

Subordination chains and univalence criteria

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Abstract

The object of the present paper is to give an univalence condition for analytic functions in the open unit disk U by using the properties for subordination chain.

1 Introduction

Let U be the open unit disk in the complex plane $\mathbb{C}$, i.e. $U = \{z \in \mathbb{C} : |z| < 1\}$. We denote by A the class of functions $f(z)$ which are analytic in U with $f(0) = 0$, $f'(0) = 1$ and by S the subclass of the class A consisting of univalent functions.

If $f(z) \in A$ and $g(z) \in S$, then $f(z)$ is said to be *subordinate* to $g(z)$ (written by $f(z) \prec g(z)$) in U if $f(U) \subset g(U)$.

A function $L : U \times [0, \infty) \rightarrow \mathbb{C}$ is said to be a *subordination chain* if $L(\cdot, t)$ is analytic and univalent in U , for all $t \in [0, \infty)$ and $L(z, s) \prec L(z, t)$, whenever $0 \leq s \leq t < \infty$.

The following result concerning subordination chains is due to Ch. Pommerenke [3].

Theorem 1 *Let $L(z, t) = a_1(t)z + \dots$ be a function from $U \times [0, \infty)$ into $\mathbb{C}$, such that:*

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(i) $L(\cdot, t)$ is analytic in U , for all $t \in [0, \infty)$.

(ii) $L(z, t)$ is a locally absolutely continuous function of t , locally uniformly with respect to $z \in U$.

(iii) $a_1(t) \neq 0$, for all $t \in [0, \infty)$ and $\lim_{t \rightarrow \infty} |a_1(t)| = \infty$.

(iv) the family of function $\left\{ \frac{L(z, t)}{a_1(t)} \right\}_{t \geq 0}$ forms a normal family in U .

Let $p : U \times [0, \infty) \rightarrow \mathbb{C}$ be an analytic function in U with $\text{Re } p(z, t) > 0$ for all $(z, t) \in U \times [0, \infty)$ and such that:

$$\frac{\partial L(z, t)}{\partial t} = zp(z, t) \frac{\partial L(z, t)}{\partial z}, \quad (1)$$

a.e. $t \in [0, \infty)$, for all $z \in U$.

Then the function $L(z, t)$ is a subordination chain in U .

2 Sufficient conditions for univalence

By using Theorem 1 we obtain an univalence condition which generalize some known univalence criteria for analytic functions in the open unit disk U .

Let $a(t)$ be a complex valued function on $[0, \infty)$ satisfying:

$$\bullet a \in C^1[0, \infty), a(0) = 1, a(t) \neq 0 \quad (2)$$

and

$$a(t) + a'(t) \neq 0, t \in [0, \infty),$$

- the modulus of $a(t)$ is increasing to ∞ . (3)

Definition 1 Let $F = F(u, v)$ be a function from $U \times \mathbb{C}$ into $\mathbb{C}$ and let $L(z, t) = F(e^{-t}z, a(t)z)$, for all $(z, t) \in U \times [0, \infty)$. We say that the function F satisfies (PA) conditions if:

(i) $L(\cdot, t)$ is analytic in U , for all $t \in [0, \infty)$.

(ii) $L(z, t)$ is a locally absolutely continuous function of t , locally uniformly with respect to $z \in U$.

(iii) the function $\frac{\partial L(z, t)}{\partial t} / z \frac{\partial L(z, t)}{\partial t}$ is analytic in $\bar{U}$, for all $t > 0$ and is analytic in U for $t = 0$.

(iv) $\frac{\partial F(0, 0)}{\partial v} \neq 0$ and $\frac{\partial F(0, 0)}{\partial u} / \frac{\partial F(0, 0)}{\partial v} \notin (-\infty, -1]$.

(v) the family of functions

$$\left\{ F(e^{-t}z, a(t)z) / \left[e^{-t} \frac{\partial F(0, 0)}{\partial u} + a(t) \frac{\partial F(0, 0)}{\partial v} \right] \right\}_{t \geq 0}$$

is a normal family in U .

