

CONVOLUTION OF CERTAIN STARLIKE FUNCTIONS WITH NEGATIVE COEFFICIENTS

M. K. AOUF(Mansoura大学) and M. NUNOKAWA(布川 護, 群馬大学)

ABSTRACT. The object of the present paper is to determine several interesting results for the modified Hadamard product (convolution) of functions belonging to the class $T_{\lambda,n}^*(A,B)$ consisting of regular functions with negative coefficients.

KEY WORDS- Regular, Starlike, Ruscheweyh derivative, convolution.

AMS (1991) Subject Classification. 30C45.

1. Introduction

Let T denote the class of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \quad (a_k \geq 0) \quad (1.1)$$

which are regular in the unit disc $U = \{z: |z| < 1\}$. If $f(z)$ defined by (1.1) and $g(z)$ defined by

$$g(z) = z - \sum_{k=2}^{\infty} b_k z^k \quad (b_k \geq 0). \quad (1.2)$$

The modified Hadamard product (convolution) of $f(z)$ and $g(z)$ is defined by the power series

$$f * g(z) = z - \sum_{k=2}^{\infty} a_k b_k z^k. \quad (1.3)$$

The n -th order Ruscheweyh derivative of $f(z)$, denoted by $D^n f(z)$, is defined by

$$D^n f(z) = \frac{z(z^{n-1} f(z))^{(n)}}{n!}, \quad (1.4)$$

where $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{N} = \{1, 2, \dots\}$. Ruscheweyh [3] determined that

$$D^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z) \quad (n \in \mathbb{N}_0). \quad (1.5)$$

It is easy to see that

$$D^n f(z) = z - \sum_{k=2}^{\infty} \delta(n,k) a_k z^k, \quad (1.6)$$

where

$$\delta(n,k) = \begin{bmatrix} n+k-1 \\ n \end{bmatrix}. \quad (1.7)$$

A function $f(z) \in T$ is said to be in the class $T_{\lambda,n}^*(A,B)$ if it satisfies the following condition

$$\left| \frac{\frac{D^{n+1}f(z)}{D^n f(z)} - 1}{\lambda(A-B) - B \left[\frac{D^{n+1}f(z)}{D^n f(z)} - 1 \right]} \right| < 1, \quad z \in U, \quad (1.8)$$

where $0 < \lambda \leq 1$, $-1 \leq B < A \leq 1$, and $-1 \leq B \leq 0$. The class $T_{\lambda,n}^*(A,B)$ was studied by Kumar and Chaudhary [1].

In order to prove our results for functions belonging to the class $T_{\lambda,n}^*(A,B)$, we shall require the following lemma given by Kumar and Chaudhary [1].

LEMMA 1. Let the function $f(z)$ defined by (1.1). Then $f(z) \in T_{\lambda,n}^*(A,B)$ if and only if

$$\sum_{k=2}^{\infty} C_{k,\lambda} \delta(n,k) a_k \leq \lambda(A-B)(n+1), \quad (1.9)$$

where

$$C_{k,\lambda} = (1-B)(k-1) + \lambda(A-B)(n+1). \quad (1.10)$$

The result is sharp.

2. Modified Hadamard Product

THEOREM 1. Let the function $f(z)$ defined by (1.1) and $g(z)$ defined by (1.2) be in the class $T_{\lambda,n}^*(A,B)$. Then $f * g(z) \in T_{\mu,n}^*(A,B)$, where

$$\mu = \frac{\lambda^2(A-B)(1-B)}{C_{2,\lambda}^2 - \lambda^2(A-B)^2(n+1)}. \quad (2.1)$$

The result is sharp.

PROOF. Employing the technique used earlier by Schild and Silverman [2]. We need to find the largest μ such that

$$\sum_{k=2}^{\infty} \frac{C_{k,\mu} \delta(n,k)}{\mu(A-B)(n+1)} a_k b_k \leq 1. \quad (2.2)$$

Since

$$\sum_{k=2}^{\infty} \frac{C_{k,\lambda} \delta(n,k)}{\lambda(A-B)(n+1)} a_k \leq 1 \quad (2.3)$$

and

$$\sum_{k=2}^{\infty} \frac{C_{k,\lambda} \delta(n,k)}{\lambda(A-B)(n+1)} b_k \leq 1, \quad (2.4)$$

by the Cauchy-Schwarz inequality we have

$$\sum_{k=2}^{\infty} \frac{C_{k,\lambda} \delta(n,k)}{\lambda(A-B)(n+1)} \sqrt{a_k b_k} \leq 1. \quad (2.5)$$

Thus it is sufficient to show that

$$\frac{C_{k,\mu} \delta(n,k)}{\mu(A-B)(n+1)} a_k b_k \leq \frac{C_{k,\lambda} \delta(n,k)}{\lambda(A-B)(n+1)} \sqrt{a_k b_k} \quad (k \geq 2), \quad (2.6)$$

that is, that

$$\sqrt{a_k b_k} \leq \frac{\mu C_{k,\lambda}}{\lambda C_{k,\mu}}. \quad (2.7)$$

Note that

$$\sqrt{a_k b_k} \leq \frac{\lambda(A-B)(n+1)}{C_{k,\lambda} \delta(n,k)} \quad (k \geq 2). \quad (2.8)$$

