

Ramification and tame characters of a finite flat representation of rank two

By

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Abstract

Let K be a complete discrete valuation field of mixed characteristic $(0, p)$ with perfect residue field F and uniformizer π . In this paper, we propose an example of the main theorem of the paper [10]. Namely, we calculate the conductor $c(\mathcal{G})$ in the sense of Abbes and Saito for a finite flat group scheme $\mathcal{G}$ over $\mathcal{O}_K$ which is reducible, killed by p and of rank p^2 , and show that the I_K -module $\mathcal{G}(\bar{K})$ contains the fundamental character of level $c(\mathcal{G})$. For this purpose, we show that the Dieudonné functor of Breuil is compatible with the base extension $K(\pi^{1/p})/K$.

§ 1. Introduction

Let K be a complete discrete valuation field of mixed characteristic $(0, p)$ with perfect residue field F , $\pi = \pi_K$ be its uniformizer, G_K be its absolute Galois group and I_K be its inertia subgroup. For $j \in \mathbb{Q}_{>0}$, we define a tame character $\theta_j : I_K \rightarrow \bar{F}^\times$ to be $\theta_{j'}^{k'}$, where k'/l' is the prime-to- p -denominator part of $j \bmod \mathbb{Z}$ ([12]). In other words, we set $\theta_j(\sigma) = (\sigma(\pi^{1/l'})/\pi^{1/l'})^{k'} \bmod \mathfrak{m}_{\bar{K}}$, where $\mathfrak{m}_{\bar{K}}$ is the maximal ideal of $\mathcal{O}_{\bar{K}}$. We refer any of $\mathbb{F}_p$ -conjugates of θ_j as the fundamental character of level j .

Let $\mathcal{G}$ be a finite flat group scheme over $\mathcal{O}_K$. When $\mathcal{G}$ is killed by p and monogenic, that is to say, when the affine algebra of $\mathcal{G}$ is generated over $\mathcal{O}_K$ by one element, it is well-known that the tame characters appearing in the I_K -module $\mathcal{G}(\bar{K})$ are determined by the slopes of the Newton polygon of a defining equation of $\mathcal{G}$, as follows.

Proposition 1.1 ([12], Proposition 10). *Let $\mathcal{G}$ be as above and write the affine algebra of $\mathcal{G}$ as $\mathcal{O}_K[T]/(f(T))$ with $f(0) = 0$. Let $s_1, \dots, s_r$ be the negatives of the slopes of the Newton polygon of $f(T)$. Then the semi-simplification of the I_K -module $\mathcal{G}(\bar{K}) \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p$ is the direct sum of fundamental characters of level s_i .*

On the other hand, for an elliptic modular form f of level N prime to p , we also have a description of the tame characters of the associated mod p Galois representation

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$\bar{\rho}_f$ ([9, Theorem 2.5, Theorem 2.6], [8, Section 4.3]). This is based on Raynaud's theory of prolongations of finite flat group schemes or the integral p -adic Hodge theory. However, for an analogous study of the associated mod p Galois representation of a Hilbert modular form over a totally real number field, we encounter a local field of higher absolute ramification index. In this case, these two theories no longer work well and we need some other techniques to study the tame characters of a Galois representation.

In this paper, we show the following theorem, which suggests that the semi-simplification of a finite flat representation can be described by the ramification jumps of its finite flat model over $\mathcal{O}_K$.

Theorem 1.2. *Let $\mathcal{G}$ be a finite flat group scheme which is reducible, killed by p and of rank p^2 . Let $c(\mathcal{G})$ be its conductor in the sense of [2], [3]. Then the I_K -module $\mathcal{G}(\bar{K})$ contains the fundamental character of level $c(\mathcal{G})$.*

To prove the main theorem, firstly we show compatibility of the theory of Breuil ([5]) with the base extension from K to $K_1 = K(\pi^{1/p})$ (Theorem 3.3). Using this theorem, we can write down a defining equation of $\mathcal{G}$ over $\mathcal{O}_{K_1}$ and calculate explicitly the tubular neighborhoods and conductor of $\mathcal{G}$ as in [10, Section 5].

In fact, we can show this more generally. In [10], we generalize Proposition 1.1 to the higher dimensional case (namely, the case where $\mathcal{G}$ is not monogenic) without any restriction on the absolute ramification index of K , on the residue field F and on $\mathcal{G}$. There we show that we can, at least for the finite flat case, determine the semi-simplification of a Galois representation using the ramification theory of Abbes and Saito ([2], [3]). The main theorem of [10] is the following, whose proof is given there by totally different method from that of Theorem 1.2 in this paper.

Theorem 1.3 ([10], Theorem 1.1).

Let $\mathcal{G}$ be a finite flat group scheme over $\mathcal{O}_K$. Write $\{\mathcal{G}^j\}_{j \in \mathbb{Q}_{\geq 0}}$ for the ramification filtration of $\mathcal{G}$ in the sense of [2] and [3]. Then the graded piece $\mathcal{G}^j(\bar{K})/\mathcal{G}^{j+}(\bar{K})$ is killed by p and the I_K -module $\mathcal{G}^j(\bar{K})/\mathcal{G}^{j+}(\bar{K}) \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p$ is the direct sum of fundamental characters of level j .

