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Continuous wavelet transforms and non-commutative Fourier analysis

By

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Abstract

We discuss continuous wavelet transforms for the semidirect product group of a unimodular (not necessarily commutative) normal subgroup N with a closed subgroup H of $\mathrm{Aut}(N)$, which is a generalization of the wavelet theory for an affine transformation group on a vector space. The operator-valued Fourier transform for N plays a substantial role in the arguments.

§ 1. Introduction

Let G be a locally compact group, and $(\pi, \mathcal{H})$ an irreducible unitary representation of G. The representation π is said to be *square-integrable* if there exists a vector $\phi \in \mathcal{H}$ for which $\int_G |(\pi(g)\phi|\phi)_{\mathcal{H}}|^2\, dg < +\infty$, where dg is a left Haar measure on G. Such ϕ is called an *admissible vector* of π. If $(\pi, \mathcal{H})$ is square-integrable, there exists a unique positive self-adjoint operator (the *Duflo-Moore operator*) C_π with the following two properties ([7], [13]):
(C1) ϕ is admissible $\Leftrightarrow$ $\phi \in \mathrm{dom}(C_\pi)$,
(C2) For $f_1, f_2 \in \mathcal{H}$ and $\phi_1, \phi_2 \in \mathrm{dom}(C_\pi)$, one has

$$\int_G (f_1|\pi(g)\phi_1)_{\mathcal{H}}\, (\pi(g)\phi_2|f_2)_{\mathcal{H}}\, dg = (f_1|f_2)_{\mathcal{H}}\, (C_\pi\phi_1|C_\pi\phi_2)_{\mathcal{H}}.$$

When G is unimodular, C_π is a scalar operator. Furthermore, if G is a compact group, then $C_\pi = (\dim \mathcal{H})^{-1/2}\mathrm{Id}$. Thus, C_π^{-2} is called *the formal degree* of π in general.

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Regarding the equality in (C2) as an identity for $f_2 \in \mathcal{H}$, we obtain

$$f_1 = \frac{1}{(C_\pi \phi_1 | C_\pi \phi_2)_{\mathcal{H}}} \int_G (f_1 | \pi(g)\phi_1)_{\mathcal{H}} \, \pi(g)\phi_2 \, dg, \tag{1.1}$$

where the integral is taken in the weak sense. For an admissible vector $\phi \in \mathrm{dom}(C_\pi)$, *the continuous wavelet transform* W_ϕ is defined as a linear map from the Hilbert space $\mathcal{H}$ into the space $C(G)$ of continuous functions on the group G defined by

$$W_\phi f(g) := (f | \pi(g)\phi)_{\mathcal{H}} \qquad (f \in \mathcal{H}, \ \ g \in G).$$

Then (1.1) tells us that the inverse formula of W_ϕ is given by

$$f = \frac{1}{c_\phi} \int_G W_\phi f(g) \, \pi(g)\phi \, dg, \tag{1.2}$$

where $c_\phi = \|C_\pi \phi\|^2_{\mathcal{H}}$.

For instance, let us consider the affine transformation group G_{aff} on the real line $\mathbb{R}$ consisting of the maps $g_{b,a} : \mathbb{R} \ni x \mapsto ax + b \in \mathbb{R}$ with $a \in \mathbb{R} \setminus \{0\}$ and $b \in \mathbb{R}$. The unitary representation L of G_{aff} is defined on the Hilbert space $L^2(\mathbb{R})$ by $L(g_{b,a})f(x) := |a|^{-1/2} f(g_{b,a}^{-1} x)$ $(x \in \mathbb{R})$. This representation is square-integrable, and the Duflo-Moore operator is given by

$$(C_L \phi)^\wedge(\xi) = \sqrt{\frac{2\pi}{|\xi|}} \hat{\phi}(\xi) \quad (\xi \in \mathbb{R}),$$

where $\hat{}$ stands for the Fourier transform given by $\hat{\phi}(\xi) := \int_{\mathbb{R}} e^{ix\xi} \phi(x) \, dx$ $(\xi \in \mathbb{R})$. Then $c_\phi = \|C_L \phi\|^2$ equals $\int_{\mathbb{R} \setminus \{0\}} |\hat{\phi}(\xi)|^2 |\xi|^{-1} \, d\xi$, and if $c_\phi < +\infty$, the equality (1.2) yields the Calderón formula

$$f = \frac{1}{c_\phi} \int_{\mathbb{R}} \int_{\mathbb{R}^\times} W_\phi f(b,a) \, L(g_{b,a})\phi \, \frac{db \, da}{|a|^2} \quad (f \in L^2(\mathbb{R})), \tag{1.3}$$

where $W_\phi f(b,a) = W_\phi f(g_{b,a}) = |a|^{-1/2} \int_{\mathbb{R}} f(x) \overline{\phi(\frac{x-b}{a})} \, dx$.

The results about the wavelet transform for G_{aff} have been generalized to a multi-dimensional affine group $G = H \ltimes \mathbb{R}^n$, where H is a closed subgroup of $GL(\mathbb{R}^n)$ (see [4], [8] and [10] for example). In this article, we consider a further generalization to the case that G is a semidirect product group $N \rtimes H$, where N is a unimodular (not necessarily commutative) group, and H is a closed subgroup of $\mathrm{Aut}(N)$ satisfying certain conditions. Although the situation becomes complicated if N is not commutative, the argument goes in parallel with the commutative case, where the operator-valued Fourier transform for N plays a substantial role instead of the ordinary Fourier transform.

