

Solvability of heat equations with hysteresis coupled with Navier-Stokes equations in 2D and 3D

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This is a prompt report of the author [33].

1 Introduction

1.1 Problem and related works

Let $T > 0$ and $\Omega \subset \mathbb{R}^N$ ($N = 2, 3$) be a bounded domain with smooth boundary Γ . We consider the following problem (P):

$$(P) \quad \begin{cases} \psi_1(\theta) \leq w \leq \psi_2(\theta) & \text{in } Q := (0, T) \times \Omega, \\ \partial w / \partial t = 0 & \text{in } Q[\psi_1(\theta) < w < \psi_2(\theta)], \\ \partial w / \partial t > 0 & \text{in } Q[w = \psi_1(\theta)], \\ \partial w / \partial t < 0 & \text{in } Q[w = \psi_2(\theta)], \\ \partial \theta / \partial t - \Delta \theta + \mathbf{v} \cdot \nabla \theta + w = f & \text{in } Q, \\ \partial \mathbf{v} / \partial t - \Delta \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \mathbf{g}(\theta) - \nabla \pi & \text{in } Q, \\ \operatorname{div} \mathbf{v} = 0 & \text{in } Q, \\ \theta = 0, \quad \mathbf{v} = 0 & \text{in } (0, T) \times \Gamma, \\ w(0) = w_0, \quad \theta(0) = \theta_0, \quad \mathbf{v}(0) = \mathbf{v}_0 & \text{in } Q, \end{cases}$$

where $w : Q \rightarrow \mathbb{R}$, $\theta : Q \rightarrow \mathbb{R}$, $\mathbf{v} : Q \rightarrow \mathbb{R}^N$ and $\pi : Q \rightarrow \mathbb{R}$ stand for the hysteresis term, the temperature, the velocity and the pressure, respectively, and these are unknown functions; $\psi_1, \psi_2 : \mathbb{R} \rightarrow \mathbb{R}$, $f : Q \rightarrow \mathbb{R}$, $\mathbf{g} : \mathbb{R} \rightarrow \mathbb{R}^N$, $w_0 : \Omega \rightarrow \mathbb{R}$, $\theta_0 : \Omega \rightarrow \mathbb{R}$ and $\mathbf{v}_0 : \Omega \rightarrow \mathbb{R}^N$ are given functions.

From a view point of physics the problem (P) describes the temperature θ , the velocity $\mathbf{v}$ and the pressure π of incompressible fluid in a bounded region Ω on a time period $[0, T]$. It is especially peculiar that the temperature will be controlled by the heat source $-w$, which is fluctuated by the present temperature. Such phenomenon comes from the temperature-dependent constraint on w :

$$\psi_1(\theta) \leq w \leq \psi_2(\theta).$$

Typical examples of ψ_1, ψ_2 are non-decreasing functions. Then such model represents e.g., phenomenon by thermostat devices. For more details, if the temperature θ rises (falls), then the heat source $-w$ will fall (rise), influenced by the obstacle functions ψ_1, ψ_2 . This means that thermostat devices cool (heat) the fluid, responding to too high (low) temperature.

Mathematically, the problem (P) is the Boussinesq system with hysteresis formulated in a quasi-variational inequality, which represents the phenomenon by thermostat devices. Boussinesq systems are dealt with in many works such as Morimoto [25], Fukao-Kenmochi [8], Kubo [20], Fukao-Kubo [10], [11], Sobajima-the author-Yokota [28], Larios-Lunasin-Titi [21], Li-Xu [22], Miao-Zheng [23], Fukao-Kenmochi [9] and the author [31]. Thermostat models for hysteresis formulated in a quasi-variational inequality are studied in e.g., Kenmochi-Koyama-Meyer [17], and other models for such hysteresis are also studied in, Kubo [19], Colli-Kenmochi-Kubo [4] and so on. Thermostat models with relay hysteresis are studied by many authors such as Glashoff-Sprekels [12], [13], Visintin [34], Kopfová-Kopf [18], Gurevich-Jäger-Skubachevskii [15] and Gurevich-Tikhomirov [16].

Recently, the author [32] showed existence for the problem (P) in the 2D case with the Navier-Stokes equation in a weak sense. That is, (P) has at least one solution (w, θ, v) satisfying

$$(1.1) \quad v \in H^1(0, T; (\mathbf{H}_\sigma^1(\Omega))^*) \cap L^\infty(0, T; \mathbf{L}_\sigma^2(\Omega)) \cap L^2(0, T; \mathbf{H}_\sigma^1(\Omega))$$

with the condition

$$v_0 \in \mathbf{L}_\sigma^2(\Omega) = D(A^0),$$

where $\mathbf{L}_\sigma^2(\Omega)$ and $\mathbf{H}_\sigma^1(\Omega)$ are roughly sets of Lebesgue and Sobolev functions satisfying divergence freeness, respectively (see Section 1.2), and

$$A : D(A) := \mathbf{H}^2(\Omega) \cap \mathbf{H}_\sigma^1(\Omega) \subset \mathbf{L}_\sigma^2(\Omega) \rightarrow \mathbf{L}_\sigma^2(\Omega)$$

is the Stokes operator, which is defined as roughly $-\Delta$ (see Section 1.2). However this result does not assert uniqueness for (P). When we try to attain uniqueness for (P), we would put (w_i, θ_i, v_i) as a solution of (P) ($i = 1, 2$). In this case, $\|w_1 - w_2\|_{L^\infty(0, T; L^\infty(\Omega))}$ is required to be estimated, and hence so is $\|\theta_1 - \theta_2\|_{L^\infty(0, T; L^\infty(\Omega))}$. Then we need an appropriate estimate for

$$(1.2) \quad \theta_1 \cdot \nabla(v_1 - v_2) \quad \text{and} \quad (\theta_1 - \theta_2) \cdot \nabla v_2.$$

This breaks down in [32] because of low regularity for solutions of the Navier-Stokes equation (see (1.1)).

The purpose of this paper is to establish existence and uniqueness for (P) with v more regular than the class (1.1). In order to decide height of regularity for v so that (1.2) can be appropriately estimated, we introduce the fractional power of the Stokes operator and its domain $D(A^\alpha)$ ($0 \leq \alpha \leq 1$) (such operator A^α is dealt with by e.g., Fujiwara [7], Fujita-Morimoto [6], Ôtani [26], Mitrea-Monnaux [24], and Guermond-Salgado [14]). In fact, we will establish existence and uniqueness for (P) in a N -dimensional domain ($N = 2, 3$), where the solution of the Navier-Stokes equation belongs the next class:

$$v \in H^1(0, T; D(A^{\frac{1-\alpha}{2}})^*) \cap L^\infty(0, T; D(A^{\frac{\alpha}{2}})) \cap L^2(0, T; D(A^{\frac{1+\alpha}{2}}))$$

with the condition

$$v_0 \in D(A^{\frac{\alpha}{2}}), \quad \frac{3(N-2)}{4} < \alpha \leq 1.$$

Here $D(A^{\frac{\alpha}{2}})$ is roughly a set of α -order differentiable functions satisfying divergence free.

1.2 Main results

First we introduce notation, starting with $H := L^2(\Omega)$, $V := H_0^1(\Omega)$, $\mathbf{H} := \mathbf{L}_\sigma^2(\Omega)$ and $\mathbf{V} := \mathbf{H}_\sigma^1(\Omega)$ with the standard inner products, respectively, where $\mathbf{L}_\sigma^2(\Omega)$ and $\mathbf{H}_\sigma^1(\Omega)$ are the closure of $\mathcal{D}_\sigma(\Omega) := \{\mathbf{v} \in \mathcal{D}(\Omega) = \mathcal{C}_0^\infty(\Omega) \mid \operatorname{div} \mathbf{v} = 0\}$ on $\mathbf{L}^2(\Omega)$ and $\mathbf{H}^1(\Omega)$, respectively. Here the dense and compact imbeddings $V \hookrightarrow H$ and $\mathbf{V} \hookrightarrow \mathbf{H} \hookrightarrow \mathbf{V}^*$ hold.

To formulate the equation for hysteresis we define the closed and convex set $K(\theta)$ and the indicator function I_θ , which are depending on $\theta \in H$, as

$$K(\theta) := \{w \in H \mid \psi_1(\theta) \leq w \leq \psi_2(\theta) \text{ a.e. on } \Omega\}, \quad \theta \in H,$$

$$I_\theta(w) := \begin{cases} 0 & w \in K(\theta), \\ \infty & w \in H \setminus K(\theta), \end{cases} \quad \theta \in H.$$

Then we introduce the subdifferential operator of ∂I_θ , which is characterized by $\xi \in \partial I_\theta(w) \Leftrightarrow (-\xi, w - z)_H \leq 0$ ($z \in K(\theta)$) for $\theta \in H$ and $w \in D(\partial I_\theta) = K(\theta)$. For details on subdifferential operators we can refer to e.g., Barbu [1], [2].

On the other hand, for formulation of the Navier-Stokes equation, we define the Stokes operator $A : D(A) \subset \mathbf{H} \rightarrow \mathbf{H}$ as $A := -P\Delta$, where $D(A) := \mathbf{H}^2(\Omega) \cap \mathbf{V}$ and $P : \mathbf{L}^2(\Omega) \rightarrow \mathbf{H}$ is the Helmholtz projection. It is well-known the operator A can be extended to the following form:

$$A : \mathbf{V} \rightarrow \mathbf{V}^*, \quad \langle A\mathbf{v}, \mathbf{z} \rangle_{\mathbf{V}^*, \mathbf{V}} := \sum_{i,j=1}^N \int_{\Omega} \frac{\partial v_j}{\partial x_i} \frac{\partial z_j}{\partial x_i} dx, \quad \mathbf{v}, \mathbf{z} \in \mathbf{V}.$$

Here we introduce the fractional power of the Stokes operator A^α ($-1 \leq \alpha \leq 1$), which is linear, unbounded and self-adjoint operator on $\mathbf{H}$. Moreover we define the Hilbert space $\mathbf{V}_\alpha$ as $\mathbf{V}_\alpha := D(A^{\frac{\alpha}{2}})$ for $0 \leq \alpha \leq 2$ and $\mathbf{V}_\alpha := \mathbf{V}_{-\alpha}^*$ for $-2 \leq \alpha < 0$ with the inner product $(\mathbf{u}, \mathbf{v})_{\mathbf{V}_\alpha} := (A^{\frac{\alpha}{2}} \mathbf{u}, A^{\frac{\alpha}{2}} \mathbf{v})_{\mathbf{H}}$, $\mathbf{u}, \mathbf{v} \in \mathbf{V}_\alpha$ for $-2 \leq \alpha \leq 2$, where $A^{\frac{\alpha}{2}} \mathbf{u} \in \mathbf{H}$ for $-2 \leq \alpha < 0$ and $\mathbf{u} \in \mathbf{V}_\alpha$ means that $(A^{\frac{\alpha}{2}} \mathbf{u}, \mathbf{z})_{\mathbf{H}} = \langle \mathbf{u}, A^{\frac{\alpha}{2}} \mathbf{z} \rangle_{\mathbf{V}_\alpha, \mathbf{V}_{-\alpha}}$ for all $\mathbf{z} \in \mathbf{H}$. Then $\mathbf{V}_\alpha$ is a set of α -order differentiable functions as follows:

$$\mathbf{V}_\alpha = \begin{cases} \mathbf{H}^\alpha(\Omega) \cap \mathbf{H}, & 0 \leq \alpha < \frac{1}{2}, \\ \mathbf{H}_0^\alpha(\Omega) \cap \mathbf{H}, & \frac{1}{2} \leq \alpha \leq 1, \\ \mathbf{H}^\alpha(\Omega) \cap \mathbf{V}, & 1 \leq \alpha \leq 2. \end{cases}$$

Here $\mathbf{H}^\alpha$ and $\mathbf{H}_0^\alpha$ are the fractional Sobolev spaces (see e.g., Demengel-Demengel [5]). Indeed, e.g., [14, Corollary 2.1] read the above characterization. For details on the fractional powers of the Stokes operator, we can refer to [7], [6] and [24]. Moreover note $\mathbf{V}_0 = \mathbf{H}$, $\mathbf{V}_1 = \mathbf{V}$ and the compact and dense imbeddings $\mathbf{V}_\alpha \hookrightarrow \mathbf{H} \hookrightarrow \mathbf{V}_{-\alpha}$ for $0 \leq \alpha \leq 1$. In this paper, we regard A as the following form for all $0 \leq \alpha \leq 1$:

$$A : \mathbf{V}_{1+\alpha} \rightarrow \mathbf{V}_{-1+\alpha},$$

$$\langle A\mathbf{v}, \mathbf{z} \rangle_{\mathbf{V}_{-1+\alpha}, \mathbf{V}_{1-\alpha}} = \left(A^{\frac{1+\alpha}{2}} \mathbf{v}, A^{\frac{1-\alpha}{2}} \mathbf{z} \right)_{\mathbf{H}}, \quad \mathbf{v} \in \mathbf{V}_{1+\alpha}, \mathbf{z} \in \mathbf{V}_{1-\alpha}.$$

Moreover we define the operator B as for all $0 \leq \alpha \leq 1$,

$$B : \mathbf{V}_\alpha \times \mathbf{V}_{1+\alpha} \rightarrow \mathbf{V}_{-1+\alpha},$$

$$\langle B(\mathbf{u}, \mathbf{v}), \mathbf{z} \rangle_{\mathbf{V}_{-1+\alpha}, \mathbf{V}_{1-\alpha}} := \int_{\Omega} ((\mathbf{u} \cdot \nabla) \mathbf{v}) \mathbf{z} \, d\mathbf{x} = \sum_{i,j=1}^N \int_{\Omega} u_i \frac{\partial v_j}{\partial x_i} z_j \, d\mathbf{x},$$

$$(\mathbf{u}, \mathbf{v}) \in \mathbf{V}_\alpha \times \mathbf{V}_{1+\alpha}, \mathbf{z} \in \mathbf{V}_{1-\alpha}.$$

Here (3.3) in Lemma 3.1 in Section 3 guarantees B operates $\mathbf{V}_\alpha \times \mathbf{V}_{1+\alpha}$ on $\mathbf{V}_{-1+\alpha}$.