Theorem 2 Let $a : [0, \infty) \rightarrow \mathbb{C}$ be a function satisfying (2) and (3). Further, suppose $F : U \times \mathbb{C}$ is a function which satisfies (PA) conditions. If

$$\left| G(z, z) + \frac{a(t) - a'(t)}{2a(t)} \right| < \frac{|a(t) + a'(t)|}{2|a(t)|}, \quad z \in U, \quad t \geq 0 \quad (4)$$

and

$$\max_{|z|=e^{-t}} \left| G\left(z, a(t) \frac{z}{|z|}\right) + \frac{a(t) - a'(t)}{2a(t)} \right| \quad (5)$$

$$\leq \frac{|a(t) + a'(t)|}{2|a(t)|}, \quad z \in U \setminus \{0\}, \quad t \geq 0,$$

where

$$G(u, v) = \frac{u}{v} \cdot \frac{\partial F(u, v)}{\partial u} / \frac{\partial F(u, v)}{\partial v}, \quad (6)$$

then $F(z, z)$ is an univalent function in U .

Proof. We wish to show that the function $L(z, t) = F(e^{-t}z, a(t)z)$ satisfies the conditions of Theorem 1 and hence $L(\cdot, t)$ is univalent in U , for all $t \in [0, \infty)$.

If $F(e^{-t}z, a(t)z) = a_1(t)z + \dots$, then

$$a_1(t) = e^{-t} \frac{\partial F(0, 0)}{\partial u} + a(t) \frac{\partial F(0, 0)}{\partial v}.$$

By using the conditions (iv) and (v) of the Definition 1 we have $a_1(t) \neq 0$ for all $t \geq 0$, $\lim_{t \rightarrow \infty} |a_1(t)| = \infty$ and the family of functions $\left\{ \frac{L(z, t)}{a_1(t)} \right\}_{t \geq 0}$ is a normal family in U . Let

$$p(z, t) = \frac{\partial L(z, t)}{\partial t} / z \frac{\partial L(z, t)}{\partial z}, \quad (z, t) \in U \times [0, \infty). \quad (7)$$

Then the condition (1) of Theorem 1 is satisfied for all $z \in U$ and $t \in [0, \infty)$. It remains to prove that the function $p(z, t)$ has a positive real part in U , for all $t \in [0, \infty)$. If

$$w(z, t) = \frac{1 - p(z, t)}{1 + p(z, t)}, \quad (z, t) \in U \times [0, \infty), \quad (8)$$

then $\operatorname{Re} p(z, t) > 0$ if and only if $|w(z, t)| < 1$. According with (6), (7), (8) we have

$$w(z, t) = \frac{2a(t)}{a(t) + a'(t)} G(e^{-t}z, a(t)z) \quad (9)$$

$$+ \frac{a(t) - a'(t)}{a(t) + a'(t)}, \quad (z, t) \in U \times [0, \infty).$$

By using the inequality (4) we obtain $|w(z, 0)| < 1$, for all $z \in U$. For $t > 0$ the function $p(z, t)$ is analytic in $\bar{U}$ and it follows

$$|w(z, t)| < \max_{|\zeta|=1} |w(\zeta, t)| = \max_{|\zeta|=1} \left| \frac{2a(t)}{a(t) + a'(t)} G(e^{-t}\zeta, a(t)\zeta) + \frac{a(t) - a'(t)}{a(t) + a'(t)} \right|.$$

If we let $z = e^{-t}\zeta$ with $|\zeta| = 1$, then $|z| = e^{-t}$ and by using (5) we have

$$|w(z, t)| < \max_{|z|=e^{-t}} \left| \frac{2a(t)}{a(t) + a'(t)} G\left(z, a(t) \frac{z}{|z|}\right) + \frac{a(t) - a'(t)}{a(t) + a'(t)} \right| \leq 1.$$

Since $L(z, t)$ satisfies all the conditions of Theorem 1, it follows that $L(z, t)$ is a subordination chain in U and $F(z, z) = L(z, 0)$ is an univalent function in U .

Remark 1

1. If $a(t) = e^t$ we obtain the univalence condition due to N.N. Pascu [1].

2. If

$$F(u, v) = f(u) + \frac{(v - u) R(u)}{1 - (v - u) Q(u)},$$

where $u = e^{-t}z$, $v = a(t)z$ and $R(z), Q(z)$ are analytic functions in U , we obtain the results concerning univalence criteria due to J.A. Pfaltzgraff [2].

References

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