The content of Sections 2–4 is essentially a summary of [16], while Section 5 is devoted to discuss a concrete example that N is the Heisenberg group. Such Heisenberg case is already studied by He-Liu [14], whereas we shall present a new example of admissible vector $\phi_{\pm,\alpha} \in L^2(N)$. In this article, we write $\mathbb{T}$ for the set of complex numbers with absolute value 1. For a Hilbert space $\mathcal{H}$, we denote by $\mathcal{B}(\mathcal{H})$, (resp. $\mathcal{B}_{\mathrm{HS}}(\mathcal{H})$, $\mathcal{B}_{\mathrm{Tr}}(\mathcal{H}), U(\mathcal{H})$) the space of bounded (resp. Hilbert-Schmidt, trace class, unitary) operators on $\mathcal{H}$.

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§ 2. Preliminaries

Let N be a separable locally compact unimodular group of type I. We denote by $\hat{N}$ the unitary dual of N, that is, the set of equivalence classes of irreducible unitary representations of N. For each $\lambda \in \hat{N}$, we take a unitary representation $(\pi_\lambda, \mathcal{H}_\lambda)$ for which λ equals the equivalence class $[\pi_\lambda]$ of π_λ. Let us fix a Haar measure ν on N. For $f \in L^1(N)$, we define the bounded operator $\pi_\lambda(f) \in \mathcal{B}(\mathcal{H}_\lambda)$ by $\pi_\lambda(f) := \int_N f(n)\pi_\lambda(n)\,d\nu(n)$. It is known that the Plancherel measure μ on $\hat{N}$ is uniquely determined by the abstract Plancherel formula [6]:

$$\int_N |f(n)|^2 d\nu(n) = \int_{\hat{N}} \|\pi_\lambda(f)\|_{\mathrm{HS}}^2 \, d\mu(\lambda) \qquad (f \in L^1(N) \cap L^2(N)), \tag{2.1}$$

where $\|\cdot\|_{\mathrm{HS}}$ stands for the Hilbert-Schmidt norm. We define the operator-valued Fourier transform $\mathbf{F} : L^2(N) \to \int_{\hat{N}}^{\oplus} \mathcal{B}_{\mathrm{HS}}(\mathcal{H}_\lambda)\,d\mu(\lambda)$ as the unitary isomorphism which is the extension of the map $L^1(N) \cap L^2(N)) \ni f \mapsto (\pi_\lambda(f))_{\lambda \in \hat{N}} \in \int_{\hat{N}}^{\oplus} \mathcal{B}_{\mathrm{HS}}(\mathcal{H}_\lambda)\,d\mu(\lambda)$. The inverse formula of $\mathbf{F}$ is given as follows:

Proposition 2.1 ([11, Theorem 4.15]). *Let $(A(\lambda))_{\lambda \in \hat{N}}$ be an element of the direct integral $\int_{\hat{N}}^{\oplus} \mathcal{B}_{\mathrm{Tr}}(\mathcal{H}_\lambda)\,d\mu(\lambda)$ of the Banach space $\mathcal{B}_{\mathrm{Tr}}(\mathcal{H}_\lambda)$. Define a function f on N by*

$$f(n) := \int_{\hat{N}} \mathrm{tr}\ A(\lambda)\pi_\lambda(n)^* \, d\mu(\lambda) \qquad (n \in N).$$

Then f belongs to $L^2(N)$ if and only if $(A(\lambda))_{\lambda \in \hat{N}} \in \int_{\hat{N}}^{\oplus} \mathcal{B}_{\mathrm{HS}}(\mathcal{H}_\lambda)\,d\mu(\lambda)$. In that case, one has $\mathbf{F}f(\lambda) = A(\lambda)$ (a.a. $\lambda \in \hat{N}$).

Let H be a closed subgroup of the automorphism group $\mathrm{Aut}(N)$ on N, and G the semidirect product group $N \rtimes H$. We write the action of h to $n \in N$ as $h \cdot n$. For $h \in H$, we have a positive number $\delta(h)$ for which $d\nu(h \cdot n) = \delta(h)\,d\nu(n)$ $(n \in N)$. Clearly, $\delta : H \to \mathbb{R}_+$ is a representation of H. Define the unitary representation L of G on $L^2(N)$ by

$$\begin{aligned} L(h)f(n_0) &:= \delta(h)^{-1/2} f(h^{-1} \cdot n_0), \\ L(n)f(n_0) &:= f(n^{-1}n_0) \qquad (f \in L^2(N),\ h \in H,\ n, n_0 \in N). \end{aligned} \tag{2.2}$$

It is easy to see that the representation L is equivalent to the induced representation $\mathrm{Ind}_H^G \mathbf{1}$, where $\mathbf{1}$ is the trivial representation of H.

We define the action of H on the unitary dual $\hat{N}$ by $h \cdot \lambda := [\pi_\lambda \circ h^{-1}]$ $(h \in H,\ \lambda \in \hat{N})$. Then we have a unitary operator $C(h,\lambda) : \mathcal{H}_\lambda \to \mathcal{H}_{h\cdot\lambda}$ with the property

$$C(h,\lambda)\pi_\lambda(h^{-1}\cdot n) = \pi_{h\cdot\lambda}(n)C(h,\lambda) \qquad (n \in N).$$

The operator $C(h,\lambda)$ is unique up to multiple by elements of $\mathbb{T}$ by Schur's lemma. Moreover, for $h, h' \in H$ and $\lambda \in \hat{N}$, we have

$$C(hh',\lambda) = s_{h,h',\lambda} C(h, h'\cdot\lambda)C(h',\lambda) \tag{2.3}$$

with $s_{h,h',\lambda} \in \mathbb{T}$. Thus we have a uniquely determined operator $D(h,\lambda) : \mathcal{B}(\mathcal{H}_\lambda) \ni T \mapsto C(h,\lambda)TC(h,\lambda)^* \in \mathcal{B}(\mathcal{H}_{h\cdot\lambda})$, which satisfies the chain rule

$$D(hh',\lambda) = D(h,h'\cdot\lambda)D(h',\lambda).$$

Using the operator-valued Fourier transform $\mathbf{F}$, we describe the representation $(L,\ L^2(N))$ of G as follows:

Proposition 2.2. *For $f \in L^2(N)$, $h \in H$ and $n \in N$, one has*

$$\mathbf{F}[L(h)f](\lambda) = \delta(h)^{1/2} D(h, h^{-1}\cdot\lambda)\mathbf{F}f(h^{-1}\cdot\lambda), \tag{2.4}$$

$$\mathbf{F}[L(n)f](\lambda) = \pi_\lambda(n)\mathbf{F}f(\lambda) \tag{2.5}$$

for almost all $\lambda \in \hat{N}$ with respect to the Plancherel measure μ.