Under the above setting we provide a definition of solutions.

Definition 1.1. A triplet $(w, \theta, \mathbf{v})$ is called a *solution* to (P) if the followings hold:

- (D1) $w \in \mathcal{C}_1(T; \theta) := \{w \in H^1(0, T; H) \mid w(t) \in K(\theta(t)) \text{ for all } t \in [0, T]\}$,
 $\theta \in \mathcal{C}_2(T) := H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; H^2(\Omega)) \cap L^\infty(0, T; L^\infty(\Omega))$,
 $\mathbf{v} \in \mathcal{C}_3(T) := H^1(0, T; \mathbf{V}_{-1+\alpha}) \cap L^\infty(0, T; \mathbf{V}_\alpha) \cap L^2(0, T; \mathbf{V}_{1+\alpha})$;
- (D2) $\begin{aligned} dw/dt + \partial I_\theta(w) &\ni 0 && \text{in } H && \text{a.e. on } (0, T), \\ d\theta/dt - \Delta \theta + \mathbf{v} \cdot \nabla \theta + w &= f && \text{in } H && \text{a.e. on } (0, T), \\ d\mathbf{v}/dt + A\mathbf{v} + B(\mathbf{v}, \mathbf{v}) &= P\mathbf{g}(\theta) && \text{in } \mathbf{V}_{-1+\alpha} && \text{a.e. on } (0, T); \end{aligned}$
- (D3) $(w(0), \theta(0), \mathbf{v}(0)) = (w_0, \theta_0, \mathbf{v}_0) \text{ in } H \times H \times \mathbf{H}.$

Now we are in a position to state the main results. Assume the following conditions:

- (A1) $\psi_1, \psi_2 \in C^1(\mathbb{R}) \cap \text{Lip}(\mathbb{R})$, $\psi_1 \leq \psi_2$ on $\mathbb{R}$;
- (A2) $f \in L^2(0, T; H) \cap L^1(0, T; L^\infty(\Omega))$, $\mathbf{g} \in \text{Lip}(\mathbb{R}; \mathbb{R}^N)$;
- (A3) $w_0 \in K(\theta_0)$, $\theta_0 \in V \cap L^\infty(\Omega)$, $\mathbf{v}_0 \in \mathbf{V}_\alpha$.

Under the above assumption with the condition

$$(1.3) \quad \frac{3(N-2)}{4} < \alpha \leq 1$$

we establish solvability of global in time solutions in 2D and local in time solutions in 3D.

Theorem 1.1. *Let $N = 2$, $0 < T < \infty$ and $0 < \alpha \leq 1$, Suppose (A1)–(A3). Then there exists a unique solution $(w, \theta, \mathbf{v})$ to (P). Furthermore, if $(w_i, \theta_i, \mathbf{v}_i)$ is a solution with the initial data $(w_{0,i}, \theta_{0,i}, \mathbf{v}_{0,i})$ ($i = 1, 2$), then continuous dependence of solutions on initial data holds:*

$$(1.4) \quad \begin{aligned} &\|w_1 - w_2\|_{L^\infty(0,T;L^\infty(\Omega))} + \|\theta_1 - \theta_2\|_{L^\infty(0,T;L^\infty(\Omega))} + \|\mathbf{v}_1 - \mathbf{v}_2\|_{L^\infty(0,T;\mathbf{V}_\alpha)} \\ &\leq C_0 (\|w_{0,1} - w_{0,2}\|_{L^\infty(\Omega)} + \|\theta_{0,1} - \theta_{0,2}\|_V + \|\theta_{0,1} - \theta_{0,2}\|_{L^\infty(\Omega)} + \|\mathbf{v}_{0,1} - \mathbf{v}_{0,2}\|_{\mathbf{V}_\alpha}), \end{aligned}$$

where $C_0 > 0$ is a constant, which increases depending on increase of $\max_{i=1,2} \|\theta_{0,i}\|_H$, $\|\theta_{0,2}\|_V$ and $\max_{i=1,2} \|\mathbf{v}_{0,i}\|_{\mathbf{V}_\alpha}$.

Theorem 1.2. Let $N = 3$, $0 < T < \infty$ and $\frac{3}{4} < \alpha \leq 1$, Suppose (A1)–(A3). Put

$$T_* = T_*(\psi_1, \psi_2, f, \mathbf{g}, \theta_0, \mathbf{v}_0) := \delta \gamma^{-\frac{4}{2\alpha-1}} \wedge T,$$

where $\delta > 0$ is a constant small enough, and $\gamma = \gamma(\psi_1, \psi_2, f, \mathbf{g}, \theta_0, \mathbf{v}_0) > 0$ is defined as

$$\gamma := \|\mathbf{v}_0\|_{V_\alpha} + \|P\mathbf{g}(0)\|_{\mathbf{H}} + \|\mathbf{g}'\|_{L^\infty(\mathbb{R})} \left(\|\theta_0\|_{L^\infty(\Omega)} + \|f\|_{L^1(0,T;L^\infty(\Omega))} + \max_{i=1,2} |\psi_i(0)| \right).$$

Then there exists a unique solution $(w, \theta, \mathbf{v})$ to (P) with $T = T_*$. Furthermore, the continuous dependence of solutions on initial data (1.4) holds where $T = T_*$ and C_0 increases depending on increase of $\max_{i=1,2} \|\theta_{0,i}\|_H$, $\|\theta_{0,2}\|_V$, $\|\theta_{0,2}\|_{L^\infty(\Omega)}$ and $\max_{i=1,2} \|\mathbf{v}_{0,i}\|_{V_\alpha}$.

Remark 1.1. Let $N = 2, 3$ and $\alpha = 1$. Let $(w, \theta, \mathbf{v})$ be a solution to (P) for some $0 < T < \infty$. In light of the well-known fact $\mathbf{H}^\perp = \{\nabla\pi \in \mathbf{L}^2(\Omega) \mid \pi \in H^1(\Omega)\}$ (see e.g., Temam [29, Theorem 1.4 in Chapter I]), there exists a function π satisfying $\nabla\pi \in L^2(0, T; \mathbf{L}^2(\Omega))$ such that $\partial\mathbf{v}/\partial t - \Delta\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{g}(\theta) - \nabla\pi$ in $\mathbf{L}^2(\Omega)$.

2 Orientation

The proof of the main results proceeds in the following three steps.

1. In Section 4 we show existence and uniqueness of solutions to

$$\begin{cases} dw/dt + \partial I_\theta(w) \ni 0 & \text{in } H \quad \text{a.e. on } (0, T), \\ d\theta/dt - \Delta\theta + \mathbf{v} \cdot \nabla\theta + w = f & \text{in } H \quad \text{a.e. on } (0, T), \\ (w(0), \theta(0)) = (w_0, \theta_0) & \text{in } H \times H \end{cases}$$

with some estimates for θ with fixed $\mathbf{v}$. Hence we have the mapping $S_1 : \mathbf{v} \mapsto \theta$.

2. In Section 5 we also establish solvability for

$$\begin{cases} d\mathbf{v}/dt + A\mathbf{v} + B(\mathbf{v}, \mathbf{v}) = P\mathbf{g}(\theta) & \text{in } \mathbf{V}_{-1+\alpha} \quad \text{a.e. on } (0, T), \\ \mathbf{v}(0) = \mathbf{v}_0 \in \mathbf{V}_\alpha & \text{in } \mathbf{H} \end{cases}$$

with estimates for $\mathbf{v}$ with fixed θ . Thus the mapping $S_2 : \theta \mapsto \mathbf{v}$ appears.

3. In Section 6 we combine the above two problems by virtue of the contraction mapping principle for the mapping $S := S_2 \circ S_1$. The cornerstone of estimates toward contractivity of S is appropriate estimates for

$$\mathbf{v}_1 \cdot \nabla(\theta_1 - \theta_2) \quad \text{or} \quad (\mathbf{v}_1 - \mathbf{v}_2) \cdot \nabla\theta_2$$

by adopting the semigroup of the Dirichlet Laplacian and its properties.

In Sections 4 and 6 we will use the following norm. For a Banach space X and $k > 0$, we introduce an equivalent norm $\|\cdot\|_{L_k^\infty(0,T;X)}$ on the Lebesgue space $L^\infty(0, T; X)$ as follows:

$$(2.1) \quad \|u\|_{L_k^\infty(0,T;X)} := \sup_{t \in (0,T)} \|u(t)\|_X e^{-kt}, \quad u \in L^\infty(0, T; X),$$

with

$$\|\cdot\|_{L^\infty(0,T;X)} e^{-kT} \leq \|\cdot\|_{L_k^\infty(0,T;X)} \leq \|\cdot\|_{L^\infty(0,T;X)}.$$

Especially, in Section 6 we adopt $\|\cdot\|_{L_k^\infty(0,T;L^\infty(\Omega))}$ as the metric function of the contraction mapping principle.

3 Estimates for the convective terms

The space V_α causes not a few complication when we estimate the convective terms $\mathbf{v} \cdot \nabla \theta$ or $B(\mathbf{u}, \mathbf{v})$. The following lemma gives estimates for the convective terms.

Lemma 3.1. *The following holds:*

$$\begin{aligned}
 (3.1) \quad & \|\mathbf{v} \cdot \nabla \theta\|_H \leq c_0 \|\mathbf{v}\|_{V_\alpha} \|\theta\|_V^\rho \|\Delta \theta\|_H^{1-\rho}, & \mathbf{v} \in V_\alpha, \theta \in H^2(\Omega) \cap V, \\
 (3.2) \quad & \|\mathbf{v} \cdot \nabla \theta\|_{L^\sigma(\Omega)} \leq c_0 \|\mathbf{v}\|_{V_\alpha} \|\theta\|_{L^\infty(\Omega)}^{1/2} \|\Delta \theta\|_H^{1/2}, & \mathbf{v} \in V_\alpha, \theta \in H^2(\Omega) \cap V, \\
 (3.3) \quad & \|B(\mathbf{u}, \mathbf{v})\|_{V_{-1+\alpha}} \leq c_0 \|\mathbf{u}\|_{V_\alpha} \|\mathbf{v}\|_{V_\alpha}^\rho \|\mathbf{v}\|_{V_{1+\alpha}}^{1-\rho}, & \mathbf{u} \in V_\alpha, \mathbf{v} \in V_{1+\alpha}, \\
 (3.4) \quad & \|B(\mathbf{u}, \mathbf{u})\|_{V_{-\frac{N}{2}+\alpha}} \leq c_0 \|\mathbf{u}\|_H^{1/2} \|\mathbf{u}\|_V^{1/2} \|\mathbf{u}\|_{V_\alpha}^{1/2} \|\mathbf{u}\|_{V_{1+\alpha}}^{1/2}, & \mathbf{u} \in V_{1+\alpha}, \\
 (3.5) \quad & \|B(\mathbf{u}, \mathbf{u})\|_{V_{-\tau}} \leq c_0 \|\mathbf{u}\|_H \|\mathbf{v}\|_{V_{1+\alpha}}, & \mathbf{u} \in H, \mathbf{v} \in V_{1+\alpha}.
 \end{aligned}$$

where $\rho \in (0, 1]$ and $\sigma, \tau \in [1, \infty]$ are defined as

$$\begin{aligned}
 \rho &:= \begin{cases} 1 - \frac{N}{2} + \alpha, & N = 2, 0 < \alpha < 1 \quad \text{or} \quad N = 3, \frac{1}{2} < \alpha \leq 1, \\ \frac{1}{2}, & N = 2, \alpha = 1. \end{cases} \\
 \sigma &:= \begin{cases} \left(\frac{3}{4} - \frac{\alpha}{N}\right)^{-1}, & N = 2, 0 < \alpha < 1 \quad \text{or} \quad N = 3, \frac{1}{2} < \alpha \leq 1, \\ 2, & N = 2, \alpha = 1, \end{cases} \\
 \tau &:= \begin{cases} \frac{N}{2} - \alpha, & N = 2, 0 < \alpha < 1 \quad \text{or} \quad N = 3, \frac{1}{2} < \alpha \leq 1, \\ \frac{1}{2}, & N = 2, \alpha = 1, \end{cases}
 \end{aligned}$$

and $c_0 > 0$ is a constant.

Proof. For simplicity use the notation $\|\cdot\|_p := \|\cdot\|_{L^p(\Omega)}$ or $\|\cdot\|_p := \|\cdot\|_{L^p(\Omega)}$ and let $c > 0$ denote certain constant. We use the Hölder inequality, the Sobolev inequality and the Gagliardo-Nirenberg inequality through the proof. At that time we choose N and α so as not to satisfy (3.1) with $\rho = 0$ nor (3.3) with $\rho = 0$. (See e.g., [5] for the Sobolev imbedding theorem with fractional orders).

