

Characterization of finite Automata by the Images and the Kernels of their Transition Functions¹

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1. Introduction

By an automaton $\mathcal{A}$, we mean here a 3-tuple (X, A, δ) , where X is a finite set (the set of states), A is a finite alphabet (the set of inputs) and δ is a mapping of $X \times A$ into X (the transition function).

As usual, A^* and A^+ denotes the free monoid and free semigroup generated by A , respectively, and δ is extended from $X \times A$ to $X \times A^*$. In this case, $\delta(x, s)$ is denoted simply by xs for $x \in X, s \in A^*$.

Let $\rho = \{(s, t) \in A^* \times A^* : xs = xt \text{ for every } x \in X\}$. Then ρ is a congruence on A^* and A^*/ρ is a finite transformation semigroup on X by defining the action of $s\rho \in A^*/\rho$ on X as $x(s\rho) = xs$. The semigroup A^*/ρ is called the *characteristic semigroup* of $\mathcal{A}$. Let $\mathcal{V}$ be a class of semigroups not necessarily a variety. Then an automaton $\mathcal{A}$ is called a $\mathcal{V}$ -type if $A^*/\rho \in \mathcal{V}$.

For $s \in A^*$, let $\text{im } s = \{xs : x \in X\} = Xs$ and $\ker s = \{(x, y) \in X \times X : xs = ys\}$, which are called the image and the kernel of s , respectively. Then $\ker s$ is an equivalence on X .

Let $\mathcal{V}$ and $\mathcal{U}$ be two classes of semigroups. Then the direct product of $\mathcal{V}$ and $\mathcal{U}$ is defined by $\mathcal{V} \times \mathcal{U} = \{V \times U : V \in \mathcal{V}, U \in \mathcal{U}\}$. Let $U \in \mathcal{U}$ and let S be a semigroup. If for each $s \in V$, there exists $U_s \in \mathcal{U}$ such that $S = \sqcup \{U_s : s \in U\}$ and $U_s \cdot U_t = \{u_s u_t : v_s \in U_s, v_t \in U_t\} \subseteq U_{st}$, then we say that S belongs to $\mathcal{V}(\mathcal{U})$, where $\sqcup$ denotes a disjoint union..

As our start, we consider the following classes of semigroups: $\mathcal{G} = \{\text{groups}\}$, $\mathcal{LZ} = \{\text{left zero semigroups } [st = s]\}$, $\mathcal{RZ} = \{\text{right zero semigroups } [st = t]\}$ and $\mathcal{SL} = \{\text{semilattices } [st = ts, s^2 = s]\}$. We first characterize, for the classes $\mathcal{G}, \mathcal{LZ} \times \mathcal{G}, \mathcal{G} \times \mathcal{RZ}$ and $\mathcal{LZ} \times \mathcal{G} \times \mathcal{RZ}$, their types automata by the images and the kernels of their transition functions. By using the results, we characterize, for $\mathcal{V} \in \{\mathcal{SL}, \mathcal{LZ}, \mathcal{RZ}\}$ and $\mathcal{U} \in \{\mathcal{G}, \mathcal{LZ} \times \mathcal{G}, \mathcal{G} \times \mathcal{RZ}, \mathcal{LZ} \times \mathcal{G} \times \mathcal{RZ}\}$, $\mathcal{V}(\mathcal{U})$ -type automaton by the same way.

¹ This is an abstract and the details will be published elsewhere

For a example, we show later that an automaton $\mathcal{A}$ is a $SL(\mathcal{LZ} \times \mathcal{G})$ -type if and only if $\text{im } st = \text{im } s \cap \text{im } t$ for every $s, t \in A^+$.

2. Preliminaries

For a set Y , $|Y|$ denotes the cardinality of Y , for an equivalence λ , $x\lambda$ denotes the λ -class containing x and $|\lambda|$ denotes the number of λ -classes. i.e., $|\lambda| = |\{x\lambda : x \in X\}|$, and for $s \in A^+$, $|s|$ denotes the length of s . Then clearly $|\text{im } s| = |\ker s|$ for every $s \in A^+$.

For $s \in A^*$, let $\text{fix } s = \{x \in X : xs = x\}$. Let $E(A^+) = \{e \in A^+ : (e, e^2) \in \rho\}$. Then it is easy to see that $e \in E(A^+)$ if and only if $\text{im } e = \text{fix } e$. and that $(e, f) \in \rho$ for $e, f \in E(A^+)$ if and only if $\text{im } e = \text{im } f$ and $\ker e = \ker f$. Since A^+/ρ is finite, for every $s \in A^+$, there exists a positive integer m such that $s^m \in E(A^+)$.