§ 3. Decomposition of $L^2(N)$

For $\lambda \in \hat{N}$, let $\mathcal{O}^*_\lambda \subset \hat{N}$ be the H-orbit through λ. Now let us assume that there exist elements $\lambda_1, \dots, \lambda_K \in \hat{N}$ satisfying the following conditions:
(H1) $\mu(\mathcal{O}^*_{\lambda_k}) > 0$ $(k = 1, \dots, K)$,
(H2) The stabilizer $H_k := \{\, h \in H\,;\ h\cdot\lambda_k = \lambda_k \,\}$ is compact for all $k = 1, \dots, K$.
(H3) The map $H/H_k \ni hH_k \mapsto h\cdot\lambda_k \in \mathcal{O}^*_{\lambda_k}$ is a homeomorphism for $k = 1, \dots, K$, where the topology on $\mathcal{O}^*_{\lambda_k}$ is induced from the Fell topology on $\hat{N}$.
(H4) $\mathcal{O}^*_{\lambda_k} \cap \mathcal{O}^*_{\lambda_l} = \emptyset$ $(k \neq l)$ and $\mu(\hat{N} \setminus \bigsqcup_{k=1}^K \mathcal{O}^*_{\lambda_k}) = 0$.
Under this assumption, we shall see in Sections 3 and 4 that the unitary representation $(L, L^2(N))$ of G is decomposed into the direct sum of countably many irreducible subrepresentations, and such subrepresentations are all square-integrable. Their Duflo-Moore operators are described by using the operator-valued Fourier transform $\mathbf{F}$.

Thanks to (2.3), we have a projective unitary representation $\tau_k : H_k \ni h \mapsto C(h,\lambda_k) \in U(\mathcal{H}_{\lambda_k})$ of the group H_k for $k = 1, \dots, K$. Since H_k is compact, we have an irreducible decomposition $\mathcal{H}_{\lambda_k} = \sum_{\alpha \in A_k}^{\oplus} \mathcal{H}_{\lambda_k,\alpha}$, where A_k is an at most countable index set. The subspaces $\mathcal{H}_{\lambda_k,\alpha}$ are finite dimensional. For $\lambda = \tilde{h} \cdot \lambda_k \in \mathcal{O}^*_{\lambda_k}$ with $\tilde{h} \in H$, we put $\mathcal{H}_{\lambda,\alpha} := C(\tilde{h}, \lambda_k)\mathcal{H}_{\lambda_k,\alpha}$ $(\alpha \in A_k)$, where the right-hand side is independent of the choice of $\tilde{h}$. Moreover we have $\mathcal{H}_\lambda = \sum_{\alpha \in A_k}^{\oplus} \mathcal{H}_{\lambda,\alpha}$, which gives an irreducible decomposition of the projective representation $\tau_\lambda : H_\lambda \ni h \mapsto C(h, \lambda) \in U(\mathcal{H}_\lambda)$ of the compact group $H_\lambda := \{ h \in H \,;\, h \cdot \lambda = \lambda \}$. On the other hand, the relation

$$C(h, \lambda)\mathcal{H}_{\lambda,\alpha} = \mathcal{H}_{h\cdot\lambda,\alpha} \tag{3.1}$$

holds for $h \in H$, $\lambda \in \mathcal{O}^*_{\lambda_k}$ and $\alpha \in A_k$ in general. Using the orthogonal projection $P_{\lambda,\alpha} : \mathcal{H}_\lambda \to \mathcal{H}_{\lambda,\alpha}$, we define

$$\mathcal{B}_{\lambda,\alpha} := \{ T \in \mathcal{B}_{\mathrm{HS}}(\mathcal{H}_\lambda) \,;\, TP_{\lambda,\alpha} = T \} . \tag{3.2}$$

If we identify $\mathcal{B}_{\mathrm{HS}}(\mathcal{H}_\lambda)$ with the tensor product $\mathcal{H}_\lambda \otimes \overline{\mathcal{H}}_\lambda$, the space $\mathcal{B}_{\lambda,\alpha}$ is nothing but $\mathcal{H}_\lambda \otimes \overline{\mathcal{H}}_{\lambda,\alpha}$. Thus we see that

$$\mathcal{B}_{\mathrm{HS}}(\mathcal{H}_\lambda) = \sum_{\alpha \in A_k}^{\oplus} \mathcal{B}_{\lambda,\alpha}, \tag{3.3}$$

while (3.1) yields

$$D(h, \lambda)\mathcal{B}_{\lambda,\alpha} = \mathcal{B}_{h\cdot\lambda,\alpha}. \tag{3.4}$$

Now we set

$$\begin{aligned} L_{k,\alpha}(N) &:= \mathbf{F}^{-1}\left(\int_{\mathcal{O}^*_{\lambda_k}}^{\oplus} \mathcal{B}_{\lambda,\alpha}\, d\mu(\lambda) \right) \\ &= \left\{ f \in L^2(N)\,; \begin{array}{ll} \mathbf{F}f(\lambda) = \mathbf{F}f(\lambda)P_{\lambda,\alpha} & (\text{a.a. } \lambda \in \mathcal{O}^*_{\lambda_k}) \\ \mathbf{F}f(\lambda) = 0 & (\text{otherwise}) \end{array} \right\} \end{aligned} \tag{3.5}$$

for $k = 1, \ldots, K$ and $\alpha \in A_k$. By (3.3) and (H4), we have

$$L^2(N) = \sum_{1 \le k \le K}^{\oplus} \sum_{\alpha \in A_k}^{\oplus} L_{k,\alpha}(N), \tag{3.6}$$

and each $L_{k,\alpha}(N)$ is G-invariant thanks to Proposition 2.2 and (3.4).

