

# Blow-up profile for a nonlinear heat equation with the Neumann boundary condition

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This paper is concerned with the nonlinear diffusion equation

$$\begin{cases} u_t = \Delta u + u^p & \text{in } \Omega \times (0, T), \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x) & x \in \bar{\Omega}, \end{cases}$$

where  $\Omega$  is a bounded smooth domain in  $\mathbf{R}^N$ ,  $\nu$  is the unit outward normal vector on  $\partial\Omega$ ,  $p > 1$  is a constant and  $u_0 \in L^\infty(\Omega)$  is a nonnegative function with  $\|u_0\|_\infty \neq 0$ . For the solution  $u(x, t)$  of the nonlinear diffusion equation, the *blow-up time*  $T$  is defined by

$$T = \sup\{\tau > 0 \mid u(x, t) \text{ is bounded in } \bar{\Omega} \times (0, \tau)\}.$$

Then,  $0 < T < +\infty$  and  $\lim_{t \rightarrow T} \|u(x, t)\|_{C(\bar{\Omega})} = +\infty$  hold. The *blow-up set* of the solution  $u(x, t)$  is defined as the set

$\{x \in \bar{\Omega} \mid \text{there is a sequence } (x_n, t_n) \text{ in } \bar{\Omega} \times (0, T) \text{ such that}$

$$(x_n, t_n) \rightarrow (x, T) \text{ and } u(x_n, t_n) \rightarrow +\infty \text{ as } n \rightarrow \infty\}.$$

This set is a nonempty closed set in  $\bar{\Omega}$ . From standard parabolic estimates, we can obtain the *blow-up profile*, which is a continuous function defined by

$$u_*(x) = \lim_{t \rightarrow T} u(x, t)$$

outside the blow-up set.

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The blow-up problem has been studied by many authors since the pioneering work due to Fujita [13]. There are a number of results for the nature of the blow-up set. For the Cauchy problem with  $(N-2)p < N+2$ , Velázquez [34] showed that the  $(N-1)$ -dimensional Hausdorff measure of the blow-up set is bounded in compact sets of  $\mathbf{R}^N$  whenever the solution is not the constant blow-up one  $(p-1)^{-\frac{1}{p-1}}(T-t)^{-\frac{1}{p-1}}$ . For the Cauchy problem or the Cauchy-Dirichlet problem in a convex domain with  $(N-2)p < N+2$ , Merle and Zaag [25] showed that for any finite set  $D \subset \Omega$ , there exists  $u_0$  such that the blow-up set is  $D$  (See also [1] and [3]). For the Cauchy problem with  $N=1$ , Herrero and Velázquez [17] showed that for any point  $\bar{x}$  in the blow-up set of a solution  $\bar{u}$  and  $\varepsilon > 0$ , there exists  $u_0$  with  $\|u_0 - \bar{u}_0\|_C \leq \varepsilon$  such that the blow-up set of  $u$  consists of a single point  $x$  with  $|x - \bar{x}| \leq \varepsilon$ . For the Cauchy-Dirichlet problem in an ellipsoid centred at the origin with  $(N-2)p < N$ , Filippas and Merle [10] showed that if the blow-up time is large, then the blow-up set consists of a single point near the origin. Also, for the Cauchy or Cauchy-Dirichlet problem with  $(N-2)p < N+2$ , the second author [27] showed the following. For any nonnegative function  $\phi \in C(\bar{\Omega})$  and  $\delta > 0$ , if  $\varepsilon > 0$  is small, then any point  $x$  in the blow-up set satisfies  $\phi(x) \geq \max_y \phi(y) - \delta$  for  $u_0 = \varepsilon^{-1}\phi$ . For the Cauchy-Neumann problem, the first author [18] showed the following. Suppose that  $\Omega = (0, \pi) \times \Omega_0$  is a cylindrical domain with a bounded smooth domain  $\Omega_0$  in  $\mathbf{R}^{N-1}$  and that a nonnegative function  $\phi \in L^\infty(\Omega)$  satisfies  $\int_\Omega \phi(x_1, x_2, \dots, x_N) \cos x_1 dx > 0$ . If  $\varepsilon > 0$  is small, then the blow-up set is contained in the base plane  $\{0\} \times \bar{\Omega}_0$  for  $u_0 = \varepsilon\phi$ . Recently, for the Cauchy-Neumann problem with  $(N-2)p < N+2$ , the first and second authors [20] obtained the following. Let  $P$  be the orthogonal projection in  $L^2(\Omega)$  onto the eigenspace corresponding to the second eigenvalue of the Laplace operator with the Neumann condition. For any nonnegative function  $\phi \in L^\infty(\Omega)$  and neighborhood  $W$  of  $\{x \in \bar{\Omega} \mid (P\phi)(x) = \max_{y \in \bar{\Omega}} (P\phi)(y)\} \cup \partial\Omega$ , if  $\varepsilon > 0$  is small, then the blow-up set is contained in  $W$  for  $u_0 = \varepsilon\phi$ . See, e.g., the references in this paper for related results or other studies on blow-up formation in  $u_t = \Delta u + u^p$ .

