

Infinite-variate extensions of Krawtchouk polynomials and zonal spherical functions over a local field

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Abstract

The multivariate Krawtchouk polynomials are orthogonal polynomials for the multinomial distribution, first defined by Griffiths in 1971. We construct infinite-variate extensions of them as complete orthogonal systems of specific weighted l^2 -spaces. We also give realizations of our infinite-variate extensions as zonal spherical functions on groups over a non-Archimedean local field. Some typical properties of Krawtchouk polynomials like duality, orthogonality and completeness are thus shed light from the point of view of zonal spherical functions.

Key Words and Phrases. Multivariate Krawtchouk polynomials, Complete orthogonal systems of weighted l^2 -spaces, Zonal spherical functions, Locally compact Abelian groups with compact group actions, Non-Archimedean local field.

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0 Introduction

The (1-variate) Krawtchouk polynomials $K_n(x; p, N)$ are classical orthogonal polynomials over finite points $x \in \{0, 1, \dots, N\}$ of degree $n \leq N$, associated with the binomial distribution for probabilistic parameters $0 < p < 1$. Griffiths [4] first gave the multivariate extensions of the Krawtchouk polynomials, which are orthogonal for the multinomial distribution.

Orthogonal polynomials often arise from the (zonal) spherical functions on specific groups. Dunkl [2] exhibited the (1-variate) Krawtchouk polynomials as spherical functions on the wreath products of symmetric groups. As a

generalization of this fact, Mizukawa [11] presented multivariate Krawtchouk polynomials as zonal spherical functions on the complex reflection groups. The author in [8] gave another realizations of Krawtchouk polynomials as follows. Let $\mathbb{F}$ be a finite field. The wreath product $G = (\mathbb{F}^\times)^N \rtimes \mathfrak{S}_N$ acts on the additive group $A = \mathbb{F}^N$. Then the zonal spherical functions of the Gelfand pair $(A \rtimes G, G)$ are given in terms of the Krawtchouk polynomials. After that, the author noticed that when we replace $\mathbb{F}$ in the above setting with a non-Archimedean local field F , then we find the zonal spherical functions in terms of a kind of “infinite-variate extensions” of the Krawtchouk polynomials. This observation is the motivation for the present paper.

The aim of this paper is two-fold. First, we construct infinite-variate extensions of Griffiths’s Krawtchouk polynomials by the generating functions. We establish their typical properties, especially a duality between variables and degree indices, orthogonality and completeness in specific weighted l^2 -spaces. Second, we give realizations of the infinite-variate extensions of Krawtchouk polynomials with special parameters as zonal spherical functions on groups over a non-Archimedean local field. Since zonal spherical functions in suitable settings are shown to have a duality, orthogonality and completeness, the realizations will shed another light on the properties established in the first part.

Let us explain our results in more detail. For an integer $\ell \geq 1$, the ℓ -variate Krawtchouk polynomials $K_n^{(\ell)}(x; U, N)$ have the index set $[\ell] = \{0, 1, \dots, \ell - 1\}$ for the variables $x = (x_i)_{i \in [\ell]}$ (and for n, U). To extend the index set $[\ell]$ to infinite ones, we have two choices, the non-negative integers $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ or all integers $\mathbb{Z}$. According to these types of indices, we introduce two kinds of infinite-variate extensions of Krawtchouk polynomials. We name the first ones $K_n^{(+)}(x; U, N)$ for $x = (x_i)_{i \in \mathbb{N}_0}$ the “half infinite extensions”, and the second ones $K_n^{(\infty)}(x; U, N)$ for $x = (x_i)_{i \in \mathbb{Z}}$ the “full infinite extensions” of Krawtchouk polynomials respectively. We remark that the half infinite extensions seem to have appeared in Griffiths [5] for the use of probabilistic applications, without mentioning their detailed properties like completeness. According to the inclusions of indices $[\ell] \subset \mathbb{N}_0 \subset \mathbb{Z}$, the Krawtchouk polynomials (or their extensions) of the smaller size are considered as specializations of the bigger ones. Conversely, the bigger ones are gotten as the limits from the smaller ones (the equalities (35), (37) and Lemmas 1.3.1, 1.3.2).

The ℓ -variate Krawtchouk polynomials satisfy their orthogonality relation (7), which has been proved by some authors by their own approaches (see [13], [6] and [7]). As an infinite-variate analogue of it, in the first place, we prove the orthogonality relation (28) of the full infinite extensions $K_n^{(\infty)}(x; U, N)$

with completeness in a specific weighted l^2 -space. Secondly, we prove the orthogonality relation (46) and completeness of the half infinite extensions $K_n^{(+)}(x; U, N)$ by use of those of $K_n^{(\infty)}(x; U, N)$. Finally, we prove the orthogonality relation (7) of finite-variate ones based on those of $K_n^{(+)}(x; U, N)$ (Theorems 1.2.5, 1.3.3 and the last part of Section 1.3).

The ℓ -variate Krawtchouk polynomials possess a duality $K_n^{(\ell)}(x; U, N) = K_x^{(\ell)}(n; {}^tU, N)$, where tU denotes the transposed matrix of U . This typical property also holds for the full infinite extensions $K_n^{(\infty)}(x; U, N)$. On the other hand, the half infinite extensions may not possess such a duality, because the domain of variables x and the set of degree indices n are different. This may be explained in the latter part of the paper, where we realize these Krawtchouk polynomials with special parameters as zonal spherical functions. The difference comes from the parameters of open orbits of two groups: one for the Abelian group and the other its dual.

We carry out the realizations in a unified way without any restrictions of index type, either finite or infinite. We assume a compact group G acts on a locally compact Abelian group A as topological group automorphisms. Then the pair $(A \rtimes G, G)$ is a Gelfand pair. For each open G -orbit $\mathcal{P}$ in the dual group $\hat{A}$, we can construct its zonal spherical function $\omega_{\mathcal{P}} \in L^2(A)^G$ as a G -invariant L^2 -function on A . They are mutually orthogonal by the L^2 -inner product on A (the relation (97)). Furthermore under an appropriate condition, the system of zonal spherical functions $\{\omega_{\mathcal{P}} \mid \mathcal{P} \subset \hat{A}, \text{ open orbit}\}$ has completeness in $L^2(A)^G$ (Proposition 2.1.7). These properties correspond to orthogonality and completeness of Krawtchouk polynomials described above.

In our specific realizations, we use a non-Archimedean local field F , its ring of integers $\mathfrak{o}$ and its maximal ideal $\mathfrak{p}$. The group $G = (\mathfrak{o}^\times)^N \rtimes \mathfrak{S}_N$ acts on $A = (\mathfrak{o}/\mathfrak{p}^\ell)^N$, $A = \mathfrak{o}^N$ or $A = F^N$. According to these actions, the zonal spherical functions are respectively given in terms of the ℓ -variate Krawtchouk polynomials, the half infinite extensions or the full finite extensions of Krawtchouk polynomials (Theorems 2.3.3, 2.4.2 and 2.5.2). Under an appropriate identification of A and $\hat{A}$, the duality of Krawtchouk polynomials is seen in the first or the third examples. In the second example $A = \mathfrak{o}^N$, no such identifications exist. Correspondingly, the half infinite extensions $K_n^{(+)}(x; U, N)$ do not have dualities.

This paper is organized as follows. Chapter 1 consisting of four sections is devoted to formal constructions of the infinite-variate extensions of Krawtchouk polynomials. In Section 1.1, we summarize 1-variate and Griffith's ℓ -variate Krawtchouk polynomials as preliminaries. In Section 1.2 and 1.3, we study properties of the full infinite extensions $K_n^{(\infty)}(x; U, N)$ and

the half infinite extensions $K_n^{(+)}(x; U, N)$ respectively. In Section 1.4, we introduce expressions of multivariate Krawtchouk polynomials as products of 1-variate ones. In Chapter 2 consisting of five sections, we give realizations of multivariate Krawtchouk polynomials. Section 2.1 organizes the setting for zonal spherical functions, and Section 2.2 gives notations for a non-Archimedean local field. In the following sections, we deal with specific examples, that is, $A = (\mathfrak{o}/\mathfrak{p}^\ell)^N$ in Section 2.3, $A = \mathfrak{o}^N$ in Section 2.4 and $A = F^N$ in Section 2.5, with the actions of $G = (\mathfrak{o}^\times)^N \rtimes \mathfrak{S}_N$ in all examples.

1 Infinite-variate extensions of Krawtchouk polynomials

1.1 Finite-variate Krawtchouk polynomials

We fix positive integers ℓ and N . Here as preliminaries, we introduce the ℓ -variate Krawtchouk polynomials first defined by Griffiths [4]¹. We denote by $[\ell] = \{0, 1, \dots, \ell - 1\}$ the index set for ℓ -variables used as $(x_i)_{i \in [\ell]} = (x_0, x_1, \dots, x_{\ell-1})$. We define the finite point set

$$\bar{X}(\ell, N) = \{x = (x_i)_{i \in [\ell]} \mid x_i \in \mathbb{N}_0, |x| \leq N\}, \quad (1)$$

where $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ denotes the set of non-negative integers, and $|x| = \sum_{i \in [\ell]} x_i$. The multinomial coefficient for $x \in \bar{X}(\ell, N)$ is defined by

$$\binom{N}{x} = \frac{N!}{(N - |x|)! \prod_i x_i!}. \quad (2)$$

The ℓ -variate Krawtchouk polynomials $K_n^{(\ell)}(x; U, N)$ are functions on $\bar{X}(\ell, N)$, with degree indices $n = (n_j)_{j \in [\ell]} \in \bar{X}(\ell, N)$ and a parameter matrix $U = (u_{i,j})_{i,j \in [\ell]}$ over $\mathbb{C}$. The definition is given by the following generating function: for given $x \in \bar{X}(\ell, N)$ and indeterminates T and $t = (t_j)_{j \in [\ell]}$,

$$\begin{aligned} \Phi^{(\ell)}(T, t; x, U, N) &:= \left(T + \sum_{j \in [\ell]} t_j \right)^{N - |x|} \prod_{i \in [\ell]} \left(T + \sum_{j \in [\ell]} u_{i,j} t_j \right)^{x_i} \\ &= \sum_{n \in \bar{X}(\ell, N)} \binom{N}{n} K_n^{(\ell)}(x; U, N) T^{N - |n|} \prod_{j \in [\ell]} t_j^{n_j}. \end{aligned} \quad (3)$$

¹We remark that Griffiths uses one more variable $x_\ell = N - |x|$ and regard $K_n^{(\ell)}(x; U, N)$ as $(\ell + 1)$ -variate. Our definition here is to emphasize the duality (4).

An expression by the Gelfand hypergeometric function is given in [12], which indicates that $K_n^{(\ell)}(x; U, N)$ are polynomials in x of the total degree $|n|$. We do not use such expression except for the case of $\ell = 1$ in (9).

They have a duality relation

$$K_n^{(\ell)}(x; U, N) = K_x^{(\ell)}(n; {}^tU, N) \quad (x, n \in \bar{X}(\ell, N)), \quad (4)$$

where tU denotes the transposed matrix of U . It follows from the symmetry of $s = (s_i)_{i \in [\ell]}$ and $t = (t_j)_{j \in [\ell]}$ in $\left(1 + \sum_{i \in [\ell]} s_i + \sum_{j \in [\ell]} t_j + \sum_{i, j \in [\ell]} u_{i,j} s_i t_j\right)^N$, the coefficient of which on $\prod_{i, j \in [\ell]} s_i^{x_i} t_j^{n_j}$ is $\binom{N}{x} \binom{N}{n} K_n^{(\ell)}(x; U, N)$.

The orthogonality relation of the ℓ -variate Krawtchouk polynomials is given as follows. We take a set of parameters $p = (p_i)_{i \in [\ell]} \in (\mathbb{R}_{>0})^\ell$ such that $|p| < 1$, and consider a weighted inner product $(u, v)_p = \sum_{i=0}^\ell u_i \bar{v}_i p_i$, for $u = (u_i)_{i=0}^\ell, v = (v_i)_{i=0}^\ell \in \mathbb{C}^{\ell+1}$ and $p_\ell = 1 - |p|$. We set $u^{(j)} = (u_{i,j})_{i=0}^\ell \in \mathbb{C}^{\ell+1}$, the j -th column vector of $U = (u_{i,j})_{i,j \in [\ell]}$ attached to one entry $u_{\ell,j} = 1$, together with $u^{(-\infty)} = (1, 1, \dots, 1) \in \mathbb{C}^{\ell+1}$, and assume the orthogonality condition

$$(u^{(j)}, u^{(k)})_p = 0 \quad \text{for } j, k \in \{-\infty, 0, 1, \dots, \ell-1\}, j \neq k. \quad (5)$$

A weight function m_p on $\bar{X}(\ell, N)$ is defined by

$$m_p(x) = \binom{N}{x} (1 - |p|)^{N-|x|} \prod_{i \in [\ell]} p_i^{x_i} \quad (x \in \bar{X}(\ell, N)), \quad (6)$$

which represents the probabilistic distribution of N balls dropped into ℓ boxes (assuming that some balls may not go into any boxes). Let $V_p(\ell, N)$ denote the space of complex valued functions on $\bar{X}(\ell, N)$ equipped with the l^2 inner product weighted by m_p . Then the system of ℓ -variate Krawtchouk polynomials $\{K_n^{(\ell)}(\cdot; U, N)\}_{n \in \bar{X}(\ell, N)}$ forms an orthogonal basis of $V_p(\ell, N)$. More specifically, the following orthogonality relation holds: for $n, m \in \bar{X}(\ell, N)$,

$$\sum_{x \in \bar{X}(\ell, N)} K_n^{(\ell)}(x; U, N) \overline{K_m^{(\ell)}(x; U, N)} m_p(x) = \delta_{nm} \binom{N}{n}^{-1} \prod_{j \in [\ell]} (u^{(j)}, u^{(j)})_p^{n_j}, \quad (7)$$

where δ denotes the Kronecker's delta. Some authors proved this relation by their own approaches (See [13], [6] and [7]). We also prove (7) in the last part

of Section 1.3 by restricting the orthogonality relation of the infinite-variate extensions of Krawtchouk polynomials.

We introduce the (1-variate) Krawtchouk polynomials $K_n(x; p; N)$ for $x, n \in \bar{X}(1, N) = \{0, 1, \dots, N\}$ and $0 < p < 1$, as the simplest case of $\ell = 1$ and $U = (1 - p^{-1})$ of the above definition:

$$K_n(x; p, N) = K_n^{(1)}(x; (1 - p^{-1}), N). \quad (8)$$

They are expressed by the Gaussian hypergeometric function ${}_2F_1$ [10]:

$$K_n(x; p, N) = {}_2F_1 \left(\begin{matrix} -n, & -x \\ & -N \end{matrix} ; \frac{1}{p} \right) = \sum_{k=0}^n \frac{(-n)_k (-x)_k}{(-N)_k k! p^k}, \quad (9)$$

where $(a)_k$ denotes the Pochhammer symbol defined by

$$(a)_0 = 1, \quad (a)_k = \prod_{i=0}^{k-1} (a + i) \quad (k \geq 1). \quad (10)$$

From this expression, we know $K_n(x; p, N)$ is a polynomial in x of degree n .

We introduce two recurrence formulae [10]:

$$K_n(x; p, N) = \frac{N - n}{N} K_n(x - 1; p, N - 1) - \frac{(1 - p)n}{pN} K_{n-1}(x - 1; p, N - 1), \quad (11)$$

$$K_n(x; p, N) - K_n(x - 1; p, N) = -\frac{n}{pN} K_{n-1}(x - 1; p, N - 1). \quad (12)$$

See [8] for a group theoretic proof of them. Conversely, the recurrence formula (12) characterizes Krawtchouk polynomials as follows:

Proposition 1.1.1. *Let p be a non-zero constant. We assume a set of functions $\{\Phi_r(s; N) \mid N \geq 1, 0 \leq s, r \leq N\}$ satisfies the following recurrence formula:*

$$\Phi_r(s; N) - \Phi_r(s - 1; N) = -\frac{r}{pN} \Phi_{r-1}(s - 1; N - 1). \quad (13)$$

Then,

$$\Phi_r(s; N) = \sum_{k=0}^r \frac{(-s)_k (-r)_k}{(-N)_k k! p^k} \Phi_{r-k}(0; N - k) \quad (14)$$

holds. Moreover if the initial values $\Phi_r(0; N)$ have the form

$$\Phi_r(0; N) = \frac{(-M)_r}{(-N)_r} \varphi(N - r), \quad (15)$$

where M is a non-negative integer and $\varphi(N-r)$ is any function relying only on $N-r$, then we have

$$\Phi_r(s; N) = \Phi_r(0; N) K_r(s; p, M). \quad (16)$$

Proof. The equality (14) is easily verified from (13) by induction. Putting (15) into (14) and by a straightforward calculation, we see (16) from (9). $\square$

1.2 Infinite-variate extensions with indices on $\mathbb{Z}$

Here we construct a class of infinite-variate extensions of the Krawtchouk polynomials, which we call “full infinite-variate extensions”. We will extend the index set $[\ell] = \{0, 1, \dots, \ell-1\}$ of the ℓ -variate Krawtchouk polynomials to all integers $\mathbb{Z}$: let

$$X(N) = \{x = (x_i)_{i \in \mathbb{Z}} \mid x_i \in \mathbb{N}_0, |x| = N\}. \quad (17)$$

We remark that each $x \in X(N)$ has only finitely many indices $i \in \mathbb{Z}$ such that $x_i \neq 0$. We take both variables $x = (x_i)_{i \in \mathbb{Z}}$ and degree indices $n = (n_j)_{j \in \mathbb{Z}}$ from $X(N)$, and a parameter set $U = (u_{i,j})_{i,j \in \mathbb{Z}}$ in the form of an infinite matrix over $\mathbb{C}$. We define the full infinite-variate extensions $K_n^{(\infty)}(x; U, N)$ of Krawtchouk polynomials by the following generating function: For given $x \in X(N)$ and indeterminates $t = (t_j)_{j \in \mathbb{Z}}$,

$$\Phi(t; x, U, N) := \prod_{i \in \mathbb{Z}} \left(\sum_{j \in \mathbb{Z}} u_{i,j} t_j \right)^{x_i} = \sum_{n \in X(N)} \binom{N}{n} K_n^{(\infty)}(x; U, N) \prod_{j \in \mathbb{Z}} t_j^{n_j}, \quad (18)$$

where $\binom{N}{n}$ is defined by the same formula as (2). To be more precise, for given $n \in X(N)$, we should regard $\Phi(t; x, U, N)$ as a polynomial in finite indeterminates $(t_j)_{n_j \neq 0}$ by putting $t_j = 0$ when $n_j = 0$, and define $K_n^{(\infty)}(x; U, N)$ as the coefficients of its expansion.

