

Some remarks on representations of fundamental generalized inverse $\ast$ -semigroups

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Abstract

W.D.Munn [4] described that every fundamental inverse semigroup can be faithfully represented by isomorphisms among principal ideals of the semilattice of idempotents of it. Also T.Imaoka [3] has given a generalization of the Preston-Vagner representation for generalized inverse $\ast$ -semigroups by using a concept of a structure sandwich set of an inverse subsemigroup of the symmetric inverse semigroup on a set.

In this paper, we shall construct a fundamental regular $\ast$ -semigroup $\mathcal{FGI}_{X(\pi)}$ on a set X with a partition $\pi : X = \Sigma\{X_i : i \in I\}$, and obtain a faithful representation of a fundamental generalized inverse $\ast$ -semigroup into $\ast$ -semigroup $\mathcal{FGI}_{X(\pi)}$ on a set $X(\pi)$.

1 Introduction

A semigroup S with a unary operation $\ast : S \rightarrow S$ is called a *regular $\ast$ -semigroup* if it satisfies followings

- (1) $(x^\ast)^\ast = x,$
- (2) $(xy)^\ast = y^\ast x^\ast,$
- (3) $xx^\ast x = x.$

Let S be a regular $\ast$ -semigroup. Then an idempotent e of S is called a *projection* if it satisfies $e^\ast = e$. For any subset A of S , denote the sets of idempotents and projections of A by $E(A)$ and $P(A)$, respectively. The following result is well-known and we use it frequently throughout this paper.

Result 1.1 (see [2]) *Let S be a regular $\ast$ -semigroup. Then we have the followings:*

- (1) $E(S) = P(S)^2;$
- (2) for any $a \in S$ and $e \in P(S)$, $a^\ast ea \in P(S);$
- (3) each $\mathcal{L}$ -class and each $\mathcal{R}$ -class have one and only one projection.

A regular $*$ -semigroup S is called a *generalized inverse $*$ -semigroup* if $E(S)$ forms a normal band, that is, $E(S)$ satisfies the identity $xyzx = xzyx$, equivalently, $P(S)$ satisfies the identity $xyzw = xzyw$.

Let S be a regular $*$ -semigroup and ρ a congruence on S . Then ρ is called a *$*$ -congruence* if $(x, y) \in \rho$ implies $(x^*, y^*) \in \rho$.

Result 1.2 (see [2]) *Let S be a regular $*$ -semigroup. Then,*

$$\mu = \{(a, b) \in S \times S : aea^* = beb^* \text{ and } a^*ea = b^*eb \text{ for all } e \in P(S)\}$$

is the maximum idempotent separating $$ -congruence on S .*

A regular $*$ -semigroup S is said to be *fundamental* if μ is the equality relation on S . The notation and terminology are those of [1] and [2] unless otherwise stated.

2 Structure sandwich sets

Let $\mathcal{I}_X$ be the symmetric inverse semigroup on a set X . For any subset A of X , 1_A means the identity mapping on A . Let $\mathcal{A}$ be an inverse subsemigroup of $\mathcal{I}_X$ and $\theta : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ a mapping. Denote the image $(\alpha, \beta)\theta$ of an ordered pair (α, β) by $\theta_{\alpha, \beta}$. Set $\mathcal{M} = \{\theta_{\alpha, \beta} : \alpha, \beta \in \mathcal{A}\}$. If $\mathcal{M}$ satisfies the following conditions:

- (1) $\theta_{\alpha, \beta}^{-1} = \theta_{\beta^{-1}, \alpha^{-1}},$
- (2) $\theta_{\alpha, \alpha^{-1}} = 1_{r(\alpha)},$
- (3) $\theta_{1_{d(\alpha)}, \alpha} = 1_{d(\alpha)},$
- (4) $\theta_{\alpha, \beta} \theta_{\beta, \gamma} = \theta_{\alpha, \beta\theta_{\beta, \gamma}},$

we call it the *structure sandwich set* of $\mathcal{A}$ determined by θ . The following lemma and theorem are obvious by the definition.

Lemma 2.1 *Let $\mathcal{M}$ be the structure sandwich set of $\mathcal{A}$ determined by θ . Then,*

$$\theta_{\alpha, 1_{r(\alpha)}} = 1_{r(\alpha)}.$$

Theorem 2.2 (see [3]) *Let $\mathcal{A}$ be an inverse subsemigroup of the symmetric inverse semigroup $\mathcal{I}_X$ on a set X , and $\mathcal{M}$ the structure sandwich set of $\mathcal{A}$ determined by a mapping $\theta : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$. Define a multiplication $\circ$ and a unary operation $*$ on $\mathcal{A}$ as follows:*

$$\begin{aligned} \alpha \circ \beta &= \alpha \theta_{\alpha, \beta}, \\ \alpha^* &= \alpha^{-1}. \end{aligned}$$

*Then $\mathcal{A}(\circ, *)$ becomes a regular $*$ -semigroup with $E(\mathcal{A}(\circ, *)) = \{\alpha \in \mathcal{A}(\circ, *) : \theta_{\alpha, \alpha} = \alpha^{-1}\}$ and $P(\mathcal{A}(\circ, *)) = \{1_{d(\alpha)} : \alpha \in \mathcal{A}(\circ, *)\}$.*

We denote $\mathcal{A}(\circ, *)$, defined above, by $\mathcal{A}(\mathcal{M})$.