§ 2. Review of the ramification theory of Abbes and Saito

Let K be a complete discrete valuation field with residue field F which may be imperfect. Set $\pi = \pi_K$ to be a uniformizer of K . The separable closure of K is denoted by $\bar{K}$ and the absolute Galois group of K by G_K . In [2] and [3], Abbes and Saito defined the ramification theory of a finite flat $\mathcal{O}_K$ -algebra of relative complete intersection. In this section, we gather the necessary definitions and briefly recall their theory.

Let A be a finite flat $\mathcal{O}_K$ -algebra and $\mathbb{A}$ be a complete Noetherian semi-local ring (with its topology defined by $\text{rad}(\mathbb{A})$) which is of formally smooth over $\mathcal{O}_K$ and whose quotient ring $\mathbb{A}/\text{rad}(\mathbb{A})$ is of finite type over F . A surjection of $\mathcal{O}_K$ -algebras $\mathbb{A} \rightarrow A$ is called an embedding if $\mathbb{A}/\text{rad}(\mathbb{A}) \rightarrow A/\text{rad}(A)$ is an isomorphism. For an embedding $(\mathbb{A} \rightarrow A)$ and $j \in \mathbb{Q}_{>0}$, the j -th tubular neighborhood of $(\mathbb{A} \rightarrow A)$ is the K -affinoid variety $X^j(\mathbb{A} \rightarrow A)$ constructed as follows. Write $j = k/l$ with k, l non-negative integers. Put $I = \text{Ker}(\mathbb{A} \rightarrow A)$ and

$$\mathcal{A}_0^{k,l} = \mathbb{A}[I^l/\pi^k]^\wedge,$$

where $\wedge$ means the π -adic completion. Then $\mathcal{A}_0^{k,l}$ is a quotient ring of the Tate algebra $\mathcal{O}_K\langle T_1, \dots, T_r \rangle$ for some r . Its generic fiber $\mathcal{A}_K^j = \mathcal{A}_0^{k,l} \otimes_{\mathcal{O}_K} K$ is independent of the choice of a representation $j = k/l$ ([3, Lemma 1.4]) and set

$$X^j(\mathbb{A} \rightarrow A) = \text{Sp}(\mathcal{A}_K^j).$$

We put $F(A) = \text{Hom}_{\mathcal{O}_K\text{-alg.}}(A, \mathcal{O}_{\bar{K}})$ and

$$F^j(A) = \varprojlim \pi_0(X^j(\mathbb{A} \rightarrow A)_{\bar{K}}).$$

Here $\pi_0(X_{\bar{K}})$ denotes the set of geometric connected components of a K -affinoid variety X and the projective limit is taken in the category of embeddings of A . Note that the projective family $\pi_0(X^j(\mathbb{A} \rightarrow A)_{\bar{K}})$ is constant ([3, Section 1.2]). These define contravariant functors F and F^j from the category of finite flat $\mathcal{O}_K$ -algebras to the category of finite G_K -sets. Moreover, there are morphisms of functors $F \rightarrow F^j$ and $F^{j'} \rightarrow F^j$ for $j' \geq j > 0$.

Suppose that A is of relative complete intersection over $\mathcal{O}_K$ and $A \otimes_{\mathcal{O}_K} K$ is étale over K . Then the natural map $F(A) \rightarrow F^j(A)$ is surjective. The family $\{F(A) \rightarrow F^j(A)\}_{j \in \mathbb{Q}_{>0}}$ is separated, exhaustive and its jumps are rational ([2, Proposition 6.4]). The conductor of A is defined to be

$$c(A) = \inf\{j \in \mathbb{Q}_{>0} \mid F(A) \rightarrow F^j(A) \text{ is an isomorphism}\}.$$

If B is the affine algebra of a finite flat group scheme $\mathcal{G}$ over $\mathcal{O}_K$ which is generically étale, then B is of relative complete intersection (for example, [5, Proposition 2.2.2]) and the theory above can all be applied to B . By the functoriality, $F^j(B)$ is endowed with a G_K -module structure ([1, Lemme 2.1.1]) and the natural map $\mathcal{G}(\bar{K}) = F(B) \rightarrow F^j(B)$ is a G_K -homomorphism. Let $\mathcal{G}^j$ denote the schematic closure ([11]) in $\mathcal{G}$ of the kernel of this homomorphism. It is called the j -th ramification filtration of $\mathcal{G}$. We refer $c(B)$ as the conductor of $\mathcal{G}$, which is denoted also by $c(\mathcal{G})$. We put

$$\mathcal{G}^{j+}(\bar{K}) = \cup_{j' > j} \mathcal{G}^{j'}(\bar{K}).$$

We write the j -th tubular neighborhood of B with respect to some embedding as $X_{\mathcal{G}}^j$ by abuse of notation.