First we see that

$$\|\mathbf{v} \cdot \nabla \theta\|_H \leq \|\mathbf{v}\|_{\left(\frac{1}{2} - \frac{\alpha}{N}\right)^{-1}} \|\nabla \theta\|_{\left(\frac{\alpha}{N}\right)^{-1}} \leq c \|\mathbf{v}\|_{V_\alpha} \|\theta\|_V^{1-\frac{N}{2}+\alpha} \|\Delta \theta\|_H^{\frac{N}{2}-\alpha}.$$

Here note that $\frac{1}{2} - \frac{\alpha}{N} \neq 0$ and $\frac{1}{2} - \frac{1}{N} < \frac{\alpha}{N} \leq \frac{1}{2}$. On the other hand, if $N = 2$ and $\alpha = 1$, then (by using the Poincaré inequality if needed) it follows that

$$\begin{aligned}
 (3.6) \quad \|\mathbf{v} \cdot \nabla \theta\|_H &\leq \|\mathbf{v}\|_4 \|\nabla \theta\|_4 \leq c \|\mathbf{v}\|_H^{1/2} \|\mathbf{v}\|_V^{1/2} \|\theta\|_V^{1/2} \|\Delta \theta\|_H^{1/2} \\
 &\leq c \|\mathbf{v}\|_V \|\theta\|_V^{1/2} \|\Delta \theta\|_H^{1/2}.
 \end{aligned}$$

Hence the desired inequality (3.1) holds. We also see that

$$\|\mathbf{v} \cdot \nabla \theta\|_{\left(\frac{3}{4} - \frac{\alpha}{N}\right)^{-1}} \leq \|\mathbf{v}\|_{\left(\frac{1}{2} - \frac{\alpha}{N}\right)^{-1}} \|\nabla \theta\|_{\left(\frac{1}{4}\right)^{-1}} \leq c \|\mathbf{v}\|_{V_\alpha} \|\theta\|_{L^\infty(\Omega)}^{1/2} \|\Delta \theta\|_H^{1/2}.$$

Here note that $\frac{1}{2} - \frac{\alpha}{N} \neq 0$. This inequality and (3.6) yield the desired inequality (3.2). Next it follows that for all $\mathbf{z} \in \mathbf{V}_{1-\alpha}$,

$$\begin{aligned} \left| \int_{\Omega} ((\mathbf{u} \cdot \nabla) \mathbf{v}) \mathbf{z} \right| &\leq \|\mathbf{u}\|_{(\frac{1}{2}-\frac{\alpha}{N})^{-1}} \|\nabla \mathbf{v}\|_{(\frac{1}{N})^{-1}} \|\mathbf{z}\|_{(\frac{1}{2}-\frac{1-\alpha}{N})^{-1}} \\ &\leq c \|\mathbf{u}\|_{\mathbf{V}_{\alpha}} \|\mathbf{v}\|_{\mathbf{V}_{\alpha}}^{1-\frac{N}{2}+\alpha} \|\mathbf{v}\|_{\mathbf{V}_{1+\alpha}}^{\frac{N}{2}-\alpha} \|\mathbf{z}\|_{\mathbf{V}_{1-\alpha}}. \end{aligned}$$

Here note that $\frac{1}{2} - \frac{\alpha}{N} \neq 0$, $\frac{1}{2} - \frac{1}{N} < \frac{1}{N} - \frac{1-\alpha}{N} \leq \frac{1}{2} + \frac{1-\alpha}{N}$ and $\frac{1}{2} - \frac{1-\alpha}{N} \neq 0$. On the other hand, if $N = 2$ and $\alpha = 1$, then we also have

$$\begin{aligned} \|(\mathbf{u} \cdot \nabla) \mathbf{v}\|_{\mathbf{H}} &\leq \|\mathbf{u}\|_4 \|\nabla \mathbf{v}\|_4 \\ &\leq c \|\mathbf{u}\|_{\mathbf{H}}^{1/2} \|\mathbf{u}\|_{\mathbf{V}}^{1/2} \|\mathbf{v}\|_{\mathbf{V}}^{1/2} \|\mathbf{v}\|_{\mathbf{V}_2}^{1/2} \leq c \|\mathbf{u}\|_{\mathbf{V}} \|\mathbf{v}\|_{\mathbf{V}}^{1/2} \|\mathbf{v}\|_{\mathbf{V}_2}^{1/2}. \end{aligned}$$

Hence the desired inequality (3.3) is obtained. Moreover it follows that for all $\mathbf{z} \in \mathbf{V}_{\frac{N}{2}-\alpha}$,

$$\begin{aligned} \left| \int_{\Omega} ((\mathbf{u} \cdot \nabla) \mathbf{u}) \mathbf{z} \right| &\leq \|\mathbf{u}\|_{(\frac{1}{2}-\frac{\alpha}{2N})^{-1}} \|\nabla \mathbf{u}\|_{(\frac{1}{2}-\frac{\alpha}{2N})^{-1}} \|\mathbf{z}\|_{(\frac{\alpha}{N})^{-1}} \\ &\leq c \|\mathbf{u}\|_{\mathbf{H}}^{1/2} \|\mathbf{u}\|_{\mathbf{V}_{\alpha}}^{1/2} \|\mathbf{u}\|_{\mathbf{V}}^{1/2} \|\mathbf{u}\|_{\mathbf{V}_{1+\alpha}}^{1/2} \|\mathbf{z}\|_{\mathbf{V}_{\frac{N}{2}-\alpha}}. \end{aligned}$$

Thus we obtain the desired inequality (3.4). Finally we see that for all $\mathbf{z} \in \mathbf{V}_{\frac{N}{2}-\alpha}$,

$$\begin{aligned} \left| \int_{\Omega} ((\mathbf{u} \cdot \nabla) \mathbf{v}) \mathbf{z} \right| &\leq \|\mathbf{u}\|_{(\frac{1}{2})^{-1}} \|\nabla \mathbf{v}\|_{(\frac{1}{2}-\frac{\alpha}{N})^{-1}} \|\mathbf{z}\|_{(\frac{\alpha}{N})^{-1}} \\ &\leq c \|\mathbf{u}\|_{\mathbf{H}} \|\mathbf{v}\|_{\mathbf{V}_{1+\alpha}} \|\mathbf{z}\|_{\mathbf{V}_{\frac{N}{2}-\alpha}}. \end{aligned}$$

Here note that $\frac{1}{2} - \frac{\alpha}{N} \neq 0$. On the other hand, if $N = 2$ and $\alpha = 1$, then it follows that for all $\mathbf{z} \in \mathbf{V}_{\frac{1}{2}}$,

$$\begin{aligned} \left| \int_{\Omega} ((\mathbf{u} \cdot \nabla) \mathbf{v}) \mathbf{z} \right| &\leq \|\mathbf{u}\|_2 \|\mathbf{v}\|_4 \|\mathbf{z}\|_4 \\ &\leq c \|\mathbf{u}\|_{\mathbf{H}} \|\mathbf{v}\|_{\mathbf{V}}^{1/2} \|\mathbf{v}\|_{\mathbf{V}_2}^{1/2} \|\mathbf{z}\|_{\mathbf{V}_{\frac{1}{2}}} \leq c \|\mathbf{u}\|_{\mathbf{H}} \|\mathbf{v}\|_{\mathbf{V}} \|\mathbf{z}\|_{\mathbf{V}_{\frac{1}{2}}}. \end{aligned}$$

Therefore we derive the desired inequality (3.5). $\square$

4 Heat equation with hysteresis

The following proposition provides solvability for the heat equation with hysteresis with some estimates in the case $N = 2, 3$.

Proposition 4.1. *Let $N = 2, 3$ and $0 < T < \infty$. Let $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ be as in Definition 1.1. Assume (A1), (A2) and (A3). Then for all $\mathbf{v} \in \mathcal{C}_3(T)$, there exists a unique solution (w, θ) satisfying $w \in \mathcal{C}_1(T; \theta)$ and $\theta \in \mathcal{C}_2(T)$ such that*

$$(H) \quad \begin{cases} dw/dt + \partial I_{\theta}(w) \ni 0 & \text{in } H \quad \text{a.e. on } (0, T), \\ d\theta/dt - \Delta \theta + \mathbf{v} \cdot \nabla \theta + w = f & \text{in } H \quad \text{a.e. on } (0, T), \\ (w(0), \theta(0)) = (w_0, \theta_0) & \text{in } H \times H, \end{cases}$$

and moreover the following holds:

$$(4.1) \quad \|\theta\|_{L^\infty(0,T;H)} \leq M_1 = M_1(\|\theta_0\|_H),$$

$$(4.2) \quad \|\theta\|_{L^\infty(0,T;L^\infty(\Omega))} \leq M_2 = M_2(\|\theta_0\|_{L^\infty(\Omega)}),$$

$$(4.3) \quad \|\theta\|_{L^\infty(0,T;V)}^2 + \|\Delta\theta\|_{L^2(0,T;H)}^2 \leq M_3 = M_3(\|\theta_0\|_V, \|\mathbf{v}\|_{L^\infty(0,T;\mathbf{V}_\alpha)}),$$

Furthermore, if (w_i, θ_i) is a solution with $\mathbf{v} = \mathbf{v}_i$, $w_0 = w_{0,i}$ and $\theta_0 = \theta_{0,i}$ ($i = 1, 2$), then the following holds for all $t \in [0, T]$:

$$(4.4) \quad \begin{aligned} & \|(\theta_1 - \theta_2)(t)\|_V^2 + \|\Delta(\theta_1 - \theta_2)\|_{L^2(0,t;H)}^2 \\ & \leq C_2 \left(\|\theta_{0,1} - \theta_{0,2}\|_V^2 + \|\mathbf{v}_1 - \mathbf{v}_2\|_{L^\infty(0,t;\mathbf{V}_\alpha)}^2 + \|w_1 - w_2\|_{L^2(0,t;H)}^2 \right), \end{aligned}$$

$$(4.5) \quad \|(w_1 - w_2)(t)\|_{L^\infty(\Omega)} \leq \|w_{0,1} - w_{0,2}\|_{L^\infty(\Omega)} + C_3 \|\theta_1 - \theta_2\|_{L^\infty(0,t;L^\infty(\Omega))}.$$

Here $M_1, M_2, M_3, C_1, C_2, C_3 > 0$ are constants. In particular,

- M_1 increases depending on increase of $\|\theta_0\|_H$. Specifically

$$(4.6) \quad M_1 := C_1 \left(\|\theta_0\|_H + \|f\|_{L^1(0,T;H)} + \max_{i=1,2} |\psi_i(0)| \right);$$

- M_2 increases depending on increase of $\|\theta_0\|_{L^\infty(\Omega)}$;
- M_3 increases depending on increase of $\|\theta_0\|_V$ and $\|\mathbf{v}\|_{L^\infty(0,T;\mathbf{V}_\alpha)}$;
- C_2 increases depending on increase of $\min_{i=1,2} \|\theta_{0,i}\|_V$ and $\max_{i=1,2} \|\mathbf{v}_i\|_{L^\infty(0,T;\mathbf{V}_\alpha)}$.

Proof. The proof would be completed by referring to the statement and the proof of [32, Lemma 3.1 and Propositions 3.2 and 5.1].

First existence and uniqueness for (H) would be obtained by almost the same argument of [32, Proof of Proposition 5.1] via [32, Lemma 3.1 and Proposition 3.2]. It suffices to only note (3.1) in Lemma 3.1 and replace the definition of k in [32, Lemma 3.1 and Proposition 3.2] with $k(t) := k_0 \int_0^t \|\mathbf{v}(r)\|_{\mathbf{V}_\alpha}^{2/\rho} dr$, where ρ is defined in Lemma 3.1.

Next letting (w, θ) be a solution to (H), we show the estimates (4.1), (4.2) and (4.3). Multiplying the second equation in (H) by $\theta(t)$, we see that for a.a. $t \in (0, T)$,

$$\|\theta(t)\|_H \frac{d}{dt} \|\theta(t)\|_H + \|\theta(t)\|_V^2 \leq (\|f(t)\|_H + \|w(t)\|_H) \|\theta(t)\|_H.$$

In view of the condition $w \in K(\theta)$ and Lipschitz continuity of ψ_1, ψ_2 integrating the above inequality implies that for all $t \in [0, T]$,

$$\begin{aligned} \|\theta(t)\|_H & \leq \|\theta_0\|_H + \|f\|_{L^1(0,t;H)} + \|w\|_{L^1(0,t;H)} \\ & \leq \|\theta_0\|_H + \|f\|_{L^1(0,t;H)} + t|\Omega|^{1/2} \max_{i=1,2} |\psi_i(0)| + \max_{i=1,2} \|\psi'_i\|_{L^\infty(\mathbb{R})} \|\theta\|_{L^1(0,t;H)}. \end{aligned}$$

Multiply it by e^{-kt} , take the supremum as $t \in (0, T)$ and note that

$$\|\theta\|_{L^1(0,t;H)} e^{-kt} = \int_0^t \|\theta(s)\|_H e^{-ks} e^{k(s-t)} ds \leq \frac{1}{k} \|\theta\|_{L_k^\infty(0,t;H)}$$

(see (2.1) for the definition of $\|\cdot\|_{L_k^\infty(0,T;H)}$). Then we deduce that

$$\|\theta\|_{L_k^\infty(0,T;H)} \leq \|\theta_0\|_H + \|f\|_{L^1(0,T;H)} + T|\Omega|^{1/2} \max_{i=1,2} |\psi_i(0)| + \frac{1}{k} \max_{i=1,2} \|\psi'_i\|_{L^\infty(\mathbb{R})} \|\theta\|_{L_k^\infty(0,T;H)}.$$

Thus the desired inequality (4.1) holds for $k > 0$ large enough. On the other hand, applying [32, Eq. (3.5) in Lemma 3.1] ($h = f - w$, $u_0 = \theta_0$ and $u = \theta$) implies that for $t \in [0, T]$,

$$\|\theta(t)\|_{L^\infty(\Omega)} \leq \|\theta_0\|_{L^\infty(\Omega)} + \|f\|_{L^1(0,t;L^\infty(\Omega))} + \|w\|_{L^1(0,t;L^\infty(\Omega))}.$$

By a similar argument toward (4.1) as above (replace H with $L^\infty(\Omega)$) we also deduce the desired inequality (4.2). Moreover apply [32, Eq. (3.4) in Lemma 3.1] ($h = f - w$, $u_0 = \theta_0$ and $u = \theta$). Then we have

$$\|\theta\|_{L^\infty(0,T;V)}^2 + \|\Delta\theta\|_{L^2(0,T;H)}^2 \leq ce^{c\|\mathbf{v}\|_{L^\infty(0,T;V_\alpha)}^{2/\rho}} \left(\|\theta_0\|_V^2 + \|f\|_{L^2(0,T;H)}^2 + \|w\|_{L^2(0,T;H)}^2 \right),$$

where $c > 0$ is a constant and ρ is defined in Lemma 3.1. Then using the condition $w \in K(\theta)$, i.e.,

$$\|w\|_{L^2(0,T;H)} \leq |Q|^{1/2} \max_{i=1,2} |\psi_i(0)| + \max_{i=1,2} \|\psi'_i\|_{L^\infty(\Omega)} \|\theta\|_{L^2(0,T;H)}$$

and plugging (4.1), we obtain the desired inequality (4.3).

