrow -\infty \text{ and } X_{\infty}^{\text{sup}} \leq y_0\}, \quad (6.21)$$

$$\{\Gamma_{\infty}^{\text{lt},f} > 0\} = \{[X_t^B \rightarrow \infty \text{ or } X_t^B \rightarrow -\infty] \text{ and } X_{\infty}^{\text{lt}} \leq l_0\}, \quad (6.22)$$

$$\{\Gamma_{\infty}^{\text{Kac},v} > 0\} = \{X_t^B \rightarrow \infty \text{ or } X_t^B \rightarrow -\infty\}, \quad (6.23)$$

$$\{\Gamma_{\infty}^{\text{Hev},\lambda} > 0\} = \{X_t^B \rightarrow -\infty\}. \quad (6.24)$$

7 Lévy penalisation revisited

Let us look at some results of K.Yano–Y.Yano–Yor [20, 21], Y.Yano [22] and Takeda–K.Yano [16] in our framework.

Let $\{Z = (Z_t)_{t \geq 0}, (P_x^Z)_{x \in \mathbb{R}}\}$ denote the canonical representation of one-dimensional strictly α -stable process of index $1 < \alpha < 2$, skewness $-1 \leq \beta \leq 1$ and scaling parameter $c_\theta > 0$:

$$P_0^Z[e^{i\lambda Z_t}] = \exp\left(-c_\theta |\lambda|^\alpha \left(1 - i\beta \operatorname{sgn}(\lambda) \tan \frac{\pi\alpha}{2}\right)\right), \quad \lambda \in \mathbb{R}. \quad (7.1)$$

(For the facts in this paragraph, see e.g. [2, Section VIII].) We assume that $1 < \alpha < 2$ so as to exclude the Brownian case and to assure that zero is regular for itself: Writing $T_0 = \inf\{t > 0 : Z_t = 0\}$ for the hitting time of zero, we have

$$P_0^Z(T_0 > 0) = 1. \quad (7.2)$$

Set $\bar{Z}_t = \sup_{s \leq t} Z_s$ and let L_t denote the local time of Z at 0. Let

$$\rho := P_0^Z(Z_1 > 0) = \frac{1}{2} + \frac{1}{\pi\alpha} \arctan\left(\beta \tan \frac{\pi\alpha}{2}\right) \in [1 - 1/\alpha, 1/\alpha] \quad (7.3)$$

and let k denote the positive constant such that

$$\lim_{y \rightarrow \infty} y^\alpha P_0^Z(\bar{Z} > y) = k. \quad (7.4)$$

We set

$$S = \{(x, y, l) \in \mathbb{R}^3 : y \geq x, l \geq 0\} \quad (7.5)$$

as the state space and consider the coordinate process $X = (X_t)_{t \geq 0} = (X_t^Z, X_t^{\sup}, X_t^{\text{lt}})_{t \geq 0}$ on the space of càdlàg paths from $[0, \infty)$ to S . We define $P_{(x,y,l)}$ by the law on $\mathbb{D}$ of $(Z, y \vee \bar{Z}, l + L)$ under P_x^Z , and adopt the notation of Section 2.

(1) Supremum penalisation. For a non-increasing function $f : \mathbb{R} \rightarrow [0, \infty)$ such that for some $-\infty < y_0 \leq \infty$ we have $f(y) > 0$ for $y \leq y_0$ and $f(y) = 0$ for $y > y_0$, and

$$\int_0^{y_0} x^{\alpha\rho-1} f(y) dy < \infty, \quad (7.6)$$

we set

$$\Gamma_t^{\sup, f} = \frac{f(X_t^{\sup})}{f(X_0^{\sup})} 1_{\{X_t^{\sup} \leq y_0\}}, \quad S^{\sup, f} = \{(x, y, l) \in S : y \leq y_0\}. \quad (7.7)$$