We can also define $K_n^{(\infty)}(x; U, N)$ by the following generating function with indeterminates $s = (s_i)_{i \in \mathbb{Z}}$ and $t = (t_j)_{j \in \mathbb{Z}}$:

$$\left(\sum_{i,j \in \mathbb{Z}} u_{i,j} s_i t_j \right)^N = \sum_{x,n \in X(N)} \binom{N}{x} \binom{N}{n} K_n^{(\infty)}(x; U, N) \prod_{i \in \mathbb{Z}} s_i^{x_i} \prod_{j \in \mathbb{Z}} t_j^{n_j}. \quad (19)$$

In fact, by comparing the coefficients of $\prod_i s_i^{x_i}$ on both sides of (19), we have (18). By the symmetry of s and t in (19), we have a duality relation

$$K_n^{(\infty)}(x; U, N) = K_x^{(\infty)}(n; {}^t U, N) \quad (x, n \in X(N)), \quad (20)$$

where ${}^tU = (u_{j,i})_{i,j \in \mathbb{Z}}$ is the transposed (infinite) matrix of U .

For $i \in \mathbb{Z}$, we denote by e_i the ‘ i -th unit vector’, that is, the element in $X(1)$ of which the i -th entry is 1 and the others are 0, or we can write

$$e_i = (\delta_{ik})_{k \in \mathbb{Z}} \in X(1). \quad (21)$$

For $x \in X(N)$ and $y \in X(N')$, we define $x + y = (x_i + y_i)_{i \in \mathbb{Z}} \in X(N + N')$ by adding in each entries. Then we have an expression

$$X(N) = \{e_{i_1} + e_{i_2} + \cdots + e_{i_N} \mid (i_1, i_2, \cdots, i_N) \in \mathbb{Z}^N\}. \quad (22)$$

The following lemma can be regarded as an infinite-variate extension of the recurrence formula (11) of the 1-variate Krawtchouk polynomials.

Lemma 1.2.1. *For $x \in X(N - 1)$, $n \in X(N)$ and $i \in \mathbb{Z}$,*

$$K_n^{(\infty)}(x + e_i; U, N) = \frac{1}{N} \sum_{j \in \mathbb{Z}} n_j u_{ij} K_{n - e_j}^{(\infty)}(x; U, N - 1). \quad (23)$$

Proof. From the definition of the generating function (18),

$$\Phi(t; x + e_i, U, N) = \Phi(t; x, U, N - 1) \cdot \left(\sum_{j \in \mathbb{Z}} u_{ij} t_j \right)$$

holds. Comparing the coefficients of $\prod_{j \in \mathbb{Z}} t_j^{n_j}$ in both sides, we have

$$\binom{N}{n} K_n^{(\infty)}(x + e_i; U, N) = \sum_{j \in \mathbb{Z}} u_{ij} \binom{N - 1}{n - e_j} K_{n - e_j}^{(\infty)}(x; U, N - 1),$$

which leads to the equation (23). $\square$

Lemma 1.2.2. *For $x \in X(N)$, $n \in X(N - 1)$ and $j \in \mathbb{Z}$,*

$$K_{n + e_j}^{(\infty)}(x; U, N) = \frac{1}{N} \sum_{i \in \mathbb{Z}} x_i u_{ij} K_n^{(\infty)}(x - e_i; U, N - 1). \quad (24)$$

Proof. It is easily derived from Lemma 1.2.1 and the duality (20). $\square$

By the use of (22), we have a symmetric expression of $K_n^{(\infty)}(x; U, N)$ as follows:

Proposition 1.2.3. For $(i_1, i_2, \dots, i_N), (j_1, j_2, \dots, j_N) \in \mathbb{Z}^N$,

$$K_{e_{j_1}+e_{j_2}+\dots+e_{j_N}}^{(\infty)}(e_{i_1}+e_{i_2}+\dots+e_{i_N}; U, N) = \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} u_{i_1 j_{\sigma(1)}} u_{i_2 j_{\sigma(2)}} \cdots u_{i_N j_{\sigma(N)}}, \quad (25)$$

where $\mathfrak{S}_N$ is the symmetric group of order N .

Proof. We verify the statement by induction on N . When $N = 1$, $K_{e_j}^{(\infty)}(e_i; U, 1) = u_{ij}$ is obvious from the definition. For the case of $N \geq 2$, by Lemma 1.2.1, we have

$$\begin{aligned} & K_{e_{j_1}+e_{j_2}+\dots+e_{j_N}}^{(\infty)}(e_{i_1}+e_{i_2}+\dots+e_{i_N}; U, N) \\ &= \frac{1}{N} \sum_{k=1}^N u_{i_1 j_k} K_{e_{j_1}+\dots+e_{j_{k-1}}+e_{j_{k+1}}+\dots+e_{j_N}}^{(\infty)}(e_{i_2}+\dots+e_{i_N}; U, N-1). \end{aligned}$$

From the assumption of induction, the right hand side is equal to

$$\frac{1}{N} \sum_{k=1}^N u_{i_1 j_k} \frac{1}{(N-1)!} \sum_{\tau_k} u_{i_2 j_{\tau_k(2)}} \cdots u_{i_N j_{\tau_k(N)}},$$

where τ_k runs over all bijections $\tau_k : \{2, \dots, N\} \rightarrow \{1, \dots, k-1, k+1, \dots, N\}$. It is equal to the right hand side of the equation (25). $\square$

We now take parameters $p = (p_i)_{i \in \mathbb{Z}}$ such that $p_i > 0$, and denote by l_p^2 the l^2 -space over $\mathbb{Z}$ equipped with a p -weighted inner product $(u, v)_p = \sum_{i \in \mathbb{Z}} u_i \bar{v}_i p_i$, $u = (u_i)_{i \in \mathbb{Z}}, v = (v_i)_{i \in \mathbb{Z}} \in l_p^2$. We define a weight function m_p over $X(N)$ by

$$m_p(x) = \binom{N}{x} \prod_{i \in \mathbb{Z}} p_i^{x_i} \quad (x \in X(N)), \quad (26)$$

and denote by $V_p(N)$ the l^2 -space on $X(N)$ with the measure $m_p(x)$ for the one element set $\{x\} \subset X(N)$. The following lemma gives a symmetric form of integrations on $X(N)$ by the measure m_p .

Lemma 1.2.4. For any non-negative or l^1 -functions ϕ on $X(N)$,

$$\sum_{x \in X(N)} \phi(x) m_p(x) = \sum_{(i_1, \dots, i_N) \in \mathbb{Z}^N} \phi(e_{i_1} + \dots + e_{i_N}) p_{i_1} \cdots p_{i_N}. \quad (27)$$

Proof. For each $x \in X(N)$, the number of N -tuples $(i_1, \dots, i_N) \in \mathbb{Z}^N$ such that $x = e_{i_1} + \dots + e_{i_N}$ seems to be $\binom{N}{x}$. This clearly proves the lemma. $\square$

We are ready to prove one of the main results of the paper, that is, orthogonality and completeness of the full infinite-variate extensions $K_n^{(\infty)}(x; U, N)$ in $V_p(N)$. We denote by $u^{(j)} = (u_{i,j})_{i \in \mathbb{Z}}$ the j -th column of U for $j \in \mathbb{Z}$.

Theorem 1.2.5. *When $\{u^{(j)}\}_{j \in \mathbb{Z}}$ is an orthogonal system of l_p^2 , then the system $\{K_n^{(\infty)}(\cdot; U, N)\}_{n \in X(N)}$ forms an orthogonal system of $V_p(N)$. More specifically, the following orthogonality relation holds: for $n, m \in X(N)$,*

$$\sum_{x \in X(N)} K_n^{(\infty)}(x; U, N) \overline{K_m^{(\infty)}(x; U, N)} m_p(x) = \delta_{nm} \binom{N}{n}^{-1} \prod_{j \in \mathbb{Z}} (u^{(j)}, u^{(j)})_p^{n_j}. \quad (28)$$

Moreover when $\{u^{(j)}\}_{j \in \mathbb{Z}}$ is a complete orthogonal system of l_p^2 , then the system $\{K_n^{(\infty)}(\cdot; U, N)\}_{n \in X(N)}$ forms a complete orthogonal system of $V_p(N)$.

Proof. Firstly we prove the orthogonality relation (28). Let $n = e_{j_1} + \dots + e_{j_N}$, $m = e_{k_1} + \dots + e_{k_N}$, where $(j_1, \dots, j_N), (k_1, \dots, k_N) \in \mathbb{Z}^N$. From Lemmas 1.2.3 and 1.2.4, we have

$$\begin{aligned} & \sum_{x \in X(N)} K_n^{(\infty)}(x; U, N) \overline{K_m^{(\infty)}(x; U, N)} m_p(x) \\ &= \sum_{(i_1, \dots, i_N) \in \mathbb{Z}^N} K_n^{(\infty)}(e_{i_1} + \dots + e_{i_N}; U, N) \overline{K_m^{(\infty)}(e_{i_1} + \dots + e_{i_N}; U, N)} p_{i_1} \dots p_{i_N} \\ &= \frac{1}{(N!)^2} \sum_{(i_1, \dots, i_N) \in \mathbb{Z}^N} \left(\sum_{\rho \in \mathfrak{S}_N} u_{i_1 j_{\rho(1)}} \dots u_{i_N j_{\rho(N)}} \right) \left(\sum_{\tau \in \mathfrak{S}_N} \overline{u_{i_1 k_{\tau(1)}}} \dots \overline{u_{i_N k_{\tau(N)}}} \right) p_{i_1} \dots p_{i_N} \\ &= \frac{1}{(N!)^2} \sum_{\rho, \tau \in \mathfrak{S}_N} (u^{(j_{\rho(1)})}, u^{(k_{\tau(1)})})_p \dots (u^{(j_{\rho(N)})}, u^{(k_{\tau(N)})})_p. \end{aligned}$$

By the orthogonality of $\{u^{(j)}\}_{j \in \mathbb{Z}}$, all terms of the right hand side are 0, except for the ones with $\rho, \tau \in \mathfrak{S}_N$ such that $j_{\rho(r)} = k_{\tau(r)}$ for all $r = 1, \dots, N$. If $n \neq m$, there are no such pairs (ρ, τ) , and if $n = m$, the number of such pairs seems to be $N! \prod_{j \in \mathbb{Z}} n_j!$. The right hand side is therefore equal to

$$\frac{\delta_{nm} \prod_{j \in \mathbb{Z}} n_j!}{N!} (u^{(j_1)}, u^{(j_1)})_p \dots (u^{(j_N)}, u^{(j_N)})_p = \delta_{nm} \binom{N}{n}^{-1} \prod_{j \in \mathbb{Z}} (u^{(j)}, u^{(j)})_p^{n_j},$$

which proves (28).

Secondly we prove the latter half of the theorem, the completeness, by induction on N . When $N = 1$, since $K_{e_j}^{(\infty)}(e_i; U, 1) = u_{ij}$, the statement

is true. Assume the case of $N - 1$ is true, and take a function $\varphi \in V_p(N)$ orthogonal to $K_n^{(\infty)}(x; U, N)$ with arbitrary $n \in X(N)$. We should show $\varphi = 0$. For any $k \in \mathbb{Z}$, we put $\varphi_k \in V_p(N - 1)$ by

$$\varphi_k(y) = \varphi(y + e_k) \quad (y \in X(N - 1)), \quad (29)$$

and for any $m \in X(N - 1)$, we define

$$\phi_m(k) = (\varphi_k, K_m^{(\infty)}(\cdot; U, N - 1))_{V_p(N-1)} \quad (k \in \mathbb{Z}). \quad (30)$$

We can derive that $\phi_m \in l_p^2$, from $\varphi \in V_p(N)$. With the use of a one-to-one correspondence

$$X(N - 1) \leftrightarrow \{x \in X(N) \mid x_k \geq 1\}, \quad y \mapsto y + e_k,$$

we can change

$$\begin{aligned} \phi_m(k) &= \sum_{y \in X(N-1)} \phi(y + e_k) \overline{K_m^{(\infty)}(y; U, N - 1)} \binom{N-1}{y} \prod_{i \in \mathbb{Z}} p_i^{y_i} \\ &= \frac{1}{p_k N} \sum_{x \in X(N)} \phi(x) x_k \overline{K_m^{(\infty)}(x - e_j; U, N)} \binom{N}{x} \prod_{i \in \mathbb{Z}} p_i^{x_i}. \end{aligned}$$

It with Lemma 1.2.2 gives for any $j \in \mathbb{Z}$,

$$\begin{aligned} (\phi_m, u^{(j)})_p &= \sum_{k \in \mathbb{Z}} \phi_m(k) \overline{u_{kj}} p_k \\ &= \frac{1}{N} \sum_{k \in \mathbb{Z}} \sum_{x \in X(N)} \varphi(x) x_k \overline{u_{kj}} \overline{K_m^{(\infty)}(x - e_k; U, N - 1)} \binom{N}{x} \prod_{i \in \mathbb{Z}} p_i^{x_i} \\ &= \sum_{x \in X(N)} \varphi(x) \overline{K_{m+e_j}^{(\infty)}(x; U, N)} \binom{N}{x} \prod_{i \in \mathbb{Z}} p_i^{x_i} \\ &= (\varphi, K_{m+e_j}^{(\infty)}(\cdot; U, N))_{V_p(N)} = 0. \end{aligned}$$

We thus have $\phi_m = 0$ from completeness of $\{u^{(j)}\}_{j \in \mathbb{Z}}$. Since m is arbitrary, we have $\varphi_k = 0$ for any $k \in \mathbb{Z}$ from completeness of $\{K_m^{(\infty)}(\cdot; U, N - 1)\}_{m \in X(N-1)}$ assumed in induction. That concludes $\varphi = 0$. $\square$

1.3 Infinite-variate extensions with indices on $\mathbb{N}_0 = \{0, 1, 2, \dots\}$

Here we introduce another class of infinite-variate extensions $K_n^{(+)}(x; U, N)$ of Krawtchouk polynomials, which we call “half infinite-variate extensions”.

The variables $x = (x_i)_{i \in \mathbb{N}_0}$ and the degree indices $n = (n_j)_{j \in \mathbb{N}_0}$ are indexed by $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. The variables x run over the set

$$X^+(N) = \{x = (x_i)_{i \in \mathbb{N}_0} \mid x_i \in \mathbb{N}_0, |x| = N\}, \quad (31)$$

on the other hand, the degree indices n run over the set

$$\bar{X}^+(N) = \{n = (n_j)_{j \in \mathbb{N}_0} \mid n_j \in \mathbb{N}_0, |n| \leq N\}. \quad (32)$$

We remark that $X^+(N) \subsetneq \bar{X}^+(N)$, thus this extensions may not possess dualities like (4) or (20). A complex parameters set $U = (u_{i,j})_{i,j \in \mathbb{N}_0}$ is indexed by $\mathbb{N}_0 \times \mathbb{N}_0$. The definition is given by the following generating function with indeterminates T and $t = (t_j)_{j \in \mathbb{N}_0}$:

$$\begin{aligned} \Phi^{(+)}(T, t; x, U, N) &:= \prod_{i \geq 0} \left(T + \sum_{j \geq 0} u_{i,j} t_j \right)^{x_i} \\ &= \sum_{n \in \bar{X}^+(N)} \binom{N}{n} K_n^{(+)}(x; U, N) T^{N-|n|} \prod_{j \geq 0} t_j^{n_j}, \end{aligned} \quad (33)$$

where $\binom{N}{n}$ is defined by the same formula as (2).

We can regard them as limit cases of the finite-variate Krawtchouk polynomials. For $x \in \bar{X}^+(N)$ and $\ell \geq 1$, we denote the restriction of indices by

$$x|_\ell = (x_i)_{i \in [\ell]} \in \bar{X}(\ell, N). \quad (34)$$

Similarly we write $U|_\ell = (u_{i,j})_{i,j \in [\ell]}$. Then we have

$$K_n^{(+)}(x; U, N) = \lim_{\ell \rightarrow \infty} K_{n|_\ell}^{(\ell)}(x|_\ell; U|_\ell, N). \quad (35)$$

In fact, when ℓ is large enough to give $x_i = 0$, $n_j = 0$ for all $i, j \geq \ell$, then the generating function (3) of the multivariate Krawtchouk polynomials coincides with (33), thus $K_{n|_\ell}^{(\ell)}(x|_\ell; U|_\ell, N)$ is equal to $K_n^{(+)}(x; U, N)$. In a similar way, the full infinite-variate extensions $K_n^{(\infty)}(x; U, N)$ can be regarded as limit cases of the half infinite ones. For $x, n \in X(N)$, parameters $U = (u_{i,j})_{i,j \in \mathbb{Z}}$ and an integer k , we set

$$x^{(k)} = (x_{k+i})_{i \in \mathbb{N}_0} \in \bar{X}^+(N) \quad (36)$$

by shifting and restricting indices. We also put $n^{(k)} = (n_{k+i})_{i \in \mathbb{N}_0}$ and $U^{(k)} = (u_{k+i, k+j})_{i,j \in \mathbb{N}_0}$. When $k < 0$ is small enough to satisfy $x^{(k)}, n^{(k)} \in X^+(N)$,

then we have $K_{n^{(k)}}^{(+)}(x^{(k)}; U^{(k)}, N)$ equal to $K_n^{(\infty)}(x; U, N)$ by comparing the generating functions (18) and (33). We may write this as

$$K_n^{(\infty)}(x; U, N) = \lim_{k \rightarrow -\infty} K_{n^{(k)}}^{(+)}(x^{(k)}; U^{(k)}, N). \quad (37)$$

Conversely we can express the finite-variate Krawtchouk polynomials $K_n^{(\ell)}(x; U, N)$ as restrictions of special cases of the half infinite-variate extensions. We regard that $\bar{X}(\ell, N) \subset \bar{X}^+(N)$ by the embedding map

$$\bar{X}(\ell, N) \hookrightarrow \bar{X}^+(N), \quad n \mapsto n \sqcup (n_j = 0)_{j \geq \ell}, \quad (38)$$

where $(n_j)_{j \in [\ell]} \sqcup (n_j)_{j \geq \ell} = (n_j)_{j \in \mathbb{N}_0}$ denotes the joint of entries. For a given parameter matrix $U = (u_{i,j})_{i,j \in [\ell]}$, we fix an extension $U^+ = (u_{i,j})_{i,j \in \mathbb{N}_0}$ such that $u_{i,j} = 1$ for $i \geq \ell$ and $0 \leq j \leq \ell - 1$, that is,

$$U^+ = \left(\frac{U}{\mathbb{1}} \middle| \begin{array}{c} * \\ * \end{array} \right), \quad (39)$$

where the vertical and horizontal lines in the right hand side divide the parts of indices $\{0, 1, \dots, \ell - 1\}$ and ones over ℓ . The part of $\mathbb{1}$ is to fill all the entries with 1, and the parts of $*$ allows any complex parameters.