Finally letting (w_i, θ_i) be a solution with $\mathbf{v} = \mathbf{v}_i$, $w_0 = w_{0,i}$ and $\theta_0 = \theta_{0,i}$ ($i = 1, 2$), we show the estimates (4.4) and (4.5). By applying [32, Eq. (3.4) of Lemma 3.1] ($h = -(\mathbf{v}_1 - \mathbf{v}_2) \cdot \nabla \theta_2 - (w_1 - w_2)$, $\mathbf{v} = \mathbf{v}_1$, $u_0 = \theta_{0,1} - \theta_{0,2}$ and $u = \theta_1 - \theta_2$) we deduce that for all $t \in [0, T]$,

$$\begin{aligned} & \|(\theta_1 - \theta_2)(t)\|_V^2 + \|\Delta(\theta_1 - \theta_2)\|_{L^2(0,t;H)}^2 \\ & \leq ce^{c\|\mathbf{v}_1\|_{L^\infty(0,T;V_\alpha)}^{2/\rho}} \left(\|\theta_{0,1} - \theta_{0,2}\|_V^2 + \|(\mathbf{v}_1 - \mathbf{v}_2) \cdot \nabla \theta_2\|_{L^2(0,t;H)}^2 + \|w_1 - w_2\|_{L^2(0,t;H)}^2 \right), \end{aligned}$$

where $c > 0$ is a constant and ρ is defined in Lemma 3.1. Here (3.1) in Lemma 3.1 and (4.3) imply

$$\begin{aligned} \|(\mathbf{v}_1 - \mathbf{v}_2) \cdot \nabla \theta_2\|_{L^2(0,t;H)}^2 & \leq c_0^2 \|\mathbf{v}_1 - \mathbf{v}_2\|_{L^\infty(0,t;V_\alpha)}^2 \|\theta_2\|_{L^\infty(0,t;V)}^{2\rho} \|\Delta\theta_2\|_{L^{2-2\rho}(0,t;H)}^{2-2\rho} \\ & \leq c_0^2 T^\rho M_3 \|\mathbf{v}_1 - \mathbf{v}_2\|_{L^\infty(0,t;V_\alpha)}^2, \end{aligned}$$

where $M_3 = M_3(\|\theta_{0,2}\|_V, \|\mathbf{v}_2\|_{L^\infty(0,T;V_\alpha)})$ is defined as (4.3). Then the desired inequality (4.4) is obtained. The estimate (4.5) is proved by a similar way as in the proof of [17, Lemma 3.1] or [32, Lemma 2.1]. Indeed, we would show $\frac{d}{dt} \|w_\pm(t)\|_H^2 \leq 0$, where

$$w_\pm(t) := \left[w_1(t) - w_2(t) \mp \|w_{0,1} - w_{0,2}\|_{L^\infty(\Omega)} \mp \max_{i=1,2} \|\psi_i(\theta_1) - \psi_i(\theta_2)\|_{L^\infty(0,T;L^\infty(\Omega))} \right]^\pm,$$

and hence the desired inequality (4.5) holds. $\square$

5 Navier-Stokes equations

In this section we provide the solvability with estimates for

$$(NS)_\alpha \quad \begin{cases} d\mathbf{v}/dt + A\mathbf{v} + B(\mathbf{v}, \mathbf{v}) = P\mathbf{g}(\theta) & \text{in } \mathbf{V}_{-1+\alpha} \quad \text{a.e. on } (0, T), \\ \mathbf{v}(0) = \mathbf{v}_0 \in \mathbf{V}_\alpha & \text{in } \mathbf{H}. \end{cases}$$

The following two propositions show solvability for $(NS)_\alpha$ with $\frac{N-2}{2} < \alpha \leq 1$ for $N = 2, 3$.

Proposition 5.1. Let $N = 2$, $0 < T < \infty$ and $0 < \alpha \leq 1$. Let $\mathcal{C}_2$ and $\mathcal{C}_3$ be as in Definition 1.1. Assume (A2) and (A3). Then for all $\theta \in \mathcal{C}_2(T)$, there exists a unique solution $\mathbf{v} \in \mathcal{C}_3(T)$ to $(\text{NS})_\alpha$. Moreover the following holds:

$$(5.1) \quad \|\mathbf{v}\|_{L^\infty(0,T;\mathbf{V}_\alpha)}^2 + \|\mathbf{v}\|_{L^2(0,T;\mathbf{V}_{1+\alpha})}^2 \leq M_4 = M_4(\|\mathbf{v}_0\|_{\mathbf{V}_\alpha}, \|\theta\|_{L^2(0,T;H)}).$$

Furthermore, if $\mathbf{v}_i$ is a solution with $\theta = \theta_i$ and $\mathbf{v}_0 = \mathbf{v}_{0,i}$ ($i = 1, 2$), then the following holds for all $t \in [0, T]$:

$$(5.2) \quad \|\mathbf{v}_1(t) - \mathbf{v}_2(t)\|_{\mathbf{V}_\alpha}^2 + \|\mathbf{v}_1 - \mathbf{v}_2\|_{L^2(0,t;\mathbf{V}_{1+\alpha})}^2 \leq C_4 \left(\|\mathbf{v}_{0,1} - \mathbf{v}_{0,2}\|_{\mathbf{V}_\alpha}^2 + \|\theta_1 - \theta_2\|_{L^2(0,t;H)}^2 \right).$$

Here $M_4, C_4 > 0$ are constants. In particular,

- M_4 increases depending on increase of $\|\mathbf{v}_0\|_{\mathbf{V}_\alpha}$ and $\|\theta\|_{L^2(0,T;H)}$;
- C_4 increases depending on increase of $\max_{i=1,2} \|\mathbf{v}_{0,i}\|_{\mathbf{V}_\alpha}$ and $\max_{i=1,2} \|\theta_i\|_{L^2(0,T;H)}$.

Proposition 5.2. Let $N = 3$, $0 < T < \infty$ and $\frac{1}{2} < \alpha \leq 1$. Let $\mathcal{C}_2$ and $\mathcal{C}_3$ be as in Definition 1.1. Assume (A2) and (A3). Put

$$T_0 = T_0(\theta, \mathbf{v}_0) := \delta \left(\|\mathbf{v}_0\|_{\mathbf{V}_\alpha} + \|Pg(0)\|_H + \|g'\|_{L^\infty(\mathbb{R})} \|\theta\|_{L^\infty(0,T;H)} \right)^{-\frac{4}{2\alpha-1}} \wedge T.$$

Then for all $\theta \in \mathcal{C}_2(T)$, there exists a unique solution $\mathbf{v} \in \mathcal{C}_3(T_0)$ to $(\text{NS})_\alpha$. Moreover the following holds:

$$(5.1)' \quad \|\mathbf{v}\|_{L^\infty(0,T_0;\mathbf{V}_\alpha)}^2 + \|\mathbf{v}\|_{L^2(0,T_0;\mathbf{V}_{1+\alpha})}^2 \leq M'_4 = M'_4(\|\mathbf{v}_0\|_{\mathbf{V}_\alpha}, \|\theta\|_{L^\infty(0,T;H)}).$$

Furthermore, if $\mathbf{v}_i$ is a solution with $\theta = \theta_i$ and $\mathbf{v}_0 = \mathbf{v}_{0,i}$ ($i = 1, 2$), then the following holds for all $t \in [0, T_0]$:

$$(5.2)' \quad \|\mathbf{v}_1(t) - \mathbf{v}_2(t)\|_{\mathbf{V}_\alpha}^2 + \|\mathbf{v}_1 - \mathbf{v}_2\|_{L^2(0,t;\mathbf{V}_{1+\alpha})}^2 \leq C'_4 \left(\|\mathbf{v}_{0,1} - \mathbf{v}_{0,2}\|_{\mathbf{V}_\alpha}^2 + \|\theta_1 - \theta_2\|_{L^2(0,t;H)}^2 \right).$$

Here $\delta, M'_4, C'_4 > 0$ are constants. In particular,

- M'_4 increases depending on increase of $\|\mathbf{v}_0\|_{\mathbf{V}_\alpha}$ and $\|\theta\|_{L^\infty(0,T;H)}$;
- C'_4 increases depending on increase of $\max_{i=1,2} \|\mathbf{v}_{0,i}\|_{\mathbf{V}_\alpha}$ and $\max_{i=1,2} \|\theta_i\|_{L^\infty(0,T;H)}$.

Remark 5.1. $T_0(\theta, \mathbf{v}_0)$ is bounded below by T_* (defined in Theorem 1.2) uniformly on $\theta \in L^\infty(0, T; H)$ which is the second part of solutions to (H) (see (4.1) with (4.6)).

Proof of Propositions 5.1 and 5.2. Let $N = 2, 3$, $0 < T < \infty$ and $\frac{N-2}{2} < \alpha \leq 1$. From Lipschitz continuity of g we see that for all $t \in [0, T]$,

$$(5.3) \quad \|Pg(\theta(t))\|_H \leq \|Pg(0)\|_H + \|g'\|_{L^\infty(\mathbb{R})} \|\theta(t)\|_H.$$

Use it when we estimate $\|P\mathbf{g}(\theta(t))\|_{\mathbf{H}}$. First using (5.4) as below, we prove the estimate (5.1) (for $N = 2$) or (5.1)' (for $N = 3$). Suppose $\mathbf{v}$ is a solution to $(\text{NS})_\alpha$ and multiply the equation in $(\text{NS})_\alpha$ by $A^\alpha \mathbf{v}$. Then we see that for a.a. $t \in (0, T)$,

$$(5.4) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{v}(t)\|_{\mathbf{V}_\alpha}^2 + \|\mathbf{v}(t)\|_{\mathbf{V}_{1+\alpha}}^2 \\ \leq (\|B(\mathbf{v}(t), \mathbf{v}(t))\|_{\mathbf{V}_{-1+\alpha}} + \|P\mathbf{g}(\theta(t))\|_{\mathbf{V}_{-1+\alpha}}) \|A^\alpha \mathbf{v}(t)\|_{\mathbf{V}_{1-\alpha}} \\ \leq (\|B(\mathbf{v}(t), \mathbf{v}(t))\|_{\mathbf{V}_{-1+\alpha}} + c_1 \|P\mathbf{g}(\theta(t))\|_{\mathbf{H}}) \|\mathbf{v}(t)\|_{\mathbf{V}_{1+\alpha}}, \end{aligned}$$

where $c_1 > 0$ is a constant. By the way note that the following estimate holds:

$$(5.5) \quad \|\mathbf{v}\|_{L^\infty(0,T;\mathbf{H})}^2 + \|\mathbf{v}\|_{L^2(0,T;\mathbf{V})}^2 \leq M_5 = M_5(\|\mathbf{v}_0\|_{\mathbf{H}}, \|\theta\|_{L^2(0,T;\mathbf{H})}),$$

where $M_5 > 0$ is a constant, which increases depending on increase of $\|\mathbf{v}_0\|_{\mathbf{H}}$, $\|\theta\|_{L^2(0,T;\mathbf{H})}$. Indeed, multiplying the equation in $(\text{NS})_0$ by $\mathbf{v}$ with the standard argument yields the inequality (5.5). For details refer to, e.g., [30, Chapter 3.1]. Note (5.3) if needed.

Now we put $N = 2$ and show (5.1). We see from (5.4) with (3.4) in Lemma 3.1 that for a.a. $t \in (0, T)$,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{v}(t)\|_{\mathbf{V}_\alpha}^2 + \|\mathbf{v}(t)\|_{\mathbf{V}_{1+\alpha}}^2 \\ \leq c_0 \|\mathbf{v}(t)\|_{\mathbf{H}}^{1/2} \|\mathbf{v}(t)\|_{\mathbf{V}}^{1/2} \|\mathbf{v}(t)\|_{\mathbf{V}_\alpha}^{1/2} \|\mathbf{v}(t)\|_{\mathbf{V}_{1+\alpha}}^{3/2} + c_1 \|P\mathbf{g}(\theta(t))\|_{\mathbf{H}} \|\mathbf{v}(t)\|_{\mathbf{V}_{1+\alpha}} \\ \leq c'_1 (\|\mathbf{v}(t)\|_{\mathbf{H}}^2 \|\mathbf{v}(t)\|_{\mathbf{V}}^2 \|\mathbf{v}(t)\|_{\mathbf{V}_\alpha}^2 + \|P\mathbf{g}(\theta(t))\|_{\mathbf{H}}^2) + \frac{1}{2} \|\mathbf{v}(t)\|_{\mathbf{V}_{1+\alpha}}^2, \end{aligned}$$

where $c'_1 > 0$ is a constant depending only on c_0 and c_1 . Using the Gronwall lemma and (5.5), we deduce that for all $t \in [0, T]$,

$$\begin{aligned} \|\mathbf{v}(t)\|_{\mathbf{V}_\alpha}^2 + \|\mathbf{v}\|_{L^2(0,t;\mathbf{V}_{1+\alpha})}^2 &\leq e^{2c'_1 \int_0^t \|\mathbf{v}(r)\|_{\mathbf{H}}^2 \|\mathbf{v}(r)\|_{\mathbf{V}}^2 dr} \left(\|\mathbf{v}_0\|_{\mathbf{V}_\alpha}^2 + 2c'_1 \|P\mathbf{g}(\theta)\|_{L^2(0,t;\mathbf{H})}^2 \right) \\ &\leq e^{2c'_1 M_5^2} \left(\|\mathbf{v}_0\|_{\mathbf{V}_\alpha}^2 + 2c'_1 \|P\mathbf{g}(\theta)\|_{L^2(0,t;\mathbf{H})}^2 \right). \end{aligned}$$

Hence the desired inequality (5.1) holds.