In this paper, we study the blow-up profile.

For large initial data  $u_0^\varepsilon = \varepsilon^{-1}\phi$ , we have the following.

**Theorem 1** ([35]) *Let  $\phi \in C^2(\bar{\Omega})$  be a positive function satisfying  $\frac{\partial \phi}{\partial \nu} = 0$  on  $\partial\Omega$ , and let  $\delta > 0$  be a constant. Then, there exists  $\varepsilon_0 > 0$  such that for any  $\varepsilon \in (0, \varepsilon_0]$ , the blow-up set of the solution  $u^\varepsilon$  with the initial data  $u_0^\varepsilon = \varepsilon^{-1}\phi$  is contained in the set  $S := \{x \in \bar{\Omega} \mid \phi(x) \geq \max_{y \in \bar{\Omega}} \phi(y) - \delta\}$  and the blow-up profile  $u_*^\varepsilon$  satisfies the inequality*

$$\left\| \varepsilon u_*^\varepsilon(x) - \left( \phi(x)^{-(p-1)} - (\max_{y \in \bar{\Omega}} \phi(y))^{-(p-1)} \right)^{-\frac{1}{p-1}} \right\|_{C(\bar{\Omega} \setminus S)} \leq \delta.$$

Theorems 2 and 3 are instability results for constant blow-up solutions.

**Theorem 2** ([36]) *Let  $f \in C(\bar{\Omega})$  be a positive function, and let  $\delta$  and  $T_0$  be positive constants. Then, there exist  $C$  and  $\varepsilon_0 > 0$  satisfying the following: For any  $\varepsilon \in (0, \varepsilon_0]$ , there exists  $u_0^\varepsilon \in C^2(\bar{\Omega})$  satisfying  $\frac{\partial u_0^\varepsilon}{\partial \nu} = 0$  on  $\partial\Omega$  and*

$$\left\| u_0^\varepsilon(x) - (p-1)^{-\frac{1}{p-1}} T_0^{-\frac{1}{p-1}} \right\|_{C^2(\bar{\Omega})} \leq C\varepsilon^{p-1}$$

*such that the blow-up time of the solution  $u^\varepsilon$  with initial data  $u^\varepsilon(x, 0) = u_0^\varepsilon(x)$  is larger than  $T_0$  and the inequality*

$$\|\varepsilon u^\varepsilon(x, T_0) - f(x)\|_{C(\bar{\Omega})} \leq \delta$$

*holds.*

**Theorem 3** ([36]) *Let  $f \in C^2(\bar{\Omega})$  be a positive function satisfying  $\frac{\partial f}{\partial \nu} = 0$  on  $\partial\Omega$ , and let  $\delta$  and  $c$  be positive constants. Then, there exist  $C$  and  $\varepsilon_0 > 0$  satisfying the following: For any  $\varepsilon \in (0, \varepsilon_0]$ , there exists  $u_0^\varepsilon \in C^2(\bar{\Omega})$  with  $\frac{\partial u_0^\varepsilon}{\partial \nu} = 0$  on  $\partial\Omega$  and  $\|u_0^\varepsilon - c\|_{C^2(\bar{\Omega})} \leq C\varepsilon^{p-1}$  such that the blow-up set of the solution  $u^\varepsilon$  with the initial data  $u_0^\varepsilon$  is contained in the set  $S := \{x \in \bar{\Omega} \mid f(x) \geq \max_{y \in \bar{\Omega}} f(y) - \delta\}$  and the blow-up profile  $u_*^\varepsilon$  satisfies the inequality*

$$\left\| \varepsilon u_*^\varepsilon(x) - \left( f(x)^{-(p-1)} - (\max_{y \in \bar{\Omega}} f(y))^{-(p-1)} \right)^{-\frac{1}{p-1}} \right\|_{C(\bar{\Omega} \setminus S)} \leq \delta.$$

Let  $\lambda_i$  be the  $i$ -th eigenvalue of  $-\Delta\varphi = \lambda\varphi$  with the Neumann boundary condition  $\frac{\partial\varphi}{\partial\nu} = 0$ , where  $0 = \lambda_1 < \lambda_2 < \lambda_3 < \dots$ . We denote the orthogonal projection in  $L^2(\Omega)$  onto the eigenspace  $X_i$  corresponding to the  $i$ -th eigenvalue by  $P_i$ . Here, we remark that  $P_1\phi = \frac{1}{|\Omega|} \int_{\Omega} \phi dx$  is a constant.