Lemma 1.3.1. *For $x \in X^+(N)$ and $n \in \bar{X}(\ell, N)$, we have*

$$K_n^{(\ell)}(x|_{\ell}; U, N) = K_n^{(+)}(x; U^+, N). \quad (40)$$

Proof. We let $t = (t_j)_{j \in [\ell]}$ and $t' = (t_j)_{j \in \mathbb{N}_0}$. The form of U^+ in (39) gives

$$\Phi^{(+)}(1, t'; x, U^+, N) = \prod_{i \in [\ell]} \left(1 + \sum_{j \geq 0} u_{i,j} t_j \right)^{x_i} \cdot \prod_{i \geq \ell} \left(1 + \sum_{j \in [\ell]} t_j + \sum_{j \geq \ell} u_{i,j} t_j \right)^{x_i}.$$

Putting $t_j = 0$ for $j \geq \ell$, we have

$$\begin{aligned} \Phi^{(+)}(1, t \sqcup (0)_{j \geq \ell}; x, U^+, N) &= \prod_{i \in [\ell]} \left(1 + \sum_{j \in [\ell]} u_{i,j} t_j \right)^{x_i} \cdot \prod_{i \geq \ell} \left(1 + \sum_{j \in [\ell]} t_j \right)^{x_i} \\ &= \Phi^{(\ell)}(1, t; x|_{\ell}, U, N). \end{aligned}$$

Compare the coefficients of $\prod_{j \in [\ell]} t_j^{n_j}$ in both sides. Then we have the equation (40). $\square$

We now express the half infinite extensions $K_n^{(+)}(x; U, N)$ as restrictions of special cases of the full infinite extensions. Here we consider the embedding

$$X^+(N) \hookrightarrow X(N), \quad x \mapsto (x_i = 0)_{i \leq -1} \sqcup x. \quad (41)$$

For $n \in X(N)$, we recall $n^{(0)} = (n_j)_{j \in \mathbb{N}_0} \in \bar{X}^+(N)$ by the notation in (36), and we denote $n_{(-1)} = (n_j)_{j \leq -1}$. We extend $U = (u_{i,j})_{i,j \in \mathbb{N}_0}$ to $U^\infty = (u_{i,j})_{i,j \in \mathbb{Z}}$ such that $u_{i,j} = 1$ for $i \geq 0$ and $j \leq -1$, that is,

$$U^\infty = \left(\frac{*}{\mathbb{1}} \middle| \frac{*}{U} \right), \quad (42)$$

where the vertical and horizontal lines in the right hand side divide the parts of negative and non-negative indices. We give the part of $\mathbb{1}$ and $*$ the same remark as (39).

Lemma 1.3.2. *For $x \in X^+(N)$ and $n \in X(N)$, we have*

$$K_{n^{(0)}}^{(+)}(x; U, N) = K_n^{(\infty)}(x; U^\infty, N). \quad (43)$$

Proof. By putting $T = \sum_{j \leq -1} t_j$ in the generating function (33), we have

$$\begin{aligned} & \prod_{i \geq 0} \left(\sum_{j \leq -1} t_j + \sum_{j \geq 0} u_{i,j} t_j \right)^{x_i} \\ &= \sum_{n^{(0)} \in \bar{X}^+(N)} \binom{N}{n^{(0)}} K_{n^{(0)}}^{(+)}(x; U, N) \left(\sum_{j \leq -1} t_j \right)^{N - |n^{(0)}|} \prod_{j \geq 0} t_j^{n_j}. \end{aligned} \quad (44)$$

Since x is in $X^+(N)$ and U^∞ has the form in (42), the left hand side of (44) is equal to the generating function $\Phi(t; x, U^\infty, N)$. By the use of an expansion formula

$$\left(\sum_{j \leq -1} t_j \right)^{N - |n^{(0)}|} = \sum_{|n_{(-1)}| = N - |n^{(0)}|} \binom{N - |n^{(0)}|}{n_{(-1)}} \prod_{j \leq -1} t_j^{n_j}$$

together with $\binom{N}{n^{(0)}} \binom{N - |n^{(0)}|}{n_{(-1)}} = \binom{N}{n}$ on the right hand side of (44), we have

$$\Phi(t; x, U^\infty, N) = \sum_{n \in X(N)} \binom{N}{n} K_n^{(+)}(x; U, N) \prod_{j \in \mathbb{Z}} t_j^{n_j},$$

which gives the equality (43). $\square$

We give orthogonality and completeness of the extensions $K_n^{(+)}(x; U, N)$ as follows. We take a parameters set $p = (p_i)_{i \in \mathbb{N}_0}$ such that $p_i > 0$ and $|p| < \infty$, and denote by $l_p^2(\mathbb{N}_0)$ the l^2 -space over $\mathbb{N}_0$ weighted by p . We define a weight function m_p on $X^+(N)$ by the similar formula as (26) such that

$$m_p(x) = \binom{N}{x} \prod_{i \geq 0} p_i^{x_i} \quad (x \in X^+(N)), \quad (45)$$

and denote by $V_p^+(N)$ the l^2 -space over $X^+(N)$ weighted by m_p . For a given parameter matrix $U = (u_{i,j})_{i,j \in \mathbb{N}_0}$, we set $u^{(j)} = (u_{i,j})_{i \in \mathbb{N}_0}$ for $j \in \mathbb{N}_0$ and $u^{(-\infty)} = (1)_{i \in \mathbb{N}_0} = (1, 1, \dots) \in l_p^2(\mathbb{N}_0)$.

Theorem 1.3.3. *When $\{u^{(j)}\}_{j \in \mathbb{N}_0 \cup \{-\infty\}}$ is an orthogonal system of $l_p^2(\mathbb{N}_0)$, then the system $\{K_n^{(+)}(\cdot; U, N)\}_{n \in \bar{X}^+(N)}$ forms an orthogonal system of $V_p^+(N)$. More specifically, the following orthogonality relation holds: for $n, m \in \bar{X}^+(N)$,*

$$\sum_{x \in X^+(N)} K_n^{(+)}(x; U, N) \overline{K_m^{(+)}(x; U, N)} m_p(x) = \delta_{nm} \binom{N}{n}^{-1} |p|^{N-|n|} \prod_{j \in \mathbb{N}_0} (u^{(j)}, u^{(j)})_p^{n_j}. \quad (46)$$

Moreover when $\{u^{(j)}\}_{j \in \mathbb{N}_0 \cup \{-\infty\}}$ is complete orthogonal system of $l_p^2(\mathbb{N}_0)$, then $\{K_n^{(+)}(\cdot; U, N)\}_{n \in \bar{X}^+(N)}$ forms a complete orthogonal system of $V_p^+(N)$.

To prove this theorem, we use Theorem 1.2.5. At the beginning, we give appropriate extensions of p and U to have indices on $\mathbb{Z}$. We define $p^\infty = (p_i)_{i \in \mathbb{Z}}$ by setting $p_i = 1$ for $i \leq -1$, or we may write

$$p^\infty = (1)_{i \leq -1} \sqcup p. \quad (47)$$

We take $U^\infty = (u_{i,j})_{i,j \in \mathbb{Z}}$ as a special case of (42), such that $u_{i,j} = 0$ ($i \leq -1$ and $i < j$), $u_{i,j} = 1$ ($j \leq -1$ and $j < i$) and $u_{i,i} = -|p| + i + 1$ ($i \leq -1$), that is,

$$U^\infty = \left(\begin{array}{ccc|ccc} \ddots & & & & & \\ & -|p|-2 & & \mathbb{O} & & \\ & & -|p|-1 & & \mathbb{O} & \\ \mathbb{1} & & & -|p| & & \\ \hline & & & & & \\ & & \mathbb{1} & & & U \end{array} \right). \quad (48)$$

We set $u^{\infty(j)} = (u_{i,j})_{i \in \mathbb{Z}}$ for $j \in \mathbb{Z}$.

Lemma 1.3.4. *If $\{u^{(j)}\}_{j \in \mathbb{N}_0 \cup \{-\infty\}}$ is an orthogonal or complete orthogonal system of $l_p^2(\mathbb{N}_0)$, then $\{u^{\infty(j)}\}_{j \in \mathbb{Z}}$ forms that of $l_{p^\infty}^2$ respectively.*

Proof. Simple calculations show that $u^{\infty(j)} \in l_{p^\infty}^2$ for $j \in \mathbb{Z}$, and the orthogonality $(u^{\infty(j)}, u^{\infty(k)})_{p^\infty} = 0$ for $j, k \in \mathbb{Z}$, $j \neq k$. In order to prove completeness, we take $v = (v_i)_{i \in \mathbb{Z}} \in l_{p^\infty}^2$ such that $(v, u^{\infty(j)})_{p^\infty} = 0$ for all $j \in \mathbb{Z}$. For $j \leq -1$, it is seen from (48) that all entries of $u^{\infty(j-1)} - u^{\infty(j)}$ are 0 except for the $(j-1)$ -th entry $(-|p|+j)$ and the j -th entry $|p|-j$. It gives that

$$0 = (v, u^{\infty(j-1)} - u^{\infty(j)})_{p^\infty} = (v_{j-1} - v_j)(-|p|+j),$$

thus $v_{j-1} = v_j$ on one hand. On the other hand, from the assumption $\sum_{j \leq -1} |v_j|^2 \leq (v, v) < \infty$, we conclude $v_j = 0$ for $j \leq -1$. As for $v^{(0)} = (v_i)_{i \geq 0}$, since $(v^{(0)}, u^{(j)})_p = (v, u^{\infty(j)})_{p^\infty} = 0$ for all $j \in \mathbb{N}_0$, and $(v^{(0)}, u^{(-\infty)})_p = (v, u^{\infty(-1)})_{p^\infty} = 0$, we have $v^{(0)} = 0$ by the completeness of $\{u^{(j)}\}_{j \in \mathbb{N}_0 \cup \{-\infty\}}$. These show $v = 0$ as desired. $\square$

Lemma 1.3.4 ensures Theorem 1.2.5 holds in $V_{p^\infty}(N)$ for the full-infinite extensions with parameter U^∞ , especially the orthogonality relation for $n, m \in X(N)$:

$$\begin{aligned} \sum_{x \in X(N)} K_n^{(\infty)}(x, U^\infty, N) \overline{K_n^{(\infty)}(x, U^\infty, N)} \binom{N}{x} \prod_{i \in \mathbb{N}_0} p_i^{x_i} \\ = \delta_{nm} \binom{N}{n}^{-1} \prod_{j \leq -1} (u^{\infty(j)}, u^{\infty(j)})_{p^\infty} \prod_{j \in \mathbb{N}_0} (u^{(j)}, u^{(j)})_p. \end{aligned} \quad (49)$$

For $n \in \bar{X}^+(N)$, we give an extension $n^\infty = (n_j)_{j \in \mathbb{Z}} \in X(N)$ by $n_j = 0$ for $j \leq -2$ and $n_{-1} = N - |n|$, or we may write

$$n^\infty = (0)_{j \leq -2} \sqcup (N - |n|)_{j=-1} \sqcup n. \quad (50)$$

For $x \in X(N)$, we use $x^{(0)} = (x_i)_{i \in \mathbb{N}_0}$ in the notation (36). Then we have

Lemma 1.3.5.

$$\begin{aligned} K_{n^\infty}^{(\infty)}(x; U^\infty, N) \\ = \begin{cases} (-|p|)^{x_{-1}} \frac{(-N+|n|)_{x_{-1}}}{(-N)_{x_{-1}}} K_n^{(+)}(x^{(0)}; U, N-x_{-1}) & (\text{if } x_i = 0, \forall i \leq -2), \\ 0 & (\text{otherwise}), \end{cases} \end{aligned}$$

where $(a)_k$ denotes the Pochhammer symbol defined in (10).

Proof. From the form of U^∞ in (48), the generating function (18) is give as

$$\Phi(t; x, U^\infty, N) = \prod_{i \leq -1} \left(\sum_{j \leq i-1} t_j + (-|p| + i + 1)t_i \right)^{x_i} \prod_{i \geq 0} \left(\sum_{j \leq -1} t_j + \sum_{j \geq 0} u_{i,j} t_j \right)^{x_i}. \quad (51)$$

If $x_i \neq 0$ for some $i \leq -2$, the coefficient of $t_{-1}^{N-|n|} \prod_{j \geq 0} t_j^{n_j}$ in the expansion of (51) is 0, thus $K_{n^\infty}^{(\infty)}(x; U^\infty, N) = 0$. Hereafter we assume $x_i = 0$ for all $i \leq -2$. Then by putting $t_j = 0$ for $j \leq -2$ and $t_{-1} = 1$ in (51), we have

$$(-|p|)^{x_{-1}} \prod_{i \geq 0} \left(1 + \sum_{j \geq 0} u_{i,j} t_j \right)^{x_i} = \sum_{n \in \bar{X}^+(N)} \binom{N}{n} K_{n^\infty}^{(\infty)}(x; U^\infty, N) \prod_{j \geq 0} t_j^{n_j}.$$

Since the left hand side is equal to $(-|p|)^{x_{-1}} \Phi^{(+)}(1, t^{(0)}; x^{(0)}, U, N - x_{-1})$, we have

$$(-|p|)^{x_{-1}} \binom{N - x_{-1}}{n} K_n^{(+)}(x^{(0)}; U, N - x_{-1}) = \binom{N}{n} K_{n^\infty}^{(\infty)}(x; U^\infty, N),$$

which induces the assertion. $\square$

We now prove the orthogonality relation (46) by induction on N . When $N = 1$, it follows from orthogonality of $\{u^{(j)}\}_{j \in \mathbb{N}_0 \cup \{-\infty\}}$. Let $N \geq 2$. For $n, m \in \bar{X}^+(N)$, we set $n^\infty, m^\infty \in X(N)$ by (50). Then the relation (49) gives

$$\begin{aligned} \sum_{x \in X(N)} K_{n^\infty}^{(\infty)}(x; U^\infty, N) \overline{K_{m^\infty}^{(\infty)}(x; U^\infty, N)} \binom{N}{x} \prod_{i \geq 0} p_i^{x_i} \\ = \delta_{n,m} \binom{N}{n}^{-1} (|p|^2 + |p|)^{N-|n|} \prod_{j \geq 0} (u^{(j)}, u^{(j)})_p^{n_j}. \end{aligned} \quad (52)$$

From Lemma 1.3.5, the left hand side of (52) is transformed to

$$\begin{aligned} \sum_{x_{-1}=0}^N \sum_{x^{(0)} \in X^+(N-x_{-1})} |p|^{2x_{-1}} \frac{(-N + |n|)_{x_{-1}} (-N + |m|)_{x_{-1}}}{(-N)_{x_{-1}}^2} \\ \times K_n^{(+)}(x^{(0)}; U, N - x_{-1}) \overline{K_m^{(+)}(x^{(0)}; U, N - x_{-1})} \frac{N!}{\prod_{i \geq -1} x_i!} \prod_{i \geq 0} p_i^{x_i} \\ = \sum_{k=0}^{N-|n|} |p|^{2k} \frac{(-N + |n|)_k (-N + |m|)_k}{(-N)_k^2} \binom{N}{k} \\ \times \left(K_n^{(+)}(\cdot; U, N - k), K_m^{(+)}(\cdot; U, N - k) \right)_{V_p^+(N-k)}, \end{aligned}$$

and by applying the assumption of induction for the terms with $k \geq 1$, it is further changed to

$$\begin{aligned}
& \left(K_n^{(+)}(\cdot; U, N), K_m^{(+)}(\cdot; U, N) \right)_{V_p^+(N)} \\
& + \delta_{nm} \sum_{k=1}^{N-|n|} |p|^{2k} \frac{(-N+|n|)_k^2}{(-N)_k^2} \binom{N}{k} \binom{N-k}{n}^{-1} |p|^{N-k-|n|} \prod_{j \geq 0} (u^{(j)}, u^{(j)})_p^{n_j} \\
& = \left(K_n^{(+)}(\cdot; U, N), K_m^{(+)}(\cdot; U, N) \right)_{V_p^+(N)} \\
& + \delta_{nm} \binom{N}{n}^{-1} ((|p|^2 + |p|)^{N-|n|} - |p|^{N-|n|}) \prod_{j \geq 0} (u^{(j)}, u^{(j)})_p^{n_j}.
\end{aligned}$$

Putting it on the left hand side of (52), we have

$$\left(K_n^{(+)}(\cdot; U, N), K_m^{(+)}(\cdot; U, N) \right)_{V_p^+(N)} = \delta_{n,m} \binom{N}{n}^{-1} |p|^{N-|n|} \prod_{j \geq 0} (u^{(j)}, u^{(j)})_p^{n_j},$$

which is the orthogonality relation (46).

