

K-FLIPS AND VARIATION OF MODULI SCHEME OF SHEAVES ON A SURFACE, II

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INTRODUCTION

We shall consider some analogy between the wall-crossing problem of moduli schemes of stable sheaves on a surface, and the minimal model program of higher-dimensional varieties. This article is a continuation of [10].

Let X be a non-singular projective surface over $\mathbb{C}$, and H an ample line bundle on X . Denote by $M(H)$ (resp. $M^s(H)$) the coarse moduli scheme of rank-two H -semistable (resp. H -stable) sheaves on X with Chern class $\alpha = (c_1, c_2) \in \text{Pic}(X) \times \mathbb{Z}$.

Let H_- and H_+ be α -generic polarizations such that just one α -wall W separates them. For $a \in [0, 1]$ one can define the a -semistability of sheaves on X and the coarse moduli scheme $M(a)$ (resp. $M^s(a)$) of rank-two a -semistable (resp. a -stable) sheaves with Chern classes α in such a way that $M(\epsilon) = M(H_+)$ and $M(1 - \epsilon) = M(H_-)$ if $\epsilon > 0$ is sufficiently small. $M(a)$ is projective over $\mathbb{C}$. Let $a_- < a_+$ be minichambers separated by only one miniwall a_0 , and denote $M_+ = M(a_+)$, $M_- = M(a_-)$ and $M_0 = M(a_0)$. There are natural morphisms $\phi_- : M_- \rightarrow M_0$ and $\phi_+ : M_+ \rightarrow M_0$ ([1], [2], [8]). One may say they are morphisms of moduli schemes coming from wall-crossing methods. Let $\phi_- : V_- \rightarrow V_0$ be a birational projective morphism such that (1) V_- is normal, (2) $-K_{V_-}$ is $\mathbb{Q}$ -Cartier and ϕ_- -ample, (3) the codimension of the exceptional set $\text{Exc}(\phi_-)$ is more than 1, and (4) the relative Picard number $\rho(V_-/V_0)$ of ϕ_- is 1. After the theory of minimal model program, we say a birational projective morphism $\phi_+ : V_+ \rightarrow V_0$ is a *K-flip* of $\phi_- : V_- \rightarrow V_0$ if (1) V_+ is normal, (2) K_{V_+} is $\mathbb{Q}$ -Cartier and ϕ_+ -ample, (3) the codimension of the exceptional set $\text{Exc}(\phi_+)$ is more than 1, and (4) the relative Picard number $\rho(V_+/V_0)$ of ϕ_+ is 1.

Theorem 0.1. *Fix a closed, finite, rational polyhedral cone $\mathcal{S} \subset \overline{\text{Amp}}(X)$ such that $\mathcal{S} \cap \partial \overline{\text{Amp}}(X) \subset \mathbb{R}_{\geq 0} \cdot K_X$. If c_2 is sufficiently large with respect to c_1 and $\mathcal{S}$, then for any α -generic polarizations H_- and H_+ in $\mathcal{S}$ separated by just one α -wall W , and for any adjacent minichambers $a_- < a_+$ separated by a miniwall a_0 we have the following.*

- (i) $M_{\pm}$ are normal and $\mathbb{Q}$ -factorial, $K_{M_{\pm}}$ are Cartier, $M_{\pm}^s$ are l.c.i., and M_- and M_+ are isomorphic in codimension 1.
- (ii) Suppose K_X does not lie in the α -wall, and that K_X and H_+ lie in the same connected components of $\text{NS}(X)_{\mathbb{R}} \setminus W$. Then $\rho(M_-/M_0) = 1$ and $\phi_+ : M_+ \rightarrow M_0$ is a *K-flip* of $\phi_- : M_- \rightarrow M_0$. This morphism ϕ_+ (resp. ϕ_-) is the contraction of an extremal ray of $\overline{\text{NE}}(M_+)$ (resp. $\overline{\text{NE}}(M_-)$), which is described in moduli theory.
- (iii) Suppose X is minimal and $\kappa(X) > 0$, which means K_X is not numerically equivalent to 0 and contained in $\overline{\text{Amp}}(X)$. Then there is a polarization, say H_X ,

contained in $\mathcal{S}$ such that no α -wall separates H_X and K_X , and the canonical divisor of $M(H_X)$ is nef.