By K.Yano–Y.Yano–Yor [21, Theorem 5.1], we see that all the assumptions of Proposition 5.1 are satisfied with $\rho(t) = t^\rho/k$ and

$$\varphi^{\sup, f}(x, y, l) = (y - x)^{\alpha\rho} + \frac{\alpha\rho}{f(y)} \int_y^{y_0} f(u)(u - x)^{\alpha\rho-1} du, \quad (x, y, l) \in S^{\sup, f}, \quad (7.8)$$

so that **(A1)** and **(A2')** are satisfied. In the same way as that of deducing (6.5), we see that $[X_t^{\sup} = X_\infty^{\sup} \text{ for large } t]$ and $[\Gamma_t^{\sup, f} \rightarrow \Gamma_\infty^{\sup, f} > 0]$ $P_{(x,y,l)}^{\sup, f}$ -a.s., which shows **(A3)**. By

(ii) of Proposition 2.1 and by the dominated convergence theorem, we obtain the following known results:

$$P_{(x,y,l)}^{\sup,f} \left(X_t^Z \rightarrow -\infty, \frac{\varphi^{\sup,f}(X_t)}{(-X_t^Z)^{\alpha\rho}} \rightarrow 1 \right) = 1. \quad (7.9)$$

Note that the special case of the supremum penalisation with the weight $1_{\{X_t^{\sup}=0\}} = \Gamma_t^{\sup,f}$ for $f(l) = 1_{\{y=0\}}$ corresponds to the conditioning to stay negative.

(2) Local time penalisation. For an integrable function $f : [0, \infty) \rightarrow [0, \infty)$ such that for some $0 \leq l_0 \leq \infty$ we have $f(l) > 0$ for $l \leq l_0$ and $f(l) = 0$ for $l > l_0$, we set

$$\Gamma_t^{\text{lt},f} = \frac{f(X_t^{\text{lt}})}{f(X_0^{\text{lt}})} 1_{\{X_t^{\text{lt}} \leq l_0\}}, \quad S^{\text{lt},f} = \{(x, y, l) \in S : l \leq l_0\}. \quad (7.10)$$

By Takeda–K.Yano [16] and by certain computation in [18, Section 5], we see that all the assumptions of Proposition 5.2 are satisfied with $r(q) = c_r q^{1/\alpha-1}$ for a certain constant $c_r > 0$ and

$$\varphi^{\text{lt},f}(x, y, l) = C_{\alpha,\beta}(1 - \beta \operatorname{sgn}(x))|x|^{\alpha-1} + \frac{1}{f(l)} \int_l^{l_0} f(u) du, \quad (x, y, l) \in S^{\text{lt},f} \quad (7.11)$$

with a certain constant $C_{\alpha,\beta} > 0$, so that **(A1)** and **(A2)** are satisfied. In the same way as that of deducing (6.5), we see that $[X_t^{\text{lt}} = X_\infty^{\text{lt}}$ for large $t]$ and $[\Gamma_t^{\text{lt},f} \rightarrow \Gamma_\infty^{\text{lt},f} > 0]$ $P_{(x,y,l)}^{\text{lt},f}$ -a.s., which shows **(A3)**. By (ii) of Proposition 2.1, we obtain

$$P_{(x,y,l)}^{\text{lt},f} \left((1 - \beta \operatorname{sgn}(X_t^Z))|X_t^Z|^{\alpha-1} \rightarrow \infty, \frac{\varphi^{\text{lt},f}(X_t)}{C_{\alpha,\beta}(1 - \beta \operatorname{sgn}(X_t^Z))|X_t^Z|^{\alpha-1}} \rightarrow 1 \right) = 1; \quad (7.12)$$

in particular,

$$P_{(x,y,l)}^{\text{lt},f} \left(X_t^Z \rightarrow -\infty, \frac{\varphi^{\text{lt},f}(X_t)}{(-X_t^Z)^{\alpha-1}} \rightarrow 2C_{\alpha,1} \right) = 1 \quad (\text{if } \beta = 1), \quad (7.13)$$

$$P_{(x,y,l)}^{\text{lt},f} \left(X_t^Z \rightarrow \infty, \frac{\varphi^{\text{lt},f}(X_t)}{(X_t^Z)^{\alpha-1}} \rightarrow 2C_{\alpha,-1} \right) = 1 \quad (\text{if } \beta = -1). \quad (7.14)$$

In the case of $-1 < \beta < 1$, we have a stronger convergence result in Takeda–K.Yano [16]:

$$P_{(x,y,l)}^{\text{lt},f} (\lim X_t^Z = \limsup X_t^Z = \limsup(-X_t^Z) = \infty) = 1 \quad \text{if } -1 < \beta < 1. \quad (7.15)$$

Note that the special case of the local time penalisation with the weight $1_{\{X_t^{\text{lt}}=0\}} = \Gamma_t^{\text{lt},f}$ for $f(l) = 1_{\{l=0\}}$ corresponds to the conditioning to avoid zero. See [17] for comparison of two types of conditionings for Lévy processes.