On the other hand, we put $N = 3$ and show (5.1)' similarly to [29, Proof of Theorem 3.11 in Chapter III]. It follows from (5.4) with (3.3) in Lemma 3.1 that for a.a. $t \in (0, T)$,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{v}(t)\|_{\mathbf{V}_\alpha}^2 + \|\mathbf{v}(t)\|_{\mathbf{V}_{1+\alpha}}^2 &\leq c_0 \|\mathbf{v}(t)\|_{\mathbf{V}_\alpha}^{1+\rho} \|\mathbf{v}(t)\|_{\mathbf{V}_{1+\alpha}}^{2-\rho} + c_1 \|P\mathbf{g}(\theta(t))\|_{\mathbf{H}} \|\mathbf{v}(t)\|_{\mathbf{V}_{1+\alpha}} \\ &\leq c'_1 \left(\|\mathbf{v}(t)\|_{\mathbf{V}_\alpha}^{2(\frac{1}{\rho}+1)} + \|P\mathbf{g}(\theta(t))\|_{\mathbf{H}}^2 \right) + \frac{1}{2} \|\mathbf{v}(t)\|_{\mathbf{V}_{1+\alpha}}^2, \end{aligned}$$

where $c'_1 > 0$ is a constant depending only on c_0 and c_1 , and $\rho := \alpha - \frac{1}{2}$ is defined in Lemma 3.1. Thus we see that for a.a. $t \in (0, T)$,

$$(5.6) \quad \frac{d}{dt} \|\mathbf{v}(t)\|_{\mathbf{V}_\alpha}^2 + \|\mathbf{v}(t)\|_{\mathbf{V}_{1+\alpha}}^2 \leq 2c'_1 \left(\|\mathbf{v}(t)\|_{\mathbf{V}_\alpha}^{2(\frac{1}{\rho}+1)} + \|P\mathbf{g}(\theta(t))\|_{\mathbf{H}}^2 \right).$$

Now we let $z(t) := \max\{\|\mathbf{v}(t)\|_{\mathbf{V}_\alpha}^2, \|\mathbf{v}_0\|_{\mathbf{V}_\alpha}^2, 2c_1''c\|P\mathbf{g}(\theta)\|_{L^\infty(0,T;\mathbf{H})}^2\}$ for $t \in [0, T]$, where $c > 0$ is a constant satisfying $\|\cdot\|_{\mathbf{V}_\alpha} \leq c\|\cdot\|_{\mathbf{V}_{1+\alpha}}$. Then for a.a. $t \in (0, T)$,

$$\frac{d}{dt}z(t) = \begin{cases} \frac{d}{dt}\|\mathbf{v}(t)\|_{\mathbf{V}_\alpha}^2 & \text{if } \|\mathbf{v}(t)\|_{\mathbf{V}_\alpha}^2 \geq \max\left\{\|\mathbf{v}_0\|_{\mathbf{V}_\alpha}^2, 2c_1''c\|P\mathbf{g}(\theta)\|_{L^\infty(0,T;\mathbf{H})}^2\right\}, \\ 0 & \text{if } \|\mathbf{v}(t)\|_{\mathbf{V}_\alpha}^2 < \max\left\{\|\mathbf{v}_0\|_{\mathbf{V}_\alpha}^2, 2c_1''c\|P\mathbf{g}(\theta)\|_{L^\infty(0,T;\mathbf{H})}^2\right\}. \end{cases}$$

Hence (5.6) implies that $\frac{d}{dt}z(t) \leq 2c_1''z(t)^{\frac{1}{\rho}+1}$ for a.a. $t \in (0, T)$. Moreover it follows that for all $\varepsilon > 0$,

$$\frac{d}{dt}(z(t) + \varepsilon)^{-\frac{1}{\rho}} = -\frac{1}{\rho}(z(t) + \varepsilon)^{-(\frac{1}{\rho}+1)} \frac{d}{dt}z(t) \geq -\frac{2c_1''}{\rho}.$$

Integrating it yields that for all $t \in [0, T_\varepsilon]$,

$$(5.7) \quad (z(t) + \varepsilon)^{-\frac{1}{\rho}} \geq (z(0) + \varepsilon)^{-\frac{1}{\rho}} - \frac{2c_1''}{\rho} \cdot T_\varepsilon \geq 2^{-\frac{1}{\rho}}(z(0) + \varepsilon)^{-\frac{1}{\rho}},$$

where $T_\varepsilon := \frac{\rho}{2c_1''}(1 - 2^{-\frac{1}{\rho}})(z(0) + \varepsilon)^{-\frac{1}{\rho}} \wedge T$. Thus taking a limit of (5.7) to the power of $-\rho$ as $\varepsilon \downarrow 0$, we see that for all $t \in [0, T_0]$,

$$(5.8) \quad \|\mathbf{v}(t)\|_{\mathbf{V}_\alpha}^2 \leq z(t) \leq 2z(0) = 2 \max\{\|\mathbf{v}_0\|_{\mathbf{V}_\alpha}^2, 2c_1''\|P\mathbf{g}(\theta)\|_{L^\infty(0,T;\mathbf{H})}^2\}.$$

Here note that

$$\lim_{\varepsilon \downarrow 0} T_\varepsilon = \delta \max\{\|\mathbf{v}_0\|_{\mathbf{V}_\alpha}^2, 2c_1''\|P\mathbf{g}(\theta)\|_{L^\infty(0,T;\mathbf{H})}^2\}^{-\frac{2}{2\alpha-1}} \wedge T \geq T_0,$$

where $\delta := \frac{2\alpha-1}{4c_1''}(1 - 2^{-\frac{2}{2\alpha-1}})$. Then by integrating (5.6) and using (5.8) we obtain the desired inequality (5.1)'.

Next letting $N = 2, 3$, we prove the estimate (5.2) (for $N = 2$) or (5.2)' (for $N = 3$). For simplicity we let T_0 (defined in the case $N = 3$) be denoted by T . Suppose $\mathbf{v}_i$ is a solution with $\theta = \theta_i$ and $\mathbf{v}_0 = \mathbf{v}_{0,i}$ to $(\text{NS})_\alpha$ ($i = 1, 2$) and take the difference between the equation for $i = 1$ and $i = 2$. For simplicity put $\theta := \theta_1 - \theta_2$, $\mathbf{v}_0 := \mathbf{v}_{0,1} - \mathbf{v}_{0,2}$ and $\mathbf{v} := \mathbf{v}_1 - \mathbf{v}_2$. Then it follows that

$$\begin{cases} d\mathbf{v}/dt + A\mathbf{v} + B(\mathbf{v}_1, \mathbf{v}) + B(\mathbf{v}, \mathbf{v}_2) = P\mathbf{g}(\theta_1) - P\mathbf{g}(\theta_2) & \text{in } \mathbf{V}_{-1+\alpha}, \\ \mathbf{v}(0) = \mathbf{v}_0 \in \mathbf{V}_\alpha & \text{in } \mathbf{H}. \end{cases}$$

Multiply it by $A^\alpha \mathbf{v}$ and use (3.3) in Lemma 3.1. Then we see that for a.a. $t \in (0, T)$,

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\mathbf{v}(t)\|_{\mathbf{V}_\alpha}^2 + \|\mathbf{v}(t)\|_{\mathbf{V}_{1+\alpha}}^2 \\ & \leq (\|B(\mathbf{v}_1(t), \mathbf{v}(t))\|_{\mathbf{V}_{-1+\alpha}} + \|B(\mathbf{v}(t), \mathbf{v}_2(t))\|_{\mathbf{V}_{-1+\alpha}} \\ & \quad + \|P\mathbf{g}(\theta_1(t)) - P\mathbf{g}(\theta_2(t))\|_{\mathbf{V}_{-1+\alpha}}) \|A^\alpha \mathbf{v}(t)\|_{\mathbf{V}_{1-\alpha}} \\ & \leq c_0 \|\mathbf{v}_1(t)\|_{\mathbf{V}_\alpha} \|\mathbf{v}(t)\|_{\mathbf{V}_\alpha}^\rho \|\mathbf{v}(t)\|_{\mathbf{V}_{1+\alpha}}^{2-\rho} + c_0 \|\mathbf{v}(t)\|_{\mathbf{V}_\alpha} \|\mathbf{v}_2(t)\|_{\mathbf{V}_\alpha}^\rho \|\mathbf{v}_2(t)\|_{\mathbf{V}_{1+\alpha}}^{1-\rho} \|\mathbf{v}(t)\|_{\mathbf{V}_{1+\alpha}} \\ & \quad + c_2 \|\mathbf{g}'\|_{L^\infty(\mathbb{R})} \|\theta(t)\|_{\mathbf{H}} \|\mathbf{v}(t)\|_{\mathbf{V}_{1+\alpha}} \\ & \leq c_2' \left(\|\mathbf{v}_1(t)\|_{\mathbf{V}_\alpha}^{\frac{2}{\rho}} \|\mathbf{v}(t)\|_{\mathbf{V}_\alpha}^2 + \|\mathbf{v}(t)\|_{\mathbf{V}_\alpha}^2 \|\mathbf{v}_2(t)\|_{\mathbf{V}_\alpha}^{2\rho} \|\mathbf{v}_2(t)\|_{\mathbf{V}_{1+\alpha}}^{2-2\rho} + \|\theta(t)\|_{\mathbf{H}}^2 \right) \\ & \quad + \frac{1}{2} \|\mathbf{v}(t)\|_{\mathbf{V}_{1+\alpha}}^2, \end{aligned}$$

where $c_2, c'_2 > 0$ are constants. In particular, c'_2 depends only on c_0, c_2 and $\|g'\|_{L^\infty(\mathbb{R})}$. From the Gronwall lemma and (5.1) (for $N = 2$) or (5.1)' (for $N = 3$) we deduce that for all $t \in [0, T]$,

$$\begin{aligned} & \|v(t)\|_{V_\alpha}^2 + \|v\|_{L^2(0,t;V_{1+\alpha})}^2 \\ & \leq \exp \left[2c'_2 \left(T \|v_1\|_{L^\infty(0,T;V_\alpha)}^{\frac{2}{\rho}} + T^\rho \|v_2\|_{L^\infty(0,T;V_\alpha)}^{2\rho} \|v_2\|_{L^2(0,T;V_{1+\alpha})}^{2-2\rho} \right) \right] \\ & \quad \times \left(\|v_0\|_{V_\alpha}^2 + \|\theta\|_{L^2(0,t;H)}^2 \right) \\ & \leq \exp \left[2c'_2 \left(TM_4(\|v_{0,1}\|_{V_\alpha}, \|\theta_1\|_{L^2(0,T;H)})^{\frac{1}{\rho}} + T^\rho M_4(\|v_{0,2}\|_{V_\alpha}, \|\theta_2\|_{L^2(0,T;H)}) \right) \right] \\ & \quad \times \left(\|v_0\|_{V_\alpha}^2 + \|\theta\|_{L^2(0,t;H)}^2 \right). \end{aligned}$$

Here, in the case $N = 3$, replace $M_4(\|v_{0,i}\|_{V_\alpha}, \|\theta_i\|_{L^2(0,T;H)})$ by $M'_4(\|v_{0,i}\|_{V_\alpha}, \|\theta_i\|_{L^\infty(0,T;H)})$ ($i = 1, 2$). Hence we obtain the desired inequality (5.2) (for $N = 2$) or (5.2)' (for $N = 3$), which also implies uniqueness for $(NS)_\alpha$.

Finally let $\theta \in \mathcal{C}_2(T)$ and $v_0 \in V_\alpha$ as in (A3). Then we prove existence for $(NS)_\alpha$ for $N = 2, 3$. We apply the Galerkin approximation similarly as in [29]. It is well-known that for a Hilbert basis $\{e_n\} \subset V$ of the topology on H and $v_{0,n} \in E_n := \text{span}\{e_1, \dots, e_n\}$ (which is the space spanned by $e_1, \dots, e_n$) there exists a solution

$$(5.9) \quad v_n(t) = \sum_{k=1}^n v_{n,k}(t) e_k \in E_n, \quad t \in [0, T],$$

where $v_{n,k}(t) \in \mathbb{R}$, such that for each $k = 1, \dots, n$,

$$(5.10) \quad \begin{cases} \langle dv_n/dt(t) + Av_n(t) + B(v_n(t), v_n(t)), e_k \rangle_{V^*, V} \\ \quad = \langle Pg(\theta(t)), e_k \rangle_{V^*, V} \quad \text{a.a. } t \in (0, T), \\ v_n(0) = v_{0,n} \in E_n. \end{cases}$$

Here we decide $\{e_n\}$ and $v_{0,n}$ as follows. By virtue of the Riesz representation theorem for V_α , we have the continuous operator $\Lambda : V_{-\alpha} \rightarrow V_\alpha$ such that

$$(\Lambda u, z)_{V_\alpha} = (u, z)_{V_{-\alpha}, V_\alpha} \quad \text{for all } z \in V_\alpha,$$

and hence the compact imbeddings $V_\alpha \hookrightarrow H \hookrightarrow V_{-\alpha}$ yield that Λ is a compact operator on H . Moreover self-adjointness of $\Lambda : H \rightarrow H$ is easily seen. Thus H has a Hilbert basis $\{e_n\}$ composed of eigenfunctions of Λ with the eigenvalues $\{\lambda_n^{-1}\}$ satisfying $\lambda_n > 0$. That is,

$$(5.11) \quad (A^{\frac{\alpha}{2}} e_n, A^{\frac{\alpha}{2}} z)_H = \lambda_n (e_n, z)_H \quad \text{for all } z \in V_\alpha.$$