Lemma 1. *Let $s, t \in A^+$. Then*

- (1) *If $|\ker st| = |\ker t|$, then $\text{im } st = \text{im } t$.*
- (2) *If $|\text{im } st| = |\text{im } s|$, then $\ker st = \ker s$.*

Lemma 2. *Let $s \in A^+$. Then the following are equivalent :*

- (1) *$\text{im } s \cap x\ker s \neq \emptyset$ for every $x \in X$.*
- (2) *$\text{im } s^m = \text{im } s$ and $\ker s^m = \ker s$ for every $m \in \mathbb{N}^+$.*
- (3) *There exists $e \in E(A^+)$ such that $\text{im } s = \text{im } e$, $\ker s = \ker e$, $(s, se) \in \rho$ and $(s, es) \in \rho$.*

Lemma 3. *The following are equivalent :*

For every $s, t \in A^+$,

- (1) *$\text{im } s \cap x\ker t \neq \emptyset$ for every $x \in X$,*
- (2) *$\text{im } st = \text{im } t$.*
- (3) *$\ker st = \ker s$.*

Let $\mathcal{A} = (X, A, \delta)$ be an automaton, and let $Y = \cup\{\text{im } a : a \in A\}$ and $\kappa = \cap\{\ker a : a \in A\}$. Then we have $Y = \cup\{\text{im } s; s \in A^+\}$ and $\kappa = \cap\{\ker s : s \in A^+\}$. In fact, if $s \in A^+$, then $s = s'a = bs''$ for some $a, b \in A, s's'' \in A^*$, so that $\text{im } s'a \subseteq \text{im } a$ and $\ker b \subseteq \ker bs''$.

Since $Ys \subseteq Y$ for every $s \in A^+$, the restriction s_Y of s to Y can be defined. Let $A_Y = \{a_Y : a \in A\}$. Then the automaton $\mathcal{A}_Y = (Y, A_Y, \delta)$ is called the *subautomaton of $\mathcal{A}$ with respect to Y* .

Let $s, t \in A^+$ and $x \in X$. Since $(xs)t_Y = (xs)t$, the action of st_Y on X is defined by $x(st_Y) = x(st)$.

Let κ be as above. Define the action of $s \in A^+$ on X/κ by $(x\kappa)s = (xs)\kappa$. Then the action is well-defined. In fact, if $x\kappa = y\kappa$, then $(x, y) \in \kappa \subseteq \ker s$, so that $xs = ys$. When the action of s is on X/κ , s is denoted by s_κ . Let $A_\kappa = \{a_\kappa : a \in A\}$. Then the automaton $\mathcal{A}_\kappa = (X/\kappa, A_\kappa, \delta)$ is called the *automaton induced from $\mathcal{A}$ by κ* .

Let $s, t \in A^+$ and $x \in X$. Then clearly $(x\ker s)s = xs$. Since $\kappa \subseteq \ker s$, we have $(x\kappa)s = xs$, so that $((x\kappa)s_\kappa)t = ((xs)\kappa)t = (xs)t$. Thus the action of $s_\kappa t$ on X is defined by $x(s_\kappa t) = x(st)$.

For an automaton $\mathcal{A} = (X, A, \delta)$, let $Im(A^+) = \{\text{im } s : s \in A^+\} = \{Y_i : i \in I\}$, i.e., for each $i \in I$, $Y_i = \text{im } s$ for some $s \in A^+$ and $\text{im } s \in Im(A^+)$ for every $s \in A^+$, and let $Ker(A^+) = \{\ker s : s \in A^+\} = \{\kappa_\mu : \mu \in M\}$, $Im(A_\kappa^+) = \{\text{im } s_\kappa : s \in A^+\} = \{Z_i : i \in I'\}$ and $Ker(A_\kappa^+) = \{\ker s_Y : s \in A^+\} = \{\kappa_\mu : \mu \in M'\}$. In this case, if $\text{im } s \cap x\ker s \neq \emptyset$ holds for every $s \in A^+$ and $x \in X$, then $Im(A^+) = Im(E(A^+))$ and $Ker(A^+) = Ker(E(A^+))$.

3. Main Results

A semigroup in $\mathcal{LZ} \times \mathcal{G}$ is called a left group whose class is denoted simply by $\mathcal{LG}$, i.e., $\mathcal{LG} = \mathcal{LZ} \times \mathcal{G}$.

Theorem 1. *Let $\mathcal{A} = (X, A, \delta)$ be an automaton. Then the following are equivalent :*

- (1) *There exists a subset Y of X such that $\text{im } a = Y$ and $Y \cap x\ker a \neq \emptyset$ for every $a \in A$ and $x \in X$.*
- (2) *There exists a subset Y of X such that $\text{im } s = Y$ for every $s \in S$.*
- (3) *$\mathcal{A}$ is a left group type.*

From Theorem 1 we obtain the following results

Corollary 1.1. *An automaton $\mathcal{A} = (X, A, \delta)$ is a $SL(\mathcal{LG})$ -type if and only if $\text{im } st = \text{im } s \cap \text{im } t$ for every $s, t \in A^+$.*

Corollary 1.2. *An automaton $\mathcal{A} = (X, A, \delta)$ is a $RZ(\mathcal{LG})$ -type if and only if $\text{im } st = \text{im } t$ for every $s, t \in A^+$.*

A semigroup in $\mathcal{G} \times \mathcal{RZ}$ is called a right group whose class is denoted by $\mathcal{RG}$, i.e., $\mathcal{RG} = \mathcal{G} \times \mathcal{RZ}$.