Example 2.1. For integers $0 \leq s_1, \dots, s_r \leq e$, let $\mathcal{G} = \mathcal{G}(s_1, \dots, s_r)$ denote the Raynaud $\mathbb{F}_{p^r}$ -vector space scheme ([11]) over $\mathcal{O}_K$ defined by the r equations

$$T_1^p = \pi^{s_1} T_2, T_2^p = \pi^{s_2} T_3, \dots, T_r^p = \pi^{s_r} T_1.$$

We set

$$j_k = (ps_k + p^2 s_{k-1} + \dots + p^k s_1 + p^{k+1} s_r + p^{k+2} s_{r-1} + \dots + p^r s_{k+1}) / (p^r - 1).$$

Then we have ([10, Theorem 5.5])

$$c(\mathcal{G}) = \sup_k j_k.$$

In this case, we see that the I_K -module $\mathcal{G}(\bar{K})$ is given by the fundamental character of level $c(\mathcal{G})$. For the proof, we refer to [10], where we take an appropriate syntomic cover of the affine algebra of $\mathcal{G}$ and compare its j -th tubular neighborhood with $X_{\mathcal{G}}^j$.

§ 3. Proof of Theorem 1.2

In this section, we assume that K is as in Section 1 and write its residue field as k in place of F , in accordance with [5].

Let $\mathcal{G}$ be a finite flat group scheme over $\mathcal{O}_K$ which is reducible, killed by p and of $\mathbb{F}_p$ -rank two. Namely, we have an exact sequence

$$0 \rightarrow \mathcal{G}(e-r) \rightarrow \mathcal{G} \rightarrow \mathcal{G}(e-s) \rightarrow 0$$

for some integers $0 \leq r, s \leq e$.

To state our result, let us recall the theory of filtered ϕ_1 -modules of Breuil ([5]). In the following, we take the divided power envelope of a W -algebra only with respect to the compatibility condition with the natural divided power structure on pW .

Let e be the absolute ramification index of K , $W = W(k)$ and σ be the Frobenius of W . We fix once and for all a uniformizer π of K . Let $E(u) = u^e - pF(u)$ be the Eisenstein polynomial of π over W and set $S = S_\pi = (W[u]^{\text{PD}})^\wedge$, where the divided power envelope of $W[u]$ is taken with respect to an ideal $(E(u))$ and $\wedge$ means the π -adic completion. The ring S is endowed with a σ -semilinear map $\phi : u \mapsto u^p$, which we also call Frobenius, and the natural filtration induced by the divided power structure. We set $\phi_1 = p^{-1}\phi|_{\text{Fil}^1 S}$ and $c = \phi_1(E(u)) \in S^\times$. We define ϕ , ϕ_1 and a filtration on $S_n = S/p^n$ similarly.

In [5], the following categories of filtered ϕ_1 -modules are defined. Set $\mathcal{M}$ to be the category consisting of following data;

- an S -module M and its S -submodule $\mathrm{Fil}^1 M$ containing $\mathrm{Fil}^1 S.M$,
- a ϕ -semilinear map $\phi_1 : \mathrm{Fil}^1 M \rightarrow M$ satisfying

$$\phi_1(s_1 m) = \phi_1(s_1) \phi(m),$$

where $s_1 \in \mathrm{Fil}^1 S$, $m \in M$ and $\phi(m) = c^{-1} \phi_1(E(u)m)$.

Let $\mathcal{M}_1$ be the full subcategory of $'\mathcal{M}$ consisting of M satisfying

- the S_1 -module M is free of finite rank,
- $\phi_1(\mathrm{Fil}^1 M)$ generates M as an S -module.

and $\mathcal{M}$ be the minimal full subcategory of $'\mathcal{M}$ which contains $\mathcal{M}_1$ and stable under extension.

The category $\mathcal{M}$ is shown to be categorically anti-equivalent to the category $(p\text{-Gr}/\mathcal{O}_K)$ of finite flat group schemes over $\mathcal{O}_K$ which is killed by some p -power ([5]). Let us recall the definition of this equivalence. Let $\mathrm{Spf}(\mathcal{O}_K)_{\mathrm{syn}}$ be the category of formally syntomic p -adic formal schemes, endowed with the Grothendieck topology generated by the surjective families of formally syntomic morphisms. Write $(\mathrm{Ab}/\mathcal{O}_K)$ for the category of abelian sheaves on $\mathrm{Spf}(\mathcal{O}_K)_{\mathrm{syn}}$. The sheaves $\mathcal{O}_{n,\pi}$ and $\mathcal{J}_{n,\pi}$ are defined by the formula

$$\mathcal{O}_{n,\pi}(\mathfrak{X}) = H_{\mathrm{crys}}^0((\mathfrak{X}_n/S_n)_{\mathrm{crys}}, \mathcal{O}_{\mathfrak{X}_n/S_n})$$

and

$$\mathcal{J}_{n,\pi}(\mathfrak{X}) = H_{\mathrm{crys}}^0((\mathfrak{X}_n/S_n)_{\mathrm{crys}}, \mathcal{J}_{\mathfrak{X}_n/S_n}),$$

where $\mathfrak{X}_n = \mathfrak{X}/p^n$. We also set $\mathcal{O}_{\infty,\pi} = \varinjlim \mathcal{O}_{n,\pi}$ and $\mathcal{J}_{\infty,\pi} = \varinjlim \mathcal{J}_{n,\pi}$. We let the crystalline Frobenius map be denoted by $\phi : \mathcal{O}_{n,\pi} \rightarrow \mathcal{O}_{n,\pi}$. We can define the natural morphism $\phi_1 : \mathcal{J}_{n,\pi} \rightarrow \mathcal{O}_{n,\pi}$ which makes the following diagram commutative.