Now we regularize $e_n \in V_\alpha$. It follows from (5.11) that for all $z \in V$,

$$\left(A^{\frac{1}{2}} A^{-\frac{1+\alpha}{2}} e_n, A^{\frac{1}{2}} z \right)_H = \left(A^{\frac{\alpha}{2}} e_n, A^{\frac{\alpha}{2}} A^{\frac{1-\alpha}{2}} z \right)_H = \left(\lambda_n e_n, A^{\frac{1-\alpha}{2}} z \right)_H = \left(\lambda_n A^{\frac{1-\alpha}{2}} e_n, z \right)_H.$$

Thus $A^{-\frac{1+\alpha}{2}}e_n$ satisfies the following:

$$\begin{cases} -\Delta \left(A^{-\frac{1+\alpha}{2}}e_n \right) + \nabla \pi = \lambda_n A^{\frac{1-\alpha}{2}}e_n & \text{in } \Omega, \\ \operatorname{div} \left(A^{-\frac{1+\alpha}{2}}e_n \right) = 0 & \text{in } \Omega, \\ A^{-\frac{1+\alpha}{2}}e_n = 0 & \text{on } \Gamma. \end{cases}$$

Apply the regularization for the above elliptic problem with $\lambda_n A^{\frac{1-\alpha}{2}}e_n \in V_{2\alpha-1} \subset H$. Then we have $A^{-\frac{1+\alpha}{2}}e_n \in V_2$, i.e., $e_n \in V_{1+\alpha} \subset V_{2\alpha}$. Therefore (5.11) yields that

$$(5.11)' \quad A^\alpha e_n = \lambda_n e_n \quad \text{in } H.$$

Moreover (5.10) has a solution (5.9), and hence for each $k = 1, \dots, n$,

$$(5.10)' \quad \begin{cases} \langle dv_n/dt(t) + Av_n(t) + B(v_n(t), v_n(t)), e_k \rangle_{V_{-1+\alpha}, V_{1-\alpha}} \\ \quad = \langle Pg(\theta(t)), e_k \rangle_{V_{-1+\alpha}, V_{1-\alpha}} \quad \text{a.a. } t \in (0, T), \\ v_n(0) = v_{0,n} \in E_n. \end{cases}$$

Now we define $v_{0,n} \in E_n$ as $v_{0,n} := P_n v_0$ where $P_n : V_{-\alpha} \rightarrow E_n$ is defined as $P_n u := \sum_{k=1}^n \langle u, e_k \rangle_{V_{-\alpha}, V_\alpha} e_k$ for $u \in V_{-\alpha}$. In light of (5.11)', P_n is the orthogonal projection on E_n of each topology on $V_{-\alpha}$, H and V_α . Then P_n would satisfy the following conditions:

$$(5.12) \quad \|P_n u\|_{V_\beta} \leq \|u\|_{V_\beta}, \quad u \in V_\beta \quad (\beta \in \{-\alpha, 0, \alpha\}),$$

$$(5.13) \quad P_n u \rightarrow u \quad \text{in } V_\alpha, \quad u \in V_\alpha.$$

The standard property of orthogonal projections implies (5.12). On the other hand, if $u \in V_{2\alpha}$, then (5.13) holds since (5.11)' yields that

$$\begin{aligned} A^\alpha P_n u &= \sum_{k=1}^n (u, e_k)_H A^\alpha e_k = \sum_{k=1}^n (u, A^\alpha e_k)_H e_k = \sum_{k=1}^n (A^\alpha u, e_k)_H e_k = P_n A^\alpha u \\ &\rightarrow A^\alpha u \quad \text{in } H. \end{aligned}$$

In the case $u \in V_\alpha$, we also have (5.13). Indeed, take arbitrary $\varepsilon > 0$. Then there is $u_\varepsilon \in V_{2\alpha}$ such that $\|u - u_\varepsilon\|_{V_\alpha} < \varepsilon$, and hence

$$\begin{aligned} \|P_n u - u\|_{V_\alpha} &\leq \|P_n(u - u_\varepsilon)\|_{V_\alpha} + \|P_n u_\varepsilon - u_\varepsilon\|_{V_\alpha} + \|u_\varepsilon - u\|_{V_\alpha} \\ &< \|P_n u_\varepsilon - u_\varepsilon\|_{V_\alpha} + 2\varepsilon. \end{aligned}$$

Therefore we obtain $\limsup_{n \rightarrow \infty} \|P_n u - u\|_{V_\alpha} \leq 2\varepsilon$, which implies (5.13).

Now multiplying the equation in (5.10)' by $v_{n,k}(t)$ and taking addition as $k = 1, \dots, n$ (namely $\sum_{k=1}^n v_{n,k}(t) \times (5.10)'$) with (5.9) implies

$$\langle dv_n/dt + Av_n + B(v_n, v_n), v_n \rangle_{V_{-1+\alpha}, V_{1-\alpha}} = \langle Pg(\theta), v_n \rangle_{V_{-1+\alpha}, V_{1-\alpha}}.$$

Similarly $\sum_{k=1}^n \lambda_k v_{n,k}(t) \times (5.10)'$ with (5.11)' implies

$$\langle dv_n/dt + Av_n + B(v_n, v_n), A^\alpha v_n \rangle_{V_{-1+\alpha}, V_{1-\alpha}} = \langle Pg(\theta), A^\alpha v_n \rangle_{V_{-1+\alpha}, V_{1-\alpha}}.$$

Therefore by noting the above two equations and almost the same calculation toward (5.1) (for $N = 2$) or (5.1)' (for $N = 3$) it follows from (5.12) with $\beta = \alpha$ that

$$\begin{aligned}\|\mathbf{v}_n\|_{L^\infty(0,T;\mathbf{V}_\alpha)}^2 + \|\mathbf{v}_n\|_{L^2(0,T;\mathbf{V}_{1+\alpha})}^2 &\leq M_4(\|\mathbf{v}_{0,n}\|_{\mathbf{V}_\alpha}, \|\theta\|_{L^2(0,T;H)}) \\ &\leq M_4(\|\mathbf{v}_0\|_{\mathbf{V}_\alpha}, \|\theta\|_{L^2(0,T;H)}).\end{aligned}$$

Here, in the case $N = 3$, replace $M_4(\|\mathbf{v}_0\|_{\mathbf{V}_\alpha}, \|\theta\|_{L^2(0,T;H)})$ by $M'_4(\|\mathbf{v}_0\|_{\mathbf{V}_\alpha}, \|\theta\|_{L^\infty(0,T;H)})$. Hence there exists subsequence of $\{\mathbf{v}_n\}$ (still denoted by $\{\mathbf{v}_n\}$) with the limit function $\mathbf{v} \in L^\infty(0,T;\mathbf{V}_\alpha) \cap L^2(0,T;\mathbf{V}_{1+\alpha})$ and

$$\begin{aligned}\mathbf{v}_n &\rightarrow \mathbf{v} \quad \text{weakly}^* \text{ in } L^\infty(0,T;\mathbf{V}_\alpha), \\ \mathbf{v}_n &\rightarrow \mathbf{v} \quad \text{weakly in } L^2(0,T;\mathbf{V}_{1+\alpha}).\end{aligned}$$

Moreover it follows from the characterization of $A : \mathbf{V}_{1+\alpha} \rightarrow \mathbf{V}_{-1+\alpha}$ and (3.3) in Lemma 3.1 that there exists $\boldsymbol{\xi} \in L^{\frac{2}{1-\rho}}(0,T;\mathbf{V}_{-1+\alpha})$ and

$$\begin{aligned}A\mathbf{v}_n &\rightarrow A\mathbf{v} \quad \text{weakly in } L^2(0,T;\mathbf{V}_{-1+\alpha}), \\ B(\mathbf{v}_n, \mathbf{v}_n) &\rightarrow \boldsymbol{\xi} \quad \text{weakly in } L^{\frac{2}{1-\rho}}(0,T;\mathbf{V}_{-1+\alpha}).\end{aligned}$$

We show $\boldsymbol{\xi} = B(\mathbf{v}, \mathbf{v})$ later. Therefore we have

$$\begin{aligned}(5.14) \quad \mathbf{h}_n &:= -A\mathbf{v}_n - B(\mathbf{v}_n, \mathbf{v}_n) + P\mathbf{g}(\theta) \\ &\rightarrow -A\mathbf{v} - \boldsymbol{\xi} + P\mathbf{g}(\theta) =: \mathbf{h} \quad \text{weakly in } L^2(0,T;\mathbf{V}_{-1+\alpha}).\end{aligned}$$

Here the equation in (5.10)' yields $v'_{n,k}(t) = \langle \mathbf{h}_n(t), \mathbf{e}_k \rangle_{\mathbf{V}_{-\alpha}, \mathbf{V}_\alpha}$, and hence

$$\frac{d}{dt}\mathbf{v}_n(t) = \sum_{k=1}^n v'_{n,k}(t)\mathbf{e}_k = \sum_{k=1}^n \langle \mathbf{h}_n(t), \mathbf{e}_k \rangle_{\mathbf{V}_{-\alpha}, \mathbf{V}_\alpha} \mathbf{e}_k = P_n \mathbf{h}_n(t).$$

Thus (5.12) with $\beta = -\alpha$ implies that

$$\|d\mathbf{v}_n/dt\|_{L^2(0,T;\mathbf{V}_{-\alpha})} = \|P_n \mathbf{h}_n\|_{L^2(0,T;\mathbf{V}_{-\alpha})} \leq \|\mathbf{h}_n\|_{L^2(0,T;\mathbf{V}_{-\alpha})}.$$

Since $\{\mathbf{h}_n\}$ is bounded in $L^2(0,T;\mathbf{V}_{-\alpha})$, so is $\{d\mathbf{v}_n/dt\}$, and hence

$$(5.15) \quad d\mathbf{v}_n/dt \rightarrow d\mathbf{v}/dt \quad \text{weakly in } L^2(0,T;\mathbf{V}_{-\alpha}).$$

Then the Lions-Aubin compact theorem (see e.g., Simon [27, Corollary 4]) yields

$$\mathbf{v}_n \rightarrow \mathbf{v} \quad \text{in } C([0,T];\mathbf{H}).$$

Moreover we have

$$B(\mathbf{v}_n, \mathbf{v}_n) \rightarrow B(\mathbf{v}, \mathbf{v}) \quad \text{weakly in } L^2(0,T;\mathbf{V}_{-\tau}),$$

where τ is defined in Lemma 3.1. Indeed, in view of (3.5) in Lemma 3.1 we see that for all $\zeta \in L^2(0, T; \mathbf{V}_\tau)$,

$$\begin{aligned} & \left| \langle B(\mathbf{v}_n, \mathbf{v}_n) - B(\mathbf{v}, \mathbf{v}), \zeta \rangle_{L^2(0, T; \mathbf{V}_{-\tau}), L^2(0, T; \mathbf{V}_\tau)} \right| \\ &= \left| \langle B(\mathbf{v}_n - \mathbf{v}, \mathbf{v}_n) + B(\mathbf{v}, \mathbf{v}_n - \mathbf{v}), \zeta \rangle_{L^2(0, T; \mathbf{V}_{-\tau}), L^2(0, T; \mathbf{V}_\tau)} \right| \\ &\leq c_0 \|\mathbf{v}_n - \mathbf{v}\|_{C([0, T]; \mathbf{H})} \|\mathbf{v}_n\|_{L^2(0, T; \mathbf{V}_{1+\alpha})} \|\zeta\|_{L^2(0, T; \mathbf{V}_\tau)} \\ &\quad + \langle B(\mathbf{v}, \mathbf{v}_n - \mathbf{v}), \zeta \rangle_{L^2(0, T; \mathbf{V}_{-\tau}), L^2(0, T; \mathbf{V}_\tau)} \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Therefore $\xi = B(\mathbf{v}, \mathbf{v})$. Now take arbitrary $\zeta \in L^2(0, T; \mathbf{V}_\alpha)$, multiply the equation in (5.10)' by $\sum_{k=1}^n (\zeta(t), \mathbf{e}_k)_\mathbf{H}$ and integrate over $[0, T]$ (namely $\int_0^T \sum_{k=1}^n (\zeta(t), \mathbf{e}_k)_\mathbf{H} \times (5.10)'$). Then we have

$$\langle d\mathbf{v}_n/dt, P_n \zeta \rangle_{L^2(0, T; \mathbf{V}_{-\alpha}), L^2(0, T; \mathbf{V}_\alpha)} = \langle \mathbf{h}_n, P_n \zeta \rangle_{L^2(0, T; \mathbf{V}_{-\alpha}), L^2(0, T; \mathbf{V}_\alpha)}.$$

Passage to the limit of the above relation with (5.13), (5.14) and (5.15) yields that

$$\langle d\mathbf{v}/dt, \zeta \rangle_{L^2(0, T; \mathbf{V}_{-\alpha}), L^2(0, T; \mathbf{V}_\alpha)} = \langle \mathbf{h}, \zeta \rangle_{L^2(0, T; \mathbf{V}_{-\alpha}), L^2(0, T; \mathbf{V}_\alpha)},$$

and hence $d\mathbf{v}/dt = \mathbf{h} \in L^2(0, T; \mathbf{V}_{-1+\alpha})$ holds from the arbitrariness of $\zeta \in L^2(0, T; \mathbf{V}_\alpha)$. This concludes existence since $\mathbf{v}$ is a solution to (NS) $_\alpha$. $\square$

Remark 5.2. Let $N = 2, 3$, $0 < T < \infty$ and $\alpha = 1$. It is well-known that (NS) $_1$ has a (strong) solution $\mathbf{v} \in H^1(0, T; \mathbf{H}) \cap L^\infty(0, T; \mathbf{V}) \cap L^2(0, T; \mathbf{V}_2)$ with an initial data $\mathbf{v}_0 \in \mathbf{V}$ (see e.g., [29, Theorem 3.10 or 3.11 in Chapter III], [30, Theorem 3.2]). Concerning the (global in time) existence in Proposition 5.1 ($N = 2$), we would prove via another approximation instead of the Galerkin approximation. Indeed, for $\mathbf{v}_0 \in \mathbf{V}_\alpha$ take $\{\mathbf{v}_{0,n}\} \in \mathbf{V}$ such that $\mathbf{v}_{0,n} \rightarrow \mathbf{v}_0$ in $\mathbf{V}_\alpha$ and consider the approximate solution $\mathbf{v}_n \in H^1(0, T; \mathbf{H}) \cap L^\infty(0, T; \mathbf{V}) \cap L^2(0, T; \mathbf{V}_2)$ with the initial data $\mathbf{v}_{0,n} \in \mathbf{V}$. Then a similar calculation guarantees the existence. However concerning Proposition 5.2 ($N = 3$), the same way toward the (local in time) existence would break down since $T_0(\theta, \mathbf{v}_{0,n})$ decreases depending on increase of $\|\mathbf{v}_{0,n}\|_\mathbf{V}$ and there is a possibility $T_0(\theta, \mathbf{v}_{0,n})$ tends to 0.