Then we prove the latter half of Theorem 1.3.3, completeness. We take $\varphi \in V_p^+(N)$ such that $\left(\varphi, K_m^{(+)}(\cdot; U, N) \right)_{V_p^+(N)} = 0$ for all $m \in \bar{X}^+(N)$. We regard that $X^+(N) \subset X(N)$ by the embedding (41), and extend to φ on $X(N)$ by $\varphi(x) = 0$ for $x \in X(N) \setminus X^+(N)$. Then for arbitrary $n \in X(N)$, we have from Lemma 1.3.2,

$$\begin{aligned}
& \left(\varphi, K_n^{(\infty)}(\cdot; U^\infty, N) \right)_{V_p(N)} = \sum_{x \in X(N)} \varphi(x) \overline{K_n^{(\infty)}(x; U^\infty, N)} \binom{N}{x} \prod_{i \in \mathbb{Z}} p_i^{x_i} \\
& = \sum_{x \in X^+(N)} \varphi(x) \overline{K_{n_0}^{(+)}(x; U, N)} \binom{N}{x} \prod_{i \geq 0} p_i^{x_i} = \left(\varphi, K_{n_0}^{(+)}(\cdot; U, N) \right)_{V_p^+(N)} = 0,
\end{aligned}$$

thus $\varphi = 0$ by Theorem 1.2.5. It completes the proof of Theorem 1.3.3.

We have a following ‘expansion formula’ as a corollary of the theorem:

Corollary 1.3.6. *For $p = (p_i)_{i \in \mathbb{N}_0}$ such that $p_i > 0$ and $|p| < \infty$, we have*

$$\sum_{x \in X^+(N)} \binom{N}{x} \prod_{i \in \mathbb{N}_0} p_i^{x_i} = |p|^N. \quad (53)$$

Proof. We take a parameter matrix $U = (u_{i,j})_{i,j \in \mathbb{N}_0}$ such that $\{u^{(j)}\}_{j \in \mathbb{N}_0 \cup \{-\infty\}}$ is an orthogonal system of the space $l_p^2(\mathbb{N}_0)$ (for example, $u_{i,j} = 0$ for $i < j$, $u_{i,j} = 1$ for $i > j$ and $u_{i,i} = -\frac{|p^{(i+1)}|}{p_i}$, where $|p^{(i+1)}| = \sum_{k \geq i+1} p_k$). Then the orthogonality relation (46) with $n = m = 0 = (0)_{j \geq 0}$ together with the fact $K_0^{(+)}(x; U, N) = 1$ proves the assertion. $\square$

At the end of this section, we give a proof of the orthogonality relation (7) of the finite-variate Krawtchouk polynomials by use of Theorem 1.3.3. We assume to be given parameters $p = (p_i)_{i \in [\ell]}$ such that $p_i > 0$ and $|p| < 1$, and a parameter matrix $U = (u_{i,j})_{i,j \in [\ell]}$ over $\mathbb{C}$ which satisfies the orthogonality condition (5). We extend p to $p^+ = (p_i)_{i \in \mathbb{N}_0}$ by

$$p_i = |p|(1 - |p|)^{i-\ell+1} \quad (i \geq \ell). \quad (54)$$

Then we remark that $|p^+| = 1$. We also extend to $U^+ = (u_{i,j})_{i,j \in \mathbb{N}_0}$ as a special case of (39), by setting $u_{i,j} = 0$ ($i < j$ and $j \geq \ell$), $u_{i,j} = 1$ ($j < i$ and $i \geq \ell$) and $u_{i,i} = 1 - |p|^{-1}$ ($i \geq \ell$), that is,

$$U^+ = \left(\begin{array}{c|ccc} U & & & \mathbb{O} \\ \hline & 1 - |p|^{-1} & & \mathbb{O} \\ \hline \mathbb{1} & & 1 - |p|^{-1} & \mathbb{O} \\ & \mathbb{1} & & \ddots \end{array} \right). \quad (55)$$

We let $u^{+(j)} = (u_{i,j})_{i \in \mathbb{N}_0}$ for $j \in \mathbb{N}_0$ and $u^{+(-\infty)} = (1)_{i \in \mathbb{N}_0}$. Then,

Lemma 1.3.7. *the system $\{u^{+(j)}\}_{j \in \mathbb{N}_0 \cup \{-\infty\}}$ forms a complete orthogonal system of $l_{p^+}^2(\mathbb{N}_0)$.*

Proof. Simple calculations shows that $u^{+(j)} \in l_{p^+}^2(\mathbb{N}_0)$ and the orthogonality $(u^{+(j)}, u^{+(k)})_{p^+} = 0$ for $j, k \in \mathbb{N}_0 \cup \{-\infty\}$, $j \neq k$. To prove completeness, we take $v = (v_i)_{i \geq 0} \in l_{p^+}^2(\mathbb{N}_0)$ such that $(v, u^{+(j)})_{p^+} = 0$ for all $j \in \mathbb{N}_0 \cup \{-\infty\}$. For arbitrary $j \geq \ell$, since all entries of $u^{+(j)} - u^{+(j+1)}$ are 0 except for the j -th entry $(1 - |p|^{-1})$ and the $(j+1)$ -th entry $|p|^{-1}$ from (55), we have

$$0 = (v, u^{+(j)} - u^{+(j+1)})_{p^+} = (-v_j + v_{j+1})(1 - |p|)^{j-\ell+2},$$

thus $v_j = v_{j+1}$ on one hand. On the other hand, for $v|_{\ell+1} = (v_i)_{i=0}^\ell \in \mathbb{C}^{\ell+1}$ and arbitrary $j \in [\ell] \cup \{-\infty\}$, we have $(v|_{\ell+1}, u^{(j)})_p = \sum_{i \in [\ell]} v_i u_{i,j} p_i + v_\ell (1 - |p|) = \sum_{i \in [\ell]} v_i u_{i,j} p_i + \sum_{i \geq \ell} v_i p_i = (v, u^{+(j)})_{p^+} = 0$. Since $\{u^{(j)}\}_{j \in [\ell] \cup \{-\infty\}}$ is a basis of $\mathbb{C}^{\ell+1}$, we have $v|_{\ell+1} = 0$, therefore $v = 0$. $\square$

We now prove the orthogonality relation (7) for $n, m \in \bar{X}(\ell, N)$. We regard as $n, m \in \bar{X}^+(N)$ by the embedding (38). Lemma 1.3.7 ensures Theorem 1.3.3, especially the orthogonality relation (46) for $K_n^{(+)}(\cdot; U^+, N)$ and $K_m^{(+)}(\cdot; U^+, N)$ in $V_{p^+}^+(N)$:

$$\sum_{y \in X^+(N)} K_n^{(+)}(y; U^+, N) \overline{K_m^{(+)}(y; U^+, N)} \binom{N}{y} \prod_{i \in \mathbb{N}_0} p_i^{y_i} = \delta_{nm} \binom{N}{n}^{-1} \prod_{j \in [\ell]} (u^{(j)}, u^{(j)})_p^{n_j}. \quad (56)$$

Putting $x = y|_\ell = (y_i)_{i \in [\ell]} \in \bar{X}(\ell, N)$ and $z = y^{(\ell)} = (y_{\ell+i})_{i \in \mathbb{N}_0} \in X^+(N - |x|)$ in each term and using Lemma 1.3.1, the left hand side of (56) is transformed to

$$\sum_{x \in \bar{X}(\ell, N)} \sum_{z \in X^+(N - |x|)} K_n^{(\ell)}(x; U, N) \overline{K_m^{(\ell)}(x; U, N)} \binom{N}{x} \binom{N - |x|}{z} \prod_{i \in [\ell]} p_i^{x_i} \prod_{i \in \mathbb{N}_0} p_{\ell+i}^{z_i}.$$

Moreover since $\sum_{z \in X^+(N - |x|)} \binom{N - |x|}{z} \prod_{i \in \mathbb{N}_0} p_{\ell+i}^{z_i} = (1 - |p|)^{N - |x|}$ from Corollary 1.3.6, we can transform (56) to

$$\begin{aligned} \sum_{x \in \bar{X}(\ell, N)} K_n^{(\ell)}(x; U, N) \overline{K_m^{(\ell)}(x; U, N)} \binom{N}{x} (1 - |p|)^{N - |x|} \prod_{i \in [\ell]} p_i^{x_i} \\ = \delta_{nm} \binom{N}{n}^{-1} \prod_{j \in [\ell]} (u^{(j)}, u^{(j)})_p^{n_j}, \end{aligned}$$

which is the orthogonality relation (7) of finite-variate Krawtchouk polynomials.

1.4 Product expressions of Krawtchouk polynomials

As described in Section 1.1, Griffiths's multivariate Krawtchouk polynomials are defined by the generating function (3). On the other hand Xu [18] also defines multivariate Krawtchouk polynomials of his own, in terms of products of 1-variate Krawtchouk polynomials. Diaconis and Griffiths [1] shows that Xu's ones are special cases of the Griffiths's ones. Here we will introduce essentially Xu's Krawtchouk polynomials in a modified way, which are used in realizing as zonal spherical functions in Chapter 2.

For ℓ -variables or numbers $x = (x_i)_{i \in [\ell]}$ and $k \in [\ell]$, we prepare the following notations:

$$|x_{(k)}| = \sum_{i \leq k} x_i, \quad |x^{(k)}| = \sum_{i \geq k} x_i. \quad (57)$$

Definition 1.4.1. For $x = (x_i)_{i \in [\ell]}$, $n = (n_i)_{i \in [\ell]} \in \bar{X}(\ell, N)$ and parameters $a = (a_i)_{i \in [\ell]}$ such that $0 < a_i < 1$, we define the ℓ -product Krawtchouk polynomials

$$K_n^{(\ell)}(x; a, N) = \frac{1}{(-N)_{|n|}} \prod_{i \in [\ell]} (-N_i)_{n_i} K_{n_i}(x_i; a_i, N_i), \quad (58)$$

$$\text{where} \quad N_i = N - |x_{(i-1)}| - |n^{(i+1)}|. \quad (59)$$

We remark that the Xu's definition [18] is given by a suitable transformation of parameters and a normalization: if we denote his ones by $\tilde{K}_n(x; p, N)$ with $p = (p_i)_{i \in [\ell]}$, $0 < p_i < 1$, $|p| < 1$ and $x, n \in \bar{X}(\ell, N)$, then the relation is

$$\tilde{K}_n(x; p, N) = \prod_{i \in [\ell]} \left(-\frac{p_i}{1 - |p_{(i)}|} \right)^{n_i} K_n^{(\ell)}(x; \left\{ \frac{p_i}{1 - |p_{(i-1)}|} \right\}_{i \in [\ell]}, N). \quad (60)$$

The polynomials $K_n^{(\ell)}(x; a, N)$ are special cases of the Griffiths's Krawtchouk polynomials (see Proposition 2 of [1] and use the relation (60)):

Proposition 1.4.2. For $a = (a_i)_{i \in [\ell]}$, we set a parameter matrix $U_a = (u_{i,j})_{i,j \in [\ell]}$ by $u_{i,j} = 0$ for $i < j$, $u_{i,j} = 1$ for $i > j$ and $u_{i,i} = 1 - a_i^{-1}$, that is,

$$U_a = \begin{pmatrix} 1 - a_0^{-1} & & & \mathbb{O} \\ & 1 - a_1^{-1} & & \\ & & \ddots & \\ \mathbb{1} & & & 1 - a_{\ell-1}^{-1} \end{pmatrix}. \quad (61)$$

Then we have

$$K_n^{(\ell)}(x; a, N) = K_n^{(\ell)}(x; U_a, N) \quad (x, n \in \bar{X}(\ell, N)). \quad (62)$$

We set probabilistic parameters $p = (p_i)_{i \in [\ell]}$ by

$$p_i = a_i \prod_{k=0}^{i-1} (1 - a_k) \quad (i \in [\ell]), \quad (63)$$

then we have $|p| = 1 - \prod_{i \in [\ell]} (1 - a_i) < 1$. We can also verify that the matrix U_a in (61) satisfies the orthogonality condition (5) and

$$(u^{(j)}, u^{(j)})_p = a_j^{-1} \prod_{k=0}^j (1 - a_k) \quad (j \in [\ell]). \quad (64)$$

Putting (63) and (64) into (7), we have the orthogonality relations of ℓ -product Krawtchouk polynomials $K_n^{(\ell)}(x; a, N)$:

Proposition 1.4.3. For $n, m \in \bar{X}(\ell, N)$,

$$\begin{aligned} \sum_{x \in \bar{X}(\ell, N)} K_n^{(\ell)}(x; a, N) K_m^{(\ell)}(x; a, N) \binom{N}{x} \prod_{i \in [\ell]} a_i^{x_i} (1 - a_i)^{N - |x_{(i)}|} \\ = \delta_{nm} \binom{N}{n}^{-1} \prod_{i \in [\ell]} a_i^{-n_i} (1 - a_i)^{|n^{(i)}|}. \end{aligned} \quad (65)$$

By the same idea, we can construct the product forms of the infinite-variate extensions of Krawtchouk polynomials. Hereafter in this section, in order to deal with two extension cases (the full and the half infinite extensions) simultaneously, we let $\mathbb{I}$ denote $\mathbb{N}_0$ or $\mathbb{Z}$. For any $x = (x_i)_{i \in \mathbb{I}}$ and $k \in \mathbb{I}$, we also use the notations $|x_{(k)}|$ and $|x^{(k)}|$ in (57).

Definition 1.4.4. We take a set of parameters $a = (a_i)_{i \in \mathbb{I}}$ such that $0 < a_i < 1$ for all $i \in \mathbb{I}$. We define for $x \in X^+(N)$ and $n \in \bar{X}^+(N)$,

$$K_n^{(+)}(x; a, N) = \frac{1}{(-N)_{|n|}} \prod_{i \in \mathbb{N}_0} (-N_i)_{n_i} K_{n_i}(x_i; a_i, N_i), \quad (66)$$

also for $x, n \in X(N)$,

$$K_n^{(\infty)}(x; a, N) = \frac{(-1)^N}{N!} \prod_{i \in \mathbb{Z}} (-N_i)_{n_i} K_{n_i}(x_i; a_i, N_i), \quad (67)$$

where $N_i = N - |x_{(i-1)}| - |n^{(i+1)}|$ is given by the same formula as (59) for both (66) and (67).

These infinite-variate extensions are also special cases of the extensions $K_n^{(+)}(x; U, N)$ and $K_n^{(\infty)}(x; U, N)$ with special parameter matrices U :

Proposition 1.4.5. For $a = (a_i)_{i \in \mathbb{I}}$, we set a parameter matrix $U_a = (u_{i,j})_{i,j \in \mathbb{I}}$ by $u_{i,j} = 0$ for $i < j$, $u_{i,j} = 1$ for $i > j$ and $u_{i,i} = 1 - a_i^{-1}$, that is,

$$U_a = \begin{pmatrix} \ddots & & \mathbb{O} \\ & 1 - a_i^{-1} & \\ \mathbb{1} & & \ddots \end{pmatrix}. \quad (68)$$

Then we have

$$K_n^{(+)}(x; a, N) = K_n^{(+)}(x; U_a, N) \quad \text{for } \mathbb{I} = \mathbb{N}_0, \quad (69)$$

or

$$K_n^{(\infty)}(x; a, N) = K_n^{(\infty)}(x; U_a, N) \quad \text{for } \mathbb{I} = \mathbb{Z}. \quad (70)$$

Proof. We consider the case of $\mathbb{I} = \mathbb{N}_0$. For any $\ell \geq 1$, Proposition 1.4.2 gives the relation

$$K_{n|\ell}^{(\ell)}(x|_\ell; a|_\ell, N) = K_{n|\ell}^{(\ell)}(x|_\ell; U_a|_\ell, N),$$

where the notation $x|_\ell$ is given in (34). As ℓ is large enough, the left hand side is equal to $K_n^{(+)}(x; a, N)$ by the definition (66), and the right hand side is equal to $K_n^{(+)}(x; U_a, N)$ by the equality (35), thus we have (69). The case of $\mathbb{I} = \mathbb{Z}$ is similarly verified by the equalities (69) and (37). $\square$

In order to give the orthogonalities of the extensions $K_n^{(+)}(x; a, N)$ or $K_n^{(\infty)}(x; a, N)$, we need to assume a condition for parameters $a = (a_i)_{i \in \mathbb{I}}$ such that

$$\lim_{i \rightarrow \infty} \prod_{k=0}^i (1 - a_k) = 0. \quad (71)$$

We set parameters $p = (p_i)_{i \in \mathbb{I}}$ by $p_i = a_i \prod_{k=0}^{i-1} (1 - a_k)$ for $i \geq 0$ and $p_i = a_i \prod_{k=i}^{-1} (1 - a_k)^{-1}$ for $i \leq -1$. Then under the condition (71), we have the following results by straightforward calculations:

$$|p^{(0)}| := \sum_{i \geq 0} p_i = 1, \quad (u^{(j)}, u^{(j)})_p = \begin{cases} a_j^{-1} \prod_{k=0}^j (1 - a_k) & (j \geq 0), \\ a_j^{-1} \prod_{k=j+1}^{-1} (1 - a_k)^{-1} & (j \leq -1). \end{cases} \quad (72)$$

Since the matrix U_a in (68) satisfies the orthogonality condition in Theorem 1.3.3 or Theorem 1.2.5, we can put these results into the orthogonality relation (46) or (28). Then,

Proposition 1.4.6. *We have for $n, m \in \bar{X}^+(N)$,*

$$\begin{aligned} \sum_{x \in X^+(N)} K_n^{(+)}(x; a, N) K_m^{(+)}(x; a, N) \binom{N}{x} \prod_{i \geq 0} a_i^{x_i} (1 - a_i)^{N - |x_{(i)}|} \\ = \delta_{nm} \binom{N}{n}^{-1} \prod_{i \geq 0} a_i^{-n_i} (1 - a_i)^{|n^{(i)}|}, \end{aligned} \quad (73)$$

and for $n, m \in X(N)$,

$$\begin{aligned} \sum_{x \in X(N)} K_n^{(\infty)}(x; a, N) K_m^{(\infty)}(x; a, N) \binom{N}{x} \prod_{i \in \mathbb{Z}} a_i^{x_i} (1 - a_i)^{\delta_{i, |i|} N - |x_{(i)}|} \\ = \delta_{nm} \binom{N}{n}^{-1} \prod_{i \in \mathbb{Z}} a_i^{-n_i} \prod_{i \leq -1} (1 - a_i)^{-N + |n^{(i)}|} \prod_{i \geq 0} (1 - a_i)^{|n^{(i)}|}. \end{aligned} \quad (74)$$

We remark for completeness of these infinite extensions. As for the case $\mathbb{I} = \mathbb{N}_0$, we can verify the system $\{u^{(j)}\}_{j \in \mathbb{N}_0 \cup \{-\infty\}}$ of columns of U_a forms a complete system of $l_p^2(\mathbb{N}_0)$, therefore the system $\{K_n^{(+)}(\cdot; a, N)\}_{n \in \bar{X}^+(N)}$ forms a complete system of $V_p^+(N)$ by Theorem 1.3.3. However for the case $\mathbb{I} = \mathbb{Z}$, we need to assume one more condition of $a = (a_i)_{i \in \mathbb{Z}}$ such that

$$\lim_{i \rightarrow -\infty} \prod_{k=i}^{-1} (1 - a_k) = 0. \quad (75)$$

Then we can show completeness of $\{u^{(j)}\}_{j \in \mathbb{Z}}$ in l_p^2 , thus we have completeness of $\{K_n^{(\infty)}(\cdot; a, N)\}_{n \in X(N)}$ in $V_p(N)$ by Theorem 1.2.5. The verifications are left to the reader.

