

ABOUT THE JAMES CONSTANT OF ABSOLUTE NORMED SPACES II

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ABSTRACT. In this note, we describe some recent results concerning James constant of absolute norms on $\mathbb{R}^2$ and the 2-dimensional Lorentz sequence spaces.

1. INTRODUCTION

A Banach space X is called *uniformly non-square* if there is a $\delta > 0$ such that if $x, y \in S_X$ then $\|x + y\|/2 \leq 1 - \delta$ or $\|x - y\|/2 \leq 1 - \delta$, where $S_X = \{x \in X : \|x\| = 1\}$. Gao and Lau [4] introduced the *James constant* of a Banach space X as follows:

$$J(X) = \sup \left\{ \min \{ \|x + y\|, \|x - y\| \} : x, y \in S_X \right\}.$$

We shall collect some properties about James constant:

- (1) For any Banach space X we have $\sqrt{2} \leq J(X) \leq 2$.
- (2) If X is a Hilbert space, then $J(X) = \sqrt{2}$.
- (3) $J(X) < 2$ if and only if X is uniformly non-square.
- (4) If $1 \leq p \leq \infty$ and $\dim L_p \geq 2$, then

$$J(L_p) = \max\{2^{1/p}, 2^{1/p'}\}$$

where $1/p + 1/p' = 1$.

In this note, we describe some recent results concerning the James constant of absolute norms on $\mathbb{R}^2$ and the 2-dimensional Lorentz sequence spaces.

A norm $\|\cdot\|$ on $\mathbb{R}^2$ is said to be *absolute* if $\|(x, y)\| = \|(|x|, |y|)\|$ for all $x, y \in \mathbb{R}$, and *normalized* if $\|(1, 0)\| = \|(0, 1)\| = 1$. The ℓ_p -norms $\|\cdot\|_p$ are

such examples:

$$\|(x, y)\|_p = \begin{cases} (|x|^p + |y|^p)^{1/p} & \text{if } 1 \leq p < \infty, \\ \max\{|x|, |y|\} & \text{if } p = \infty. \end{cases}$$

Let AN_2 be the family of all absolute normalized norms on $\mathbb{R}^2$. Bonsall and Duncan [2] showed that for any absolute normalized norm on $\mathbb{R}^2$ there corresponds a continuous convex function on $[0, 1]$ with some appropriate conditions as follows. Let Ψ_2 be the family of all continuous convex functions on $[0, 1]$ such that $\psi(0) = \psi(1) = 1$ and $\max\{1 - t, t\} \leq \psi(t) \leq 1$. Then AN_2 and Ψ_2 are in a one to one correspondence under the equation

$$(1) \quad \psi(t) = \|(1 - t, t)\| \quad (0 \leq t \leq 1).$$

Indeed, for any $\|\cdot\| \in AN_2$ we put ψ as (1). Then $\psi \in \Psi_2$. Also, for all $\psi \in \Psi_2$ we define

$$\|(x, y)\|_\psi = \begin{cases} (|x| + |y|)\psi\left(\frac{|y|}{|x| + |y|}\right) & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Then $\|\cdot\|_\psi \in AN_2$, and $\|\cdot\|_\psi$ satisfies (1). From this result, we can consider many non- ℓ_p -type norms easily. The functions which correspond with the ℓ_p -norms $\|\cdot\|_p$ on $\mathbb{R}^2$ are

$$\psi_p(t) = \begin{cases} \{(1 - t)^p + t^p\}^{1/p} & \text{if } 1 \leq p < \infty, \\ \max\{1 - t, t\} & \text{if } p = \infty. \end{cases}$$

For $0 < \omega < 1$ and $1 \leq q < \infty$, the 2-dimensional Lorentz sequence space $d^{(2)}(\omega, q)$ is $\mathbb{R}^2$ with the norm

$$\|x\|_{\omega, q} = (x_1^{*q} + \omega x_2^{*q})^{1/q}, \quad x = (x_1, x_2) \in \mathbb{R}^2,$$

where (x_1^*, x_2^*) is the nonincreasing rearrangement of $(|x_1|, |x_2|)$; that is, $x_1^* = \max\{|x_1|, |x_2|\}$ and $x_2^* = \min\{|x_1|, |x_2|\}$.

Note here that the norm $\|\cdot\|_{\omega,q}$ of $d^{(2)}(\omega, q)$ is a symmetric absolute normalized norm on $\mathbb{R}^2$, and the corresponding convex function is given by

$$\psi_{\omega,q}(t) = \begin{cases} ((1-t)^q + \omega t^q)^{1/q} & \text{if } 0 \leq t \leq 1/2, \\ (t^q + \omega(1-t)^q)^{1/q} & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

2. JAMES CONSTANT OF ABSOLUTE NORMALIZED NORMS ON $\mathbb{R}^2$

For a norm $\|\cdot\|$ on $\mathbb{R}^2$, we write $J(\|\cdot\|)$ for $J((\mathbb{R}^2, \|\cdot\|))$. Mitani and Saito [6] characterized the James constant of $(\mathbb{R}^2, \|\cdot\|_\psi)$ in terms of ψ .