6 Proof of the main theorems

In this section $e^{t\Delta}$ denotes the semigroup of the Dirichlet Laplacian Δ for $t \in [0, T]$. See e.g., Cazenave-Haraux [3] for such semigroup and its properties.

Lemma 6.1. *For all $\xi \in L^p(0, T; L^q(\Omega))$ with*

$$(6.1) \quad \frac{1}{p} + \frac{N}{2} \cdot \frac{1}{q} < 1$$

the following estimate holds for $t \in [0, T]$:

$$\int_0^t \|e^{(t-s)\Delta} \xi(s)\|_{L^\infty(\Omega)} ds \leq c_0 t^{1-\frac{1}{p}-\frac{N}{2q}} \|\xi\|_{L^p(0, t; L^q(\Omega))},$$

where $c_0 > 0$ is a constant.

Proof. The standard estimate for the heat kernel and the Hölder inequality yield that

$$\begin{aligned} \int_0^t \|e^{(t-s)\Delta} \xi(s)\|_{L^\infty(\Omega)} ds &\leq c \int_0^t (t-s)^{-\frac{N}{2} \cdot \frac{1}{q}} \|\xi(s)\|_{L^q(\Omega)} ds \\ &\leq c \left(\int_0^t (t-s)^{-\frac{N}{2} \cdot \frac{1}{q} \cdot p'} \right)^{1/p'} ds \|\xi\|_{L^p(0,t;L^q(\Omega))}, \end{aligned}$$

where $c > 0$ is a constant. Here the necessary and sufficient condition for integrability of $(t-s)^{-\frac{N}{2} \cdot \frac{1}{q} \cdot p'}$ on $(0, t)$ is that $-\frac{N}{2} \cdot \frac{1}{q} \cdot p' > -1$, namely (6.1), and hence the desired inequality is obtained. $\square$

Proof of Theorems 1.1 and 1.2. Let $N = 2, 3$, $0 < T < \infty$ and $\frac{3(N-2)}{4} < \alpha \leq 1$. Suppose (A1)–(A3). Even if $N = 3$, we let T_* be denoted with T for simplicity. Fixing $\theta \in L^\infty(0, T; L^\infty(\Omega))$, we see from Proposition 5.1 (for $N = 2$) or Proposition 5.2 (for $N = 3$) that there exists a unique solution $\mathbf{v}(=: S_1(\theta))$ to the Navier-Stokes equation. On the other hand, Proposition 4.1 gives a unique solution $(w, \theta)(=: (S'_2(\mathbf{v}), S_2(\mathbf{v})))$ to the heat equation with the hysteresis with fixed $\mathbf{v}$. That is, Proposition 5.1 or 5.2 and Proposition 4.1 provide the following mappings:

$$\begin{aligned} S_1 : \theta \in X(T) &\mapsto \mathbf{v} \in \mathcal{C}_3(T) & (\mathbf{v} \text{ is the solution to (NS)}_\alpha \text{ for } \theta), \\ S_2 : \mathbf{v} \in \mathcal{C}_3(T) &\mapsto \theta \in X(T) & (\theta \text{ is the second part of the solution to (H) for } \mathbf{v}), \\ S'_2 : \mathbf{v} \in \mathcal{C}_3(T) &\mapsto w \in \mathcal{C}_1(T; \theta) & (w \text{ is the first part of the solution to (H) for } \mathbf{v}), \end{aligned}$$

where $X(T) \subset L^\infty(0, T; L^\infty(\Omega))$ is defined below. Moreover we consider the well-defined mapping

$$S := S_2 \circ S_1 : \bar{\theta} \in X(T) \mapsto \mathbf{v} \in \mathcal{C}_3(T) \mapsto \tilde{\theta} \in X(T).$$

In other words, for fixed $\bar{\theta}$ there exists a unique solution $(w, \tilde{\theta}, \mathbf{v})$ such that

$$\begin{cases} dw/dt + \partial I_{\tilde{\theta}}(w) \ni 0 & \text{in } H & \text{a.e. on } (0, T), \\ d\tilde{\theta}/dt - \Delta \tilde{\theta} + \mathbf{v} \cdot \nabla \tilde{\theta} + w = f & \text{in } H & \text{a.e. on } (0, T), \\ d\mathbf{v}/dt + A\mathbf{v} + B(\mathbf{v}, \mathbf{v}) = P\mathbf{g}(\bar{\theta}) & \text{in } \mathbf{V}_{-1+\alpha} & \text{a.e. on } (0, T), \\ (w(0), \theta(0), \mathbf{v}(0)) = (w_0, \theta_0, \mathbf{v}_0) & \text{in } H \times H \times \mathbf{H}. \end{cases}$$

In order to establish existence we apply the contraction mapping principle with the complete metric space $(X(T), d)$ as

$$\begin{aligned} X(T) &:= \{\theta \in L^\infty(0, T; L^\infty(\Omega)) \mid \|\theta\|_{L^\infty(0,T;H)} \leq M_1(\|\theta_0\|_H)\}, \\ d(\theta_1, \theta_2) &:= \|\theta_1 - \theta_2\|_{L_k^\infty(0,T;L^\infty(\Omega))}, \end{aligned}$$

where $M_1(\|\theta_0\|_H) > 0$ is defined as (4.6) in Proposition 4.1 and $\|\cdot\|_{L_k^\infty(0,T;L^\infty(\Omega))}$ is defined as (2.1) with $k > 0$ large enough. Note the relation $S_1(X(T)) \subset \mathcal{C}_3(T)$ as above. Actually, in the case $N = 3$, the relation $S_1(X(T_*)) \subset \mathcal{C}_3(T_*)$ eventually holds since the relation $\theta \in X(T_*)$ implies $T_* \leq T_0(\theta, \mathbf{v}_0)$, and hence $S_1(\theta) \in \mathcal{C}_3(T_0(\theta, \mathbf{v}_0)) \subset \mathcal{C}_3(T_*)$. (From now on we let T_* be denoted by T for simplicity.) Now let $\bar{\theta} \in X(T)$ and put $\mathbf{v} := S_1(\bar{\theta})$,

$\tilde{\theta} := S_2(\mathbf{v})$ and $w := S'_2(\mathbf{v})$. Then Proposition 5.1 (for $N = 2$) or 5.2 (for $N = 3$) implies

$$(6.2) \quad \begin{aligned} \|\mathbf{v}\|_{L^\infty(0,T;\mathbf{V}_\alpha)}^2 + \|\mathbf{v}\|_{L^2(0,T;\mathbf{V}_{1+\alpha})}^2 &\leq M_4(\|\mathbf{v}_0\|_{\mathbf{V}_\alpha}, \|\tilde{\theta}\|_{L^2(0,T;H)}) \\ &\leq M_4(\|\mathbf{v}_0\|_{\mathbf{V}_\alpha}, T^{1/2}M_1(\|\theta_0\|_H)) \\ &=: M_4'' = M_4''(\|\theta_0\|_H, \|\mathbf{v}_0\|_{\mathbf{V}_\alpha}). \end{aligned}$$

Here, in the case $N = 3$, replace $M_4(\|\mathbf{v}_0\|_{\mathbf{V}_\alpha}, T^{1/2}M_1(\|\theta_0\|_H))$ by $M_4'(\|\mathbf{v}_0\|_{\mathbf{V}_\alpha}, M_1(\|\theta_0\|_H))$. Hence M_4'' increases depending on increase of $\|\theta_0\|_H$ and $\|\mathbf{v}_0\|_{\mathbf{V}_\alpha}$. Moreover Proposition 4.1 yields

$$(6.3) \quad \|\tilde{\theta}\|_{L^\infty(0,T;H)} \leq M_1(\|\theta_0\|_H),$$

$$(6.4) \quad \|\tilde{\theta}\|_{L^\infty(0,T;L^\infty(\Omega))} \leq M_2(\|\theta_0\|_{L^\infty(\Omega)}),$$

$$(6.5) \quad \begin{aligned} \|\tilde{\theta}\|_{L^\infty(0,T;V)}^2 + \|\Delta\tilde{\theta}\|_{L^2(0,T;H)}^2 &\leq M_3(\|\theta_0\|_V, \|\mathbf{v}\|_{L^\infty(0,T;\mathbf{V}_\alpha)}) \\ &\leq M_3(\|\theta_0\|_V, M_4''(\|\theta_0\|_H, \|\mathbf{v}_0\|_{\mathbf{V}_\alpha})^{1/2}) \\ &=: M_3' = M_3'(\|\theta_0\|_V, \|\mathbf{v}_0\|_{\mathbf{V}_\alpha}), \end{aligned}$$

where M_3' increases depending on increase of $\|\theta_0\|_V$ and $\|\mathbf{v}_0\|_{\mathbf{V}_\alpha}$. Thus the estimate (6.3) yields the relation $\tilde{\theta} \in X(T)$, and hence guarantees $S(X(T)) \subset X(T)$.

Now we show contractivity of S . Let $\bar{\theta}_1, \bar{\theta}_2 \in X$ and put $\mathbf{v}_i := S_1(\bar{\theta}_i)$, $\tilde{\theta}_i := S_2(\mathbf{v}_i)$ and $w_i := S'_2(\mathbf{v}_i)$ ($i = 1, 2$). For simplicity put $w := w_1 - w_2$, $\tilde{\theta} := \tilde{\theta}_1 - \tilde{\theta}_2$, $\bar{\theta} := \bar{\theta}_1 - \bar{\theta}_2$, $\mathbf{v} := \mathbf{v}_1 - \mathbf{v}_2$, $w_0 := w_{0,1} - w_{0,2}$, $\theta_0 := \theta_{0,1} - \theta_{0,2}$ and $\mathbf{v}_0 := \mathbf{v}_{0,1} - \mathbf{v}_{0,2}$. Here in view of the estimate (4.5) and the estimate (5.2) (for $N = 2$) or (5.2)' (for $N = 3$) we see that for $t \in [0, T]$:

$$(6.6) \quad \|w(t)\|_{L^\infty(\Omega)} \leq \|w_0\|_{L^\infty(\Omega)} + C_3\|\tilde{\theta}\|_{L^\infty(0,t;L^\infty(\Omega))},$$

$$(6.7) \quad \|\mathbf{v}(t)\|_{\mathbf{V}_\alpha}^2 + \|\mathbf{v}\|_{L^2(0,t;\mathbf{V}_{1+\alpha})}^2 \leq C_4'' \left(\|\mathbf{v}_0\|_{\mathbf{V}_\alpha}^2 + \|\tilde{\theta}\|_{L^2(0,t;H)}^2 \right),$$

where

$$C_4'' := \begin{cases} C_4(\max_{i=1,2} \|\mathbf{v}_{0,i}\|_{\mathbf{V}_\alpha}, T^{1/2} \max_{i=1,2} M_1(\|\theta_{0,i}\|_H)), & N = 2, \\ C_4'(\max_{i=1,2} \|\mathbf{v}_{0,i}\|_{\mathbf{V}_\alpha}, \max_{i=1,2} M_1(\|\theta_{0,i}\|_H)), & N = 3, \end{cases}$$

which increases depending on increase of $\max_{i=1,2} \|\theta_{0,i}\|_H$ and $\max_{i=1,2} \|\mathbf{v}_{0,i}\|_{\mathbf{V}_\alpha}$. Moreover plugging (6.7) into the estimate (4.4) implies that for all $t \in [0, T]$,

$$(6.8) \quad \begin{aligned} \|\tilde{\theta}(t)\|_V^2 + \|\Delta\tilde{\theta}\|_{L^2(0,t;H)}^2 &\leq C_2 \left(\|\theta_0\|_V^2 + \|\mathbf{v}\|_{L^\infty(0,t;\mathbf{V}_\alpha)}^2 + \|w\|_{L^2(0,t;H)}^2 \right) \\ &\leq C_2' \left(\|\theta_0\|_V^2 + \|\mathbf{v}_0\|_{\mathbf{V}_\alpha}^2 + \|\tilde{\theta}\|_{L^2(0,t;H)}^2 + \|w\|_{L^2(0,t;H)}^2 \right), \end{aligned}$$

where

$$\begin{aligned} C_2' &:= C_2(\min_{i=1,2} \|\theta_{0,i}\|_V, \max_{i=1,2} M_4''(\|\theta_{0,i}\|_H, \|\mathbf{v}_{0,i}\|_{\mathbf{V}_\alpha})^{1/2}) \\ &\quad \times \left(C_4''(\max_{i=1,2} \|\theta_{0,i}\|_H, \max_{i=1,2} \|\mathbf{v}_{0,i}\|_{\mathbf{V}_\alpha}) \vee 1 \right). \end{aligned}$$