The greater part of this result has already appeared in [10, Theorem 1.1.]. In Section 1, we shall prove the remaining part of this theorem which has not appeared in [10], that largely is the statement about the $\mathbb{Q}$ -factoriality of $M_{\pm}$ and $\rho(M_{\pm}/M_0)$. The author was not aware of this part at the time of writing [10]. There is some application; suppose X is minimal and $\kappa(X) > 0$, and fix a polarization L on X . If c_2 is sufficiently large with respect to c_1 and L , then one can observe a moduli-theoretic analogue of the minimal model program of $M(L)$. Here “analogue” means that singularities of $M(H_X)$ are not considered. About this analogy, see Introduction in [10] for detail. We remark that a K -flip differs from a Thaddeus-type flip in [8].

In Section 2, we give some notes about extremal faces of $\overline{NE}(M(H)) \subset N_1(M(H))$, where H is an α -generic polarization. We shall point out that some extremal faces with $\dim \geq 2$ can appear in $\overline{NE}(M(H))$ when H gets closer to more than one α -wall.

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Notation. All schemes are locally of finite type over $\mathbb{C}$ or, more generally, an algebraically closed field of characteristic zero. For a projective scheme V over $\mathbb{C}$, $\text{Num}(V)$ means $\text{Pic}(V)$ modulo numerical equivalence. For any coherent sheaf E on V , $\text{Ext}_V^i(E, E)^0$ means the kernel of trace map $\text{Ext}_V^i(E, E) \rightarrow H^i(\mathcal{O}_V)$.

1. PROOF OF THEOREM

There is a union of hyperplanes $W \subset \text{Amp}(X)$ called α -walls in the ample cone $\text{Amp}(X)$ such that $M(H) = M(H, \alpha)$ changes only when H passes through α -walls ([9]). A polarization on X is called α -generic if no α -wall contains it. Now fix a closed, finite, rational polyhedral cone $\mathcal{S} \subset \overline{\text{Amp}}(X)$ as in Theorem 0.1. Refer to [1, Section 3] about the a -stability, minichambers and miniwalls, which appeared in Introduction.

Lemma 1.1. *If c_2 is sufficiently large with respect to c_1 and $\mathcal{S}$, then for any α -generic polarizations H_- and H_+ in $\mathcal{S}$ separated by just one α -wall W , and for any adjacent minichambers a_- and a_+ separated by a miniwall a_0 , (i) $M_{\pm}$ are normal, (ii) $K_{M_{\pm}}$ are Cartier, (iii) $M_{\pm}^s$ are l.c.i., (iv) M_- and M_+ are isomorphic in codimension 4, and (v) our natural birational map $M_- \cdots > M_+$ induces $\text{Pic}(M_-^s) \simeq \text{Pic}(M_+^s)$.*

Proof. Fix a polarization $L \in \mathcal{S}$. If c_2 is sufficiently large w.r.t. c_1 and L , then $M(L)$ is normal, $M^s(L)$ is of expected dimension, and the codimension of $\text{Sing}'(M(L))$ in $M(L)$ is greater than 4 by [5] and [11], where $\text{Sing}'(M(L)) \subset M(L)$ is the closed subset consisting of sheaves E such that E is not L - μ -stable or that $\text{Ext}_X^2(E, E)^0 \neq 0$. One can check (iv) in a similar way to [10, Lemma 2.4.]. Now we compare $M(L)$ with M_+ . By (iv) and the deformation theory of simple sheaves, M_+^s is of expected dimension so it is l.c.i., and