Theorem 1 ([6]). *Let $\psi \in \Psi_2$. If ψ is symmetric with respect to $t = 1/2$, then*

$$J(\|\cdot\|_\psi) = \max_{0 \leq t \leq 1/2} \frac{2-2t}{\psi(t)} \psi\left(\frac{1}{2-2t}\right).$$

Example 2. Let $1 \leq p \leq \infty$ and $1/p + 1/p' = 1$. Then

$$(2) \quad J(\|\cdot\|_p) = \max\{2^{1/p}, 2^{1/p'}\}.$$

Indeed, we define a function f on $[0, 1/2]$ as follows:

$$\begin{aligned} f(t) &= \frac{2-2t}{\psi_p(t)} \psi_p\left(\frac{1}{2-2t}\right) \\ &= \left(\frac{1 + (1-2t)^p}{(1-t)^p + t^p}\right)^{1/p}. \end{aligned}$$

If $1 \leq p \leq 2$, then f is the maximum at $t = 0$ and

$$J(\|\cdot\|_{\psi_p}) = f(0) = 2^{1/p}.$$

If $p \geq 2$, then f is the maximum at $t = 1/2$ and

$$J(\|\cdot\|_{\psi_p}) = f(1/2) = 2^{1/p'}.$$

Thus we obtain (2).

Example 3. Let $1/2 \leq \lambda \leq 1$. We define a function φ_λ as

$$\varphi_\lambda(t) = \max\{1 - t, t, \lambda\}.$$

Then it is obvious that $\varphi_\lambda \in \Psi_2$. The corresponding absolute normalized norm $\|\cdot\|_{\varphi_\lambda}$ is

$$\|\cdot\|_{\varphi_\lambda} = \max\{\|\cdot\|_\infty, \lambda\|\cdot\|_1\}.$$

Then

$$J(\|\cdot\|_{\varphi_\lambda}) = \begin{cases} 1/\lambda & \text{if } 1/2 \leq \lambda \leq 1/\sqrt{2}, \\ 2\lambda & \text{if } 1/\sqrt{2} \leq \lambda \leq 1. \end{cases}$$

3. JAMES CONSTANT OF 2-DIMENSIONAL LORENTZ SEQUENCE SPACES

Kato and Maligranda [5] calculated $d^{(2)}(\omega, q)$ in the case where $q \geq 2$, that is, they proved that if $0 < \omega < 1$ and $q \geq 2$, then

$$J(d^{(2)}(\omega, q)) = 2 \left(\frac{1}{1 + \omega} \right)^{1/q}.$$

However, from Theorem 1 we obtain the following.

Lemma 4. For $0 < \omega < 1$ and $1 \leq q < \infty$,

$$J(d^{(2)}(\omega, q)) (= J(\|\cdot\|_{\psi_{\omega,q}})) = \max_{0 \leq t \leq 1/2} \frac{2 - 2t}{\psi_{\omega,q}(t)} \psi_{\omega,q} \left(\frac{1}{2 - 2t} \right)$$

holds.

By using this lemma, we calculate $J(d^{(2)}(\omega, q))$ in the case where $1 \leq q < 2$.

Theorem 5 ([9], cf. [6, 12]). Let $1 \leq q < 2$. (i) If $0 < \omega \leq (\sqrt{2} - 1)^{2-q}$, then

$$J(d^{(2)}(\omega, q)) = 2 \left(\frac{1}{1 + \omega} \right)^{1/q}.$$

(ii) If $(\sqrt{2} - 1)^{2-q} < \omega < 1$, then there exists a unique solution s_0 of the equation

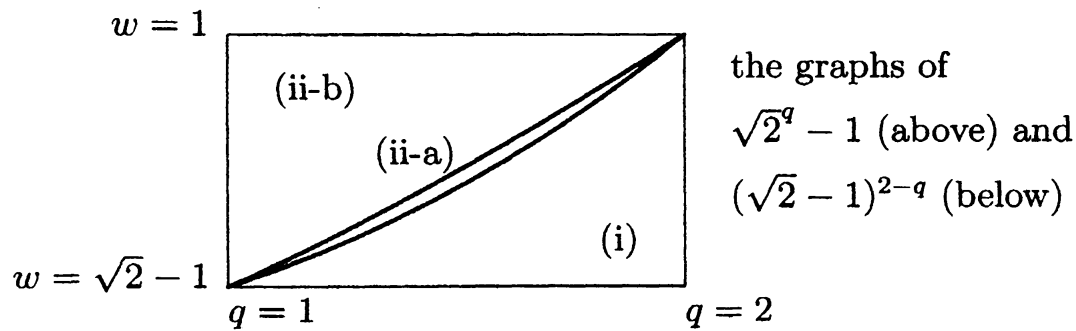
$$(1 + s_0)^{q-1} (1 - \omega s_0^{q-1}) = \omega (1 - s_0)^{q-1} (1 + \omega s_0^{q-1}), \quad 0 < s_0 < \omega^{1/(2-q)}.$$

(ii-a) If $(\sqrt{2} - 1)^{2-q} < \omega \leq \sqrt{2}^q - 1$, then

$$J(d^{(2)}(\omega, q)) = \max \left\{ \left(\frac{2(1 + s_0)^{q-1}}{1 + \omega s_0^{q-1}} \right)^{1/q}, 2 \left(\frac{1}{1 + \omega} \right)^{1/q} \right\}.$$

(ii-b) If $\sqrt{2}^q - 1 < \omega < 1$, then

$$J(d^{(2)}(\omega, q)) = \left(\frac{2(1 + s_0)^{q-1}}{1 + \omega s_0^{q-1}} \right)^{1/q}.$$



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