§ 4. Main results

Fixing a left Haar measure d_H on H, we define a left Haar measure d_G on G by $d_G(nh) := \delta(h)^{-1} d\nu(n) d_H(h)$ $(n \in N,\, h \in H)$. Let us consider the square-integrability of the matrix coefficients $(f|L(g)\phi)$ $(f, \phi \in L_{k,\alpha}(N),\ g \in G)$ of the representation $(L, L_{k,\alpha}(N))$ of G. By (2.1) and (2.5), we have

$$(4.1)\qquad \begin{aligned}(f|L(n)L(h)\phi) &= \int_{\mathcal{O}^*_{\lambda_k}} \mathrm{tr}\ \mathbf{F}f(\lambda)\mathbf{F}[L(n)L(h)\phi](\lambda)^*\, d\mu(\lambda)\\ &= \int_{\mathcal{O}^*_{\lambda_k}} \mathrm{tr}\ (\mathbf{F}f(\lambda)\mathbf{F}[L(h)\phi](\lambda)^*)\pi_\lambda(n)^*\, d\mu(\lambda).\end{aligned}$$

We note that the operator $\mathbf{F}f(\lambda)\mathbf{F}[L(h)\phi](\lambda)^*$ is of trace class because both $\mathbf{F}f(\lambda)$ and $\mathbf{F}[L(h)\phi](\lambda)$ are Hilbert-Schmidt operators. Furthermore, $(\mathbf{F}f(\lambda)\mathbf{F}[L(h)\phi](\lambda)^*)_{\lambda\in\mathcal{O}^*_{\lambda_k}}$ belongs to $\int^{\oplus}_{\mathcal{O}^*_{\lambda_k}} \mathcal{B}_{\mathrm{Tr}}(\mathcal{H}_\lambda)\, d\mu(\lambda)$ because

$$(4.2)\qquad \begin{aligned}&\int_{\mathcal{O}^*_{\lambda_k}} \|\mathbf{F}f(\lambda)\mathbf{F}[L(h)\phi](\lambda)^*\|_{\mathrm{Tr}}\, d\mu(\lambda)\\ &\le \int_{\mathcal{O}^*_{\lambda_k}} \|\mathbf{F}f(\lambda)\|_{\mathrm{HS}}\, \|\mathbf{F}[L(h)\phi](\lambda)\|_{\mathrm{HS}}\, d\mu(\lambda)\\ &\le \left\{\int_{\mathcal{O}^*_{\lambda_k}} \|\mathbf{F}f(\lambda)\|^2_{\mathrm{HS}}\, d\mu(\lambda)\right\}^{1/2} \left\{\int_{\mathcal{O}^*_{\lambda_k}} \|\mathbf{F}[L(h)\phi](\lambda)\|^2_{\mathrm{HS}}\, d\mu(\lambda)\right\}^{1/2}\\ &= \|f\|\,\|L(h)\phi\| = \|f\|\,\|\phi\| < +\infty,\end{aligned}$$

where $\|\cdot\|_{\mathrm{Tr}}$ denotes the Trace norm. Now we assume that

$$\int_G |(f|L(g)\phi)|^2\, d_G(g) = \int_H \int_N |(f|L(n)L(h)\phi)|^2 \delta(h)^{-1} d\nu(n)\, d_H(h) < +\infty.$$

Then $\int_N |(f|L(n)L(h)\phi)|^2\, d\nu(n)$ is finite for almost all $h \in H$. Thus, applying Proposition 2.1, we see from (4.1) and (2.1) that

$$(4.3)\qquad \int_N |(f|L(n)L(h)\phi)|^2\, d\nu(n) = \int_{\mathcal{O}^*_{\lambda_k}} \|\mathbf{F}f(\lambda)\mathbf{F}[L(h)\phi](\lambda)^*\|^2_{\mathrm{HS}}\, d\mu(\lambda) \quad (\text{a.a. } h \in H).$$

Therefore the integral $\int_G |(f|L(g)\phi)|^2\, d_G(g)$ is equal to

$$(4.4)\qquad \int_H \int_{\mathcal{O}^*_{\lambda_k}} \|\mathbf{F}f(\lambda)\mathbf{F}[L(h)\phi](\lambda)^*\|^2_{\mathrm{HS}}\, \delta(h)^{-1}\, d\mu(\lambda)\, d_H(h).$$

Now we observe that

$$(4.5)\qquad \begin{aligned}&\int_H \|\mathbf{F}f(\lambda)\mathbf{F}[L(h)\phi](\lambda)^*\|^2_{\mathrm{HS}}\, \delta(h)^{-1}\, d_H(h)\\ &= \int_H \int_{H_\lambda} \Big(\mathrm{tr}\ \mathbf{F}f(\lambda)\mathbf{F}[L(h_1h)\phi](\lambda)^*\mathbf{F}[L(h_1h)\phi](\lambda)\mathbf{F}f(\lambda)^*\Big)\, \delta(h_1h)^{-1}\, dh_1 d_H(h),\end{aligned}$$

where dh_1 is the normalized Haar measure on the compact group H_λ. Since $L(h_1h)\phi$ belongs to $L_{k,\alpha}(N)$, we have by (3.5)

$$P_{\lambda,\alpha}\mathbf{F}[L(h_1h)\phi](\lambda)^*\mathbf{F}[L(h_1h)\phi](\lambda)P_{\lambda,\alpha} = \mathbf{F}[L(h_1h)\phi](\lambda)^*\mathbf{F}[L(h_1h)\phi](\lambda),$$