(*) The universality classes of Lévy penalisation. By (7.9), it holds that

$$\{\Gamma_\infty^{\sup,f} > 0\} = \{X_t^Z \rightarrow -\infty \text{ and } X_\infty^{\sup} \leq y_0\} \quad \text{up to } \mathcal{P}_{(x,y,l)}^{\sup,f}\text{-null sets} \quad (7.16)$$

in any case of $-1 \leq \beta \leq 1$.

(*1) Consider the case of $-1 < \beta < 1$. By (7.15), it holds that

$$\{\Gamma_\infty^{\text{lt},g} > 0\} = \{\lim X_t^Z = \limsup X_t^Z = \limsup(-X_t^Z) = \infty \text{ and } X_\infty^{\text{lt}} \leq y_0\} \quad (7.17)$$

up to $\mathcal{P}_{(x,y,l)}^{\text{lt},g}$ -null sets.

This shows that the two σ -finite measures $\mathcal{P}_{(x,y,l)}^{\text{sup},f}$ and $\mathcal{P}_{(x,y,l)}^{\text{lt},g}$ are singular to each other. Note that (7.9) and (7.15) imply

$$P_{(x,y,l)}^{\text{sup},f} \left(\frac{\varphi^{\text{lt},g}(X_t)}{\varphi^{\text{sup},f}(X_t)} \rightarrow 0 \right) = 1 \quad (7.18)$$

because $\alpha\rho > \alpha - 1$, so that the assumption of Theorem 4.1 is not satisfied.

(*2) Consider the case of $\beta = 1$, the spectrally positive case. Take $\mathcal{E}_t = \exp(X_0^{\text{sup}} - X_t^{\text{sup}})$ as a special case of (1) with $f(y) = e^{-y}$. Then, since $\alpha\rho = \alpha - 1$, all the assumptions of Theorem 4.1 are satisfied with $\mathcal{E}$ and $\Gamma = \Gamma^{\text{sup},f}$ or $\Gamma^{\text{lt},g}$, so that we conclude as a new result that

$$\mathcal{P}_{(x,y,l)}^\Gamma = 1_{\{\Gamma_\infty > 0\}} \cdot \mathcal{P}_{(x,y,l)}^\mathcal{E} \quad \text{for all } (x, y, l) \in S^\Gamma. \quad (7.19)$$

It holds up to $\mathcal{P}_{(x,y,l)}^\mathcal{E}$ -null sets that

$$\mathbb{D} = \{X_t^Z \rightarrow -\infty\}, \quad (7.20)$$

and that the event $\{\Gamma_\infty > 0\}$ becomes

$$\{\Gamma_\infty^{\text{lt},g} > 0\} = \{X_t^Z \rightarrow -\infty \text{ and } X_\infty^{\text{lt}} \leq l_0\}. \quad (7.21)$$

(*3) Consider the case of $\beta = -1$, the spectrally negative case. Then

$$\{\Gamma_\infty^{\text{lt},g} > 0\} = \{X_t^Z \rightarrow \infty \text{ and } X_\infty^{\text{lt}} \leq l_0\} \quad \text{up to } \mathcal{P}_{(x,y,l)}^{\text{lt},g}\text{-null sets}, \quad (7.22)$$

which shows that $\mathcal{P}_{(x,y,l)}^{\text{sup},f}$ and $\mathcal{P}_{(x,y,l)}^{\text{lt},g}$ are singular to each other.

8 Langevin penalisation revisited

Let us look at some results of Profeta [10] in our framework.