$$(1) \quad \text{codim}(\text{Sing}'(M_+), M_+) > 4.$$

Thereby M_+^s is normal. Since $H_{\pm}$ are α -generic and $a_{\pm}$ are minichambers, if a rank-two sheaf E with Chern classes α is a_- -semistable and not a_+ -semistable, then E

is H -semistable for any polarization H , and so our birational map $M_+ \cdots > M_-$ is isomorphic near $M_+ \setminus M_+^s$. Thus M_+ is normal near $M_+ \setminus M_+^s$, and accordingly M_+ itself is normal. Item (v) follows item (iv) and (1) because of Fact 1.3 below. Last, M_+ is the GIT quotient of an open subset R_+ of some Quot-scheme on X . Let $\mathcal{E}$ be a universal family of R_+ on $X \times R_+$. Since a_+ is not a miniwall, one can check that the line bundle $\det R\mathcal{H}om_{p_2}(\mathcal{E}, \mathcal{E})$ on R_+ descends to a line bundle on M_+ , that equals K_{M_+} . $\square$

Next we recall a fact concerning $\text{Pic}(M_+^s)$ from [6]. For a moment we assume M_+^s has a universal family $\mathcal{E}$ on $X \times M_+^s$. Let K be the Grothendieck group of $X \times X$ and let $\tilde{K}$ be the kernel of $\xi : K \rightarrow \mathbb{Z}$, that is defined by $\xi(C) = \chi(C \boxtimes \pi_1^* \mathcal{E} \boxtimes \pi_2^* \mathcal{E})$. Here $\boxtimes$ denotes the tensor product of complexes. Let $\sigma : X \times X \rightarrow X \times X$ be the map exchanging factor and let $\text{Pic}(X \times X)^\sigma$ be the subgroup consisting of line bundles invariant under σ . The map $\psi : \tilde{K} \rightarrow \text{Pic}(M_+)$ defined by

$$(2) \quad \psi(C) = \det((p_1)_!(p_{23}^*(C) \boxtimes p_{12}^* \mathcal{E} \boxtimes p_{13}^* \mathcal{E})) \quad (C \in \tilde{K})$$

induces a homomorphism

$$(3) \quad \Phi_\pm : \text{Pic}(X \times X)^\sigma \oplus \mathbb{Z} \longrightarrow \text{Pic}(M_\pm^s) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{12}],$$

as explained in [6, p. 132]. One can define Φ also when M_+^s do not necessarily admit a universal family.

Proposition 1.2. *Let $a_\pm$ be a minichamber satisfying assumptions in Lemma 1.1. If c_2 is sufficiently large with respect to c_1 and $\mathcal{S}$, then*

$$(4) \quad \Phi_\pm \otimes \mathbb{Q} : \text{Pic}(X \times X)^\sigma \otimes \mathbb{Q} \oplus \mathbb{Q} \rightarrow \text{Pic}(M_\pm^s) \otimes \mathbb{Q}$$

is isomorphic.

Proof. One can verify this from Lemma 1.1 (v) and by reading [6] (especially Lemma 3.10.) carefully. $\square$

Before the proof of Theorem 0.1, recall a useful fact at [SGA2, p.132].

Fact 1.3. *Let W be any quasi-projective and l.c.i. scheme with $\text{codim}(\text{Sing}(W), W) \geq 4$. Then for any closed subset $\Lambda \subset W$ of codimension at least two, the restriction map $\text{Pic}(W) \rightarrow \text{Pic}(W \setminus \Lambda)$ is an isomorphism.*

Now we shall prove two propositions; those and [10] end the proof of Theorem 0.1.