To describe a duality of ℓ -product Krawtchouk polynomials, we consider sorting of the degree indices n .

Lemma 1.4.7. *For a permutation $\sigma \in \mathfrak{S}_\ell$ of letters $\{0, 1, \dots, \ell - 1\}$ and a parameter matrix $U = (u_{i,j})_{i,j \in [\ell]}$, we define ${}^\sigma U = (u_{i,\sigma^{-1}(j)})_{i,j \in [\ell]}$. Then we have for $x, n = (n_j)_{j \in [\ell]} \in \bar{X}(\ell, N)$,*

$$K_n^{(\ell)}(x; {}^\sigma U, N) = K_{(\sigma n)}^{(\ell)}(x; U, N), \quad (76)$$

where ${}^\sigma n = (n_{\sigma(j)})_{j \in [\ell]}$.

Proof. From the generating function (3), we have

$$\begin{aligned} \Phi^{(\ell)}(1, t; x, {}^\sigma U, N) &= \left(1 + \sum_{j \in [\ell]} t_j\right)^{N-|x|} \prod_{i \in [\ell]} \left(1 + \sum_{j \in [\ell]} u_{i,\sigma^{-1}(j)} t_j\right)^{x_i} \\ &= \left(1 + \sum_{j \in [\ell]} t_{\sigma(j)}\right)^{N-|x|} \prod_{i \in [\ell]} \left(1 + \sum_{j \in [\ell]} u_{i,j} t_{\sigma(j)}\right)^{x_i} = \Phi^{(\ell)}(1, {}^\sigma t; x, U, N). \end{aligned}$$

Comparing the coefficients of $\prod_{j \in [\ell]} t_j^{n_j}$, we have (76). $\square$

For any $y = (y_i)_{i \in [\ell]}$, we denote the sorting in reverse order by

$$\overleftarrow{y} = (y_{\ell-i-1})_{i \in [\ell]}. \quad (77)$$

Then the polynomials $K_{\overleftarrow{n}}^{(\ell)}(x; a, N)$ are also special cases of the Griffiths's Krawtchouk polynomials:

Proposition 1.4.8. *For $a = (a_i)_{i \in [\ell]}$, we set a parameter matrix $\hat{U}_a = (u_{i,j})_{i,j \in [\ell]}$ by $u_{i,j} = 0$ for $i + j < \ell - 1$, $u_{i,j} = 1$ for $i + j > \ell - 1$ and $u_{i,j} = 1 - a_i^{-1}$ for $i + j = \ell - 1$, that is,*

$$\hat{U}_a = \begin{pmatrix} \mathbb{O} & & & 1 - a_0^{-1} \\ & \ddots & & 1 - a_1^{-1} \\ & & \ddots & \\ 1 - a_{\ell-1}^{-1} & & & \mathbb{1} \end{pmatrix}. \quad (78)$$

Then we have

$$K_{\overleftarrow{n}}^{(\ell)}(x; a, N) = K_n^{(\ell)}(x; \hat{U}_a, N). \quad (79)$$

Proof. We take the permutation $\sigma \in \mathfrak{S}_\ell$ in reverse order, that is, $\sigma(i) = \ell - i - 1$ ($0 \leq i \leq \ell - 1$). Then we note that ${}^\sigma U_a = \hat{U}_a$ and ${}^\sigma n = \overleftarrow{n}$. From Lemma 1.4.7 and Proposition 1.4.2, we have

$$K_{\overleftarrow{n}}^{(\ell)}(x; a, N) = K_{\overleftarrow{n}}^{(\ell)}(x; U_a, N) = K_n^{(\ell)}(x; \hat{U}_a, N). \quad \square$$

Since the matrix $\hat{U}_a$ in (78) satisfies ${}^t \hat{U}_a = \hat{U}_{\overleftarrow{a}}$, the duality (4) immediately induces a duality of $K_{\overleftarrow{n}}^{(\ell)}(x; a, N)$ such that

$$K_{\overleftarrow{n}}^{(\ell)}(x; a, N) = K_{\overleftarrow{x}}^{(\ell)}(n; \overleftarrow{a}, N) \quad (x, n \in \bar{X}(\ell, N)). \quad (80)$$

By the same idea, the full infinite-variate extensions $K_n^{(\infty)}(x; a, N)$ are also modified to have a duality. For any $y = (y_i)_{i \in \mathbb{Z}}$, we let $\overleftarrow{y} = (y_{-i})_{i \in \mathbb{Z}}$.

Proposition 1.4.9. *For $a = (a_i)_{i \in \mathbb{Z}}$, we set a parameter matrix $\hat{U}_a = (u_{i,j})_{i,j \in \mathbb{Z}}$ by $u_{i,j} = 0$ for $i + j < 0$, $u_{i,j} = 1$ for $i + j > 0$ and $u_{i,j} = 1 - a_i^{-1}$ for $i + j = 0$, that is,*

$$\hat{U}_a = \left(\begin{array}{c|c} \mathbb{O} & \begin{matrix} 1 - a_{-1}^{-1} & \ddots \end{matrix} \\ \hline \begin{matrix} 1 - a_0^{-1} \\ \vdots \end{matrix} & \mathbb{1} \end{array} \right), \quad (81)$$

where the vertical and horizontal lines divide the parts of negative and non-negative indices. Then we have

$$K_{\overleftarrow{n}}^{(\infty)}(x; a, N) = K_n^{(\infty)}(x; \hat{U}_a, N). \quad (82)$$

This proposition is verified similarly to Proposition 1.4.8. We leave details to the reader. Anyway, since the matrix $\hat{U}_a$ in (81) satisfies ${}^t\hat{U}_a = \hat{U}_{\overline{a}}$, we have a duality of $K_n^{(\infty)}(x; a, N)$ from (20), such that

$$K_n^{(\infty)}(x; a, N) = K_x^{(\infty)}(n; \overline{a}, N) \quad (x, n \in X(N)). \quad (83)$$

2 Realizations of Krawtchouk polynomials as zonal spherical functions

2.1 Zonal spherical functions on locally compact Abelian groups

In this section, we prepare basic facts about zonal spherical functions within a required range in realizations of Krawtchouk polynomials in the following sections. Let A be a locally compact Hausdorff Abelian group, G be a compact Hausdorff group acting on A continuously as group automorphisms. Here we define zonal spherical functions of the Gelfand pair $(A \rtimes G, G)$ as G -invariant (and mainly L^2 -) functions on A . We take the Haar measures μ on A and ν on G such that $\nu(G) = 1$.

Lemma 2.1.1. *The action $g : A \rightarrow A$ for any $g \in G$ preserves the measure μ .*

Proof. For each $g \in G$, we define a measure μ_g on A by

$$\mu_g(E) = \mu(g(E)) \quad (E \subset A, \mu\text{-measurable}).$$

Since μ_g is also the Haar measure on A , there exists $r_g > 0$ such that $\mu_g = r_g \mu$. We need to show $r_g = 1$ for all $g \in G$. We take an open set $U \subset A$ such that the closure $\bar{U}$ is compact. Then $W = \{g(a) \mid g \in G, a \in U\}$ satisfies that $g(W) = W$ for all $g \in G$ and $\bar{W}$ is compact. By Urysohn's lemma, we can take μ -mesurable sets $E, F \subset A$ such that $E \subset W \subset F$, $\mu(E) > 0$ and $\mu(F) < \infty$. We now assume $r_g \neq 1$ for some $g \in G$. Since $r_{g^{-1}} = r_g^{-1}$, we can suppose $r_g < 1$ by replacing g with g^{-1} if necessary. Then there exists $k \in \mathbb{N}$ such that $r_g^k \mu(F) < \mu(E)$. We have

$$\mu(E) \leq \mu(g^k F) = r_g^k \mu(F) < \mu(E),$$

which is a contradiction. □

We introduce the Fourier transformation on A . Let $\hat{A}$ denote the dual group of A , that is, $\hat{A} = \{\xi : A \rightarrow \mathbb{T} \mid \text{continuous group homomorphism}\}$,

equipped with pointwise multiplications on the torus $\mathbb{T} \subset \mathbb{C}^\times$. It is known that $\hat{A}$ is also a locally compact Abelian group by the compact-open topology. For $\varphi \in L^1(A)$, we can define the Fourier transformation $\mathcal{F}\varphi \in C(\hat{A})$, a continuous function on $\hat{A}$, as follows:

$$\mathcal{F}\varphi(\xi) = \int_A \varphi(a) \overline{\xi(a)} d\mu(a) \quad (\xi \in \hat{A}). \quad (84)$$

It is known that $\hat{A}$ has the uniquely normalized Haar measure $\hat{\mu}$, called the dual measure of μ , which gives the inverse transformation $\bar{\mathcal{F}}$ of $\mathcal{F}$ as follows. For $\psi \in L^1(\hat{A})$, we define $\bar{\mathcal{F}}\psi \in C(A)$ by

$$\bar{\mathcal{F}}\psi(a) = \int_{\hat{A}} \psi(\xi) \xi(a) d\hat{\mu}(\xi) \quad (a \in A), \quad (85)$$

then the following fact holds (the Plancherel's theorem, see [3]).

Fact 2.1.2. *There uniquely exists a unitary isomorphism $\mathcal{F} : L^2(A) \rightarrow L^2(\hat{A})$ which coincides with the Fourier transformation (84) on $L^1(A) \cap L^2(A)$, and its inverse $\mathcal{F}^{-1} : L^2(\hat{A}) \rightarrow L^2(A)$ coincides with the inverse transformation $\bar{\mathcal{F}}$ in (85) on $L^1(\hat{A}) \cap L^2(\hat{A})$.*

We consider the restrictions of $\mathcal{F}$ and $\bar{\mathcal{F}}$ on the subspaces of G -invariant functions as follows. The action of G on A and its contragredient action on $\hat{A}$ induce the permutation representations of G on $L^1(A)$ and $L^1(\hat{A})$ by Lemma 2.1.1, and on $C(A)$ and $C(\hat{A})$ by the continuity of the actions. Then we have

Lemma 2.1.3. *The transformations $\mathcal{F} : L^1(A) \rightarrow C(\hat{A})$ and $\bar{\mathcal{F}} : L^1(\hat{A}) \rightarrow C(A)$ are commutative with G -actions.*

Proof. We take $g \in G$, $\varphi \in L^1(A)$ and $\xi \in \hat{A}$, and show $g(\mathcal{F}\varphi)\xi = \mathcal{F}(g(\varphi))\xi$. By Lemma 2.1.1, we have

$$g(\mathcal{F}\varphi)\xi = \int_A \varphi(a) \overline{\xi(g(a))} d\mu(a) = \int_A \varphi(g^{-1}(b)) \overline{\xi(b)} d\mu(b) = \mathcal{F}(g(\varphi))\xi.$$

Similarly we can show about the inverse transformation. □

Lemma 2.1.3 implies that $\mathcal{F}$ and $\bar{\mathcal{F}}$ preserve G -invariance of functions, so we can restrict them to the G -invariant subspaces:

$$\mathcal{F} : L^1(A)^G \rightarrow C(\hat{A})^G, \quad \bar{\mathcal{F}} : L^1(\hat{A})^G \rightarrow C(A)^G. \quad (86)$$

We now put $\mathbb{G} = A \rtimes G$, the semidirect product group. A unitary $\mathbb{G}$ -representation V is called class- $\mathbf{1}_G$ when the G -invariant subspace V^G is one

dimensional. Then there exists an element $v_0 \in V^G$ such that $(v_0, v_0)_V = 1$, uniquely up to multiplications by complex numbers of the absolute value 1, which we call the G -invariant unit element of V . We can easily verify that the convolution algebra $L^1(G \backslash \mathbb{G}/G) = L^1(A)^G$ is commutative, as A is Abelian. This fact is called that $(\mathbb{G}, G)$ is a Gelfand pair, which is equivalent to the condition that any irreducible unitary $\mathbb{G}$ -representation V is either class- $\mathbf{1}_G$ or $V^G = 0$ (see [15] or [17]).

Irreducible class- $\mathbf{1}_G$ unitary $\mathbb{G}$ -representations are called the spherical representations of $(\mathbb{G}, G)$. For each spherical representation V , the zonal spherical function ω_V is defined as the diagonal matrix element of V with respect to the G -invariant unit element v_0 , that is,

$$\omega_V(\alpha) = (\alpha(v_0), v_0)_V \quad (\alpha \in \mathbb{G}). \quad (87)$$

It is easy to see that ω_V is a continuous bi- G -invariant non-zero function on $\mathbb{G}$, so we can regard that $\omega_V \in C(A)^G$.

For each $\xi \in \hat{A}$, we can construct a spherical representation $W_\xi \subset L^2(G)$ of the pair $(\mathbb{G}, G)$ as follows. We define $\mathbb{G}$ -action τ_ξ on $L^2(G)$ by

$$\tau_\xi(a, g)\varphi = \phi_{\xi, a} \cdot R_g\varphi \quad ((a, g) \in \mathbb{G}, \varphi \in L^2(G)), \quad (88)$$

where $\phi_{\xi, a}$ is a continuous function on G defined by

$$\phi_{\xi, a}(h) = \xi(h(a)) \quad (h \in G), \quad (89)$$

and $R_g\varphi$ denotes the right g -shift of φ , that is, $R_g\varphi(h) = \varphi(hg)$ ($h \in G$). It is easy to verify that $(\tau_\xi, L^2(G))$ is a class- $\mathbf{1}_G$ unitary representation with a G -invariant unit element 1 (the constant function on G). We set $W_\xi = \overline{\text{Span}\{\phi_{\xi, a} \mid a \in A\}} \subset L^2(G)$, where overline means the closure in $L^2(G)$. Then (τ_ξ, W_ξ) is a spherical representation of $(\mathbb{G}, G)$ by the following lemma.

Lemma 2.1.4. *For any class- $\mathbf{1}_G$ unitary $\mathbb{G}$ -representation V with a G -invariant unit element v_0 , the subspace $W = \overline{\text{Span}\{\alpha(v_0) \mid \alpha \in \mathbb{G}\}}$ is a spherical representation of $(\mathbb{G}, G)$.*

Proof. Obviously, W is a closed class- $\mathbf{1}_G$ subrepresentation of V . We prove the irreducibility of W . Let $U \subset W$ be a closed subrepresentation of W , and $U^\perp$ denote its orthogonal complement in W . If we express $v_0 = v_0^\parallel + v_0^\perp$ ($v_0^\parallel \in U$, $v_0^\perp \in U^\perp$), then we have $v_0^\parallel, v_0^\perp \in W^G$. Since W is class- $\mathbf{1}_G$, we have $v_0^\parallel = 0$ or $v_0^\perp = 0$, that is, $v_0 \in U$ or $v_0 \in U^\perp$. Then by the definition of W , we have $W \subset U$ or $W \subset U^\perp$, therefore $U = W$ or $U = 0$, which proves the irreducibility of W . $\square$

By the definition (87), the zonal spherical function $\omega_\xi \in C(A)^G$ associated with W_ξ is given as a continuous G -invariant function on A as follows:

$$\omega_\xi(a) = \int_G \phi_{\xi,a} d\nu = \int_G \xi(g(a)) d\nu(g) \quad (a \in A). \quad (90)$$

These zonal spherical functions appear as ‘kernel functions’ of the G -invariant Fourier transformation and its inverse (86) as follows:

Proposition 2.1.5. (i) For $\varphi \in L^1(A)^G$ and $\xi \in \hat{A}$,

$$\mathcal{F}\varphi(\xi) = \int_A \varphi(a) \overline{\omega_\xi(a)} d\mu(a). \quad (91)$$

(ii) For $\psi \in L^1(\hat{A})^G$ and $a \in A$,

$$\bar{\mathcal{F}}\psi(a) = \int_{\hat{A}} \psi(\xi) \omega_\xi(a) d\hat{\mu}(\xi). \quad (92)$$

Proof. Since $\mathcal{F}\varphi$ is G -invariant as seen in Lemma 2.1.3, we have

$$\mathcal{F}\varphi(\xi) = \int_G \mathcal{F}\varphi(g(\xi)) d\nu(g) = \int_A \varphi(a) \int_G \overline{\xi(g^{-1}(a))} d\nu(g) d\mu(a).$$

From the formula of $\omega_\xi(a)$ in (90), its right hand side is equal to that of (91). We can prove (92) similarly. $\square$

We note that when $\xi, \eta \in \hat{A}$ are in the same G -orbit $\mathcal{P} \subset \hat{A}$, then W_ξ and W_η are unitary equivalent. In fact, if $\eta = \xi \circ h$ for some $h \in G$, the left h -shift L_h is a unitary intertwiner from $(\tau_\xi, L^2(G))$ to $(\tau_\eta, L^2(G))$, and $L_h(W_\xi) = W_\eta$. We thus denote by $W_{\mathcal{P}}$ the unitary equivalent class of spherical representations including all W_ξ with $\xi \in \mathcal{P}$. The associated zonal spherical function $\omega_{\mathcal{P}}$ is uniquely determined by $\omega_{\mathcal{P}} = \omega_\xi$ for any $\xi \in \mathcal{P}$.