Let $\{(B, A), (W_{(b,a)}(b,a)_{(b,a) \in \mathbb{R}^2})\}$ denote the canonical representation of the two-dimensional diffusion $(B, A) = (B_t, A_t)_{t \geq 0}$ where B is a Brownian motion starting from b and

$$A_t = a + \int_0^t B_u du. \quad (8.1)$$

This two-dimensional diffusion is a special case of the *Langevin process* and the process A is called the *integrated Brownian motion*. Set $\bar{A}_t := \sup_{s \leq t} A_s$.

We set

$$S = \{(b, a, y) \in \mathbb{R}^3 : y \geq a\} \quad (8.2)$$

as the state space and consider the coordinate process

$$X = (X_t)_{t \geq 0} = (X_t^B, X_t^A, X_t^{\sup})_{t \geq 0} \quad (8.3)$$

on the space of càdlàg paths from $[0, \infty)$ to S . We define $P_{(b,a,y)}$ by the law on $\mathbb{D}$ of $(B, A, y \vee \bar{A})$ under $W_{(b,a)}$, and adopt the notation of Section 2.

We recall the confluent hypergeometric function (see [1, Chapter 13]):

$$U(\alpha, \beta, z) = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-zu} u^{\alpha-1} (1+u)^{\beta-\alpha-1} du, \quad \alpha > 0, \beta \in \mathbb{R}, z > 0. \quad (8.4)$$

It is easy to see that

$$\frac{d}{dz}(z^\alpha U(\alpha, \beta, z)) = -\alpha(\beta - \alpha - 1)z^{\alpha-1}U(\alpha + 1, \beta, z). \quad (8.5)$$

The following asymptotics are taken from [1, Formulae 13.5.2 and 13.5.8]:

$$\lim_{z \rightarrow \infty} z^\alpha U(\alpha, \beta, z) = 1 \quad (\beta \in \mathbb{R}), \quad \lim_{z \downarrow 0} z^{\beta-1} U(\alpha, \beta, z) = \frac{\Gamma(\beta-1)}{\Gamma(\alpha)} \quad (1 < \beta < 2). \quad (8.6)$$

(1) Conditioning to stay negative. We write $\tau^A = \inf\{t > 0 : X_t^A \geq 0\}$ for the exit time from $(-\infty, 0)$ for the process X^A and set

$$\Gamma_t^A = 1_{\{\tau^A > t\}}, \quad S^A = \{(b, a, y) \in S : y < 0\} = \{(b, a, y) \in \mathbb{R}^3 : a \leq y < 0\}. \quad (8.7)$$

By modifying Profeta [10, Theorem 5], we see that all the assumptions of Proposition 5.1 are satisfied with $\rho(t) = c_1 t^{1/4}$ for a certain constant $c_1 > 0$ and

$$\varphi^A(b, a, y) = h(-a, -b), \quad (b, a, y) \in S^A, \quad (8.8)$$

with a continuous function $h : (0, \infty) \times \mathbb{R} \rightarrow (0, \infty)$ given as

$$h(x, y) = \begin{cases} (\frac{9}{2}x)^{1/6} z^{1/3} U(\frac{1}{6}, \frac{4}{3}, z) = y^{1/2} z^{1/6} U(\frac{1}{6}, \frac{4}{3}, z) & (y > 0), \\ \frac{1}{6}(\frac{9}{2}x)^{1/6} z^{1/3} U(\frac{7}{6}, \frac{4}{3}, z) e^{-z} = \frac{1}{6}|y|^{1/2} z^{1/6} U(\frac{7}{6}, \frac{4}{3}, z) e^{-z} & (y < 0), \end{cases} \quad (8.9)$$

for $x > 0$ and $z = \frac{2|y|^3}{9x}$, so that **(A1)** and **(A2')** are satisfied. Moreover, **(A3)** is also satisfied and

$$P_{(b,a,y)}^A(X_t^B \rightarrow -\infty \text{ and } X_t^A \rightarrow -\infty) = 1. \quad (8.10)$$

