

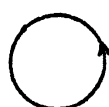
Geometry of polysymbols

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Introduction

We discuss a multiple generalization of the classical tame symbol on a Riemann surface. Let us start to explain a motivation of our work which is coming from the analogies between knots and primes.

$$\text{knot } K : S^1 = K(\mathbb{Z}, 1) \hookrightarrow \mathbb{R}^3 \quad \longleftrightarrow \quad \text{prime } \text{Spec}(\mathbb{F}_p) = K(\hat{\mathbb{Z}}, 1) \hookrightarrow \text{Spec}(\mathbb{Z})$$



$\longleftrightarrow$

Frobenius

double cover
 $Y \rightarrow X = \mathbb{R}^3 \setminus K$

$\longleftrightarrow$

double étale cover
 $Y \rightarrow X = \text{Spec}(\mathbb{Z}[\frac{1}{p}])$

For a 2-component link $K \cup L$,
monodromy around L in $\text{Gal}(Y/X)$ $\longleftrightarrow$
= linking number $\text{lk}(K, L) \bmod 2$

For 2 odd primes p, q ,
Frobenius over q in $\text{Gal}(Y/X)$
= Legendre symbol $\left(\frac{p}{q}\right)$

where we set $p^* := (-1)^{\frac{p-1}{2}} p$.

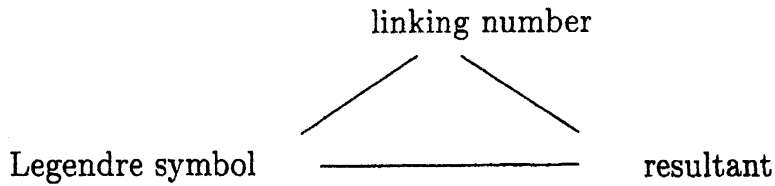
Now, let us consider the polynomial ring $\mathbb{F}_l[X]$ (l being an odd prime) in place of $\mathbb{Z}$. For monic irreducible polynomials $f, g \in \mathbb{F}_l[X]$ s.t. $(f, g) = 1$, we have:

$$\begin{aligned} \left(\frac{f}{g}\right) = 1 &\Leftrightarrow \exists h \in \mathbb{F}_l[X] \text{ s.t. } h^2 \equiv f \bmod g \\ &\Leftrightarrow f \bmod g \in (\mathbb{F}_l[X]/(g))^2 \\ &\Leftrightarrow \prod_{g(\beta)=0} f(\beta) \in (\mathbb{F}_l^\times)^2. \end{aligned}$$

where $R(f, g) := \prod_{g(\beta)=0} f(\beta) = \prod_{f(\alpha)=g(\beta)=0} (\beta - \alpha)$ is the resultant of f and g . Hence we have

$$\left(\frac{f}{g}\right) = R(f, g)^{\frac{l-1}{2}}.$$

Note that the resultant $R(f, g)$ can be defined for any ground field k and $f, g \in k[X]$. Summing up, we have the following analogies:



The symbol in the title means the tame symbol over $\mathbb{C}$ defined by

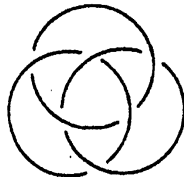
$$\{f, g\}_x := (-1)^{\text{ord}_x(f)\text{ord}_x(g)} \frac{f^{\text{ord}_x(g)}}{g^{\text{ord}_x(f)}}$$

for $f, g \in \mathbb{C}(X)$ and $x \in \mathbb{C} \cup \{\infty\}$. The connection with the resultant is given by

$$R(f, g) = \prod_{g(\beta)=0} \{f, g\}_\beta$$

for $f, g \in \mathbb{C}[X]$ and hence the tame symbol plays a local and more basic role like the Hilbert symbol.