For small initial data  $u_0^\varepsilon = \varepsilon\phi$ , the first and second authors already showed Propositions 4 and 5 below.

**Proposition 4** ([20]) *Let  $\phi \in L^\infty(\Omega)$  be a nonnegative function with  $\|\phi\|_\infty \neq 0$ . Then, there exist a constant  $\varepsilon_0 > 0$  and a family  $\{(t^\varepsilon, \delta^\varepsilon)\}_{\varepsilon \in (0, \varepsilon_0]} \subset \mathbb{R}^2$  such that the solution  $u^\varepsilon$  with the initial data  $u_0^\varepsilon = \varepsilon\phi$  and its blow-up time  $T^\varepsilon$  satisfy  $\lim_{\varepsilon \rightarrow +0} t^\varepsilon = 1$ ,  $\lim_{\varepsilon \rightarrow +0} \varepsilon^{p-1} T^\varepsilon = (p-1)^{-1} (P_1\phi)^{-(p-1)}$ ,  $\lim_{\varepsilon \rightarrow +0} \varepsilon^{p-1} e^{\lambda_2 T^\varepsilon} \delta^\varepsilon = (p-1)^{-1} (P_1\phi)^{-p}$  and*

$$\lim_{\varepsilon \rightarrow +0} \left\| \frac{t^\varepsilon}{\delta^\varepsilon} \left( 1 - (p-1)^{\frac{1}{p-1}} t^{\frac{1}{p-1}} u^\varepsilon(x, T^\varepsilon - 1) \right) - e^{\lambda_2} \left( (\max_{y \in \bar{\Omega}} (P_2\phi)(y)) - (P_2\phi)(x) \right) \right\|_{L^\infty(\Omega)} = 0.$$

**Proposition 5** ([19]) *Let  $\phi \in L^\infty(\Omega)$  be a nonnegative function with  $\|\phi\|_\infty \neq 0$ . Then, there exist  $C$  and  $\varepsilon_0 > 0$  such that for any  $\varepsilon \in (0, \varepsilon_0]$ , the solution  $u^\varepsilon$  with the initial data  $u_0^\varepsilon = \varepsilon\phi$  and its blow-up time  $T^\varepsilon$  satisfy  $u^\varepsilon(x, t) \leq C(T^\varepsilon - t)^{-\frac{1}{p-1}}$  for all  $(x, t) \in \bar{\Omega} \times [T^\varepsilon - 1, T^\varepsilon)$ .*

We obtain the following as a corollary of the propositions above.

**Theorem 6** ([21]) *Let  $\phi \in L^\infty(\Omega)$  be a nonnegative function with  $\|\phi\|_\infty \neq 0$ , and let  $\delta > 0$  be a constant. Then, there exists  $\varepsilon_0 > 0$  such that for any  $\varepsilon \in (0, \varepsilon_0]$ , the blow-up set of the solution  $u^\varepsilon$  with the initial data  $u_0^\varepsilon = \varepsilon\phi$  is contained in the set  $S := \{x \in \bar{\Omega} \mid (P_2\phi)(x) \geq \max_{y \in \bar{\Omega}} (P_2\phi)(y) - \delta\}$ . Further, the blow-up time  $T^\varepsilon$  and the blow-up profile  $u_*^\varepsilon$  satisfy the inequality*

$$\left| \varepsilon^{p-1} T^\varepsilon - (p-1)^{-1} (P_1\phi)^{-(p-1)} \right| + \left\| \varepsilon^{-1} e^{-\frac{\lambda_2 T^\varepsilon}{p-1}} u_*^\varepsilon(x) - (p-1)^{-\frac{1}{p-1}} (P_1\phi)^{\frac{p}{p-1}} \left( (\max_{y \in \bar{\Omega}} (P_2\phi)(y)) - (P_2\phi)(x) \right)^{-\frac{1}{p-1}} \right\|_{C(\bar{\Omega} \setminus S)} \leq \delta.$$

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