which means that we can regard $\mathbf{F}[L(h_1h)\phi](\lambda)^*\mathbf{F}[L(h_1h)\phi](\lambda)$ as a linear operator on the finite dimensional vector space $\mathcal{H}_{\lambda,\alpha}$. Furthermore, applying Schur's lemma to the representation $(\tau_\lambda, \mathcal{H}_{\lambda,\alpha})$ of H_λ, we get

$$\begin{aligned}\int_{H_\lambda} & \delta(h_1h)^{-1}\mathbf{F}[L(h_1h)\phi](\lambda)^*\mathbf{F}[L(h_1h)\phi](\lambda)\,dh_1 \\ &= (\dim \mathcal{H}_{\lambda,\alpha})^{-1}\delta(h)^{-1}\|\mathbf{F}[L(h)\phi](\lambda)\|_{\mathrm{HS}}^2 P_{\lambda,\alpha} \in \mathrm{End}(\mathcal{H}_{\lambda,\alpha}),\end{aligned}$$

see [16, pp. 43–44] for the detail. Note that $\dim \mathcal{H}_{\lambda,\alpha} = \dim \mathcal{H}_{\lambda_k,\alpha}$, which we denote by $n_{k,\alpha}$ in what follows. By (4.5) and (3.5), the integral (4.4) equals

$$\begin{aligned}&\frac{1}{n_{k,\alpha}}\int_{\mathcal{O}^*_{\lambda_k}}\int_H \Big(\mathrm{tr}\ \mathbf{F}f(\lambda)P_{\lambda,\alpha}\mathbf{F}f(\lambda)^*\Big)\|\mathbf{F}[L(h)\phi](\lambda)\|_{\mathrm{HS}}^2\delta(h)^{-1}\,d_H(h)\,d\mu(\lambda) \\ &= \frac{1}{n_{k,\alpha}}\int_{\mathcal{O}^*_{\lambda_k}}\int_H \|\mathbf{F}f(\lambda)\|_{\mathrm{HS}}^2\|\mathbf{F}[L(h)\phi](\lambda)\|_{\mathrm{HS}}^2\delta(h)^{-1}\,d_H(h)\,d\mu(\lambda),\end{aligned}$$

and the right-hand side is rewritten as

$$\frac{1}{n_{k,\alpha}}\int_{\mathcal{O}^*_{\lambda_k}} \|\mathbf{F}f(\lambda)\|_{\mathrm{HS}}^2\Big(\int_H \|\mathbf{F}\phi(h^{-1}\cdot\lambda)\|_{\mathrm{HS}}^2\,d_H(h)\Big)d\mu(\lambda)$$

by (2.4). Thanks to [16, Proposition 3], the integral $\int_H \|\mathbf{F}\phi(h^{-1}\cdot\lambda)\|_{\mathrm{HS}}^2\,d_H(h)$ does not depend on $\lambda \in \mathcal{O}^*_{\lambda_k}$, and it is equal to $\int_{\mathcal{O}^*_{\lambda_k}} \|\mathbf{F}\phi(\lambda)\|_{\mathrm{HS}}^2 D_k(\lambda)\,d\mu(\lambda)$ with a certain H-relatively invariant function D_k on $\mathcal{O}^*_{\lambda_k}$. Therefore we have

$$\begin{aligned}+\infty > \int_G |(f|L(g)\phi)|^2\,d_G(g) &= \int_{\mathcal{O}^*_{\lambda_k}} \|\mathbf{F}f(\lambda)\|_{\mathrm{HS}}^2 d\mu(\lambda)\cdot\frac{1}{n_{k,\alpha}}\int_H \|\mathbf{F}\phi(h^{-1}\cdot\lambda)\|_{\mathrm{HS}}^2\,d_H(h) \\ &= \|f\|^2\cdot\frac{1}{n_{k,\alpha}}\int_{\mathcal{O}^*_{\lambda_k}} \|\mathbf{F}\phi(\lambda)\|_{\mathrm{HS}}^2 D_k(\lambda)\,d\mu(\lambda).\end{aligned}$$

In particular, if $f \neq 0$, then

$$c(\phi) := \frac{1}{n_{k,\alpha}}\int_{\mathcal{O}^*_{\lambda_k}} \|\mathbf{F}\phi(\lambda)\|_{\mathrm{HS}}^2 D_k(\lambda)\,d\mu(\lambda) < +\infty \tag{4.6}$$

and

$$\int_G |(f|L(g)\phi)|^2\,d_G(g) = c(\phi)\|f\|^2. \tag{4.7}$$

Conversely, if ϕ satisfies the condition (4.6), the integral (4.4) converges for any $f \in L_{k,\alpha}(N)$, so that the right-hand side of (4.3) converges for almost all $h \in H$. Therefore Proposition 2.1 implies (4.3), so that we get (4.7) again and thus, $\int_G |(f|L(g)\phi)|^2\, d_G(g) < +\infty$.

Theorem 4.1. *The unitary representation $(L, L_{k,\alpha}(N))$ of G is irreducible.*

Proof. Let $\mathcal{L}$ be a nonzero invariant subspace of $L_{k,\alpha}(N)$. The orthogonal complement $\mathcal{L}^\perp \subset L_{k,\alpha}(N)$ is also invariant. We take $\phi \in \mathcal{L} \setminus \{0\}$ and $f \in \mathcal{L}^\perp$. By (4.7) we have

$$0 = \int_G |(f|L(g)\phi)|^2\, d_G(g) = c(\phi)\|f\|^2,$$

which implies $f = 0$. Actually, the argument is valid even if $c(\phi)$ is not finite. Therefore $\mathcal{L}^\perp = \{0\}$ and Theorem 4.1 is proved. □

Furthermore, we deduce from (4.6) and (4.7) the following result.