$$\begin{array}{ccc} \mathcal{J}_{n,\pi} & \xrightarrow{\phi_1} & \mathcal{O}_{n,\pi} \\ \uparrow & & \downarrow \times p \\ \mathcal{J}_{n+1,\pi} & \xrightarrow{\phi} & \mathcal{O}_{n+1,\pi} \end{array}$$

Let $\mathcal{G} \in (p\text{-Gr}/\mathcal{O}_K)$ and $M \in \mathcal{M}$. Define

$$\mathrm{Mod}_K(\mathcal{G}) = \mathrm{Hom}_{(\mathrm{Ab}/\mathcal{O}_K)}(\mathcal{G}, \mathcal{O}_{\infty,\pi})$$

and

$$\mathrm{Gr}_K(M) = \mathrm{Hom}_{'\mathcal{M}}(M, \mathcal{O}_{\infty,\pi}).$$

Then the main theorem of [5] is the following.

Theorem 3.1 ([5]). *The functor Gr_K defines an anti-equivalence of categories $\mathcal{M} \rightarrow (p\text{-Gr}/\mathcal{O}_K)$ and its quasi-inverse is Mod_K .*

Now let us return to our $\mathcal{G}$. Let $M = \mathrm{Mod}_K(\mathcal{G})$ be the filtered ϕ_1 -module of $\mathcal{G}$. Replacing K with an unramified extension, we may assume that we have an exact sequence in $\mathcal{M}$

$$0 \rightarrow M(s) \rightarrow M \rightarrow M(r) \rightarrow 0,$$

where $M(s)$ is the filtered ϕ_1 -module defined by $M(s) = S_1 e$, $\mathrm{Fil}^1 M(s) = u^s S_1 e$ and $\phi_1(u^s e) = e$. By [7, Lemma 5.2.2], we may assume that $\tilde{M} = M/\mathrm{Fil}^p S.M$ is of the following type;

- $\tilde{M} = \tilde{S}_1 e_0 \oplus \tilde{S}_1 e_1$, where $\tilde{S}_1 = k[u]/(u^{ep})$
- $\mathrm{Fil}^1 \tilde{M} = \langle u^s e_0, u^r e_1 + f e_0 \rangle$, where $f \in u^{\sup(0, r+s-e)} \tilde{S}_1$
- $\phi_1(u^s e_0) = e_0$ and $\phi_1(u^r e_1 + f e_0) = e_1$.

Put $m = v_u(f)$. Then we have the following theorem.

Theorem 3.2. *If $s, m \geq r$, then $c(\mathcal{G}) = p(e-r)/(p-1)$. Otherwise, $c(\mathcal{G})$ is equal to*

$$\begin{cases} \sup(p(e-r)/(p-1), p(e-s)/(p-1)) & \text{if } m \geq (ps-r)/(p-1), \\ p(e-r)/(p-1) + (r-m) & \text{if } m < (ps-r)/(p-1). \end{cases}$$

Moreover, the I_K -module $\mathcal{G}(\bar{K})$ contains the fundamental character of level $c(\mathcal{G})$.

To prove this theorem, we first write down a defining equation of $\mathcal{G}$. This is possible after taking a base extension from K to $K_1 = K(\pi_1)$, where $\pi_1 = \pi^{1/p}$ ([5, Proposition 3.1.2]) and using the theorem below.

Theorem 3.3. *Let $S' = S_{\pi_1}$ be the p -adic completion of the divided power envelope constructed as S starting from $E_1(v) = E(v^p) \in W[v]$ and consider a map $S \rightarrow S'$ defined by $u \mapsto v^p$. Then we have a canonical isomorphism of filtered ϕ_1 -modules*

$$\mathrm{Mod}_{K_1}(\mathcal{G} \otimes_{\mathcal{O}_K} \mathcal{O}_{K_1}) \simeq \mathrm{Mod}_K(\mathcal{G}) \otimes_S S'.$$

Then we can calculate the tubular neighborhoods of $\mathcal{G} \times_{\mathcal{O}_K} \mathcal{O}_{K_1}$ and its conductor. As for the assertion on the tame character, we know from [6, Theorem 3.4.3] that it suffices to consider $\mathcal{G} \times_{\mathcal{O}_K} \mathcal{O}_{K_1}$. We can easily check from the shape of a defining equation of $\mathcal{G}$ over $\mathcal{O}_{K_1}$ that if $m > (ps-r)/(p-1)$ and $s < r$, then the I_{K_1} -module $\mathcal{G}(\bar{K})$ splits. Hence this assertion follows from the assertion on the conductor.