Consequently, we need only to prove that

$$\frac{\lambda(A-B)(n+1)}{C_{k,\lambda} \delta(n,k)} \leq \frac{\mu C_{k,\lambda}}{\lambda C_{k,\mu}} \quad (k \geq 2). \quad (2.9)$$

or, equivalently that,

$$\mu \leq \frac{\lambda^2 (A-B) (1-B) (n+1) (k-1)}{C_{k,\lambda}^2 \delta(n,k) - \lambda^2 (A-B)^2 (n+1)^2} \quad (2.10)$$

Since

$$D(k) = \frac{\lambda^2 (A-B) (1-B) (n+1) (k-1)}{C_{k,\lambda}^2 \delta(n,k) - \lambda^2 (A-B)^2 (n+1)^2} \quad (2.11)$$

is an increasing function of k ($k \geq 2$), letting $k = 2$ in (2.11) we obtain

$$\mu \leq D(2) = \frac{\lambda^2 (A-B) (1-B)}{C_{2,\lambda}^2 - \lambda^2 (A-B)^2 (n+1)}, \quad (2.12)$$

which completes the proof of Theorem 1.

Finally, by taking the functions

$$f(z) = g(z) = z - \frac{\lambda(A-B)}{C_{2,\lambda}} z^2 \quad (2.13)$$

we can see that the result in Theorem 1 is sharp.

COROLLARY 1. For $f(z)$ and $g(z)$ as in Theorem 1, we have

$$h(z) = z - \sum_{k=2}^{\infty} \sqrt{a_k b_k} z^k \quad (2.14)$$

belongs to the class $T_{\lambda,n}^*(A,B)$.

The result follows from the inequality (2.5). It is sharp for the same functions as in Theorem 1.

THEOREM 2. Let the function $f(z)$ defined by (1.1) be in the class $T_{\lambda,n}^*(A,B)$ and the function $g(z)$ defined by (1.2) be in the class $T_{\beta,n}^*(A,B)$, then $f * g(z) \in T_{\zeta,n}^*(A,B)$, where

$$\zeta = \frac{\lambda\beta(A-B)(1-B)}{C_{2,\lambda}C_{2,\beta} - \lambda\beta(A-B)^2(n+1)}. \quad (2.15)$$

The result is sharp.

PROOF. Proceeding as in the proof of Theorem 1, we get

$$\zeta \leq E(k) = \frac{\lambda\beta(A-B)(1-B)(n+1)(k-1)}{C_{k,\lambda}C_{k,\beta}\delta(n,k) - \lambda\beta(A-B)^2(n+1)^2}. \quad (2.16)$$

Since the function $E(k)$ is an increasing function of k ($k \geq 2$), letting $k = 2$ in (2.16) we obtain

$$\zeta \leq E(2) = \frac{\lambda\beta(A-B)(1-B)}{C_{2,\lambda}C_{2,\beta} - \lambda\beta(A-B)^2(n+1)}, \quad (2.17)$$

which evidently proves Theorem 2.

Further, taking

$$f(z) = z - \frac{\lambda(A-B)}{C_{2,\lambda}} z^2 \quad (2.18)$$

and

$$g(z) = z - \frac{\beta(A-B)}{C_{2,\beta}} z^2 \quad (2.19)$$

we can show that the result of Theorem 2 is sharp.

COROLLARY 2. Let the functions $f(z)$, $g(z)$, $h(z)$ be in the class $T_{\lambda,n}^*(A,B)$, then $f * g * h(z) \in T_{\eta,n}^*(A,B)$, where

$$\eta = \frac{\lambda^3(A-B)^2(1-B)}{C_{2,\lambda}^3 - \lambda^3(A-B)^3(n+1)} \quad (2.20)$$

The result is best possible for the functions

$$f(z) = g(z) = h(z) = z - \frac{\lambda(A-B)}{C_{2,\lambda}} z^2. \quad (2.21)$$

PROOF. From Theorem 1, we have $f * g(z) \in T_{\mu,n}^*(A,B)$, where μ is given by (2.1). We use now Theorem 2, we get $f * g * h(z) \in T_{\eta,n}^*(A,B)$, where

$$\eta = \frac{\lambda\mu(A-B)(1-B)}{C_{2,\lambda}C_{2,\mu} - \lambda\mu(A-B)^2(n+1)} = \frac{\lambda^3(A-B)^2(1-B)}{C_{2,\lambda}^3 - \lambda^3(A-B)^3(n+1)}.$$

This completes the proof of Corollary 2.