Theorem 4.2. *The representation $(L, L_{k,\alpha}(N))$ is square-integrable, whose Duflo-Moore operator $C_{k,\alpha}$ is given by*

$$\mathbf{F}[C_{k,\alpha}\phi](\lambda) = \sqrt{\frac{D_k(\lambda)}{n_{k,\alpha}}}\mathbf{F}\phi(\lambda) \qquad (\phi \in L_{k,\alpha}(N),\ \lambda \in \mathcal{O}^*_{\lambda_k}).$$

A classification of the representations $(L, L_{k,\alpha}(N))$ is also given in [16]. The result is as follows:

Theorem 4.3 ([16, Theorem 4])**.** *The unitary representations $(L, L_{k,\alpha}(N))$ and $(L, L_{k',\alpha'}(N))$ of G are equivalent if and only if $k = k'$ and the projective representations $(\tau_k, \mathcal{H}_{k,\alpha})$ and $(\tau_k, \mathcal{H}_{k,\alpha'})$ of H_k are equivalent, that is, there exists an isometry $A : \mathcal{H}_{k,\alpha} \to \mathcal{H}_{k,\alpha'}$ such that $\tau_k(h) \circ A = A \circ \tau_k(h)$ for all $h \in H_k$.*

Keeping the decomposition (3.6) in mind, and applying (1.2) to each representation $(L, L_{k,\alpha}(N))$, we obtain

Theorem 4.4. *For each $k = 1, \ldots, K$ and $\alpha \in A_k$, take admissible vectors $\phi_{k,\alpha} \in \mathrm{dom}(C_{k,\alpha}) \subset L_{k,\alpha}(N)$. Then for all $f \in L^2(N)$ one has*

$$f = \sum_{k=1}^{K} \sum_{\alpha \in A_k} \frac{1}{\|C_{k,\alpha}\phi_{k,\alpha}\|^2} \int_G W_{\phi_{k,\alpha}} f(g)\, L(g)\phi_{k,\alpha}\, d_G(g),$$

where $W_{\phi_{k,\alpha}} f(g) = (f|L(g)\phi_{k,\alpha})$.

§ 5. Example

As an illustrative example, we shall consider the case that the unimodular group N is the Heisenberg group of $(2\ell+1)$-dimension and H is isomorphic to $\mathbb{R}_+ \times U(\ell)$. The continuous wavelet transform for this case is first considered by He-Liu [14].

Let N be the Lie group consisting of elements $n(z, c)$ $(z \in \mathbb{C}^\ell,\ c \in \mathbb{R})$ with multiplication rule

$$n(z, c)n(z', c') := (z + z', c + c' + \Im(z|z')) \qquad (z, z' \in \mathbb{C}^\ell,\ c, c' \in \mathbb{R}),$$

where $(z|z') := \sum_{k=1}^{\ell} z_k \bar{z}'_k$. Define a Haar measure $d\nu$ on N by $d\nu(n(z, c)) := dm(z)dc$, where dm is the standard Euclidean measure on $\mathbb{C}^\ell$.

For $\zeta \in \mathbb{C}^\ell$, we define the unitary character χ_ζ by $\chi_\zeta(n(z, c)) := e^{i\Re(z|\zeta)}$. For $\lambda > 0$, we define

$$\mathcal{H}_\lambda := \left\{ \varphi : \mathbb{C}^\ell \to \mathbb{C} \text{ (holomorphic)} \,;\, \|\varphi\|^2 = \frac{\lambda^\ell}{\pi^\ell} \int_{\mathbb{C}^\ell} |\varphi(w)|^2 e^{-\lambda|w|^2}\, dm(w) < +\infty \right\},$$

and $\mathcal{H}_{-\lambda} := \{\, \overline{\varphi} \,;\, \varphi \in \mathcal{H}_\lambda \,\}$. These $\mathcal{H}_{\pm\lambda}$ are Hilbert spaces, on which we define the irreducible unitary representations $\pi_{\pm\lambda}$ of N by

$$\pi_\lambda(n(z, c))\varphi(w) := e^{-i\lambda c + \lambda(w|z) - \lambda|z|^2/2}\varphi(w - z) \qquad (\varphi \in \mathcal{H}_\lambda),$$
$$\pi_{-\lambda}(n(z, c))\varphi(w) := e^{i\lambda c + \lambda(z|w) - \lambda|z|^2/2}\varphi(w - z) \qquad (\varphi \in \mathcal{H}_{-\lambda}).$$

The Stone-von Neumann theorem states that every irreducible unitary representation of N is equivalent to one of χ_ζ $(\zeta \in \mathbb{C}^\ell)$ and π_λ $(\lambda \in \mathbb{R} \setminus \{0\})$, so that $\hat{N}$ can be identified with $\mathbb{C}^\ell \sqcup (\mathbb{R} \setminus \{0\})$. For $f_1, f_2 \in L^1(N) \cap L^2(N)$, we have

$$(f_1|f_2) = \frac{2^{\ell-1}}{\pi^{\ell+1}} \int_{\mathbb{R}\setminus\{0\}} (\pi_\lambda(f_1)|\pi_\lambda(f_2))_{\mathrm{HS}}\, |\lambda|^\ell d\lambda$$

by [9, Chapter I, section 5], which implies that the Plancherel measure μ on $\hat{N}$ is given by $\mu(\mathbb{C}^\ell) = 0$ and $d\mu(\lambda) = 2^{\ell-1}\pi^{-\ell-1}|\lambda|^\ell d\lambda$ $(\lambda \in \mathbb{R} \setminus \{0\})$.