We use the following notation for the characteristic functions: for any pair of a set and its subset $Y \subset Z$,

$$1_Y(x) = 1_Y^{\text{on}Z}(x) = \begin{cases} 1 & (x \in Y), \\ 0 & (x \in Z \setminus Y). \end{cases} \quad (93)$$

We mainly deal with open G -orbits in A and $\hat{A}$ below, so we set

$$\Omega = \{\mathcal{O} \subset A \mid \text{open } G\text{-orbit}\} \quad \text{and} \quad \hat{\Omega} = \{\mathcal{P} \subset \hat{A} \mid \text{open } G\text{-orbit}\}. \quad (94)$$

Since any $\mathcal{O} \in \Omega$ and $\mathcal{P} \in \hat{\Omega}$ are open and compact, the characteristic functions $1_{\mathcal{O}} = 1_{\mathcal{O}}^{\text{on}A}$, $1_{\mathcal{P}} = 1_{\mathcal{P}}^{\text{on}\hat{A}}$ are continuous, in particular μ - or $\hat{\mu}$ -measurable,

and they have finite measures $0 < \mu(\mathcal{O}) < \infty$, $0 < \hat{\mu}(\mathcal{P}) < \infty$. By Proposition 2.1.5, we have

$$\mathcal{F}1_{\mathcal{O}}(\xi) = \int_{\mathcal{O}} \overline{\omega_{\xi}(a)} d\mu(a) = \mu(\mathcal{O}) \overline{\omega_{\xi}(\mathcal{O})} \quad (\xi \in \hat{A}), \quad (95)$$

$$\bar{\mathcal{F}}1_{\mathcal{P}}(a) = \int_{\mathcal{P}} \omega_{\xi}(a) d\hat{\mu}(\xi) = \hat{\mu}(\mathcal{P}) \omega_{\mathcal{P}}(a) \quad (a \in A). \quad (96)$$

Proposition 2.1.6. *When $\mathcal{P} \in \hat{\Omega}$, we have $\omega_{\mathcal{P}} \in L^2(A)$. Furthermore, the following orthogonal relation holds:*

$$(\omega_{\mathcal{P}}, \omega_{\mathcal{P}'})_{L^2(A)} = \delta_{\mathcal{P}, \mathcal{P}'} \frac{1}{\hat{\mu}(\mathcal{P})} \quad (\mathcal{P}, \mathcal{P}' \in \hat{\Omega}). \quad (97)$$

Proof. From (96), we know $\omega_{\mathcal{P}} = \frac{1}{\hat{\mu}(\mathcal{P})} \bar{\mathcal{F}}1_{\mathcal{P}}$. A remark that $1_{\mathcal{P}} \in L^1(\hat{A}) \cap L^2(\hat{A})$ and Fact 2.1.2 thus induce $\omega_{\mathcal{P}} \in L^2(A)$. We also have

$$(\omega_{\mathcal{P}}, \omega_{\mathcal{P}'})_{L^2(A)} = \frac{1}{\hat{\mu}(\mathcal{P})\hat{\mu}(\mathcal{P}')} (1_{\mathcal{P}}, 1_{\mathcal{P}'})_{L^2(\hat{A})} = \delta_{\mathcal{P}, \mathcal{P}'} \frac{1}{\hat{\mu}(\mathcal{P})}.$$

□

When the set $\hat{\Omega}$ has a good condition, the zonal spherical functions have a completeness in $L^2(A)^G$:

Proposition 2.1.7. *We assume*

$$\hat{\mu} \left(\hat{A} \setminus \bigcup_{\mathcal{P} \in \hat{\Omega}} \mathcal{P} \right) = 0, \quad (98)$$

that is, orbits belonging to $\hat{\Omega}$ occupy almost everywhere in $\hat{A}$. Then the system $\{\omega_{\mathcal{P}} \mid \mathcal{P} \in \hat{\Omega}\}$ forms a complete orthogonal system of the Hilbert space $L^2(A)^G$.

Proof. Under the condition (98), we can verify that $L^2(\hat{A})^G = \overline{\text{Span}\{1_{\mathcal{P}} \mid \mathcal{P} \in \hat{\Omega}\}}$. The isomorphism $\mathcal{F}^{-1}$ in Fact 2.1.2 is restricted to G -invariant functions such as $\mathcal{F}^{-1} : L^2(\hat{A})^G \rightarrow L^2(A)^G$, thus by (96) we have $L^2(A)^G = \overline{\text{Span}\{\omega_{\mathcal{P}} \mid \mathcal{P} \in \hat{\Omega}\}}$, which completes the proof. □

We describe a duality of the zonal spherical functions:

Proposition 2.1.8. *We assume that there exists a topological group isomorphism $\Theta : A \rightarrow \hat{A}$, $a \mapsto \Theta_a$ which satisfies the following conditions:*

- (i) Θ preserves open G -orbits, that is, for any subset $\mathcal{O} \subset A$,
 $\mathcal{O} \in \Omega$ if and only if $\Theta(\mathcal{O}) \in \hat{\Omega}$.
- (ii) $\Theta_a(b) = \Theta_b(a)$ holds for arbitrary $a, b \in A$.

Then we have a duality of the zonal spherical functions: for open G -orbits $\mathcal{O} \in \Omega$ and $\mathcal{P} \in \hat{\Omega}$,

$$\omega_{\mathcal{P}}(\mathcal{O}) = \omega_{\Theta(\mathcal{O})}(\Theta^{-1}(\mathcal{P})). \quad (99)$$

Proof. We remark that there exists a real number $r > 0$ such that $\mu(E) = r \cdot \hat{\mu}(\Theta(E))$ for any μ -measurable set $E \subset A$, since $E \mapsto \hat{\mu}(\Theta(E))$ is the Haar measure on A . We fix $\xi \in \mathcal{P}$ and let $\xi = \Theta_b$ by an element $b \in \Theta^{-1}(\mathcal{P})$. Then we have from (95),

$$\begin{aligned} \omega_{\mathcal{P}}(\mathcal{O}) &= \frac{1}{\mu(\mathcal{O})} \overline{\mathcal{F}1_{\mathcal{O}}(\xi)} = \frac{1}{\mu(\mathcal{O})} \int_{\mathcal{O}} \xi(a) d\mu(a) \\ &= \frac{1}{\mu(\mathcal{O})} \int_{\mathcal{O}} \Theta_a(b) d\mu(a) = \frac{1}{r \cdot \hat{\mu}(\Theta(\mathcal{O}))} \int_{\Theta(\mathcal{O})} \eta(b) r d\hat{\mu}(\eta) = \omega_{\Theta(\mathcal{O})}(\Theta^{-1}(\mathcal{P})). \quad \square \end{aligned}$$

2.2 Notations for a non-Archimedean local field

In the following three sections, we take some groups concerning a non-Archimedean local field as examples of the setting in the previous section. We will see the zonal spherical functions are expressed in terms of the multivariate Krawtchouk polynomials or their infinite-variate extensions.

Let F be a non-Archimedean local field, $v : F \rightarrow \mathbb{Z} \cup \{\infty\}$ be the discrete valuation, $\mathfrak{o} = \{a \in F \mid v(a) \geq 0\}$ be the ring of integers of F , and $\mathfrak{p} = \{a \in F \mid v(a) \geq 1\}$ be the maximal ideal of $\mathfrak{o}$. The residue field $\mathfrak{o}/\mathfrak{p}$ has the finite order q , a power of a prime. Then for $\ell \geq 1$, the residue ring $\mathfrak{o}/\mathfrak{p}^\ell$ is a finite ring of the order q^ℓ .

We deal with examples for the additive Abelian groups $A = (\mathfrak{o}/\mathfrak{p}^\ell)^N$, $\mathfrak{o}^N$ and F^N . The actions on A are commonly given by the wreath product group $G = (\mathfrak{o}^\times)^N \rtimes \mathfrak{S}_N$, where $\mathfrak{o}^\times$ is the multiplicative group of $\mathfrak{o}$ and $\mathfrak{S}_N$ is the symmetric group of order N , as follows:

$$\begin{aligned} (c, \sigma)a &= (c_1 a_{\sigma^{-1}(1)}, \dots, c_N a_{\sigma^{-1}(N)}), \\ \text{where } c &= (c_k)_{k=1}^N \in (\mathfrak{o}^\times)^N, \sigma \in \mathfrak{S}_N, a = (a_k)_{k=1}^N \in A. \end{aligned} \quad (100)$$

The orbits of these actions and their measures are given by the following lemma (we leave out the proof, which may be obvious from the action (100)):

Lemma 2.2.1. *We assume that $\mathfrak{o}^\times$ acts on a set S , and the orbits decomposition is given by $S = \bigsqcup_{i \in \mathbb{I} \sqcup \{\infty\}} S_i$, where $\mathbb{I}$ is an index set (either finite or infinite), and ∞ is just a symbol. We set*

$$\bar{X}(\mathbb{I}, N) = \{x = (x_i)_{i \in \mathbb{I}} \mid x_i \in \mathbb{N}_0, |x| \leq N\}. \quad (101)$$

Then the action of $G = (\mathfrak{o}^\times)^N \rtimes \mathfrak{S}_N$ on $A = S^N$ given by (100) has the following orbits decomposition: $S^N = \bigsqcup_{x \in \bar{X}(\mathbb{I}, N)} \mathcal{O}(x)$,

$$\mathcal{O}(x) = \left\{ (a_k)_{k=1}^N \in S^N \mid \forall i \in \mathbb{I}, \#\{k \mid a_k \in S_i\} = x_i \right\}. \quad (102)$$

Moreover if S has a measure μ such that all $\mathfrak{o}^\times$ -orbits S_i are measurable, then the product measure $\mu^{\times N}$ on S^N gives the measures of G -orbits by

$$\mu^{\times N}(\mathcal{O}(x)) = \binom{N}{x} \prod_{i \in \mathbb{I}} \mu(S_i)^{x_i} \times \mu(S_\infty)^{N-|x|} \quad (x \in \bar{X}(\mathbb{I}, N)). \quad (103)$$

2.3 ℓ -variate Krawtchouk polynomials as zonal spherical functions on $(\mathfrak{o}/\mathfrak{p}^\ell)^N$

We fix a positive integer $\ell \geq 1$, and denote $R = \mathfrak{o}/\mathfrak{p}^\ell$. Here we consider an example that $G = G_N = (\mathfrak{o}^\times)^N \rtimes \mathfrak{S}_N$ acts on $A = R^N$ by (100). Since A is finite, we can take the counting measure as the Haar measure on A , and give A the discrete topology.

The valuation v on F induces the valuation $v : R \rightarrow [\ell] \cup \{\infty\}$ by

$$v(a + \mathfrak{p}^\ell) = \begin{cases} v(a) & \text{if } v(a) \in [\ell] \\ \infty & \text{if } v(a) \geq \ell \end{cases} \quad (a \in \mathfrak{o}), \quad (104)$$

where $[\ell] = \{0, 1, \dots, \ell - 1\}$ (Recall the notation in Section 1.1). This valuation gives the $\mathfrak{o}^\times$ -orbits decomposition $R = \bigsqcup_{i \in [\ell] \cup \{\infty\}} R_i$, $R_i = \{a \in R \mid v(a) = i\}$. Since $R_i = \mathfrak{p}^i/\mathfrak{p}^\ell - \mathfrak{p}^{i+1}/\mathfrak{p}^\ell$, we have

$$|R_i| = q^{\ell-i-1}(q-1) \quad (i \in [\ell]). \quad (105)$$

Lemma 2.2.1 gives the G -orbits decomposition $R^N = \bigsqcup_{x \in \bar{X}(\ell, N)} \mathcal{O}(x)$,

$$\mathcal{O}(x) = \{(a_k)_{k=1}^N \in R^N \mid \forall i \in [\ell], \#\{k \mid a_k \in R_i\} = x_i\}, \quad (106)$$

where the parameter set $\bar{X}(\ell, N)$ is defined in (1). Cardinalities of the orbits are from (103) and (105),

$$|\mathcal{O}(x)| = \binom{N}{x} (q-1)^{|x|} q^{(\ell-1)|x| - \Delta(x)} \quad (x \in \bar{X}(\ell, N)), \quad (107)$$

where and in the following contents, we use a notation

$$\Delta(x) = \sum_i i x_i. \quad (108)$$

On the dual group $\hat{R}$, we define the ‘dual valuation’ $r : \hat{R} \rightarrow \{-\infty\} \cup [\ell]$,

$$r(1) = -\infty, \quad r(\xi) = \max\{v(a) \mid a \in R, \xi(a) \neq 1\} \quad (\xi \neq 1). \quad (109)$$

Lemma 2.3.1. *The orbits decomposition with respect to the contragredient $\mathfrak{o}^\times$ -action on $\hat{R}$ is given by $\hat{R} = \bigsqcup_{j \in \{-\infty\} \cup [\ell]} \hat{R}_j$, $\hat{R}_j = \{\xi \in \hat{R} \mid r(\xi) = j\}$, and the cardinalities of orbits are*

$$|\hat{R}_j| = q^j(q-1) \quad (j \in [\ell]). \quad (110)$$

Proof. We note that $\hat{R}_{\ell-1}$ is not empty. In fact, using the canonical surjection $\pi_{\ell, \ell-1} : R = (\mathfrak{o}/\mathfrak{p}^\ell) \rightarrow (\mathfrak{o}/\mathfrak{p}^{\ell-1})$, we can define an injection $\pi_{\ell, \ell-1}^* : (\mathfrak{o}/\mathfrak{p}^{\ell-1})^\wedge \rightarrow \hat{R}$, $\xi \mapsto \xi \circ \pi_{\ell, \ell-1}$ and verify that $\text{Image}(\pi_{\ell, \ell-1}^*) = \cup_{j \leq \ell-2} \hat{R}_j$. It induces $|\hat{R}_{\ell-1}| = |R| - |\mathfrak{o}/\mathfrak{p}^{\ell-1}| = q^{\ell-1}(q-1)$. We fix one of $\theta \in \hat{R}_{\ell-1}$ and define a group isomorphism $\Theta : R \rightarrow \hat{R}$, $a \mapsto \Theta_a$ by

$$\Theta_a(b) = \theta(ab) \quad (b \in A). \quad (111)$$

Then we can verify that $\Theta(R_i) = \hat{R}_{\ell-i-1}$ for $i \in [\ell] \cup \{\infty\}$ and Θ preserves $\mathfrak{o}^\times$ -orbits. These observations conclude that $\hat{R} = \bigsqcup_{j \in \{-\infty\} \cup [\ell]} \hat{R}_j$ is the $\mathfrak{o}^\times$ -orbits decomposition, and $|\hat{R}_j| = |R_{\ell-j-1}| = q^j(q-1)$ for $j \in [\ell]$ from (105). $\square$

The dual group $\hat{A} = (R^N)^\wedge$ is identified with $(\hat{R})^N$ by the following correspondence:

$$\xi(a) = \prod_{k=1}^N \xi_k(a_k) \quad \left(\xi = (\xi_k)_{k=1}^N \in (\hat{R})^N, a = (a_k)_{k=1}^N \in R^N \right). \quad (112)$$

Lemma 2.2.1 thus gives the G -orbits decomposition $(R^N)^\wedge = \bigsqcup_{n \in \bar{X}(\ell, N)} \mathcal{P}(n)$,

$$\mathcal{P}(n) = \{(\xi_k)_{k=1}^N \in (\hat{R})^N \mid \forall j \in [\ell], \#\{k \mid \xi_k \in \hat{R}_j\} = n_j\}. \quad (113)$$

From (103), cardinalities of the orbits are given by

$$|\mathcal{P}(n)| = \binom{N}{n} (q-1)^{|n|} q^{\Delta(n)} \quad (n \in \bar{X}(\ell, N)). \quad (114)$$

In general, we introduce the following lemma about the dual measure. We remark if an Abelian group A is compact, then $\hat{A}$ has desctete topology (see [14] etc.).

Lemma 2.3.2. *For the Haar measure μ on a compact Abelian group A , the dual measure of μ on $\hat{A}$ is given by*

$$\frac{1}{\mu(A)} \times (\text{counting measure}). \quad (115)$$

Proof. We know there exists $r > 0$ such that $\hat{\mu} = r \times (\text{counting measure})$. We denote by 1_1 the characteristic function of 1 on $\hat{A}$. Then since

$$\bar{\mathcal{F}}1_1(a) = \sum_{\xi \in \hat{A}} 1_1(\xi) \xi(a) \hat{\mu}(\{\xi\}) = \hat{\mu}(\{1\}) = r \quad (a \in A),$$

we have

$$1 = \mathcal{F}(\bar{\mathcal{F}}1_1)(1) = \int_{a \in A} \bar{\mathcal{F}}1_1(a) \overline{1(a)} d\mu(a) = r \cdot \mu(A),$$

which proves the lemma. $\square$

Under these notations, we denote by ω_n the zonal spherical function associated with the orbit $\mathcal{P}(n)$, and by $\omega_n(x)$ its value on the orbit $\mathcal{O}(x)$, where $x, n \in \bar{X}(\ell, N)$. In this example there exists a group isomorphism $\Theta : R^N \rightarrow (R^N)^\wedge$, $a \mapsto \Theta_a$ which satisfies the conditions in Proposition 2.1.8. We fix $\theta \in \hat{R}_{\ell-1}$, and use the symmetric non-degenerate bilinear form $\langle a | b \rangle = \sum_{k=1}^N a_k b_k$ on R^N . For $a \in R^N$, we define $\Theta_a \in (R^N)^\wedge$ by $\Theta_a(b) = \theta(\langle a | b \rangle)$, $b \in R^N$. Then for any $x \in \bar{X}(\ell, N)$,

$$\Theta(\mathcal{O}(x)) = \mathcal{P}(\overleftarrow{x}) \quad (116)$$

holds, where $\overleftarrow{x} = (x_{\ell-i-1})_{i \in [\ell]}$ is in reverse order as defined in (77). In fact, we can regard $\Theta_a = (\Theta_{a_k})_{k=1}^N \in (\hat{R})^N$, where Θ_{a_k} 's are defined in (111). As remarked in the proof of Lemma 2.3.1, we have $r(\Theta_{a_k}) = \ell - v(a_k) - 1$, which induces (116). We know from (116) that Θ satisfies the condition (i) in Proposition 2.1.8, which thus gives the following duality relation of the zonal spherical functions:

$$\omega_{\overleftarrow{n}}(x) = \omega_{\overleftarrow{x}}(n) \quad (x, n \in \bar{X}(\ell, N)). \quad (117)$$

The main objective of this section is to express $\omega_n(x)$ in terms of the ℓ -variate Krawtchouk polynomials:

Theorem 2.3.3. *For $x, n \in \bar{X}(\ell, N)$, we have*

$$\omega_n(x) = K_n^{(\ell)}(x; \frac{q-1}{q}, N), \quad (118)$$

where the right hand side is the ℓ -procuct Krawtchouk polynomial in Definition 1.4.1 with a 'constant parameter' $\frac{q-1}{q} = (\frac{q-1}{q})_{i \in [\ell]}$.