We omit the calculation here as it is just the same as in the proof of [10, Theorem 5.5]. In the rest of this paper, we prove Theorem 3.3.

Lemma 3.4. *The S -module S' is free of finite rank.*

Proof. The $W[u]$ -algebra $W[v]$ is free of finite rank. We have

$$(E(u))W[v] = (E_1(v)).$$

Therefore $W[v]^{\text{PD}} = W[u]^{\text{PD}} \otimes_{W[u]} W[v]$ from [4, Proposition 3.21] and $W[u]^{\text{PD}} \rightarrow W[v]^{\text{PD}}$ is also free of finite rank. Thus

$$(W[v]^{\text{PD}})^{\wedge} = (W[u]^{\text{PD}})^{\wedge} \otimes_{W[u]^{\text{PD}}} W[v]^{\text{PD}}.$$

This concludes the proof. $\square$

Let the categories of filtered ϕ_1 -modules over S' be denoted by $'\mathcal{M}'$ and $\mathcal{M}'$. From the lemma above, we can define a filtered ϕ_1 -module structure on $M' = M \otimes_S S'$ for any $M \in '\mathcal{M}'$ by $\text{Fil}^1 M' = (\text{Fil}^1 M) \otimes_S S'$ and $\phi_{1,M'} = \phi_1 \otimes \phi$. If $M \in \mathcal{M}$, then we have $M' \in \mathcal{M}'$.

For a presheaf $\mathcal{F}$ on $\text{Spf}(\mathcal{O}_K)_{\text{syn}}$, we let $\mathcal{F}|_{\mathcal{O}_{K_1}}$ denote the restriction of $\mathcal{F}$ to $\text{Spf}(\mathcal{O}_{K_1})_{\text{syn}}$. If $\mathcal{F}$ is a sheaf on $\text{Spf}(\mathcal{O}_K)_{\text{syn}}$, then $\mathcal{F}|_{\mathcal{O}_{K_1}}$ is also a sheaf on $\text{Spf}(\mathcal{O}_{K_1})_{\text{syn}}$. By [5, Corollaire 2.3.3], we have the following exact sequences in $(\text{Ab}/\mathcal{O}_{K_1})$.

$$(1) \quad 0 \rightarrow \mathcal{O}_{r,\pi}|_{\mathcal{O}_{K_1}} \xrightarrow{\times p^s} \mathcal{O}_{r+s,\pi}|_{\mathcal{O}_{K_1}} \rightarrow \mathcal{O}_{s,\pi}|_{\mathcal{O}_{K_1}} \rightarrow 0$$

$$(2) \quad 0 \rightarrow \mathcal{J}_{r,\pi}|_{\mathcal{O}_{K_1}} \xrightarrow{\times p^s} \mathcal{J}_{r+s,\pi}|_{\mathcal{O}_{K_1}} \rightarrow \mathcal{J}_{s,\pi}|_{\mathcal{O}_{K_1}} \rightarrow 0$$

Consider an $\mathcal{O}_{K_1}$ -algebra

$$\mathfrak{A}' = \mathcal{O}_{K_1} \langle X'_1, \dots, X'_r \rangle / (f_1, \dots, f_s),$$

where $\mathcal{O}_{K_1} \langle X'_1, \dots, X'_r \rangle$ is the π -adic completion $\mathcal{O}_{K_1}[X'_1, \dots, X'_r]^{\wedge}$ and $f_1, \dots, f_s$ is a transversally regular sequence in that ring. Then $\text{Spf}(\mathfrak{A}') \in \text{Spf}(\mathcal{O}_{K_1})_{\text{syn}}$. Put

$$\mathfrak{A}'_i = \mathcal{O}_{K_1} \langle X_0'^{p^{-i}}, \dots, X_r'^{p^{-i}} \rangle / (X_0' - \pi_1, f_1, \dots, f_s)$$

and $\mathfrak{A}'_{\infty} = \varinjlim \mathfrak{A}'_i$. Note that the formal scheme $\text{Spf}(\mathfrak{A}'_i)$ is a covering of $\text{Spf}(\mathfrak{A}')$ in $\text{Spf}(\mathcal{O}_{K_1})_{\text{syn}}$. The W -algebra $\mathfrak{A}'_i$ is isomorphic to

$$\begin{aligned} & \mathcal{O}_K[T] / (T^p - \pi) \langle X_0'^{p^{-i}}, \dots, X_r'^{p^{-i}} \rangle / (X_0' - T, f_1, \dots, f_s) \\ &= W[u, T] / (E(u), T^p - u) \langle X_0'^{p^{-i}}, \dots, X_r'^{p^{-i}} \rangle / (X_0' - T, f_1, \dots, f_s) \\ &= W \langle X_0'^{p^{-i}}, \dots, X_r'^{p^{-i}} \rangle / (E(X_0'^p), f_1, \dots, f_s). \end{aligned}$$

We also set

$$\mathfrak{A}'_{\infty} = \mathfrak{A}'_{\infty} / p = k[X_0'^{p^{-\infty}}, \dots, X_r'^{p^{-\infty}}] / (X_0'^{ep}, \bar{f}_1, \dots, \bar{f}_s),$$

where $\bar{f}_i$ is the image of f_i . Put $\mathcal{O}_{r,\pi}(\mathfrak{A}'_{\infty}) = \varinjlim \mathcal{O}_{r,\pi}(\mathfrak{A}'_i)$ and $\mathcal{J}_{r,\pi}(\mathfrak{A}'_{\infty}) = \varinjlim \mathcal{J}_{r,\pi}(\mathfrak{A}'_i)$.