For $a > 0$ and $u \in U(\ell)$, define $h(a, u) \in \mathrm{Aut}(N)$ by $h(a, u) \cdot n(z, c) := n(auz, a^2 c)$. Then we have

$$h(a, u)\, h(a', u') = h(aa', uu') \quad (a, a' > 0,\ u, u' \in U(\ell)),$$

so that $H := \{\, h(a, u) \,;\, a > 0, u \in U(\ell) \,\}$ is a subgroup of $\mathrm{Aut}(N)$. For $h = h(a, u) \in H$, we have $\delta(h) = a^{2\ell+2}$, and we define the representation $(L, L^2(N))$ of $G = N \rtimes H$ by (2.2). The action of H on $\hat{N}$ is described as

$$h(a, u) \cdot \zeta = a^{-1}u\zeta \qquad (\zeta \in \mathbb{C}^\ell),$$
$$h(a, u) \cdot \lambda = a^{-2}\lambda \qquad (\lambda \in \mathbb{R} \setminus \{0\}).$$

In particular, the intertwining operator $C(h,\lambda) : \mathcal{H}_\lambda \to \mathcal{H}_{a^{-2}\lambda}$ is given by

$$C(h,\lambda)\varphi(w) := \varphi(a^{-1}u^{-1}w) \quad (\varphi \in \mathcal{H}_\lambda).$$

We denote by $\mathcal{O}^*_\pm$ the H-orbit through $\pm 1 \in \mathbb{R} \setminus \{0\} \subset \hat{N}$. Then $\mathcal{O}^*_\pm = \{\, \lambda\,;\, \pm\lambda > 0 \,\}$, and the conditions (H1)–(H4) are satisfied for the two orbits. In particular, the stabilizer H_λ at any $\lambda \in \mathcal{O}^*_+ \sqcup \mathcal{O}^*_-$ equals the compact group $\{\, h(1,u)\,;\, u \in U(\ell) \,\} \simeq U(\ell)$. For a non-negative integer $\alpha \in \mathbb{Z}_{\geq 0}$, let $\mathcal{P}_\alpha(\mathbb{C}^\ell)$ be the space of holomorphic polynomials of degree α on $\mathbb{C}^\ell$, and $\overline{\mathcal{P}}_\alpha(\mathbb{C}^\ell)$ the space $\{\, \overline{\varphi}\,;\, \varphi \in \mathcal{P}_\alpha(\mathbb{C}^\ell) \,\}$. Then we have for $\lambda > 0$

$$\mathcal{H}_\lambda = \sum_{\alpha=0}^{\infty}{}^{\oplus}\, \mathcal{P}_\alpha(\mathbb{C}^\ell), \quad \mathcal{H}_{-\lambda} = \sum_{\alpha=0}^{\infty}{}^{\oplus}\, \overline{\mathcal{P}}_\alpha(\mathbb{C}^\ell),$$

which give the irreducible decomposition of the representation $(\tau_{\pm\lambda}, \mathcal{H}_{\pm\lambda})$ of $H_{\pm\lambda}$ respectively. Let $P_{\lambda,\alpha} : \mathcal{H}_\lambda \to \mathcal{P}_\alpha(\mathbb{C}^\ell)$ and $P_{-\lambda,\alpha} : \mathcal{H}_{-\lambda} \to \overline{\mathcal{P}}_\alpha(\mathbb{C}^\ell)$ be the orthogonal projections, and set

$$L_{\pm,\alpha}(N) := \left\{ f \in L^2(N)\,;\, \begin{array}{ll} \mathbf{F}f(\lambda) = \mathbf{F}f(\lambda)P_{\lambda,\alpha} & (\text{a.a. } \lambda \in \mathcal{O}^*_\pm) \\ \mathbf{F}f(\lambda) = 0 & (\text{otherwise}) \end{array} \right\}.$$

Then an irreducible decomposition of the unitary representation $(L, L^2(N))$ of G is given by

$$L^2(N) = \sum_{\alpha=0}^{\infty}{}^{\oplus}\, L_{+,\alpha}(N) \oplus \sum_{\alpha=0}^{\infty}{}^{\oplus}\, L_{-,\alpha}(N). \tag{5.1}$$

The decomposition is multiplicity-free thanks to Theorem 4.3.

We define a Haar measure d_H on H by $d_H(h(a,u)) := a^{-1}\, da\, du$, where du is the normalized Haar measure on $U(\ell)$. For $\lambda_0 \in \mathcal{O}^*_\pm$ and a measurable function $p : \mathcal{O}^*_\pm \to \mathbb{R}$, we observe

$$\begin{aligned}\int_H p(h^{-1}\cdot\lambda_0)d_H(h) &= \int_0^{+\infty} p(a^2\lambda_0)\,\frac{da}{a} = \int_{\mathcal{O}^*_\pm} p(\lambda)\frac{d\lambda}{2|\lambda|} \\ &= \int_{\mathcal{O}^*_\pm} p(\lambda)\cdot 2\Big(\frac{\pi}{2|\lambda|}\Big)^{\ell+1} d\mu(\lambda).\end{aligned}$$

Thus, setting $D_\pm(\lambda) := 2\Big(\frac{\pi}{2|\lambda|}\Big)^{\ell+1}$ $(\lambda \in \mathcal{O}^*_\pm)$, the Duflo-Moore operator $C_{\pm,\alpha}$ of the representation $(L,\, L_{\pm,\alpha}(N))$ is given by $\mathbf{F}[C_{\pm,\alpha}\phi](\lambda) = \sqrt{\frac{D_\pm(\lambda)}{n_\alpha}}\mathbf{F}\phi(\lambda)$ $(\phi \in L_{+,\alpha}(N),\, \lambda \in \mathcal{O}^*_\pm)$, where $n_\alpha = \dim_{\mathbb{C}} \mathcal{P}_\alpha(\mathbb{C}^\ell) = \binom{\alpha+\ell-1}{\alpha}$.