C'_2 increases depending on increase of $\max_{i=1,2} \|\theta_{0,i}\|_H$, $\min_{i=1,2} \|\theta_{0,i}\|_V$, $\max_{i=1,2} \|\mathbf{v}_{0,i}\|_{\mathbf{V}_\alpha}$. Moreover we estimate $\|\tilde{\theta}\|_{L^\infty(0,T;L^\infty(\Omega))}$ as follows. By taking the difference of the heat equations, we see that

$$\begin{cases} d\tilde{\theta}/dt - \Delta\tilde{\theta} + \mathbf{v}_1 \cdot \nabla\tilde{\theta} + \mathbf{v} \cdot \nabla\tilde{\theta}_2 + w = 0 & \text{in } H \text{ a.e. on } (0, T), \\ \tilde{\theta}(0) = \theta_0 & \text{in } H, \end{cases}$$

and hence we obtain the following integral equation for $t \in [0, T]$:

$$\tilde{\theta}(t) = e^{t\Delta}\theta_0 - \int_0^t e^{(t-s)\Delta}(\mathbf{v}_1 \cdot \nabla\tilde{\theta})(s) ds - \int_0^t e^{(t-s)\Delta}(\mathbf{v} \cdot \nabla\tilde{\theta}_2)(s) ds - \int_0^t e^{(t-s)\Delta}w(s) ds.$$

Here we apply Lemma 6.1 to $\mathbf{v}_1 \cdot \nabla\tilde{\theta}, \mathbf{v} \cdot \nabla\tilde{\theta}_2 \in L^4(0, T; L^\sigma(\Omega))$ and $w \in L^2(0, T; L^\infty(\Omega))$, where σ is defined in Lemma 3.1. See (3.2) in Lemma 3.1 and note that

$$\frac{1}{4} + \frac{N}{2} \cdot \frac{1}{\sigma} < 1 \iff \frac{1}{4} + \frac{N}{2} \cdot \left(\frac{3}{4} - \frac{\alpha}{N} \right) < 1 \iff \alpha > \frac{3(N-2)}{4}.$$

This is exactly the condition assumed as (1.3). Then it follows that for all $t \in [0, T]$,

$$\begin{aligned} \|\tilde{\theta}(t)\|_{L^\infty(\Omega)} &\leq \|\theta_0\|_{L^\infty(\Omega)} + \int_0^t \|e^{(t-s)\Delta}(\mathbf{v}_1 \cdot \nabla\tilde{\theta})(s)\|_{L^\infty(\Omega)} ds \\ &\quad + \int_0^t \|e^{(t-s)\Delta}(\mathbf{v} \cdot \nabla\tilde{\theta}_2)(s)\|_{L^\infty(\Omega)} ds + \int_0^t \|e^{(t-s)\Delta}w(s)\|_{L^\infty(\Omega)} ds \\ &\leq \|\theta_0\|_{L^\infty(\Omega)} \\ &\quad + c_0 \left[t^{\frac{3}{4}-\frac{1}{\sigma}} \left(\|\mathbf{v}_1 \cdot \nabla\tilde{\theta}\|_{L^4(0,t;L^\sigma)} + \|\mathbf{v} \cdot \nabla\tilde{\theta}_2\|_{L^4(0,t;L^\sigma)} \right) + t^{\frac{1}{2}} \|w\|_{L^2(0,t;L^\infty(\Omega))} \right]. \end{aligned}$$

By using (3.2) in Lemma 3.1 with (6.2) and (6.8) we see that for all $t \in [0, T]$,

$$\begin{aligned} \|\mathbf{v}_1 \cdot \nabla\tilde{\theta}\|_{L^4(0,t;L^\sigma(\Omega))} &\leq c_0 \|\mathbf{v}_1\|_{L^\infty(0,t;\mathbf{V}_\alpha)} \|\tilde{\theta}\|_{L^\infty(0,t;L^\infty(\Omega))}^{1/2} \|\Delta\tilde{\theta}\|_{L^\infty(0,t;H)}^{1/2} \\ &\leq c_0 (M_4'')^{1/2} (C_2')^{1/4} \|\tilde{\theta}\|_{L^\infty(0,t;L^\infty(\Omega))}^{1/2} \left(\|\theta_0\|_V^2 + \|\mathbf{v}_0\|_{\mathbf{V}_\alpha}^2 + \|\bar{\theta}\|_{L^2(0,t;H)}^2 + \|w\|_{L^2(0,t;H)}^2 \right)^{1/4} \\ &\leq C_2'' \|\tilde{\theta}\|_{L^\infty(0,t;L^\infty(\Omega))}^{1/2} \left(\|\theta_0\|_V + \|\mathbf{v}_0\|_{\mathbf{V}_\alpha} + \|\bar{\theta}\|_{L^2(0,t;H)} + \|w\|_{L^2(0,t;H)} \right)^{1/2}, \end{aligned}$$

where

$$C_2'' := c_0 M_4'' (\|\theta_{0,1}\|_H, \|\mathbf{v}_{0,1}\|_{\mathbf{V}_\alpha})^{1/2} \times C_2' (\max_{i=1,2} \|\theta_{0,i}\|_H, \min_{i=1,2} \|\theta_{0,i}\|_V, \max_{i=1,2} \|\mathbf{v}_{0,i}\|_{\mathbf{V}_\alpha})^{1/4}.$$

C_2'' increases depending on increase of $\max_{i=1,2} \|\theta_{0,i}\|_H$, $\min_{i=1,2} \|\theta_{0,i}\|_V$, $\max_{i=1,2} \|\mathbf{v}_{0,i}\|_{\mathbf{V}_\alpha}$. Similarly, it follows from (3.2) in Lemma 3.1 with (6.4), (6.5) and (6.7) that for all $t \in [0, T]$,

$$\begin{aligned} (6.9) \quad \|\mathbf{v} \cdot \nabla\tilde{\theta}_2\|_{L^4(0,t;L^\sigma(\Omega))} &\leq c_0 \|\mathbf{v}\|_{L^\infty(0,t;\mathbf{V}_\alpha)} \|\tilde{\theta}_2\|_{L^\infty(0,t;L^\infty(\Omega))}^{1/2} \|\Delta\tilde{\theta}_2\|_{L^2(0,t;H)}^{1/2} \\ &\leq c_0 (C_4'')^{1/2} M_2^{1/2} (M_3')^{1/4} \left(\|\mathbf{v}_0\|_{\mathbf{V}_\alpha}^2 + \|\bar{\theta}\|_{L^2(0,t;H)}^2 \right)^{1/2} \\ &\leq C_4''' (\|\mathbf{v}_0\|_{\mathbf{V}_\alpha} + \|\bar{\theta}\|_{L^2(0,t;H)}), \end{aligned}$$

where

$$C_4''' := c_0 C_4'' (\max_{i=1,2} \|\theta_{0,i}\|_H, \max_{i=1,2} \|\mathbf{v}_{0,i}\|_{\mathbf{V}_\alpha})^{1/2} \times M_2(\|\theta_{0,2}\|_{L^\infty(\Omega)})^{1/2} \times M_3'(\|\theta_{0,2}\|_V, \|\mathbf{v}_{0,2}\|_{\mathbf{V}_\alpha})^{1/4},$$

i.e., C_4''' increases depending on increase of $\max_{i=1,2} \|\theta_{0,i}\|_H$, $\|\theta_{0,2}\|_V$, $\|\theta_{0,2}\|_{L^\infty(\Omega)}$ and $\max_{i=1,2} \|\mathbf{v}_{0,i}\|_{\mathbf{V}_\alpha}$. Therefore by combining the above three inequalities it follows that for all $t \in [0, T]$,

$$\|\tilde{\theta}(t)\|_{L^\infty(\Omega)} \leq C_5 (\|\theta_0\|_V + \|\theta_0\|_{L^\infty(\Omega)} + \|\mathbf{v}_0\|_{\mathbf{V}_\alpha} + \|\bar{\theta}\|_{L^2(0,t;H)} + \|w\|_{L^2(0,t;L^\infty(\Omega))}),$$

where $C_5 > 0$ is a constant, which increases depending on increase of

$$\begin{aligned} & C_2'' (\max_{i=1,2} \|\theta_{0,i}\|_H, \min_{i=1,2} \|\theta_{0,i}\|_V, \max_{i=1,2} \|\mathbf{v}_{0,i}\|_{\mathbf{V}_\alpha}), \\ & C_4''' (\max_{i=1,2} \|\theta_{0,i}\|_H, \|\theta_{0,2}\|_V, \|\theta_{0,2}\|_{L^\infty(\Omega)}, \max_{i=1,2} \|\mathbf{v}_{0,i}\|_{\mathbf{V}_\alpha}), \end{aligned}$$

and hence on increase of $\max_{i=1,2} \|\theta_{0,i}\|_H$, $\|\theta_{0,2}\|_V$, $\|\theta_{0,2}\|_{L^\infty(\Omega)}$ and $\max_{i=1,2} \|\mathbf{v}_{0,i}\|_{\mathbf{V}_\alpha}$. Moreover in view of (6.6) we see that for all $t \in [0, T]$,

$$\begin{aligned} \|w(t)\|_{L^\infty(\Omega)} & \leq \|w_0\|_{L^\infty(\Omega)} \\ & + C_3 C_5 (\|\theta_0\|_V + \|\theta_0\|_{L^\infty(\Omega)} + \|\mathbf{v}_0\|_{\mathbf{V}_\alpha} + \|\bar{\theta}\|_{L^2(0,t;H)} + \|w\|_{L^2(0,t;L^\infty(\Omega))}). \end{aligned}$$

Multiplying the above two inequalities by e^{-kt} and taking the supremum as $t \in (0, T)$ (see (2.1) for the definition $\|\cdot\|_{L_k^\infty(0,T;L^\infty(\Omega))}$), we deduce that

$$\begin{aligned} \|\tilde{\theta}\|_{L_k^\infty(0,T;L^\infty(\Omega))} & \leq C_5 \left(\|\theta_0\|_V + \|\theta_0\|_{L^\infty(\Omega)} + \|\mathbf{v}_0\|_{\mathbf{V}_\alpha} + \frac{1}{(2k)^{1/2}} \|\bar{\theta}\|_{L_k^\infty(0,T;L^\infty(\Omega))} \right. \\ & \quad \left. + \frac{1}{(2k)^{1/2}} \|w\|_{L_k^\infty(0,T;L^\infty(\Omega))} \right), \end{aligned}$$

$$\begin{aligned} \|w\|_{L_k^\infty(0,T;L^\infty(\Omega))} & \leq \|w_0\|_{L^\infty(\Omega)} \\ & + C_3 C_5 \left(\|\theta_0\|_V + \|\theta_0\|_{L^\infty(\Omega)} + \|\mathbf{v}_0\|_{\mathbf{V}_\alpha} + \frac{1}{(2k)^{1/2}} \|\bar{\theta}\|_{L_k^\infty(0,T;L^\infty(\Omega))} \right. \\ & \quad \left. + \frac{1}{(2k)^{1/2}} \|w\|_{L_k^\infty(0,T;L^\infty(\Omega))} \right). \end{aligned}$$

Then taking $k > 0$ large enough for example, $k := \frac{9}{2}(C_5 \vee C_5 C_3)^2$, we see from the above two inequalities that

$$\begin{aligned} \|\tilde{\theta}\|_{L_k^\infty(0,T;L^\infty(\Omega))} & \leq C_5 (\|\theta_0\|_V + \|\theta_0\|_{L^\infty(\Omega)} + \|\mathbf{v}_0\|_{\mathbf{V}_\alpha}) \\ & \quad + \frac{1}{3} (\|\bar{\theta}\|_{L_k^\infty(0,T;L^\infty(\Omega))} + \|w\|_{L_k^\infty(0,T;L^\infty(\Omega))}) \\ \|w\|_{L_k^\infty(0,T;L^\infty(\Omega))} & \leq \frac{3}{2} \|w_0\|_{L^\infty(\Omega)} \\ & \quad + \frac{3}{2} C_5 C_3 (\|\theta_0\|_V + \|\theta_0\|_{L^\infty(\Omega)} + \|\mathbf{v}_0\|_{\mathbf{V}_\alpha}) + \frac{1}{2} \|\bar{\theta}\|_{L_k^\infty(0,T;L^\infty(\Omega))}. \end{aligned}$$

By combining the above two inequalities it follows that

$$(6.10) \quad \|\tilde{\theta}\|_{L_k^\infty(0,T;L^\infty(\Omega))} \leq C'_5 (\|w_0\|_{L^\infty(\Omega)} + \|\theta_0\|_V + \|\theta_0\|_{L^\infty(\Omega)} + \|\mathbf{v}_0\|_{V_\alpha}) \\ + \frac{1}{2} \|\bar{\theta}\|_{L_k^\infty(0,T;L^\infty(\Omega))},$$

where $C'_5 := C_5 + \frac{1}{2} + \frac{1}{2}C_5C_3$, which increases depending on increase of $\max_{i=1,2} \|\theta_{0,i}\|_H$, $\|\theta_{0,2}\|_V$, $\|\theta_{0,2}\|_{L^\infty(\Omega)}$ and $\max_{i=1,2} \|\mathbf{v}_{0,i}\|_{V_\alpha}$.

Finally we conclude the proof. (6.10) with $w_{0,1} = w_{0,2}$, $\theta_{0,1} = \theta_{0,2}$, $\mathbf{v}_{0,1} = \mathbf{v}_{0,2}$ yields

$$d(\tilde{\theta}_1, \tilde{\theta}_2) = \|\tilde{\theta}_1 - \tilde{\theta}_2\|_{L_k^\infty(0,T;L^\infty(\Omega))} \leq \frac{1}{2} \|\bar{\theta}_1 - \bar{\theta}_2\|_{L_k^\infty(0,T;L^\infty(\Omega))} = \frac{1}{2} d(\bar{\theta}_1, \bar{\theta}_2),$$

i.e., $S : X(T) \rightarrow X(T)$ is a contraction mapping. By virtue of the contraction mapping principle, there exists $\theta \in X(T)$ such that $S(\theta) = \theta$. Moreover put $\mathbf{v} := S_1(\theta)$, $w := S'_2(\theta)$. Then $\theta = S_2(\mathbf{v})$, and hence $(w, \theta, \mathbf{v})$ is a solution to (P). This guarantees existence for (P). On the other hand, continuous dependence of solutions on initial data (1.4) would be proved by almost the same calculation (see [33]), and hence this completes the proof. $\square$

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