THEOREM 3. Let the functions $f(z)$, $g(z)$ defined by (1.1) and (1.2), respectively, be in the class $T_{\lambda,n}^*(A,B)$. Then the function $h(z)$ defined by

$$h(z) = z - \sum_{k=2}^{\infty} (a_k^2 + b_k^2) z^k \quad (2.22)$$

belongs to the class $T_{\varphi,n}^*(A,B)$, where

$$\varphi = \frac{2\lambda^2(A-B)(1-B)}{C_{2,\lambda}^2 - 2\lambda^2(A-B)^2(n+1)}. \quad (2.23)$$

The result is sharp for the functions $f(z) = g(z)$ defined by (2.13).

PROOF. By virtue of Lemma 1, we obtain

$$\sum_{k=2}^{\infty} \left\{ \frac{C_{k,\lambda} \delta(n,k)}{\lambda(A-B)(n+1)} \right\}^2 a_k^2 \leq \left\{ \sum_{k=2}^{\infty} \frac{C_{k,\lambda} \delta(n,k)}{\lambda(A-B)(n+1)} a_k \right\}^2 \leq 1 \quad (2.24)$$

and

$$\sum_{k=2}^{\infty} \left\{ \frac{C_{k,\lambda} \delta(n,k)}{\lambda(A-B)(n+1)} \right\}^2 b_k^2 \leq \left\{ \sum_{k=2}^{\infty} \frac{C_{k,\lambda} \delta(n,k)}{\lambda(A-B)(n+1)} b_k \right\}^2 \leq 1. \quad (2.25)$$

It follows from (2.24) and (2.25) that

$$\sum_{k=2}^{\infty} \frac{1}{2} \left\{ \frac{C_{k,\lambda} \delta(n,k)}{\lambda(A-B)(n+1)} \right\}^2 (a_k^2 + b_k^2) \leq 1. \quad (2.26)$$

Therefore, we need to find the largest φ such that

$$\frac{C_{k,\varphi} \delta(n,k)}{\varphi(A-B)(n+1)} \leq \frac{1}{2} \left\{ \frac{C_{k,\lambda} \delta(n,k)}{\lambda(A-B)(n+1)} \right\}^2 \quad (k \geq 2),$$

that is, that

$$\varphi \leq \frac{2(k-1)\lambda^2(A-B)(1-B)(n+1)}{C_{k,\lambda}^2 \delta(n,k) - 2\lambda^2(A-B)^2(n+1)^2}. \quad (2.27)$$

Since

$$G(k) = \frac{2(k-1)\lambda^2(A-B)(1-B)(n+1)}{C_{k,\lambda}^2 \delta(n,k) - 2\lambda^2(A-B)^2(n+1)^2} \quad (2.28)$$

is an increasing function of k ($k \geq 2$), we readily have

$$\varphi \leq G(2) = \frac{2\lambda^2(A-B)(1-B)}{C_{2,\lambda}^2 - 2\lambda^2(A-B)^2(n+1)^2}, \quad (2.29)$$

which completes the proof of Theorem 3.

THEOREM 4. Let the function $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$ be in the class $T_{\lambda,n}^*(A,B)$ and $g(z) = z - \sum_{k=2}^{\infty} b_k z^k$ with $|b_k| \leq 1$, $k = 2, 3, \dots$, then $f * g(z) \in T_{\lambda,n}^*(A,B)$.

PROOF. Since

$$\begin{aligned} \sum_{k=2}^{\infty} C_{k,\lambda} \delta(n,k) |a_k b_k| &= \sum_{k=2}^{\infty} C_{k,\lambda} \delta(n,k) a_k |b_k| \\ &\leq \sum_{k=2}^{\infty} C_{k,\lambda} \delta(n,k) a_k \leq \lambda(A-B)(n+1) \end{aligned}$$

by Lemma 1 it follows that $f * g(z) \in T_{\lambda,n}^*(A,B)$.

COROLLARY 3. If $f(z) \in T_{\lambda,n}^*(A,B)$ and $g(z) = z - \sum_{k=2}^{\infty} b_k z^k$ with $0 \leq b_k \leq 1$, $k = 2, 3, \dots$, then $f * g(z) \in T_{\lambda,n}^*(A,B)$.

References

- [1] V. Kumar and A. M. Chaudhary, On a class of certain starlike functions with negative coefficients, Proc. Nat. Acad. Sci. India, 58 (A), II, (1988), 293-301.

- [2] A. Schild and H. Silverman, Convolution of univalent functions with negative coefficients, Ann. Univ. Mariae Curie-Sklodowska Sect. A, 29 (1975), 99-107.
- [3] St. Ruscheweyh, New criteria for univalent functions, Proc. Amer. Math. Soc. 49 (1975), 109-115.

M. K. Aouf

Department of Mathematics
Faculty of Science
University of Mansoura
Mansoura, Egypt.

M. Nunokawa

Faculty of Education
Gunma University
4-2, Aramaki-machi, Maebashi
Gunma 371, Japan.