3 Construction

Let X be a set and $\pi : X = \Sigma\{X_i : i \in I\}$ a partition of X . In this case, we denote X by $X(\pi)$. A subset A of X is called a π -singleton subset of $X(\pi)$ if $|A \cap X_i| \leq 1$ for all $i \in I$. A mapping $\alpha \in \mathcal{I}_X$ is called a π -singleton bijection of $X(\pi)$ if $d(\alpha)$ and $r(\alpha)$ are π -singleton subsets of $X(\pi)$. Denote the set of all π -singleton bijections of $X(\pi)$ by $\mathcal{FGI}_{X(\pi)}$. The following lemma is clear.

Lemma 3.1 *The set $\mathcal{FGI}_{X(\pi)}$, defined above, is an inverse subsemigroup of $\mathcal{I}_X$.*

For any $\alpha, \beta \in \mathcal{FGI}_{X(\pi)}$, define a mapping $\theta_{\alpha, \beta}$ as follows:

$$\begin{aligned} d(\theta_{\alpha, \beta}) &= \{e \in r(\alpha) : \text{there exist } i \in I \text{ and } f \in d(\beta) \text{ such that } e, f \in X_i\}, \\ r(\theta_{\alpha, \beta}) &= \{f \in d(\beta) : \text{there exist } i \in I \text{ and } e \in r(\alpha) \text{ such that } e, f \in X_i\}. \\ e\theta_{\alpha, \beta} &= f, \text{ where } r(\alpha) \cap X_i = \{e\} \text{ and } d(\beta) \cap X_i = \{f\}. \end{aligned}$$

Proposition 3.2 *The set $\mathcal{M} = \{\theta_{\alpha, \beta} : \alpha, \beta \in \mathcal{FGI}_{X(\pi)}\}$ is the structure sandwich set of $\mathcal{FGI}_{X(\pi)}$ determined by a mapping $\theta : (\alpha, \beta) \mapsto \theta_{\alpha, \beta}$. Therefore $\mathcal{FGI}_{X(\pi)}(\mathcal{M})$ is a regular $*$ -semigroup. Furthermore $\mathcal{FGI}_{X(\pi)}(\mathcal{M})$ becomes a fundamental generalized inverse $*$ -semigroup.*

Proof. We can prove the proposition by similar argument of Lemma 3.4 and Lemma 3.5 of [3].

4 Representation

Let S be a fundamental generalized inverse $*$ -semigroup. In this section, we denote $E(S)$ and $P(S)$ simply by E and P , respectively. Let $E \sim \Sigma\{E_i : i \in I\}$ be the structure decomposition of the normal band E . Then $\pi : P = \Sigma\{P_i : i \in I\}$ is a partition of P , where $P_i = P(E_i)$. Then $\mathcal{FGI}_{P(\pi)}(\mathcal{M})$, constructed in the preceding section, is a fundamental generalized inverse $*$ -semigroup.

Lemma 4.1 *For any $a \in S$, $Sa \cap P (= Sa^*a \cap P)$ is a π -singleton subset of $P(\pi)$.*

Proof. Put $a^*a = e$ and assume that $f, g \in Se \cap P_i$. Then there exist $x, y \in S$ such that $f = xe$ and $g = ye$. Hence $f = efe$ and $g = ege$. Since E_i is a rectangular band and E is a normal band, $f = fgf = efefef = egefege = gfg = g$. This shows that $Sa \cap P$ is a π -singleton subset of $P(\pi)$.

Lemma 4.2 *For any $a \in S$, let $\tau_a : Sa^* \cap P \rightarrow Sa \cap P$ be a mapping defined by*

$$e\tau_a = a^*ea.$$

Then, for any $a \in S$, $\tau_a \in \mathcal{FGI}_{P(\pi)}(\mathcal{M})$ and $\tau_a^{-1} = \tau_{a^}$.*

Proof. By Lemma 4.1, $d(\tau_a)$ and $r(\tau_a)$ are π -singleton subsets of $P(\pi)$. Let e and f are any elements of $Sa^* \cap P$ such that $e\tau_a = f\tau_a$, that is, $a^*ea = a^*fa$. Then there exist $x, y \in S$ such that $e = xa^*$ and $f = ya^*$. Thus $e = e^*e = ax^*xa^* = aa^*ax^*xa^*aa^* = aa^*eaa^* = aa^*faa^* = aa^*f^*faa^* = aa^*ay^*ya^*aa^* = ay^*ya^* = f^*f = f$. This implies that τ_a is injective. Next, let f be any element of $Sa \cap P$. Then there exists $x \in S$ such that $f = xa$. Put $e = aa^*x^*xaa^*$. Then $e \in Sa^* \cap P$ and $e\tau_a = a^*ea = a^*(aa^*x^*xaa^*)a = a^*x^*xa = f$. This implies that τ_a is bijective. The last statement of the lemma is obvious.