Proposition 1.4. *Let a_+ be a minichamber satisfying assumptions in Lemma 1.1. Suppose c_2 is so large with respect to c_1 and $\mathcal{S}$ that $M_\pm$ are normal, $M_\pm^s$ are l.c.i., $\text{codim}(\text{Sing}(M_\pm^s), M_\pm^s) \geq 4$, $\text{codim}(M_\pm \setminus M_\pm^s, M_\pm) \geq 2$, and the homomorphisms at (4) are isomorphic. Then $M_\pm$ are $\mathbb{Q}$ -factorial.*

Proof. First remark that assumptions in this proposition holds for $c_2 \gg 0$ from Lemma 1.1, Proposition 1.2, [11], and [3, Theorem 9.1.2.]. We shall verify this only for M_+ . Let U be the open set $M_+ \setminus \text{Sing}(M_+)$ in M_+ . If $\text{Cl}(M_+)$ means its divisor class group generated by Weil divisors, then we have

$$\text{Cl}(M_+) \longrightarrow \text{Cl}(U) \simeq \text{Pic}(U) \longrightarrow \text{Pic}(M_+^s),$$

where the first map is restriction, the second map is isomorphism since U is smooth, and the third map is an extension map, which is assured by Fact 1.3. Next, we have the following diagram.

$$\begin{array}{ccc} \bar{\Phi}_+ \otimes \mathbb{Q} : \text{Pic}(X \times X)^\sigma \otimes \mathbb{Q} \oplus \mathbb{Q} & \longrightarrow & \text{Pic}(M_+) \otimes \mathbb{Q} \\ \parallel & & \downarrow \\ \Phi_+ \otimes \mathbb{Q} : \text{Pic}(X \times X)^\sigma \otimes \mathbb{Q} \oplus \mathbb{Q} & \longrightarrow & \text{Pic}(M_+^s) \otimes \mathbb{Q}, \end{array}$$

where $\bar{\Phi}_+$ is defined at the equation (1.13) in [6] since $H_\pm$ are α -generic and a_+ is not a miniwall, and the second column is a restriction map. Proposition 1.2 implies that the second column is surjective. On the other hand, the assumptions in this proposition implies that the second column is injective. As a result we get a homomorphism $\text{Cl}(M_+) \rightarrow \text{Pic}(M_+) \otimes \mathbb{Q}$. Thus we end the proof. $\square$

For a projective morphism f , we define $N_1(f)$ and $\overline{\text{NE}}(f)$ according to [4, Example 2.16], an *extremal ray* or *extremal face* of $\overline{\text{NE}}(f)$ according to [4, Definition 1.15], and the *contraction of an extremal ray or face* according to [4, Definition 1.25].

Proposition 1.5. *Let $a_\pm$ be minichambers as in Theorem 0.1. Suppose c_2 is sufficiently so large with respect to c_1 and $\mathcal{S}$ that conclusions in Lemma 1.1 and Proposition 1.2 hold good. Then we have the following. Let t be any point in $\phi_+(\text{Exec}(\phi_+)) \subset M_0$, and let $l \simeq \mathbf{P}^1$ be any line in $\phi_+^{-1}(t) \simeq \mathbf{P}^{N_t}$. Then $\mathbb{R}_{\geq 0} \cdot l$ is an extremal ray of $\overline{\text{NE}}(M_+)$, and ϕ_+ is the contraction of this extremal ray. In particular $\rho(M_+/M_0) = 1$. The similar statement holds also for $\phi_- : M_- \rightarrow M_0$.*

Proof. We check it for a_+ ; the proof is the same for a_- . For simplicity suppose that M_+^s has a universal family $\mathcal{E}$ on $X \times M_+$, but the proof goes in a similar way for general case. The set

$$(5) \quad M_+ \supset P_+ = \{[E] \mid E \text{ is not } a_- \text{-semistable}\}$$

is contained in M_+^s since we consider rank-two case. Take a point $t \in \phi_+(P_+)$. By Proposition 2.1. in [10], it holds that $\phi_+^{-1}(t) \simeq \mathbf{P}^N$, and there is a nontrivial exact sequence on $X \times \mathbf{P}^N$