Let us prove this fact, as the part $[X_t^B \rightarrow -\infty]$ was not mentioned in [10]. By the formulae (8.6), we see that both $z^{1/6}U(\frac{1}{6}, \frac{4}{3}, z)$ and $z^{1/6}U(\frac{7}{6}, \frac{4}{3}, z)e^{-z}$ are bounded in $z > 0$, we obtain $h(x, y) \leq c_2|y|^{1/2}$ for some constant $c_2 > 0$. It holds $P_{(b,a,y)}^A$ -a.s. that, by (ii) of Proposition 2.1,

$$\varphi^A(X_t) = h(-X_t^A, -X_t^B) \rightarrow \infty, \quad (8.11)$$

which yields $[|X_t^B| \rightarrow \infty]$. But $[P_{(b,a,y)}^A(X_t^B \rightarrow \infty) = 0]$, since $[X_t^B \rightarrow \infty]$ implies $[X_t^A = a + \int_0^t X_s^B ds \rightarrow \infty]$, which contradicts the fact that $X_0^A = a < 0$ and $\tau^A = \infty$ by (i) of Proposition 2.1. Hence we obtain (8.10).

(2) Supremum penalisation. Let $f : \mathbb{R} \rightarrow [0, \infty)$ be a continuous function such that for some $-\infty < y_0 \leq 0$, we have $f(y) > 0$ for $y \leq y_0$ and $f(y) = 0$ for $y > y_0$. Set

$$\Gamma_t^{\text{sup},f} = \frac{f(X_t^A)}{f(X_0^A)} 1_{\{X_t^A \leq y_0\}}, \quad S^{\text{sup},f} = \{(b, a, y) \in S : y \leq y_0\} \\ = \{(b, a, y) \in \mathbb{R}^3 : a \leq y < y_0\}. \quad (8.12)$$

By Profeta [10, Proposition 18 and Theorem 19], we see that all the assumptions of Proposition 5.1 are satisfied with $\rho(t) = c_1 t^{1/4}$ and

$$\varphi^{\text{sup},f}(b, a, y) = h(y - a, -b) + \frac{1}{f(y)} \int_y^{y_0} f(w) \frac{\partial}{\partial w} h(w - a, -b) dw, \quad (b, a, y) \in S^{\text{sup},f}, \quad (8.13)$$

so that **(A1)** and **(A2')** are satisfied. By a similar argument to that deducing (6.5), we see that $[X_t^{\text{sup}} = X_\infty^{\text{sup}} \text{ for large } t]$ $P_{(b,a,y)}^{\text{sup},f}$ -a.s., and that $[\Gamma_t^{\text{sup},f} \rightarrow \Gamma_\infty^{\text{sup},f} > 0]$ $P_{(b,a,y)}^{\text{sup},f}$ -a.s., which shows **(A3)**. By the fact that $\frac{\partial h}{\partial w} \geq 0$, we have

$$\varphi^{\text{sup},f}(b, a, y) \leq \left(\sup_{y \leq w \leq y_0} f(w) \right) h(y_0 - a, -b). \quad (8.14)$$

By a similar argument after (8.11), and by (ii) of Proposition 2.1, we can deduce

$$P_{(b,a,y)}^{\text{sup},f}(X_t^B \rightarrow -\infty \text{ and } X_t^A \rightarrow -\infty) = 1. \quad (8.15)$$

(*) The universality class of Langevin penalisation. We would like to compare the three unweighted measures $\mathcal{P}_{(b,a,y)}^A$, $\mathcal{P}_{(b,a,y)}^{\text{sup},f}$ and $\mathcal{P}_{(b,a,y)}^B$. Here we write $\tau^B = \inf\{t > 0 : X_t^B \geq 0\}$ for the exit time from $(-\infty, 0)$ for the Brownian motion X^B and set

$$\Gamma_t^B = 1_{\{\tau^B > t\}}, \quad S^B = \{(b, a, y) \in S : b < 0\}. \quad (8.16)$$

The penalisation for the weight Γ^B is nothing else but the conditioning to stay negative for the Brownian motion, so that we obtain $\varphi^B(b, a, y) = -b$. The penalized probability $P_{(b,a,y)}^B$ is the minus times 3-dimensional Bessel process and the corresponding unweighted measure is given as $\mathcal{P}_{(b,a,y)}^B = (-b)P_{(b,a,y)}^B$. Since $X_t^A = a + \int_0^t X_u^B du$, we obtain

$$P_{(b,a,y)}^B(X_t^B \rightarrow -\infty \text{ and } X_t^A \rightarrow -\infty) = 1. \quad (8.17)$$

We prove the following proposition with conjectured assumptions.