Lemma 3.5. *There exists a canonical isomorphism*

$$\mathcal{O}_{n,\pi}(\mathfrak{A}'_\infty) = H_{\text{crys}}^0(\mathfrak{A}'_\infty/p^n/S_n) \rightarrow (W_n(A'_\infty) \otimes_{W_n, \sigma^n} W_n[u])^{\text{PD}}.$$

Here the divided power envelope is taken with respect to the kernel of a surjection

$$\begin{aligned} W_n(A'_\infty) \otimes_{W_n, \sigma^n} W_n[u] &\rightarrow \mathfrak{A}'_\infty/p^n \\ (x_0, \dots, x_{n-1}) \otimes 1 &\mapsto \sum_{k=0}^{n-1} p^k \hat{x}_k^{p^{n-k}} \\ 1 \otimes u &\mapsto X'_0{}^p \end{aligned}$$

where $\hat{x}_k$ denotes a lifting of x_k in $\mathfrak{A}'_\infty/p^n$.

Proof. We repeat exactly the same argument as in [5, Lemme 2.3.2]. Indeed, this surjection induces a PD-thickening

$$(W_n(A'_\infty) \otimes_{W_n, \sigma^n} W_n[u])^{\text{PD}} \rightarrow \mathfrak{A}'_\infty/p^n$$

of $\mathfrak{A}'_\infty/p^n$ over S_n and thus we have the natural projection

$$H_{\text{crys}}^0(\mathfrak{A}'_\infty/p^n/S_n) \rightarrow (W_n(A'_\infty) \otimes_{W_n, \sigma^n} W_n[u])^{\text{PD}}.$$

Its inverse map is defined as follows. For any affine PD-thickening $U \rightarrow T$ of $\mathfrak{A}'_\infty/p^n$ over S_n , we have a map

$$\begin{aligned} (W_n(A'_\infty) \otimes_{W_n, \sigma^n} W_n[u])^{\text{PD}} &\rightarrow \Gamma(U, \mathcal{O}_U) \\ (x_0, \dots, x_{n-1}) \otimes 1 &\mapsto \sum_{k=0}^{n-1} p^k \hat{t}_k^{p^{n-k}} \\ 1 \otimes u &\mapsto u, \end{aligned}$$

where $\hat{t}_k$ is a lifting of x_k in $\Gamma(T, \mathcal{O}_T)$. This is a well-defined ring homomorphism, patches in the non-affine case and induces the inverse map of the natural projection. $\square$

Let us define a morphism $\Psi_M : \text{Gr}_K(M)|_{\mathcal{O}_{K_1}} \rightarrow \text{Gr}_{K_1}(M')$ of $(\text{Ab}/\mathcal{O}_{K_1})$ as follows. For any $\mathfrak{X}' \in \text{Spf}(\mathcal{O}_{K_1})_{\text{syn}}$, we want to set

$$\Psi_{M, \mathfrak{X}'} : \text{Hom}_{\mathcal{M}'}(M, \mathcal{O}_{n,\pi}(\mathfrak{X}')) \rightarrow \text{Hom}_{\mathcal{M}'}(M \otimes_S S', \mathcal{O}_{n,\pi_1}(\mathfrak{X}'))$$

by $f \mapsto (m \otimes s' \mapsto s' \cdot \text{pr}_{\mathfrak{X}'}^*(f(m)))$, where

$$\text{pr}_{\mathfrak{X}'}^* : \mathcal{O}_{n,\pi}(\mathfrak{X}') = H_{\text{crys}}^0(\mathfrak{X}'_n/S_n) \rightarrow H_{\text{crys}}^0(\mathfrak{X}'_n/S'_n) = \mathcal{O}_{n,\pi_1}(\mathfrak{X}')$$

is the natural pull-back. The map $\text{pr}_{\mathfrak{X}}^*$, respects the filtration. To show the compatibility with ϕ_1 , note that we have a diagram

$$\begin{array}{ccccccc} \mathcal{J}_{n+1,\pi}|_{\mathcal{O}_{K_1}} & \longrightarrow & \mathcal{J}_{n,\pi}|_{\mathcal{O}_{K_1}} & \xrightarrow{\phi_1} & \mathcal{O}_{n,\pi}|_{\mathcal{O}_{K_1}} & \xrightarrow{\times p} & \mathcal{O}_{n+1,\pi}|_{\mathcal{O}_{K_1}} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{J}_{n+1,\pi_1} & \longrightarrow & \mathcal{J}_{n,\pi_1} & \xrightarrow{\phi_1} & \mathcal{O}_{n,\pi_1} & \xrightarrow{\times p} & \mathcal{O}_{n+1,\pi_1}, \end{array}$$