$$(6) \quad 0 \longrightarrow \pi_1^* F \otimes \mathcal{O}_{\mathbf{P}^N}(1) \longrightarrow \mathcal{E}|_{\phi_+^{-1}(t)} \otimes \pi_2^* L \longrightarrow \pi_1^* G \longrightarrow 0,$$

where F and G are coherent sheaves on X , which depends on the choice of t , and L is a line bundle on $\phi_+^{-1}(t)$. Let $l \simeq \mathbf{P}^1$ be a line in $\phi_+^{-1}(t)$. Then (6) implies that $ch(\mathcal{E}|_l) = ch(E) + \mathcal{O}_l(1) \cdot ch(F)$ in $A(X \times l)$, where E is a rank-two sheaf with Chern classes α . Let C be a class in $\tilde{K}$. Because of the definition of $\tilde{K}$ and the G.R.R. theorem, we have

$$\begin{aligned} \deg(\psi(C) \cdot l) &= [p_{1*}(ch(p_{23}^* C \boxtimes p_{12}^* \mathcal{E}|_l \boxtimes p_{13}^* \mathcal{E}|_l) \cdot p_{23}^* td(X \times X))]_{1,l \times X \times X} \cdot \mathcal{O}_l(1) \\ &= [p_{1*}(p_{23}^* ch(C) \cdot \{p_2^* ch(E) + p_1^* \mathcal{O}_l(1) \cdot p_2^* ch(F)\} \cdot \\ &\quad \{p_3^* ch(E) + p_1^* \mathcal{O}_l(1) \cdot p_3^* ch(F)\} \cdot p_{23}^* td(X \times X))]_{1,l \times X \times X} \cdot \mathcal{O}_l(1) \\ &= [ch(C) \cdot td(X \times X) \cdot \{\pi_1^* ch(F) \pi_2^* ch(E) + \pi_2^* ch(F) \pi_1^* ch(E)\}]_{0, X \times X} \\ &= \chi(X \times X, C \boxtimes (\pi_1^* F \boxtimes \pi_2^* E + \pi_2^* F \boxtimes \pi_1^* E)). \end{aligned}$$

By the projection formula and again by the definition of $\tilde{K}$, the last term equals

$$\begin{aligned} & \chi(X \times X, C \boxtimes \{\pi_1^*(F + G + F - G) \boxtimes \pi_2^*(E) + \pi_2^*(F + G + F - G) \boxtimes \pi_1^*(E)\}) / 2 \\ &= \chi(X \times X, C \boxtimes \{\pi_1^*(F - G) \boxtimes \pi_2^*(E) + \pi_2^*(F - G) \boxtimes \pi_1^*(E)\}) / 2 = \\ & [\pi_1^*td(X) \cdot \pi_2^*td(X) \cdot ch(C) \cdot \{\pi_1^*ch(F - G) \cdot \pi_2^*ch(E) + \pi_2^*ch(F - G) \cdot \pi_1^*ch(E)\}]_0 / 2 = \\ & [\{\pi_{1*}(ch(C) \cdot \pi_2^*(td(X)ch(E))) + \pi_{2*}(ch(C) \cdot \pi_1^*(td(X)ch(E)))\} \cdot td(X)ch(F - G)]_0 / 2. \end{aligned}$$

From [1, Section 3], if we denote $\xi = c_1(F) - c_1(G) \in \text{NS}(X)$, $n = c_2(F)$ and $m = c_2(G)$, then $W^\xi = \{H \in \text{Amp}(X) | H \cdot \xi = 0\}$ equals W and one can check that $td(X) \cdot ch(F - G) = (0, \xi, (a_0 - 1)(H_+ - H_-) \cdot \xi)$. Thereby one can verify that

$$(7) \quad \deg(\psi(C) \cdot l) = [\{\pi_{1*}(C \cdot \pi_2^*td(X)) + \pi_{2*}(C \cdot \pi_1^*td(X))\}^1 + (a_0 - 1) \{\pi_{1*}(C \cdot \pi_2^*td(X)) + \pi_{2*}(C \cdot \pi_1^*td(X))\}^0 \cdot (H_+ - H_-)] \cdot \xi / 2.$$