Proposition 8.1. Set $Z_t = \frac{(-X_t^B)^3}{(-X_t^A)}$. Then the following assertions hold:

(i) Suppose the following conjecture is true:

$$Z_t \xrightarrow[t \rightarrow \infty]{P_{(b,a,y)}^A} \infty \text{ and } Z_t \xrightarrow[t \rightarrow \infty]{P_{(b,a,y)}^{\sup,f}} \infty \text{ for } (b,a,y) \in S^{\sup,f}. \quad (8.18)$$

Then $\mathcal{P}_{(b,a,y)}^{\sup,f}$ and $\mathcal{P}_{(b,a,y)}^A$ coincide for $(b,a,y) \in S^{\sup,f}(\subset S^A)$.

(ii) Suppose the following conjecture is true:

$$Z_t \xrightarrow[t \rightarrow \infty]{P_{(b,a,y)}^A} \infty \text{ and } Z_t \xrightarrow[t \rightarrow \infty]{P_{(b,a,y)}^B} \infty \text{ for } (b,a,y) \in S^A \cap S^B. \quad (8.19)$$

Then $\mathcal{P}_{(b,a,y)}^A$ and $\mathcal{P}_{(b,a,y)}^B$ are singular to each other for $(b,a,y) \in S^A \cap S^B$.

Proof. (i) Set $Z_t^{\sup} = \frac{(-X_t^B)^3}{(X_t^{\sup} - X_t^A)}$. Then $Z_t \xrightarrow[t \rightarrow \infty]{P} \infty$ both for $P = P_{(b,a,y)}^A$ and for $P = P_{(b,a,y)}^{\sup,f}$. Since $X_t^B < 0$ for large t , we have

$$\frac{h(X_t^{\sup} - X_t^A, -X_t^B)}{h(-X_t^A, -X_t^B)} = \frac{(Z_t^{\sup})^{1/6} U(\frac{1}{6}, \frac{4}{3}, Z_t^{\sup})}{(Z_t)^{1/6} U(\frac{1}{6}, \frac{4}{3}, Z_t)} \xrightarrow[t \rightarrow \infty]{P} 1 \quad (8.20)$$

by the assumption. Noting that (8.5) implies

$$\frac{\partial}{\partial x} h(x, y) = c_3 x^{-5/6} \cdot z^{7/6} U(\frac{7}{6}, \frac{4}{3}, z) \leq c_4 x^{-5/6}, \quad x, y > 0, \quad z = \frac{2|y|^3}{9x} \quad (8.21)$$

for some constants $c_3, c_4 > 0$, we obtain

$$\frac{\varphi^{\sup,f}(X_t)}{\varphi^A(X_t)} \xrightarrow[t \rightarrow \infty]{P} 1 \quad (8.22)$$

both for $P = P_{(b,a,y)}^A$ and for $P = P_{(b,a,y)}^{\sup,f}$. We may now apply Theorem 4.1 for $\mathcal{E} = \Gamma^A$ and $\Gamma = \Gamma^{\sup,f}$, and thus we obtain the desired result.

(ii) By the assumption, we have

$$R_t := \frac{\Gamma_t^A \varphi^A(X_t)}{\varphi^B(X_t)} = \frac{\Gamma_t^A \cdot (-X_t^B)^{1/2} \cdot (Z_t)^{1/6} U(\frac{1}{6}, \frac{4}{3}, Z_t)}{(-X_t^B)} \xrightarrow[t \rightarrow \infty]{P} 0 \quad (8.23)$$

both for $P = P_{(b,a,y)}^A$ and for $P = P_{(b,a,y)}^B$. By the same argument of Theorem 4.1 with $\mathcal{E} = \Gamma^B$ and $\Gamma = \Gamma^A$, we obtain

$$P_{(b,a,y)}^B \left[F_s \cdot \frac{R_t}{1 + R_t + \Gamma_t^B} \right] = \frac{\varphi^A(b, a, y)}{\varphi^B(b, a, y)} P_{(b,a,y)}^A \left[F_s \cdot \frac{\Gamma_t^B}{1 + R_t + \Gamma_t^B} \right]. \quad (8.24)$$