Finally, we give an example of admissible vector in $L_{\pm,\alpha}(N)$. We set

$$\phi_{\pm,\alpha}(n(z,c)) := \int_{\mathcal{O}^*_\pm} |\lambda| e^{-|\lambda|}\, \mathrm{tr}\, P_{\lambda,\alpha}\pi_\lambda(n(z,c))^*\, d\mu(\lambda).$$

Then $(|\lambda|e^{-|\lambda|}P_{\lambda,\alpha})_{\lambda\in\mathcal{O}^*_\pm}$ belongs to both of the direct integrals $\int^\oplus_{\mathcal{O}^*_\pm}\mathcal{B}_{\mathrm{Tr}}(\mathcal{H}_\lambda)\,d\mu(\lambda)$ and $\int^\oplus_{\mathcal{O}^*_\pm}\mathcal{B}_{\mathrm{HS}}(\mathcal{H}_\lambda)\,d\mu(\lambda)$. Applying Proposition 2.1, we see that $\phi_{\pm,\alpha}\in L^2(N)$ and that

$$\mathbf{F}\phi_{\pm,\alpha}(\lambda)=\begin{cases}|\lambda|e^{-|\lambda|}P_{\lambda,\alpha} & (\text{a.a. } \lambda\in\mathcal{O}^*_\pm),\\ 0 & (\text{otherwise}).\end{cases}$$

Thus $\phi_{\pm,\alpha}\in L_{\pm,\alpha}(N)$. Furthermore we have

$$\begin{aligned}\|C_{\pm,\alpha}\phi_{\pm,\alpha}\|^2&=n_\alpha^{-1}\int_{\mathcal{O}^*_\pm}\|\mathbf{F}\phi_{\pm,\alpha}(\lambda)\|^2_{\mathrm{HS}}\frac{d\lambda}{2|\lambda|}=n_\alpha^{-1}\int_{\mathcal{O}^*_\pm}n_\alpha|\lambda|^2e^{-2|\lambda|}\frac{d\lambda}{2|\lambda|}\\&=\frac{1}{2}\int_0^{+\infty}e^{-2\lambda}\lambda\,d\lambda=\frac{1}{8},\end{aligned}$$

so that $\phi_{\pm,\alpha}$ is admissible.

The function $\phi_{\pm,\alpha}$ can be calculated explicitly. By definition, we have $\phi_{-,\alpha}=\overline{\phi}_{+,\alpha}$, so that we shall consider only $\phi_{+,\alpha}$. For non-negative integers k and α, the Laguerre polynomial $L^k_\alpha(s)$ is defined by $L^k_\alpha(s):=\frac{e^ss^{-k}}{\alpha!}(\frac{d}{ds})^\alpha[e^{-s}s^{k+\alpha}]$. By [3, Proposition 6.2], we have

$$\mathrm{tr}\ P_{\lambda,\alpha}\pi_\lambda(n(z,c))^*=e^{(ic-|z|^2/2)\lambda}L^{\ell-1}_\alpha(\lambda|z|^2).$$

Thus

$$\phi_{+,\alpha}(n(z,c))=\frac{2^{\ell-1}}{\pi^{\ell+1}}\int_0^{+\infty}e^{-(1-ic+|z|^2/2)\lambda}L^{\ell-1}_\alpha(\lambda|z|^2)\lambda^{\ell+1}\,d\lambda.$$

On the other hand, for a parameter $|r|<1$, we have $\sum_{\alpha=0}^\infty r^\alpha L^{\ell-1}_\alpha(s)=(1-r)^{-\ell}e^{-\frac{rs}{1-r}}$. Therefore

$$\begin{aligned}\sum_{\alpha=0}^\infty r^\alpha\phi_{+,\alpha}(n(z,c))&=\frac{2^{\ell-1}}{\pi^{\ell+1}}\int_0^{+\infty}e^{-(1-ic+|z|^2/2)\lambda}(1-r)^{-\ell}e^{-\frac{r|z|^2\lambda}{1-r}}\lambda^{\ell+1}\,d\lambda\\&=\frac{2^{\ell-1}}{\pi^{\ell+1}}(1-r)^{-\ell}\cdot(\ell+1)!\Big\{(1-ic+|z|^2/2)+\frac{r|z|^2}{1-r}\Big\}^{-(\ell+2)}\\&=\frac{2^{\ell-1}(\ell+1)!}{\pi^{\ell+1}}(1-ic+|z|^2/2)^{-(\ell+2)}\\&\qquad\times(1-r)^2\left(1+\frac{-1+ic+|z|^2/2}{1-ic+|z|^2/2}\cdot r\right)^{-(\ell+2)}.\end{aligned}$$

Putting $\theta:=\frac{-1+ic+|z|^2/2}{1-ic+|z|^2/2}$, we have by the binomial theorem

$$\begin{aligned}&(1-r)^2(1+\theta r)^{-(\ell+2)}\\&=\sum_{\alpha=0}^\infty\frac{(-1)^\alpha(\alpha+\ell-1)!}{(\ell+1)!\alpha!}\{\alpha(\alpha-1)\theta^{\alpha-2}+2\alpha(\alpha+\ell)\theta^{\alpha-1}+(\alpha+\ell)(\alpha+\ell+1)\theta^\alpha\}r^\alpha.\end{aligned}$$

Hence

$$\begin{aligned}
&\phi_{+,\alpha}(n(z,c)) \\
&= \frac{2^{\ell-1}}{\pi^{\ell+1}}(1 - ic + |z|^2/2)^{-(\ell+2)} \\
&\qquad \times \frac{(-1)^\alpha(\alpha+\ell-1)!}{\alpha!}\{\alpha(\alpha-1)\theta^{\alpha-2} + 2\alpha(\alpha+\ell)\theta^{\alpha-1} + (\alpha+\ell)(\alpha+\ell+1)\theta^\alpha\}.
\end{aligned}$$

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