Lemma 4.3 For any $a, b \in S$, $\theta_{\tau_a, \tau_b} = \tau_a^*abb^*$.

Proof. Let e be any element of $d(\theta_{\tau_a, \tau_b})$. Then $e \in Sa \cap P$ and there exist $i \in I$ and $f \in Sb^* \cap P$ such that $e, f \in P_i$. Hence there exist $x, y \in S$ such that $e = xa$ and $f = yb^*$. Thus $e = efe = xayb^*xa = xayb^*bb^*xaa^*a = xayb^*xabb^*a^*a \in Sbb^*a^*a \cap P = d(\tau_a^*abb^*)$.

Conversely let e be any element of $d(\tau_a^*abb^*) = Sbb^*a^*a \cap P$. Then there exists $x \in S$ such that $e = xbb^*a^*a$. Hence $e \in Sa \cap P$. Put $f = bb^*a^*ax^*xa^*abb^*$. Then $f \in Sb^* \cap P$ and it is clear that e and f are contained in a same P_i . Therefore $e \in d(\theta_{\tau_a, \tau_b})$ and $e\tau_a^*abb^* = bb^*a^*aea^*abb^* = bb^*a^*ax^*xa^*abb^* = f = e\theta_{\tau_a, \tau_b}$.

Lemma 4.4 Define a mapping $\phi : S \rightarrow \mathcal{FGI}_{P(\pi)}(\mathcal{M})$ by

$$a\phi = \tau_a.$$

Then ϕ is a $*$ -monomorphism.

Proof. To prove that ϕ is a homomorphism it is sufficient to show that, for any $a, b \in S$, $d(\tau_a \circ \tau_b) = d(\tau_{ab})$. Let a and b be any elements of S . Then,

$$\begin{aligned} d(\tau_a \circ \tau_b) &= d(\tau_a \tau_a^*abb^* \tau_b) \quad (\text{by Lemma 4.3}) \\ &= \{Sa \cap P \cap (Sa^*abb^* \cap P \cap Sb^*)\tau_a^{-1}abb^*\} \tau_a^{-1} \\ &= \{Sa \cap P \cap (Sa^*abb^* \cap P)\tau_a^{-1}abb^*\} \tau_a^{-1} \\ &= (Sa \cap P \cap Sbb^*a^*a)\tau_a^{-1} \quad (\text{since } \tau_a^{-1}abb^* = \tau_{bb^*a^*a}) \\ &= (Sbb^*a^*a \cap P)\tau_a^{-1}. \end{aligned}$$

Hence let e be any element of $d(\tau_a \circ \tau_b)$. Then there exists $f \in Sbb^*a^*a \cap P$ such that $e\tau_a = f$, that is $a^*ea = f$. Since $e \in Sa^*$ and $f \in Sbb^*a^*a$, there exist $x, y \in S$ such that $e = xa^*$ and $f = ybb^*a^*a$. Thus $e = e^*e = ax^*xa^* = aa^*(ax^*xa^*)aa^* = a(a^*ea)a^* = afa^* = aybb^*a^* \in Sb^*a^* \cap P = d(\tau_{ab})$.

Conversely let e be any element of $d(\tau_{ab}) = Sb^*a^* \cap P$. Then there exists $x \in S$ such that $e = xb^*a^*$. Hence $e\tau_a = xb^*a^*\tau_a = a^*xb^*a^*a \in Sbb^*a^*a \cap P$. Thus $e \in (Sbb^*a^*a \cap P)\tau_a^{-1}$ and so $d(\tau_a \circ \tau_b) = d(\tau_{ab})$.

Let a and b be any elements of S such that $a\phi = b\phi$, that is, $\tau_a = \tau_b$. Then it is easy to show that $aa^* = bb^*$ and $a^*a = b^*b$. Hence assume that $(a, b) \notin \mu$. Then there exists $e \in P$ such that $aea^* \neq beb^*$ or $a^*ea \neq b^*eb$.

If $aea^* \neq beb^*$, we have $aea^*\tau_a = beb^*\tau_b$, since $a^*aea^*a = b^*beb^*b$. This contradicts that $\tau_a = \tau_b$ is injective. Similarly, if $a^*ea \neq b^*eb$, we have $aa^*ea^*\tau_a = bb^*ebb^*\tau_b$, that is, $a^*ea = b^*eb$. This contradicts the hypothesis. Therefore, $(a, b) \in \mu$. Since S is fundamental, we have that $a = b$. Thus ϕ is injective.

By Lemma 4.2, ϕ is compatible with a unary operation $*$ and so ϕ is a $*$ -monomorphism.

Now we have the following theorem.

Theorem 4.5 *Every fundamental generalized inverse $*$ -semigroup has a faithful representation into $\mathcal{FGI}_{X(\pi)}$ on a set $X(\pi)$.*

References

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