Letting $t \rightarrow \infty$, we obtain $P_{(b,a,y)}^A(\Gamma_\infty^B > 0) = 0$. Since $P_{(b,a,y)}^B(\Gamma_\infty^B > 0) = 1$, we obtain the desired result. $\square$

9 Appendix: Extension of transformed probability measures

We discuss in general extension of the transformed probability measures given by local absolute continuity like (2.11). Recall that $\mathbb{D}$ is the space of càdlàg paths from $[0, \infty)$ to a locally compact separable metric space S and X is the coordinate process on $\mathbb{D}$.

Theorem 9.1. *Let P be a probability measure on $(\mathbb{D}, \sigma(X))$ and let $(M_t)_{t \geq 0}$ be a non-negative martingale such that $P[M_t] = 1$ for all $t \geq 0$. Then there exists a unique probability measure Q on $(\mathbb{D}, \sigma(X))$ such that*

$$Q|_{\mathcal{F}_t^X} = M_t \cdot P|_{\mathcal{F}_t^X}, \quad t \geq 0, \quad (9.1)$$

where $\mathcal{F}_t^X = \sigma(X_s : s \leq t)$ is the natural filtration of X .

Proof. Since $\bigcup_{t \geq 0} \mathcal{F}_t^X$ is a π -system generating $\sigma(X)$, uniqueness of Q follows immediately from Dynkin's π - λ theorem.

Let us prove existence of Q . For $n \in \mathbb{N}$, let $\mathbb{D}_n$ denote the space of càdlàg paths from $[n-1, n)$ to S , equipped with the σ -field $\mathcal{B}_n$ generated by the coordinate process on $\mathbb{D}_n$. We thus see that $\mathbb{D}$ is the product space of $\{\mathbb{D}_n\}$:

$$\mathbb{D} = \prod_{n=1}^{\infty} \mathbb{D}_n, \quad \sigma(X) = \sigma\left(\prod_{k=1}^n B_k \times \prod_{k=n+1}^{\infty} \mathbb{D}_k : B_1 \in \mathcal{B}_1, \dots, B_n \in \mathcal{B}_n; n \in \mathbb{N}\right). \quad (9.2)$$

Let μ_n denote the law on $\mathbb{D}_1 \times \dots \times \mathbb{D}_n$, the space of càdlàg paths from $[0, n)$ to S , of $(X_t)_{0 \leq t < n}$ under $M_n \cdot P|_{\mathcal{F}_n^X}$. We then see that $\{\mu_n\}$ is a projective sequence:

$$\mu_{n+1}(\cdot \times \mathbb{D}_{n+1}) = \mu_n, \quad n \in \mathbb{N}. \quad (9.3)$$

We may apply Daniell's extension theorem (cf. [7, Theorem 6.14]) to see that there exists a sequence of random variables $\{\xi_n\}$ defined on a probability space $(\Omega', \mathcal{F}', P')$ such that ξ_n for each n takes values in $\mathbb{D}_n$ and the joint distribution of $(\xi_1, \dots, \xi_n)$ under P' for each n coincides with μ_n .

We now define Q by the law on $\mathbb{D}$ of $(\xi_1, \xi_2, \dots)$ under P' . For any $A \in \mathcal{F}_n^X$ for each $n \in \mathbb{N}$, we can find $B \subset \mathbb{D}_1 \times \dots \times \mathbb{D}_n$ which belongs to $\sigma(\prod_{k=1}^n B_k : B_1 \in \mathcal{B}_1, \dots, B_n \in \mathcal{B}_n)$ such that $A = \{(X_t)_{0 \leq t < n} \in B\}$, so that we obtain

$$Q(A) = P'((\xi_1, \dots, \xi_n) \in B) = \mu_n(B) = P[M_n; (X_t)_{0 \leq t < n} \in B] = P[M_n; A]. \quad (9.4)$$

We thus conclude that Q is as desired. $\square$

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