Now we shall show that $\text{rk} N_1(M_+/M_0) = 1$. If we pick two points t_1 and t_2 in $\phi_+(P_+)$, then $\phi_+^{-1}(t_i) \simeq \mathbf{P}^{N_i}$ for $i = 1, 2$. Fix lines $l_i \subset \phi_+^{-1}(t_i)$. Then there are exact sequences on $X \times l_i$

$$0 \longrightarrow \pi_1^*F_i \otimes \mathcal{O}_{\mathbf{P}^N}(1) \longrightarrow \mathcal{E}|_{l_i} \otimes \pi_2^*L_i \longrightarrow \pi_1^*G_i \longrightarrow 0,$$

where F_i and G_i are coherent sheaves on X , and L_i is a line bundle on l_i , for $i = 1, 2$. Since the wall defined by $\xi_i = c_1(F_i) - c_1(G_i)$ equals W for $i = 1, 2$, there is a rational number r such that $\xi_1 = r\xi_2$ in $\text{Num}(X)$. Then (4) and (7) imply that $l_1 \equiv r \cdot l_2$ in $N_1(M_+/M_0)$. As a result, we have $\overline{\text{NE}}(\phi_+) = \mathbb{R}_{\geq 0} \cdot l$.

Now $\mathbb{R}_{\geq 0} \cdot l$ is an extremal ray of $\overline{\text{NE}}(M_+)$. Indeed, let $u_i \in \overline{\text{NE}}(M_+)$ ($i = 1, 2$) satisfy that $u_1 + u_2 \in \mathbb{R}_{\geq 0} \cdot l$. Then, for any $H \in \text{Amp}(M_0)$, $0 = (u_1 + u_2) \cdot \phi_+^*(H) = u_1 \cdot \phi_+^*(H) + u_2 \cdot \phi_+^*(H)$. Since $u_i \in \overline{\text{NE}}(M_+)$, we have $u_i \cdot \phi_+^*(H) \geq 0$, and hence $u_i \cdot \phi_+^*(H) = 0$ for $i = 1, 2$. Recall that, by Example-Exercise 3-5-1 in [7], a natural inclusion $N_1(\phi_+) \subset N_1(M_+)$ identifies $\overline{\text{NE}}(\phi_+)$ with

$$\{z \in \overline{\text{NE}}(M_+) \mid z \cdot \phi_+^*(H) = 0 \text{ for any } H \in \text{Amp}(M_0)\}.$$

Thereby $u_i \in \overline{\text{NE}}(\phi_+) = \mathbb{R}_{\geq 0} \cdot l$.

Last, ϕ_+ is the contraction of $\mathbb{R}_+ \cdot l$. Indeed, for any irreducible curve $C \subset M_+$, one can verify that $\phi_+(C)$ is a point if and only if $C \in \mathbb{R}_+ \cdot l$ by using arguments above. Also it holds that $\phi_{+*}(\mathcal{O}_{M_+}) \simeq \mathcal{O}_{M_0}$, since one can show that M_0 is normal from conclusions in Lemma 1.1 and Serre's criterion of normality, and so we conclude the proof of this proposition. $\square$

2. SOME EXTREMAL FACES OF $M(H)$

Now we suppose that a polarization H_+ is α -generic and contained in an α -chamber $\mathcal{C}$, with which two different α -walls W_1 and W_2 contact, that a polarization H_0 is contained in $W_1 \cap W_2 \cap \overline{\mathcal{C}}$, and that no α -wall except W_1 and W_2 contains H_0 . Similarly to [1, Section 3], for $a \in [0, 1]$ one can define the a -stability of a coherent sheaf on X and the moduli scheme $M(a)$ of a -semistable rank-two sheaves on X with fixed Chern classes in such a way that $M(1) = M(H_0)$ and $M(\epsilon) = M(H_+)$ if $\epsilon \geq 0$ is sufficiently small. Let $a_\pm$ be minichambers separated by just one miniwall a_0 . Then Proposition 2.1 below says that $\rho(M_+/M_0)$ can be greater than 1, $\overline{\text{NE}}(M_+)$ can have an extremal face with $\dim \geq 2$, and so $\overline{\text{NE}}(M_+)$ can admit a “polyhedral-like part”.