Now, it is known in knot theory that there is a higher order generalization of the linking number for a link. For example, the Borromean ring $K_1 \cup K_2 \cup K_3$ has the linking numbers $\text{lk}(K_i, K_j) = 0$ for all $i \neq j$ and the triple linking number $\text{lk}(K_1, K_2, K_3) = \pm 1$:



There are known two ways to construct the higher linking numbers; 1) Massey's higher order cup products ([Ma]) 2) Milnor's invariants defined as unipotent monodromies ([Mi]). For $\mathbb{Z}$, both constructions are known ([Mo]). The following question was asked by M. Kapranov:

Question. Is there Massey-Milnor type construction for a multiple $\{f_1, \dots, f_n\}_x$ (or $R(f_1, \dots, f_n)$) under a certain condition ?

Our result is to give the Massey type construction of a line bundle with holomorphic flat connection $\langle f_1, \dots, f_n \rangle$, called a polysymbol, on a Riemann surface so that $\{f_1, \dots, f_n\}_x$ is obtained as its holonomy along a loop encircling x , and also to give a global geometric construction of $\langle f_1, \dots, f_n \rangle$.

In the following, we shall discuss only triple symbols, i.e, the case $n = 3$, though we have similar results for any n ([MT]). Let us fix the notations:

$\overline{X} :=$ a closed Riemann surface,

$f_1, f_2, f_3 \in \mathbb{C}(\overline{X})$,

$X := \overline{X} \setminus \cup_{i=1}^3 \text{Supp}(f_i)$.

1. Massey product construction

For $p \in \mathbb{N}$, the Deligne complex is defined by

$$\mathbb{Z}(p)_D := ((2\pi\sqrt{-1})^p \mathbb{Z} \rightarrow \mathcal{O}_X \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \dots \xrightarrow{d} \Omega_X^{p-1})$$

which is quasi-isomorphic to

$$(\mathcal{O}_X^\times \xrightarrow{d\log} \Omega_X^1 \xrightarrow{d} \dots \xrightarrow{d} \Omega_X^{p-1})[-1].$$

The Deligne cohomology is then defined by the hypercohomology groups

$$\mathbb{H}^q(X, \mathbb{Z}(p)_D) = \mathbb{H}^{q-1}(X, \mathcal{O}_X^\times \xrightarrow{d\log} \Omega_X^1 \xrightarrow{d} \dots \xrightarrow{d} \Omega_X^{p-1}), \quad q \geq 1.$$

We compute Deligne cohomology groups in terms of Čech cohomology and so we take an open cover $\mathcal{U} = \cup_a U_a$ of X . (For the properties of the Deligne cohomology, we refer to [Br], [EV]).

(1) For $p = q = 1$, we have

$$\begin{array}{ccc} H^0(X, \mathcal{O}_X^\times) & \simeq & \mathbb{H}^1(X, \mathbb{Z}(1)_D) \\ f & \leftrightarrow & [(2\pi\sqrt{-1}n_{ab}, (\log f)_a)] \end{array}$$

where $(\log f)_a$ means a branch of $\log f$ on U_a with $(\log f)_b - (\log f)_a = 2\pi\sqrt{-1}n_{ab}$ on $U_{ab} = U_a \cap U_b$.

Let $\text{Pic}^\nabla(X)$ denote the group of isomorphism classes of line bundles with holomorphic (flat) connection on X .

(2) For $p = q = 2$, we have

$$\begin{aligned} \mathbb{H}^2(X, \mathbb{Z}(2)_D) &\simeq \mathbb{H}^1(X, \mathcal{O}_X^\times \xrightarrow{\text{dlog}} \Omega_X^1) \simeq \text{Pic}^\nabla(X) \\ [((2\pi\sqrt{-1})^2 n_{abc}, (\log f)_{ab}, \Omega_a)] &\leftrightarrow [(\exp \frac{1}{2\pi\sqrt{-1}} (\log f)_{ab}, \frac{1}{2\pi\sqrt{-1}} \Omega_a)] \\ &\quad [(\xi_{ab}, \omega_a)] \leftrightarrow [(L, \nabla)] \end{aligned}$$

where ξ_{ab} gives the transition function of a $\mathbb{C}$ -line bundle L on U_{ab} and local 1-forms ω_a 's define a connection ∇ on L .