We prove this theorem by use of a recurrence formula in the form of (13). The following lemma is a fundamental framework which induces these recurrence formulae (see [8]).

Lemma 2.3.4. *Let A and B be finite Abelian groups and $\pi : A \rightarrow B$ be a group homomorphism. We denote by $\mathcal{F}_A : C(A) \rightarrow C(\hat{A})$ and $\mathcal{F}_B : C(B) \rightarrow C(\hat{B})$ be the Fourier transformations. We remark since A is finite, $C(A) = L^1(A)$ is the space of all complex-valued functions on A . We fix an arbitrary character $\zeta \in \hat{A}$, and define a transformation $\pi_* : C(A) \rightarrow C(B)$ by*

$$\pi_*\varphi(b) = \sum_{a \in \pi^{-1}(b)} \overline{\zeta(a)}\varphi(a) \quad (\varphi \in C(A), b \in B). \quad (119)$$

On the other hand, we define $\hat{\pi} : \hat{B} \rightarrow \hat{A}$ by

$$\hat{\pi}\xi = (\xi \circ \pi) \cdot \zeta \quad (\xi \in \hat{B}), \quad (120)$$

and let $\hat{\pi}^* : C(\hat{A}) \rightarrow C(\hat{B})$, $\hat{\pi}^*\psi = \psi \circ \hat{\pi}$. Then $\hat{\pi}^* \circ \mathcal{F}_A = \mathcal{F}_B \circ \pi_*$ holds, that is, the following diagram is commutative:

$$\begin{array}{ccc} C(A) & \xrightarrow{\mathcal{F}_A} & C(\hat{A}) \\ \pi_* \downarrow & \circlearrowleft & \downarrow \hat{\pi}^* \\ C(B) & \xrightarrow{\mathcal{F}_B} & C(\hat{B}) \end{array} \quad (121)$$

Proof. For arbitrary $\varphi \in C(A)$ and $\xi \in \hat{B}$, we have

$$\begin{aligned} \hat{\pi}^*(\mathcal{F}_A\varphi)(\xi) &= \mathcal{F}_A\varphi(\hat{\pi}\xi) = \sum_{a \in A} \overline{\xi(\pi(a))\zeta(a)}\varphi(a) \\ &= \sum_{b \in B} \sum_{a \in \pi^{-1}(b)} \overline{\xi(b)\zeta(a)}\varphi(a) = \sum_{b \in B} \overline{\xi(b)}\pi_*\varphi(b) = \mathcal{F}_B(\pi_*\varphi)(\xi). \quad \square \end{aligned}$$

From here on in this section, in order to specify the size N , we denote the $G = G_N$ -orbits by $\mathcal{O}(x) = \mathcal{O}(x; N)$ and $\mathcal{P}(n) = \mathcal{P}(n; N)$, the zonal spherical functions by $\omega_n(x) = \omega_n(x; N)$, the Fourier transformation on R^N by $\mathcal{F} = \mathcal{F}_N$ and so on. We apply Lemma 2.3.4 as $A = R^N$ and $B = R^{N-1}$ in this example. We let $\pi : R^N \rightarrow R^{N-1}$, $(a_k)_{k=1}^N \mapsto (a_k)_{k=1}^{N-1}$, a projection map. We fix $\theta \in \hat{R}_{\ell-1}$, and define $\zeta = (1, \dots, 1, \theta) \in \hat{R}^N$, that is, $\zeta((a_k)_{k=1}^N) = \theta(a_N)$. For these settings, we define π_* and $\hat{\pi}^*$ in Lemma 2.3.4, then we have the following commutative diagram:

$$\begin{array}{ccc} C(R^N) & \xrightarrow{\mathcal{F}_N} & C(\hat{R}^N) \\ \pi_* \downarrow & \circlearrowleft & \downarrow \hat{\pi}^* \\ C(R^{N-1}) & \xrightarrow{\mathcal{F}_{N-1}} & C(\hat{R}^{N-1}) \end{array} \quad (122)$$

Lemma 2.3.5. *For any G_N -invariant function φ on R^N , the transformation $\pi_*\varphi$ is a G_{N-1} -invariant function on R^{N-1} , and we have*

$$\pi_*\varphi(y) = \varphi(y) - \varphi(y + e_{\ell-1}) \quad (y \in \bar{X}(\ell, N-1)), \quad (123)$$

where $\varphi(x)$ denotes the value of φ on the orbit $\mathcal{O}(x; N)$ for any $x \in \bar{X}(\ell, N)$.

Proof. For any $b = (b_k)_{k=1}^{N-1} \in R^{N-1}$, we have

$$\pi_*\varphi(b) = \sum_{a \in R} \overline{\theta(a)} \varphi(b, a) = \varphi(b, 0) + \sum_{i \in [\ell]} \sum_{a \in R_i} \overline{\theta(a)} \varphi(b, a), \quad (124)$$

where we denote $(b, a) = (b_1, b_2, \dots, b_{N-1}, a)$. With a remark that

$$\sum_{a \in R_i} \theta(a) = \begin{cases} 0 & (0 \leq i \leq \ell-2), \\ -1 & (i = \ell-1), \end{cases}$$

we have from (124) that $\pi_*\varphi(b) = \varphi(b, 0) - \varphi(b, a)$, $a \in R_{\ell-1}$, which proves the lemma. $\square$

Lemma 2.3.6. *For any G_N -invariant function ψ on $\hat{R}^N$, the transformation $\hat{\pi}^*\psi$ is a G_{N-1} -invariant function on $\hat{R}^{N-1}$, and we have*

$$\hat{\pi}^*\psi(m) = \psi(m + e_{\ell-1}) \quad (m \in \bar{X}(\ell, N-1)). \quad (125)$$

Proof. For any $\xi = (\xi_k)_{k=1}^{N-1} \in \mathcal{P}(m; N-1)$, we have

$$\hat{\pi}(\xi) = (\xi \circ \pi) \cdot \zeta = (\xi_1, \dots, \xi_{N-1}, \theta) \in \mathcal{P}(m + e_{\ell-1}; N),$$

which induces the lemma. $\square$

Lemma 2.3.7. *For $x \in \bar{X}(\ell, N-1)$ and $n \in \bar{X}(\ell, N)$, we have a recurrence formula*

$$\omega_n(x + e_{\ell-1}; N) - \omega_n(x; N) = -\frac{q}{q-1} \frac{n_{\ell-1}}{N} \omega_{n-e_{\ell-1}}(x; N-1). \quad (126)$$

Proof. From the diagram (122), we have

$$\pi_* \bar{\mathcal{F}}_N 1_n(x) = \bar{\mathcal{F}}_{N-1} \hat{\pi}^* 1_n(x), \quad (127)$$

where $1_n \in C(\hat{R}^N)$ is the characteristic function of the orbit $\mathcal{P}(n; N)$, and (x) denotes the value on the orbit $\mathcal{O}(x; N-1)$. By (96) and Lemma 2.3.2, we have $\bar{\mathcal{F}}_N 1_n = \frac{|\mathcal{P}(n; N)|}{|R^N|} \omega_n(\cdot; N)$, then by Lemma 2.3.5, the left hand side of (127) is equal to $\frac{|\mathcal{P}(n; N)|}{|R^N|} (\omega_n(x; N) - \omega_n(x + e_{\ell-1}; N))$. On the other hand, by Lemma 2.3.6 we have $\hat{\pi}^* 1_n = 1_n(\cdot + e_{\ell-1}) = 1_{n-e_{\ell-1}}$, and by (96), the right hand side of (127) is equal to $\frac{|\mathcal{P}(n-e_{\ell-1}; N-1)|}{|R^{N-1}|} \omega_{n-e_{\ell-1}}(x; N-1)$. Substituting $|R^N| = q^{\ell N}$ and (114), we have the recurrence formula (126). $\square$

We fix $x', n' \in \bar{X}(\ell - 1, N)$. For $0 \leq s \leq N - |x'|$ and $0 \leq r \leq N - |n'|$, we set

$$\Phi_r(s; N) = \omega_{n' \sqcup (r)}(x' \sqcup (s); N), \quad (128)$$

where $(y_i)_{i \in [\ell-1]} \sqcup (y_{\ell-1}) = (y_i)_{i \in [\ell]}$ denotes joint of entries. Then from (126), $\Phi_r(s; N)$ is the solution of the recurrence formula (13) with the parameter $p = \frac{q-1}{q}$:

$$\Phi_r(s; N) - \Phi_r(s-1; N) = -\frac{q}{q-1} \frac{r}{N} \Phi_{r-1}(s-1; N-1). \quad (129)$$

We need the initial values $\Phi_r(0, N) = \omega_{n' \sqcup (r)}(x' \sqcup (0); N)$.

Lemma 2.3.8. *For $x \in \bar{X}(\ell, N)$ such that $x_{\ell-1} = 0$ and $m \in \bar{X}(\ell, N-1)$, we have*

$$\omega_{m+e_{\ell-1}}(x; N) = \frac{N - |x|}{N} \omega_m(x; N-1). \quad (130)$$

Proof. By the diagram (122), we have

$$\hat{\pi}^* \mathcal{F}_N 1_x(m) = \mathcal{F}_{N-1} \pi_* 1_x(m). \quad (131)$$

From (95), we have $\mathcal{F}_N 1_x(n) = |\mathcal{O}(x; N)| \overline{\omega_n(x; N)}$, and from Lemma 2.3.6, the left hand side of (131) is equal to $|\mathcal{O}(x; N)| \overline{\omega_{m+e_{\ell-1}}(x; N)}$. On the other hand, by Lemma 2.3.5 and the assumption that $x_{\ell-1} = 0$, we have $\pi_* 1_x = 1_x$, and by (95), the right hand side of (131) is equal to $|\mathcal{O}(m; N-1)| \overline{\omega_m(x; N-1)}$. Substituting (107), we have (130). $\square$

By Lemma 2.3.8 we have

$$\begin{aligned} \omega_{n' \sqcup (r)}(x' \sqcup (0); N) &= \frac{-N + |x'|}{-N} \omega_{n' \sqcup (r-1)}(x' \sqcup (0); N-1) \\ &= \frac{(-N + |x'|)_2}{(-N)_2} \omega_{n' \sqcup (r-2)}(x' \sqcup (0); N-2) \\ &= \dots \\ &= \frac{(-N + |x'|)_r}{(-N)_r} \omega_{n' \sqcup (0)}(x' \sqcup (0); N-r). \end{aligned} \quad (132)$$

Lemma 2.3.9. *Let A, B be locally compact Abelian groups. We assume a compact group G acts on both A and B as topological group automorphisms, and there exists a topological group homomorphism $\mathfrak{f} : A \rightarrow B$ commutative with G -actions. Then for $\xi \in \hat{B}$, clearly we have $\xi \circ \mathfrak{f} \in \hat{A}$. We denote by $\omega_{\xi \circ \mathfrak{f}}^{(A)}$ and $\omega_{\xi}^{(B)}$ the corresponding zonal spherical functions on A and B respectively. Under these settings, we have*

$$\omega_{\xi \circ \mathfrak{f}}^{(A)} = \omega_{\xi}^{(B)} \circ \mathfrak{f} \quad (\text{on } A). \quad (133)$$

Proof. For any $a \in A$, we have from by the formula (90),

$$\omega_{\xi}^{(B)}(\mathfrak{f}(a)) = \int_G \xi(g(\mathfrak{f}(a)))dg = \int_G \xi(\mathfrak{f}(g(a)))dg = \omega_{\xi \circ \mathfrak{f}}^{(A)}(a).$$

□

We apply Lemma 2.3.9 for $A = (\mathfrak{o}/\mathfrak{p}^\ell)^N$, $B = (\mathfrak{o}/\mathfrak{p}^{\ell-1})^N$, $G = G_N$ and $\mathfrak{f} : (\mathfrak{o}/\mathfrak{p}^\ell)^N \rightarrow (\mathfrak{o}/\mathfrak{p}^{\ell-1})^N$, $(a_k + \mathfrak{p}^\ell)_{k=1}^N \mapsto (a_k + \mathfrak{p}^{\ell-1})_{k=1}^N$. We note that if $\xi \in \mathcal{P}(n')$ then $\xi \circ \mathfrak{f} \in \mathcal{P}(n' \sqcup (0))$, and if $a \in \mathcal{O}(x' \sqcup (s))$ then $\mathfrak{f}(a) \in \mathcal{O}(x')$ for any $s \geq 0$. The lemma thus gives the equation

$$\omega_{n' \sqcup (0)}(x' \sqcup (s)) = \omega_{n'}(x'). \quad (134)$$

From (132) and (134), we have

$$\Phi_r(0; N) = \frac{(-N + |x'|)_r}{(-N)_r} \omega_{n'}(x'; N - r). \quad (135)$$

By the use of Proposition 1.1.1 for the recurrence formula (129) and the initial values (135), we have

$$\omega_{n' \sqcup (r)}(x' \sqcup (s); N) = \frac{(-N + |x'|)_r}{(-N)_r} \omega_{n'}(x'; N - r) \cdot K_r(s; \frac{q-1}{q}, N - |x'|). \quad (136)$$

We can prove Theorem 2.3.3 by (136) and induction on ℓ .

At the end of this section, we confirm correspondences between properties of Krawtchouk polynomials and zonal spherical functions. Obviously the duality relation (117) of zonal spherical functions is a special case of the duality (80) of Krawtchouk polynomials. As for the orthogonality relation, by (97) we have

$$\sum_{x \in \bar{X}(\ell, N)} \omega_n(x) \overline{\omega_m(x)} |\mathcal{O}(x)| = \delta_{nm} \frac{|R^N|}{|\mathcal{P}(n)|} \quad (n, m \in \bar{X}(\ell, N)). \quad (137)$$

Substituting (107), (114) and Theorem 2.3.3, it turns into

$$\begin{aligned} \sum_{x \in \bar{X}(\ell, N)} K_n^{(\ell)}(x; \frac{q-1}{q}, N) K_m^{(\ell)}(x; \frac{q-1}{q}, N) \binom{N}{x} (q-1)^{|x|} q^{(\ell-1)|x| - \Delta(x)} \\ = \delta_{nm} \binom{N}{n}^{-1} (q-1)^{-|n|} q^{\ell N - \Delta(n)} \quad (n, m \in \bar{X}(\ell, N)), \end{aligned}$$

which is the orthogonality relation (65) of Krawtchouk polynomials for the constant parameter $a = (\frac{q-1}{q})_{i \in [\ell]}$.

2.4 Half infinite-variate extensions of Krawtchouk polynomials as zonal spherical functions on $\mathfrak{o}^N$

In this section, we deal with the example that $G = (\mathfrak{o}^\times)^N \rtimes \mathfrak{S}_N$ acts on $A = \mathfrak{o}^N$ by (100). The zonal spherical functions of this example are given in terms of the half infinite-variate extensions $K_n^{(+)}(x; U, N)$ of Krawtchouk polynomials. Again we can observe that the orthogonality and completeness of the zonal spherical functions correspond to the properties of the extensions $K_n^{(+)}(x; U, N)$. However since there are no topological group isomorphisms $\Theta : A \rightarrow \hat{A}$, we cannot have the duality (99). It may correspond to the fact that extensions $K_n^{(+)}(x; U, N)$ do not have dualities.

The $\mathfrak{o}^\times$ -orbits in $\mathfrak{o}$ are determined by the valuation $v : \mathfrak{o} \rightarrow \mathbb{N}_0 \cup \{\infty\}$, that is, $\mathfrak{o} = \bigsqcup_{i \in \mathbb{N}_0 \cup \{\infty\}} \mathfrak{o}_i$, $\mathfrak{o}_i = \{a \in \mathfrak{o} \mid v(a) = i\}$. By Lemma 2.2.1, the G -orbits decomposition is given by $\mathfrak{o}^N = \bigsqcup_{x \in \bar{X}^+(N)} \mathcal{O}(x)$,

$$\mathcal{O}(x) = \{(a_k)_{k=1}^N \in \mathfrak{o}^N \mid \forall i \in \mathbb{N}_0, \#\{k \mid a_k \in \mathfrak{o}_i\} = x_i\}, \quad (138)$$

where the parameter set $\bar{X}^+(N)$ is defined in (32). We use the Haar measure μ on $\mathfrak{o}$ normalized as $\mu(\mathfrak{o}) = 1$. Then we know the following measures (see [16] etc.)

$$\mu(\mathfrak{o}_i) = \frac{q-1}{q^{i+1}} \quad (i \in \mathbb{N}_0), \quad \mu(\mathfrak{o}_\infty) = 0. \quad (139)$$

The Haar measure on $\mathfrak{o}^N$ is given by the product measure $\mu^{\times N} = \mu \times \cdots \times \mu$. Then we have the following measures of G -orbits on $\mathfrak{o}^N$ by Lemma 2.2.1:

$$\mu^{\times N}(\mathcal{O}(x)) = \begin{cases} \binom{N}{x} \frac{(q-1)^N}{q^{\Delta(x)+N}} & (|x| = N), \\ 0 & (|x| < N), \end{cases} \quad (140)$$

where $\Delta(x)$ is the notation defined in (108). If and only if $|x| < N$, namely some entries a_k of $a \in \mathcal{O}(x)$ are 0, then the measure of the orbit $\mathcal{O}(x)$ is 0, thus not open. So all open G -orbits are parametrized by $X^+(N)$ defined by (31).