Let $P_+ \subset M_+^s$ be the set defined at (5). Every member $E \in P_+$ has a Harder-Narasimhan filtration with respect to a_- , that is given by a nontrivial exact sequence

$$0 \longrightarrow F \longrightarrow E \longrightarrow G \longrightarrow 0,$$

and then one can check that the wall defined by $\xi(E) := c_1(F) - c_1(G) \in \text{NS}(X)$ equals W_1 or W_2 because of the way to derive $a_{\pm}$ from $H_{\pm}$. For $j = 1, 2$, we define a set

$$P_+ \supset P_+^{(j)} = \{[E] \in P_+ \mid \text{the wall defined by } \xi(E) \text{ equals } W_j\}.$$

Then, from the uniqueness of a_- -HNF, $P_+^{(j)}$ is a union of some connected components of P_+ , and it holds that $P_+^{(1)} \cap P_+^{(2)} = \emptyset$.

Proposition 2.1. *Suppose that both $P_+^{(1)}$ and $P_+^{(2)}$ are non-empty. Then $\overline{\text{NE}}(M_+)$ has a two-dimensional extremal face spanned by $\mathbb{R}_{\geq 0} \cdot l_1$ and $\mathbb{R}_{\geq 0} \cdot l_2$, where $l_j \simeq \mathbf{P}^1$ is a line contained in $\phi_+^{-1}(t_j) \simeq \mathbf{P}^{N_j}$ with some $t_j \in \phi_+(P_+^{(j)})$, for $j = 1, 2$. The morphism ϕ_+ is the contraction of this extremal face.*

Proof. If a sheaf $E_j \in M_+^s$ is a member of $l_j \subset P_+$, then one can check that $\mathbb{R} \cdot \xi(E_1)$ does not contain $\xi(E_2)$ in $\text{Num}(X)$ since $W_1 \neq W_2$. Thus it follows from (7) that the ray $\mathbb{R}_{\geq 0} \cdot l_1$ does not contain l_2 in $N_1(M_+)$. In a similar way to the proof of Proposition 1.5, we can check that (i) $\overline{\text{NE}}(\phi_+) = \mathbb{R}_{\geq 0} \cdot l_1 + \mathbb{R}_{\geq 0} \cdot l_2$, (ii) this is a two-dimensional extremal face of $\overline{\text{NE}}(M_+)$, and (iii) ϕ_+ is the contraction of this extremal face. $\square$

Similarly, suppose that different α -walls W_j ($1 \leq j \leq N$) contact with an α -chamber $\mathcal{C}$ containing H_+ and satisfy that $\cap_{j=1}^N W_j \cap \overline{\mathcal{C}}$ is non-empty. Then $\rho(M_+/M_0)$ can be N or more, and $\overline{\text{NE}}(M_+)$ can have an extremal face with $\dim \geq N$.

Remark 2.2. There does exist an example of a surface X , a class α with $4c_2 - c_1^2 \gg 0$, an α -chamber $\mathcal{C}$, two α -walls W_1 and W_2 , an α -generic polarization H_+ , a polarization H_0 , a minichamber a_+ and a miniwall a_0 such that both $P_+^{(1)}$ and $P_+^{(2)}$ are non-empty. We leave it to the reader to find such examples. In rank-two case, the definition of α -walls is rather numerical. Hence if one grasps the structure of $\text{Amp}(X)$, then it may be just a calculating exercise to find such an example. Remark that, when X is an Abelian surface, $\text{Amp}(X)$ is just a connected component of the big cone of X .

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