The Deligne complexes have the product $\mathbb{Z}(p)_D \times \mathbb{Z}(q)_D \rightarrow \mathbb{Z}(p+q)_D$ which induces the cup product

$$\mathbb{H}^q(X, \mathbb{Z}(p)_D) \times \mathbb{H}^{q'}(X, \mathbb{Z}(q)_D) \xrightarrow{\cup} \mathbb{H}^{q+q'}(X, \mathbb{Z}(p+q)_D).$$

By (1) and (2), $f_1 \cup f_2$ determines the isomorphism class of line bundles with holomorphic connection on X , which we denote by $\langle f_1, f_2 \rangle$. Deligne ([D]) interpreted the tame symbol $\{f_1, f_2\}_x$ as the holonomy of $\langle f_1, f_2 \rangle$ along a loop l based at x_0 and encircling x :

$$\{f_1, f_2\}_x = \exp \frac{1}{2\pi\sqrt{-1}} \left(\int_l \log f_1 \frac{df_2}{f_2} - \log f_2(x_0) \int_l \frac{df_1}{f_1} \right).$$

Now, assume that $f_1 \cup f_2 = f_2 \cup f_3 = 0$ in the following so that

$$\begin{aligned} \alpha_1 \cup \alpha_2 &= \partial \alpha_{12}, & \alpha_2 \cup \alpha_3 &= \partial \alpha_{23}, & \exists \alpha_{ij} &\in C^1(\mathcal{U}, \mathbb{Z}(2)_D) \\ \Leftrightarrow (\log f_1)_a \frac{df_2}{f_2} &= \frac{df_{12}}{f_{12}}, & (\log f_2)_a \frac{df_3}{f_3} &= \frac{df_{23}}{f_{23}}, & \exists f_{ij} &\in H^0(X, \mathcal{O}_X^\times) \\ &+ \text{some equations} \end{aligned}$$

We fix branches $(\log f_{ij})_a$'s on U_a 's so that we have a unique cohomology class of the Massey product $[\alpha_1 \cup \alpha_{23} + \alpha_{12} \cup \alpha_3] \in \mathbb{H}^2(X, \mathbb{Z}(3)_D)$. We then define $\langle f_1, f_2, f_3 \rangle$ as the corresponding isomorphism class of line bundles with holomorphic connection under the isomorphisms:

$$\mathbb{H}^2(X, \mathbb{Z}(3)_D) \simeq \mathbb{H}^2(X, \mathbb{Z}(2)_D) \simeq \text{Pic}^\nabla(X).$$

Let $x_0 \in X$ be a base point and put $a_i := \frac{1}{2\pi\sqrt{-1}} \log f_i(x_0)$, $a_{ij} := \frac{1}{(2\pi\sqrt{-1})^2} \log f_{ij}(x_0)$ and $\omega_i := \frac{1}{2\pi\sqrt{-1}} \frac{df_i}{f_i}$.

Theorem 1. For $[l] \in \pi_1(X, x_0)$, the holonomy of $\langle f_1, f_2, f_3 \rangle$ along l is

$$\exp(2\pi\sqrt{-1}m_{123}(l)),$$

where $m_{123}(l)$ is given by

$$m_{123}(l) = \int_l \omega_1 \omega_2 \omega_3 + a_1 \int_l \omega_2 \omega_3 + a_{12} \int_l \omega_3 - \int_l \omega_1 a_{23} - \int_l \omega_1 \omega_2 a_3 - a_1 \int_l \omega_2 a_3 + \int_l \omega_1 a_2 a_3.$$

Here $\int_l \omega_{i_1} \cdots \omega_{i_k}$ denotes the iterated integral ([C])

$$\int_{0 \leq t_1 < \cdots < t_k \leq 1} F_1(t_1) \cdots F_k(t_k) dt_1 \cdots dt_k, \quad l^*(\omega_{i_j}) = F_j(t_j) dt_j.$$

For $x \in \overline{X}$, we define $\{f_1, f_2, f_3\}_x$ by the holonomy of $\langle f_1, f_2, f_3 \rangle$ along a loop encircling x .