We define the dual valuation $r : \hat{\mathfrak{o}} \rightarrow \mathbb{N}_0 \cup \{-\infty\}$ by

$$r(\xi) = \max\{v(a) \mid a \in \mathfrak{o}, \xi(a) \neq 1\}, \quad \text{and } r(1) = -\infty. \quad (141)$$

Lemma 2.4.1. *The $\mathfrak{o}^\times$ -orbits in $\hat{\mathfrak{o}}$ are given by $\hat{\mathfrak{o}} = \bigsqcup_{j \in \mathbb{N}_0 \cup \{-\infty\}} \hat{\mathfrak{o}}_j$, $\hat{\mathfrak{o}}_j = \{\xi \in \hat{\mathfrak{o}} \mid r(\xi) = j\}$. Each orbit $\hat{\mathfrak{o}}_j$ is a finite set of the cardinality*

$$|\hat{\mathfrak{o}}_j| = q^j(q-1) \quad (j \in \mathbb{N}_0), \quad |\hat{\mathfrak{o}}_{-\infty}| = 1. \quad (142)$$

Proof. For any $\ell \geq 1$, using the canonical surjection $\pi_\ell : \mathfrak{o} \rightarrow \mathfrak{o}/\mathfrak{p}^\ell = R$, we can define an injection $\pi_\ell^* : \hat{R} \rightarrow \hat{\mathfrak{o}}, \xi \mapsto \xi \circ \pi_\ell$. We can verify that $\pi_\ell^*(\hat{R}_j) = \hat{\mathfrak{o}}_j$ for any $j \in [\ell]$, where $\hat{R}_j \subset \hat{R}$ is an $\mathfrak{o}^\times$ -orbit defined in Lemma 2.3.1. Moreover since π_ℓ^* is commutative to $\mathfrak{o}^\times$ -actions, we have any $\hat{\mathfrak{o}}_j$ are $\mathfrak{o}^\times$ -orbits in $\hat{\mathfrak{o}}$. Since $|\hat{\mathfrak{o}}_j| = |\hat{R}_j|$, we have (142) from (110). $\square$

By the identification $(\mathfrak{o}^N)^\wedge = (\hat{\mathfrak{o}})^N$, Lemma 2.2.1 gives G -orbit decomposition $(\mathfrak{o}^N)^\wedge = \bigsqcup_{n \in \bar{X}^+(N)} \mathcal{P}(n)$,

$$\mathcal{P}(n) = \{ (\xi_k)_{k=1}^N \in (\hat{\mathfrak{o}})^N \mid \forall j \in \mathbb{N}, \#\{k \mid \xi_k \in \hat{\mathfrak{o}}_j\} = n_j \}, \quad (143)$$

and the cardinalities

$$|\mathcal{P}(n)| = \binom{N}{n} (q-1)^{|n|} q^{\Delta(n)} \quad (144)$$

from (103) and (142). We remark that since $\mu^{\times N}(\mathfrak{o}^N) = 1$ and by Lemma 2.3.2, the dual measure $(\mu^{\times N})^\wedge$ is the counting measure, so those of orbits $\mathcal{P}(n)$ are given by (144).

For $x \in X^+(N)$, $n \in \bar{X}^+(N)$, we denote by $\omega_n(x)$ the value on the orbit $\mathcal{O}(x)$ of the zonal spherical function associated with the orbit $\mathcal{P}(n)$. Since the topology of $(\mathfrak{o}^N)^\wedge$ is discrete, $\mathcal{P}(n)$'s are open orbits for all $n \in \bar{X}^+(N)$. This observation tells us that this example satisfies the condition (98), thus Proposition 2.1.7 assures that the system of zonal spherical functions $\{\omega_n \mid n \in \bar{X}^+(N)\}$ forms a complete orthogonal system of $L^2(\mathfrak{o}^N)^G$. We write down the orthogonality relation of zonal spherical functions (97) using measures (140) and (144): for $n, m \in \bar{X}^+(N)$,

$$\sum_{x \in X^+(N)} \omega_n(x) \overline{\omega_m(x)} \binom{N}{x} q^{-\Delta(x)} = \delta_{nm} \binom{N}{n}^{-1} \frac{q^{N-\Delta(n)}}{(q-1)^{N+|n|}}. \quad (145)$$

The zonal spherical functions are given as follows:

Theorem 2.4.2. *For $x \in X^+(N)$ and $n \in \bar{X}^+(N)$, we have*

$$\omega_n(x) = K_n^{(+)}(x; \frac{q-1}{q}, N), \quad (146)$$

where the right hand side is the product form defined in (66) with a ‘constant parameter’ $\frac{q-1}{q} = (\frac{q-1}{q})_{i \in \mathbb{N}_0}$.

Proof. We take sufficiently large ℓ such that $x_i = 0$ and $n_j = 0$ for all $i, j \geq \ell$, and denote by $x|_\ell = (x_i)_{i \in [\ell]}$, $n|_\ell = (n_j)_{j \in [\ell]} \in \bar{X}(\ell, N)$. We apply Lemma 2.3.9 for the canonical surjection $\mathfrak{f} = \pi_\ell^N : \mathfrak{o}^N \rightarrow (\mathfrak{o}/\mathfrak{p}^\ell)^N$. Then we have

$$\omega_n(x) = \omega_{n|_\ell}(x|_\ell) = K_{n|_\ell}^{(\ell)}(x|_\ell; \frac{q-1}{q}, N)$$

from Theorem 2.3.3. By the equality (35), we have the conclusion (146). $\square$

We can verify the orthogonality relation (73) of the half infinite-variate extensions of Krawtchouk polynomials with the constant parameter $a = (\frac{q-1}{q})_{i \in \mathbb{N}_0}$ coincides with (145). The completeness of zonal spheridal functions mentioned above corresponds to that of the half infinite-variate extensions of Krawtchouk polynomials (Proposition 1.3.3).

2.5 Full infinite-variate extensions of Krawtchouk polynomials as zonal spherical functions on F^N

Here we take the Abelian group $A = F^N$ with the action (100) of $G = (\mathfrak{o}^\times)^N \rtimes \mathfrak{S}_N$. The $\mathfrak{o}^\times$ -orbits in F are determined by the valuation $v : F \rightarrow \mathbb{Z} \cup \{\infty\}$ as $F = \bigsqcup_{i \in \mathbb{Z} \cup \{\infty\}} F_i$, $F_i = \{a \in F \mid v(a) = i\}$. We take the Haar measure μ on F normarized as $\mu(\mathfrak{o}) = 1$. We know

$$\mu(F_i) = \frac{q-1}{q^{i+1}} \quad (i \in \mathbb{Z}), \quad \mu(F_\infty) = 0. \quad (147)$$

The Haar measure $\mu^{\times N}$ on F^N is given as the product measure of the Haar measures μ . Lemma 2.2.1 gives the G -orbits decomposition $F^N = \bigsqcup_{x \in \bar{X}(N)} \mathcal{O}(x)$,

$$\mathcal{O}(x) = \{(a_k)_{k=1}^N \in F^N \mid \#\{k \mid a_k \in F_i\} = x_i \text{ for all } i \in \mathbb{Z}\}, \quad (148)$$

using a parameter set

$$\bar{X}(N) = \{x = (x_i)_{i \in \mathbb{Z}} \mid x_i \in \mathbb{N}_0, |x| \leq N\}, \quad (149)$$

with the following measures of the orbits:

$$\mu^{\times N}(\mathcal{O}(x)) = \begin{cases} \binom{N}{x} \frac{(q-1)^N}{q^{\Delta(x)+N}} & (|x| = N), \\ 0 & (|x| < N), \end{cases} \quad (150)$$

where $\Delta(x)$ is the notation defined by (108). Among these orbits, if and only if $|x| = N$, $\mathcal{O}(x)$ is open. So all open G -orbits are parametrized by $X(N)$ defined by (17).

The dual group $\hat{F}$ has the dual valuation $r : \hat{F} \rightarrow \{-\infty\} \cup \mathbb{Z}$ defined by

$$r(\xi) = \max\{v(a) \mid a \in F, \xi(a) \neq 1\}, \quad \text{and } r(1) = -\infty. \quad (151)$$

The $\mathfrak{o}^\times$ -orbits decomposition in $\hat{F}$ is given by this valuation such as $\hat{F} = \bigsqcup_{j \in \mathbb{Z} \cup \{-\infty\}} \hat{F}_j$, $\hat{F}_j = \{\xi \in \hat{F} \mid r(\xi) = j\}$. In fact, using a fixed $\theta \in \hat{F}_0$, we define $\Theta : F \rightarrow \hat{F}$, $a \mapsto \Theta_a$ by $\Theta_a(b) = \theta(ab)$, $b \in F$. Then we can easily see that $\Theta(F_i) = \hat{F}_{-i}$, and it preserves G -orbits. We can identify $(F^N)^\wedge$ with $(\hat{F})^N$. Lemma 2.2.1 thus gives the G -orbits decomposition $(F^N)^\wedge = \bigsqcup_{n \in \bar{X}(N)} \mathcal{P}(n)$,

$$\mathcal{P}(n) = \{(\xi_k)_{k=1}^N \in (\hat{F})^N \mid \forall j \in \mathbb{Z}, \#\{k \mid \xi_k \in \hat{F}_j\} = n_j\}. \quad (152)$$

We can define a topological group isomorphism $\Theta : F^N \rightarrow (F^N)^\wedge$, $a \mapsto \Theta_a$ which satisfies the conditions in Proposition 2.1.8 as follows. Using a fixed $\theta \in \hat{F}_0$ and the symmetric non-degenerate bilinear form $\langle a \mid b \rangle = \sum_{k=1}^N a_k b_k$ on F^N , we define

$$\Theta_a(b) = \theta(\langle a \mid b \rangle) \quad (b \in F^N). \quad (153)$$

Then for any $x \in \bar{X}(N)$,

$$\Theta(\mathcal{O}(x)) = \mathcal{P}(\overleftarrow{x}) \quad (154)$$

holds, where $\overleftarrow{x} = (x_{-i})_{i \in \mathbb{Z}}$ is in reverse order. In fact, we can regard $\Theta_a = (\Theta_{a_k})_{k=1}^N \in (\hat{F})^N$, and $r(\Theta_{a_k}) = -v(a_k)$, so we have (154). It follows that among G -orbits $\mathcal{P}(n)$ in (152), open orbits are ones parametrized by $n \in X(N)$.

For $x, n \in X(N)$, we denote by $\omega_n(x)$ the value of the zonal spherical function associated with the orbit $\mathcal{P}(n)$ on the orbit $\mathcal{O}(x)$. Proposition 2.1.8 together with (154) gives the following duality relation of the zonal spherical functions:

$$\omega_{\overleftarrow{n}}(x) = \omega_{\overleftarrow{x}}(n) \quad (x, n \in X(N)). \quad (155)$$

To determine the dual measure $(\mu^{\times N})^\wedge$ on $(F^N)^\wedge$, we prepare the following lemma in a general setting:

Lemma 2.5.1. *Let A be a locally compact Abelian group, μ be the Haar measure on A and $C \subset A$ be a compact subgroup such that $\mu(C) = 1$. We assume there exists a topological group isomorphism $\Theta : A \mapsto \hat{A}$, $a \mapsto \Theta_a$.*

Then $B = \{b \in A \mid \forall c \in C, \Theta_c(b) = 1\}$ is an open compact subgroup of A , and the dual measure on $\hat{A}$ is given by

$$\hat{\mu}(E) = \frac{1}{\mu(B)} \mu(\Theta^{-1}(E)) \quad (E \subset \hat{A}). \quad (156)$$

Proof. Since $\mu \circ \Theta^{-1}$ is the Haar measure on $\hat{A}$, there exists $r > 0$ such that $\hat{\mu} = r \cdot (\mu \circ \Theta^{-1})$. Then we have $\bar{\mathcal{F}}1_{\Theta(C)} = r \cdot 1_B$. In fact, we have for $a \in A$,

$$\bar{\mathcal{F}}1_{\Theta(C)}(a) = \int_{\Theta(C)} \xi(a) d\hat{\mu}(\xi) = r \int_C \Theta_c(a) d\mu(c) = \begin{cases} r & (a \in B), \\ 0 & (\text{otherwise}), \end{cases}$$

where the last equality is due to the orthogonality of characters on C , between the trivial character 1 and the one such that $c \mapsto \Theta_c(a)$, which is trivial if and only if $a \in B$. The property of (inverse) Fourier transformation thus induces that 1_B is a continuous function on A vanishing at infinity, which means that B is open and compact. Using any fixed $c \in C$, We concludes

$$1 = \mathcal{F}(\bar{\mathcal{F}}1_{\Theta(C)})(\Theta_c) = r \mathcal{F}1_B(\Theta_c) = r \int_B \overline{\Theta_c(b)} d\mu(b) = r\mu(B),$$

therefore $r = \frac{1}{\mu(B)}$. $\square$

We apply this lemma as $C = \mathfrak{o}^N \subset F^N$ and the isomorphism Θ in (153). Then we can easily verify that $B = \{b \in F^N \mid \forall c \in \mathfrak{o}^N, \Theta_b(c) = 1\} = \mathfrak{p}^N$. Then from (150) and a remark that $\mu^{\times N}(\mathfrak{p}^N) = \frac{1}{q^N}$, we have for $n \in \bar{X}(N)$,

$$(\mu^{\times N})^\wedge(\mathcal{P}(n)) = \frac{1}{\mu^{\times N}(\mathfrak{p}^N)} \mu^{\times N}(\mathcal{O}(\overleftarrow{n})) = \begin{cases} \binom{N}{n} q^{\Delta(n)} (q-1)^N & (|n| = N), \\ 0 & (|n| < N). \end{cases} \quad (157)$$

We write down the orthogonality relation of zonal spherical functions (97) using measures (150) and (157):

$$\sum_{x \in X(N)} \omega_n(x) \overline{\omega_m(x)} \binom{N}{x} q^{-\Delta(x)} = \delta_{nm} \binom{N}{n}^{-1} \frac{q^{N-\Delta(n)}}{(q-1)^{2N}} \quad (n, m \in X(N)). \quad (158)$$

This example also satisfies the condition (98). In fact, by (157) and since clearly the cardinality of G -orbits in $(F^N)^\wedge$ is countable, we have

$(\mu^{\times N})^\wedge \left(\bigsqcup_{|n| < N} \mathcal{P}(n) \right) = 0$. Proposition 2.1.7 thus assures that the system $\{\omega_n \mid n \in X(N)\}$ of zonal spherical functions associated with open orbits forms a copcomplete orthogonal system of $L^2(F^N)^G$.

The zonal spherical functions are given as follows:

Theorem 2.5.2. *For $x, n \in X(N)$, we have*

$$\omega_n(x) = K_n^{(\infty)}(x; \frac{q-1}{q}, N), \quad (159)$$

where the right hand side is the procuct form defined in (67) with a ‘constant parameter’ $\frac{q-1}{q} = (\frac{q-1}{q})_{i \in \mathbb{Z}}$.

To prove the theorem, we note a property of zonal spherical functions, the invariance under parallel translations as follows:

Lemma 2.5.3. *For any $x, n \in X(N)$ and $k \in \mathbb{Z}$, we denote $x^k = (x_{k+i})_{i \in \mathbb{Z}}$, $n^k = (n_{k+i})_{i \in \mathbb{Z}} \in X(N)$. Then we have*

$$\omega_n(x) = \omega_{n^k}(x^k). \quad (160)$$

Proof. We denote by ϖ a generator of the ideal $\mathfrak{p}$, and apply Lemma 2.3.9 for $\mathfrak{f} : F^N \rightarrow F^N$, $a \mapsto \varpi^k a$. Then since $\mathfrak{f}(\mathcal{O}(x)) = \mathcal{O}(x^k)$ and $\xi \circ \mathfrak{f} \in \mathcal{P}(n)$ for $\xi \in \mathcal{P}(n^k)$, we have the assertion. $\square$

We prove Theorem 2.5.2 firstly for a special case that $x_i = 0$ for all $i \leq -1$. Then we can regard $x \in X^+(N) \subset X(N)$ by the embedding (41). We use the notation $n^{(0)} = (n_j)_{j \in \mathbb{N}_0} \in \bar{X}^+(N)$ in (36). We apply Lemma 2.3.9 for the embedding map $\mathfrak{f} : \mathfrak{o}^N \hookrightarrow F^N$. Then since $\mathfrak{f}(\mathcal{O}(x)) = \mathcal{O}(x)$ and $\xi \circ \mathfrak{f} \in \mathcal{P}(n^{(0)})$ for $\xi \in \mathcal{P}(n)$, we have

$$\omega_n(x) = \omega_{n^{(0)}}(x) = K_{n^{(0)}}^{(+)}(x; \frac{q-1}{q}, N) = K_n^{(\infty)}(x; \frac{q-1}{q}, N)$$

by Theorem 2.4.2 and Lemma 1.3.2. Secondly we prove the theorem for a general $x \in X(N)$. We take $k < 0$ small enough to satisfy $x_i = 0$ for all $i < k$. Then from the first case for $x^k = (x_{k+i})_{i \in \mathbb{Z}} \in X^+(N) \subset X(N)$ and $n^k = (n_{k+j})_{j \in \mathbb{Z}} \in X(N)$, we have $\omega_{n^k}(x^k) = K_{n^k}^{(\infty)}(x^k; \frac{q-1}{q}, N)$. Moreover we have $\omega_n(x) = \omega_{n^k}(x^k)$ by Lemma 2.5.3, and $K_n^{(\infty)}(x; \frac{q-1}{q}, N) = K_{n^k}^{(\infty)}(x^k; \frac{q-1}{q}, N)$ by Definition 1.4.4, which proves Theorem 2.5.2.

We can see correspondence between the duality relations (155) of zonal spherical functions and (83) of full infinite-variate extensions of Krawtchouk polynomials. We can also verify the orthogonality relation (74) of Krawtchouk polynomials with the constant parameter $a = (\frac{q-1}{q})_{i \in \mathbb{Z}}$ coincides with (158). The completeness of zonal spheridal functions mentioned above corresponds to that of the full infinite-variate extensions of Krawtchouk polynomials (Theorem 1.2.5).

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