Theorem 2. *Let $\mathcal{A} = (X, A, \delta)$ be an automaton. Then the following are equivalent :*

- (1) *There exists an equivalence κ on X such that $\ker a = \kappa$ and $\text{im } a \cap x\kappa \neq \emptyset$ for every $a \in A$.*
- (2) *There exists an equivalence κ on X such that $\ker s = \kappa$ for every $s \in A^+$.*
- (3) *$\mathcal{A}$ is a right group type.*

Corollary 2.1. *An automaton $\mathcal{A} = (X, A, \delta)$ is a $SL(\mathcal{RG})$ -type if and only if $\ker st = \ker s \vee \ker t$ for every $s, t \in A^+$.*

Corollary 2.2. *An automaton $\mathcal{A} = (X, A, \delta)$ is a $\mathcal{LZ}(\mathcal{RG})$ -type if and only if $\text{im } st = \text{im } t$ for every $s, t \in A^+$.*

From Corollaries 1.2 and 2.2 we obtain :

Corollary 2.3. *An automaton $\mathcal{RZ}(\mathcal{LG})$ -type if and only if it is $\mathcal{LZ}(\mathcal{RG})$ -type.*

Remark. It can be easily show that $\mathcal{LZ}(\mathcal{LG}) = \mathcal{LZ}(\mathcal{G}) = \mathcal{LG}$ and $\mathcal{RZ}(\mathcal{RG}) = \mathcal{RZ}(\mathcal{G}) = \mathcal{RG}$.

Theorem 3. *Let $\mathcal{A} = (X, A, \delta)$ be an automaton. Then the following are equivalent :*

- (1) *There exist a subset Y of X and an equivalence κ on X such that $\text{im } a = Y$ and $\ker a = \kappa$ for every $a \in A$ and $Y \cap x\kappa \neq \emptyset$ for every $x \in X$.*
- (2) *There exist a subset Y of X and an equivalence κ on X such that $\text{im } s = Y$ and $\ker s = \kappa$ for every $s \in A^+$,*
- (3) *$\mathcal{A}$ is a group-type.*

A semigroup in $SL(\mathcal{G})$ is called a *Cliford semigroup*,

Corollary 3.1. *An automaton $\mathcal{A} = (X, A, \delta)$ is a Cliford smigroup type if and only if $\text{im } st = \text{im } s \cap \text{im } t$ and $\ker st = \ker s \vee \ker t$ for every $s, t \in A^+$.*

Theorem 4. *Let $\mathcal{A} = (X, A, \delta)$ be an automaton, and Let $Y = \cup\{\text{im } a : a \in A\}, \kappa = \cap\{\ker a : a \in A\}$. Suppose that $\text{im } s \cap x\ker s \neq \emptyset$ for every $s \in A^+, x \in X$. Then the following are equivalent:*

- (1) *$\mathcal{A}$ is a $\mathcal{LZ} \times \mathcal{G} \times \mathcal{RZ}$ -type.*
- (2) *$\ker s_Y = \ker t_Y$ for every $s, t \in A^+$.*
- (3) *$\text{im } s_\kappa = \text{im } t_\kappa$ for every $s, t \in A^+$.*

Corollary 4.1. *With the assumption of Theorem 4, the following are equivalent :*

- (1) *$\mathcal{A}$ is a $SL(\mathcal{LZ} \times \mathcal{G} \times \mathcal{RZ})$ -type.*

(2) $\ker s_Y t_Y = \ker s_Y \vee \ker t_Y$ for every $s, t \in A^+$.

(3) $\text{im } s_\kappa t_\kappa = \text{im } s_\kappa \cap \text{im } t_\kappa$ for every $s, t \in A^+$.

Suppose that an automaton $\mathcal{A}$ is a $\mathcal{LZ} \times \mathcal{G} \times \mathcal{RZ}$ -type. As is seen in the proof of Theorem 4, $A^+/\rho = \{(i, g, \mu) : i \in I, g \in G, \mu \in M\}$. For $i \in I$ and $\mu \in M$, let $A_i/\rho = \{(i, g, \mu) : g \in G, \mu \in M\}$ and $A_\mu = \{(i, g, \mu) : i \in I, g \in G\}$, respectively. Then $A_i/\rho \in \mathcal{RG}$ and $A_\mu/\rho \in \mathcal{LG}$. For $s\rho = (i, g, \nu)$, $t\rho = (j, h, \mu)$, since $(st)\rho = (i, gh, \mu)$, by Theorems 1 and 2, we have $\ker st = \ker s$ and $\text{im } st = \text{im } t$. Thus we obtain ;

Corollary 4.2. *If an automaton $\mathcal{A}$ is a $\mathcal{LZ} \times \mathcal{G} \times \mathcal{RZ}$ -type, then it is a $\mathcal{RZ}(\mathcal{LG})$ -type. The converse is not true,*

There is a simple example that a $\mathcal{RZ}(\mathcal{LG})$ -type automaton which is not a $\mathcal{LZ} \times \mathcal{G} \times \mathcal{RZ}$ -type.

References

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