where the vertical arrows are the pull-backs and the left and right squares are commutative. The composites of the horizontal maps are ϕ . From the exact sequences (1) and (2), we see that the middle square is also commutative. In other words, the map $\text{pr}_{\mathfrak{X}}^*$ is compatible with ϕ_1 . Therefore, we get a morphism of $(\text{Ab}/\mathcal{O}_{K_1})$

$$\Psi_M : \text{Gr}_K(M)|_{\mathcal{O}_{K_1}} \rightarrow \text{Gr}_{K_1}(M').$$

Theorem 3.6. *The canonical map Ψ_M is an isomorphism.*

Proof.

As the functor Gr_K is exact, by devissage we may assume that $pM = 0$. The sheaves of both sides come from finite flat group schemes $\text{Gr}_K(M) \times_{\mathcal{O}_K} \mathcal{O}_{K_1}$ and $\text{Gr}_{K_1}(M')$. Thus the bijectivity can be checked after taking the functor Mod_{K_1} . In other words, it suffices to show that

$$\Phi_M : M' = M \otimes_S S' \rightarrow \text{Hom}_{(\text{Ab}/\mathcal{O}_{K_1})}(\text{Hom}_{\mathcal{M}}(M, \mathcal{O}_{1,\pi}|_{\mathcal{O}_{K_1}}), \mathcal{O}_{1,\pi_1}),$$

defined by $m \otimes s' \mapsto (f \mapsto s' \cdot \text{pr}^*(f(m)))$ is an isomorphism of $\mathcal{M}'$. Here pr^* denotes the pull-back map $\mathcal{O}_{1,\pi}|_{\mathcal{O}_{K_1}} \rightarrow \mathcal{O}_{1,\pi_1}$.

We have $\text{rank}_{S'_1}(M \otimes_S S') = \text{rank}_{S_1}(M)$ and

$$\begin{aligned} & \text{rank}_{S'_1}(\text{Hom}_{(\text{Ab}/\mathcal{O}_{K_1})}(\text{Hom}_{\mathcal{M}}(M, \mathcal{O}_{1,\pi}|_{\mathcal{O}_{K_1}}), \mathcal{O}_{1,\pi_1})) \\ &= \text{rank}_{S'_1}(\text{Mod}_{K_1}(\text{Gr}_K(M) \times_{\mathcal{O}_K} \mathcal{O}_{K_1})) = \text{rank}_{S_1}(M). \end{aligned}$$

By [5, Lemme 3.3.2], it suffices to show $\text{Ker}(\Phi_M) \subseteq \text{Fil}^p S'_1 M'$.

Take an adapted basis $\{e_1, \dots, e_d\}$ of M . Let $m = \sum_{i=1}^d s'_i e_i$ be an element of $\text{Ker}(\Phi_M)$. Consider the affine algebra R_M of $\text{Gr}_K(M)$ and the element $f \in \text{Hom}_{S'_1}^{\mathcal{M}}(M, \mathcal{O}_{1,\pi}(R_M)) \simeq \text{Gr}_K(M)(R_M)$ corresponding to id_{R_M} . Then, from the proof of [5, Proposition 3.1.5], we have

$$f(e_i) \equiv \bar{X}_{i,0} + u\bar{X}_{i,1} + \dots + u^{p-1}\bar{X}_{i,p-1} \pmod{\mathcal{J}_{1,\pi}^{[p]}(R_M)},$$

where $X_{i,0}, \dots, X_{i,p-1}$ are the generators of R_M as in [5, p.507] and $\bar{X}_{i,k}$ is the image of $X_{i,k}$ in R_M/p . Here we regard $\bar{X}_{i,k}$ as an element of $\mathcal{O}_{1,\pi}(R_M)$ by the natural map $(R_M/p) \otimes_{k,\sigma} k[u] \rightarrow \mathcal{O}_{1,\pi}(R_M)$. Let us write f_1 for the image of f by the natural map

$$\text{Hom}_{\mathcal{M}}(M, \mathcal{O}_{1,\pi}(R_M)) \rightarrow \text{Hom}_{\mathcal{M}}(M, \mathcal{O}_{1,\pi}(R'_M)),$$

where $R'_M = R_M \otimes_{\mathcal{O}_K} \mathcal{O}_{K_1}$. As $m \in \text{Ker}(\Phi_M)$, we have