Theorem 2. (Reciprocity) $\prod_{x \in \overline{X}} \{f_1, f_2, f_3\}_x = 1$.

2. Global geometric construction

We give a global geometric construction of $\langle f_1, f_2, f_3 \rangle$, which generalizes those by S. Bloch ([Bl]) and R. Hain ([H]) for $\langle f_1, f_2 \rangle$.

We set:

$$N_4(R) := \left\{ \begin{pmatrix} 1 & x_1 & x_{12} & x_{123} \\ 0 & 1 & x_2 & x_{23} \\ 0 & 0 & 1 & x_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid x_* \in R \right\} \quad (R : \text{a ring}),$$

$$C_4 := \left\{ \begin{pmatrix} 1 & 0 & 0 & z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid z \in \mathbb{C} \right\},$$

$$P := N_4(\mathbb{Z}) \setminus N_4(\mathbb{C}), \quad B := N_4(\mathbb{Z}) \setminus N_4(\mathbb{C}) / C_4.$$

The projection $P \rightarrow B$ is a principal $\mathbb{C}^\times$ -bundle. The 1-form

$$\begin{aligned} \theta &:= dx_{123} - x_{12} dx_3 - x_1 dx_{23} + x_1 x_2 dx_3 \\ &= (1, 4)\text{-entry of } x^{-1} dx \end{aligned}$$

is left $N_4(\mathbb{Z})$ -invariant, right $\mathbb{C}^\times$ -invariant and a Maurer-Cartan form along fibers so that it boils down to a connection form on P . Fixing a base point $x_0 \in X$ and $N_4(\mathbb{Z})AC_4 \in B$ where

$$A := \begin{pmatrix} 1 & a_1 & a_{12} & a_{123} \\ 0 & 1 & a_2 & a_{23} \\ 0 & 0 & 1 & a_3 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

we define a holomorphic map

$$T(f_1, f_2, f_3) : X \rightarrow B$$

by

$$T(f_1, f_2, f_3) := N_4(\mathbb{Z})A \begin{pmatrix} 1 & \int_{\gamma_x} \omega_1 & \int_{\gamma_x} \omega_1 \omega_2 & \int_{\gamma_x} \omega_1 \omega_2 \omega_3 \\ 0 & 1 & \int_{\gamma_x} \omega_2 & \int_{\gamma_x} \omega_2 \omega_3 \\ 0 & 0 & 1 & \int_{\gamma_x} \omega_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} C_4$$

where γ_x is a path from x_0 to x . The map $T(f_1, f_2, f_3)$ is shown to be independent of the choice of γ_x . By computing the holonomy of $T(f_1, f_2, f_3)^*(P, \theta)$, we have

Theorem 3. $\langle f_1, f_2, f_3 \rangle =$ the isomorphism class of $T(f_1, f_2, f_3)^*(P, \theta)$.

Remark. We expect that $\langle f_1, f_2, f_3 \rangle$ would be an obstruction to a variation of the mixed Hodge structure V on X so that the weight filtration $V \supset W_3 \supset W_2 \supset W_1 \supset 0$ satisfies the properties:

- (i) $V/W_3 = \mathbb{Z}$, $W_3/W_2 = \mathbb{Z}(1)$, $W_2/W_1 = \mathbb{Z}(2)$, $W_1 = \mathbb{Z}(3)$,
- (ii) V/W_1 and W_3 are classified respectively by

$$\begin{aligned} T(f_1, f_2) : X &\longrightarrow N_3(\mathbb{Z}) \backslash N_3(\mathbb{C}) / C_3 \\ T(f_2, f_3) : X &\longrightarrow N_3(\mathbb{Z}) \backslash N_3(\mathbb{C}) / C_3, \end{aligned}$$

where $T(f_i, f_{i+1})$ ($i = 1, 2$) is defined in a similar manner to $T(f_1, f_2, f_3)$.

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