$$\sum s'_i \text{pr}_{R'_M}^*(f_1(e_i)) = 0.$$

Let $\bar{X}'_{i,k}$ be the image of $\bar{X}_{i,k}$ by the natural map $(R'_M/p) \otimes_{k,\sigma} k[v] \rightarrow \mathcal{O}_{1,\pi_1}(R'_M)$. Now we claim that $\text{pr}_{R'_M}^*(\bar{X}_{i,k}) = \bar{X}'_{i,k}$. It is sufficient to show this equality on an appropriate syntomic cover of R'_M . Thus we may consider $\text{pr}_{R'_{M,\infty}}^* : \mathcal{O}_{1,\pi}(R'_{M,\infty}) \rightarrow \mathcal{O}_{1,\pi_1}(R'_{M,\infty})$, where $R'_{M,\infty}$ is the ring constructed from $\mathfrak{A}' = R'_M$ as in the proof of Lemma 3.5. Then the composite

$$((R'_{M,\infty}/p) \otimes_{k,\sigma} k[u])^{\text{PD}} \xrightarrow{\text{pr}^*} H_{\text{crys}}^0((R'_{M,\infty}/p)/S'_1) \xrightarrow{\text{can.}} ((R'_{M,\infty}/p) \otimes_{k,\sigma} k[v])^{\text{PD}}$$

maps $1 \otimes u$ to $1 \otimes v^p$ and $r \otimes 1$ to $\hat{r}^p \otimes 1$, where $\hat{r}$ is a lifting of r by the canonical surjection $((R'_{M,\infty}/p) \otimes_{k,\phi} k[v])^{\text{PD}} \rightarrow R'_{M,\infty}/p$. We may take $\hat{r}$ to be $r^{1/p} \otimes 1$. Thus the claim follows.

Now we have

$$\sum_{i=1}^d s'_i (\bar{X}'_{i,0} + v^p \bar{X}'_{i,1} + \cdots + v^{p(p-1)} \bar{X}'_{i,p-1}) = 0$$

in $\mathcal{O}_{1,\pi_1}(R'_M)/\mathcal{J}_{1,\pi_1}^{[p]}(R'_M)$. This equation also holds in

$$\mathcal{O}_{1,\pi_1}(R'_{M,\infty})/\mathcal{J}_{1,\pi_1}^{[p]}(R'_{M,\infty})$$

and its subring

$$(R'_{M,\infty}/p)[v]/(v^p - X'_0) = (R'_{M,\infty}/p)[v]/(v^p - \pi_1)$$

(see [5, Lemme 2.3.2]). As $R'_{M,\infty}$ is the direct limit of syntomic covers of R'_M , R'_M/p is a subring of $R'_{M,\infty}/p$. Thus the equation

$$\sum_{i=1}^d s'_i (\bar{X}'_{i,0} + v^p \bar{X}'_{i,1} + \cdots + v^{p(p-1)} \bar{X}'_{i,p-1}) = 0$$

holds in $(R'_M/p)[v]/(v^p - \pi_1)$. Let us write $\bar{s}'_i$ for $s'_i \bmod v \in k$. Taking mod v , we have

$$\sum_{i=1}^d \bar{s}'_i \bar{X}'_{i,0} = 0$$

in $(R'_M/p)[v]/(v, v^p - \pi_1) = R'_M/\pi_1 = R_M/\pi$. From the proof of [5, Proposition 3.1.1], we know that $X_{1,0}, \dots, X_{d,0}$ are linearly independent over k in R_M/π . Thus $\bar{s}'_i = 0$ and $s'_i \in vS'_1 + \text{Fil}^p S'_1$ for all i . Take $s'^{(1)}_i \in S'_1$ satisfying $s'_i - vs'^{(1)}_i \in \text{Fil}^p S'_1$. Then we have

$$v \sum_{i=1}^d s'^{(1)}_i (\bar{X}'_{i,0} + v^p \bar{X}'_{i,1} + \cdots + v^{p(p-1)} \bar{X}'_{i,p-1}) = 0$$

in $(R'_M/p)[v]/(v^p - \pi_1)$. However,

$$R'_M/p \simeq (\mathcal{O}_{K_1}/p)^{\oplus N} \simeq (k[T]/(T^{ep}))^{\oplus N}$$

for some N and

$$(k[T]/(T^{ep}))[v]/(v^p - T) \simeq k[v]/(v^{ep^2}).$$

Thus $(R'_M/p)[v]/(v^p - \pi_1)$ is finite flat over $k[v]/(v^{ep^2})$, and we have

$$\sum_{i=1}^d s'_i{}^{(1)} (\bar{X}'_{i,0} + v^p \bar{X}'_{i,1} + \cdots + v^{p(p-1)} \bar{X}'_{i,p-1}) \in v^{ep^2-1} (R'_M/p)[v]/(v^p - \pi_1).$$

Taking mod v and repeating this procedure show $s'_i \in v^{ep^2} S'_1 + \text{Fil}^p S'_1 = \text{Fil}^p S'_1$. In other words, $m \in \text{Fil}^p S'_1 M'$. This concludes the proof. $\square$

Remark 3.7. In general, let L be a totally ramified extension over K of degree e' whose uniformizer is denoted by π_L . When we define $S_L = S_{\pi_L}$ as above, there exists a morphism $S \rightarrow S_L$ respecting the filtration and ϕ_1 if and only if $\pi_L^{e'} = \pi_{\zeta_{p-1}}^i$ for some i .

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